# Exact completions and toposes

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Doctor of Philosophy
University of Edinburgh
2000

A mi mamá y a mi papá.

# Abstract

Toposes and quasi-toposes have been shown to be useful in mathematics, logic and computer science. Because of this, it is important to understand the different ways in which they can be constructed.

Realizability toposes and presheaf toposes are two important classes of toposes. All of the former and many of the latter arise by adding “good” quotients of equivalence relations to a simple category with finite limits. This construction is called the *exact completion* of the original category. Exact completions are not always toposes and it was not known, not even in the realizability and presheaf cases, when or why toposes arise in this way.

Exact completions can be obtained as the composition of two related constructions. The first one assigns to a category with finite limits, the “best” regular category (called its *regular completion*) that embeds it. The second assigns to a regular category the “best” exact category (called its *ex/reg completion*) that embeds it. These two constructions are of independent interest. There are quasi-toposes that arise as regular completions and toposes, such as those of sheaves on a locale, that arise as ex/reg completions but which are not exact completions.

We give a characterization of the categories with finite limits whose exact completions are toposes. This provides a very simple way of presenting realizability toposes, it allows us to give a very simple characterization of the presheaf toposes whose exact completions are themselves toposes and also to find new examples of toposes arising as exact completions.

We also characterize universal closure operators in exact completions in terms of topologies, in a way analogous to the case of presheaf toposes and Grothendieck topologies. We then identify two “extreme” topologies in our sense and give simple conditions which ensure that the regular completion of a category is the category of separated objects for one of these topologies. This connection allows us to derive good properties of regular completions such as local cartesian closure. This, in turn, is part of our study of when a regular completion is a quasi-topos.

The second extreme topology gives rise, as its category of sheaves, to the category of what we call *complete* equivalence relations. We then characterize the locally cartesian closed regular categories whose associated category of complete equivalence relations is a topos. Moreover, we observe that in this case the topos is nothing but the ex/reg completion of the original category.

# Acknowledgements

My supervisor Alex Simpson provided first the opportunity to come to Edinburgh by accepting me as a supervised student. He then spent, during these three years, a huge amount of time in the different aspects of my supervision. His clever insights and ideas have been fundamental for the research reported in this thesis.

Gabriel Baum and Gustavo Rossi (Laboratorio de Investigación y Formación en Informática Avanzada (LIFIA), Universidad Nacional de La Plata, Argentina) made possible my coming to Edinburgh. Gabriel Baum also acted as my supervisor in Argentina and his help was essential for my stay in Edinburgh.

Adriana Compagnoni, Daniele Zardo, Martín Escardó and Healfdene Goguen took care of me when I arrived and we have spent many enjoyable moments together. Except for Daniele, I met them at the Laboratory for Foundations of Computer Science (LFCS) where the following people were also particularly kind and helpful: Samson Abramsky (who was my second supervisor), Margaret Davis, José Espirito Santo, Sibylle Froeschle, Eleanor Kerse, John Longley, Gordon Plotkin, John Power and Daniele Turi. And I would like to mention Dilsun Kirli especially.

Near the LFCS, the librarians were always cheerful and effective.

I would also like to thank Angus Macintyre for the many hours that he spent showing me beautiful pieces of mathematics. His depth and breadth of knowledge and his teaching vocation greatly enriched my stay in Scotland.

For different periods between October 1997 and December 2000 I was funded by the following institutions: Ministerio de Cultura y Educación de la Nación (Argentina), Departamento de Informática de la Universidad Nacional de La Plata, Fundación Antorchas (Argentina)/British Council and Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET, Argentina). During the three years an important part of the university fees was covered by an Overseas Research Student (ORS) scholarship from the Committee of Vice-chancellors and Principals of the Universities of the United Kingdom (CVCP). The support of all these institutions was essential for my studies.

Mis padres han sido siempre los más afectuosos, inteligentes y lindos. Y mis amigas y amigos de verdad también. El contacto con ellos y su ayuda siempre han sido preciosos.

# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Matías Menni)

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# Chapter 1

# Introduction

A category with finite limits is called *exact* if, intuitively, every equivalence relation in it has a "good" quotient. For any category with finite limits $\mathbf{C}$ it is possible to build the "best" exact category that embeds it. This category (unique up to equivalence) is called the *exact completion* of $\mathbf{C}$ and it is denoted by $\mathbf{C}_{ex}$.

Many categories of interest to mathematics, logic and computer science arise in this way. This is a very nice conceptual fact, but there is also something intriguing about it, because some of these categories have a lot more structure than "just" good quotients of equivalence relations.

We want to understand how this happens in interesting situations in practice. Indeed, a key fact that motivates much of this thesis is that there are *toposes* that arise as exact completions. Toposes have a lot of structure and are particularly useful and interesting so the problem of characterizing those $\mathbf{C}$ such that $\mathbf{C}_{ex}$ is a topos presents itself naturally. One of the achievements of this thesis is a solution to this problem.

We will discuss in more detail the motivation and contents of the thesis after the historical perspective below (Section 1.1) where I believe they can be better understood.

## 1.1 History

The history of topos theory and of regular and exact categories will not be discussed here. But in order to enter into the mood of this section let us mention some early references. For topos theory see [4, 56, 57] and for regular and exact categories see [6].

The contents of the section are divided in two. The first part is a chronological perspective on the events, results and problems that concern us in this thesis. We start with the conception of realizability toposes and then we emphasize the work intended to understand them in terms of universal constructions. Scattered along the chronology, at the times when they were recognized or proved, several problems and results are mentioned. The results motivate further problems and all these problems are then summarized in the second part of the section. Providing solutions to these problems is the motivation for the thesis.

The reader may then find the first part of the section slightly disjointed after a first reading. But the summary of open problems will have a historical basis which will make them easier to understand. With the motivation of the thesis then understood, a second reading of the chronological perspective will be a lot clearer.

In their 1979 paper [27], Fourman and Scott presented the topos of sheaves on a locale as a category of non-standard equivalence relations. The essential idea behind this presentation is attributed to Higgs and referenced to a 1973 preprint [34]. I have been unable to obtain a copy of this reference but it seems that it can be considered the starting point of the research reported here.

This construction inspired that of *realizability toposes*. Indeed, in his 1982 paper [39], Hyland attributes to Powell and Scott the idea of looking at realizability in a model theoretic way and uses this idea to explain the construction of the *effective topos* **Eff** in a way resembling Higgs' construction. In this paper, it is shown that the category **Set** of sets arises as the full subcategory of sheaves for the double negation topology. Moreover, the full subcategory of $\neg\neg$-separated objects (a *quasi-topos*) is given a very explicit and simple presentation. Nowadays, this quasi-topos is denoted by **Ass** and its objects are called *assemblies*.

These constructions by Higgs and Hyland motivated the invention of *tripos theory*. In 1980-81, tripos theory was presented in Pitts' thesis [94] and in his joint paper with Hyland and Johnstone [42]. Using tripos theory it is possible to treat uniformly the construction of toposes of sheaves on a locale and the construction of realizability toposes. For any category **C** with enough structure and equipped with a tripos, there is a functor (called the *constant objects* functor in [94]) that embeds **C** into the topos induced by the tripos. In the case of **Eff**, this embedding is just the embedding of $\neg\neg$-sheaves. On the other hand, in the case of sheaves on a locale, this functor is the left adjoint to the global sections functor.

Since then, realizability toposes have found diverse applications in logic and computer science, especially in the semantics of logics and programming languages [40, 43, 100, 92, 66, 85, 84, 88, 68, 67, 65, 11, 12].

In 1981, the *regular completion* and *exact completion* constructions for a category with finite limits appeared in C. Magno's thesis [72] (see also [19] and [16]).

A few years later, in their 1988 paper [17], Carboni, Freyd and Scedrov showed that **Eff** is the solution to a universal problem. This is a very nice and important conceptual fact so let us discuss it in more detail. As Carboni writes in [16], the idea pursued in the paper loc. cit. is that, "to understand a quite problematic construction like that of the effective topos, one should look for some universal property that the construction may enjoy". The universal problem they work with is that of adding good quotients of equivalence relations to a *regular* category. This is a different construction from the exact completion of a category with finite limits, so let us call it the *ex/reg completion*. Ex/reg completions had been known for a long time as they appear already in Lawvere's 1973 Peruggia notes [54], but they seem to have appeared in print for the first time in [17] where it is shown that the effective topos is the ex/reg completion of the (much simpler) category of assemblies (see also [32] 2.169). That is, **Eff** is the ex/reg completion of its full subcategory of separated objects for the double negation topology.

In [32] (see 2.16(12) and 2.227) it is also shown that the topos of sheaves on a locale is also the ex/reg completion of a simpler category. This is, of course, the essence of the presentation in [27].

These nice conceptual facts already provide an alternative presentation of the toposes involved. Let us concentrate on **Eff**. The idea is to introduce it by first presenting the category of assemblies, then build its ex/reg completion and finally show that it is a topos.

But the fact that **Eff** is an ex/reg completion also suggests the possibility of an even simpler presentation. If we could find a useful sufficient condition on a regular category for its ex/reg completion to be a topos, this would allow us to show that **Eff** (the ex/reg completion of **Ass**) is a topos without the need to build the completion and calculate with it. That is, we would only need to check that the sufficient condition is satisfied in the category of assemblies which is a lot simpler to present and understand. Of course, such sufficient conditions could also ease the task of finding new examples and counter-examples. In any case, such a sufficient condition was not looked for in [17, 32]. Related to this, it should also be mentioned that in 1995, McLarty presented in [75] necessary and sufficient conditions for an ex/reg completion to have power objects. But in order to check these conditions in practice one still has to build the ex/reg completion. So it was not possible to present **Eff** by checking some simple conditions on the category of assemblies.

In their 1990 paper [96], Robinson and Rosolini attribute to Joyal, Carboni and Magno the characterization of the exact categories arising as exact completions of a category with finite limits. They use this result to show that **Eff** is the exact completion of the category of *partitioned assemblies*. Again, as in the case of ex/reg completions, this result suggested that it should be possible to present **Eff** by checking some, hopefully simple, conditions on a simple category. But, as before, this was not achieved. Another fact should be also mentioned: the axiom of choice was used to prove that **Eff** is an exact completion.

In 1994, Longley introduced in his thesis [66] a natural notion of morphism between partial combinatory algebras. In order to relate these to morphisms between the associated categories of assemblies he used the embedding of **Set** (recall the constant objects functor) in a curious way. First, he defined using this embedding the notion of a *cartesian map* in a category of assemblies. Intuitively, this map is like a regular mono but it need not be injective. Then, he shows that there exists an object $D$ (which he called *generic*) such that for every assembly $X$, there exists a cartesian map $X \longrightarrow D$. This is an interesting and peculiar property, but it seems not to have prompted further study.

In 1995, Carboni collected the main results on regular, exact, ex/reg completions and their relation to **Eff** in his survey paper [16]. In particular, he showed that the category of assemblies is the regular completion of the category of partitioned assemblies. In this paper Carboni also noted that the exact completion of a topos is not a topos in general and he asked for a characterization of the toposes whose exact completions are again toposes.

Here it should be mentioned Lawvere's 1996 paper [62] where the *proof-theoretic power set functor* was introduced. The connection with regular and exact completions is not mentioned loc. cit. but we can briefly state it here. For any object $X$ in a category with finite limits $\mathbf{C}$, the proof-theoretic power set functor applied to $X$ can be described as the poset of subobjects of $X$ in the regular or exact completion of $\mathbf{C}$.

In his 1997 paper [84] on extensional realizability, van Oosten presented the topos $\mathcal{A}$. He builds the topos using tripos theory and shows, using the same idea used by Robinson and Rosolini, that $\mathcal{A}$ is the exact completion of the category of assemblies.

Also in 1997, D. Scott distributed a manuscript presenting the category of *equilogical spaces* (see [102] and [8], see also Section 2.3.5 where the original terminology is slightly modified). Two of its main virtues are that it embeds the category **Top** of topological spaces and that it is locally cartesian closed (it is actually a quasi-topos).

In 1998, Carboni and Rosolini discovered a characterization of the categories with finite limits whose exact completions are locally cartesian closed (see [21]). This result can be used to show that the exact completion of **Top** (and also the exact completion of the full subcategory of $T_0$ topological spaces) is locally cartesian closed (see also [98]). Moreover, in their joint paper [13] with Birkedal and Scott, they used this result to show that **Equ** is locally cartesian closed by presenting it as the regular completion of ($T_0$) topological spaces.

Also in 1998, Rosolini showed that **Equ** also arises as the category of $\neg\neg$-separated objects of the exact completion of the category of $T_0$ topological spaces (see [101]).

In 1999, Longley introduced a typed version of the notion of a partial combinatory algebra in [68] and described how to build a category of assemblies $\mathbf{Ass}(\mathbf{A})$ over a such a structure $\mathbf{A}$. Shortly after, Lietz and Streicher showed that the ex/reg completion of $\mathbf{Ass}(\mathbf{A})$ is a topos if and only if the typed structure $\mathbf{A}$ is equivalent, in a suitable sense, to an untyped structure. Their proof uses the notion of a *generic mono* (a mono $\tau$ such that every other mono arises as a pullback of $\tau$ along a not necessarily unique map) and of the constant-objects embedding of **Set** into the category $\mathbf{Ass}(\mathbf{A})$ which they see as an inclusion of *codiscrete* objects. Related to this, it should be mentioned that Lawvere had already advocated for a conceptual use of codiscrete or *chaotic* objects in other areas of mathematics (see for example [59, 55, 61, 63]).

It was very nice to contemplate these very different classes of categories arising as solutions to the universal problems of finding regular, exact and ex/reg completions. But it was also clear that the phenomenon was not completely understood. Indeed, let us summarize some of the open problems indicated above.

1. Many toposes arise as the exact completion of a category with finite limits. For example, many presheaf toposes, realizability toposes and also variants of van Oosten's topos $\mathcal{A}$. On the other hand, a characterization of the categories with finite limits that give rise to toposes was not available. Such a characterization would not only ease the presentation of some of these toposes. It should also be useful, for example, to answer Carboni's question on toposes whose exact completions are themselves toposes and also to find new examples and counterexamples.

2. Similar problems relate quasi-toposes and regular completions on the one hand and toposes and ex/reg completions on the other. The first problem is illustrated by the categories of assemblies and of equilogical spaces. The second is suggested by realizability toposes (as ex/reg completions of as-

semblies) and also by toposes of sheaves over a complete Heyting algebra (frame).

3. Many of the examples we have mentioned share a number of curious properties: chaotic objects, generic objects and generic monos. These should be related by the fact that the underlying category gives rise to a topos through a completion process.

4. Assemblies and equilogical spaces are examples of regular completions. But they also arise as categories of $\neg\neg$-separated objects of the associated exact completions. So it is natural to wonder what is the relation between categories of separated objects and regular completions. But more generally we could ask for a treatment of universal closure operators in exact completions in the spirit of the analogous case for presheaf toposes in terms of Grothendieck topologies.

The main motivation of the thesis is to provide a clearer picture of these phenomena and questions.

## 1.2 Overview of the contents

Chapters 2 and 3 are mainly a review of the basic material and examples. In Chapter 2 we review the definitions of regular, exact, lextensive and related classes of categories and introduce the examples that will be used in the rest of the thesis. In Chapter 3 we present the constructions of coproduct, regular, exact and ex/reg completions and state their main properties.

The first original work is presented in Chapter 4 where we give sufficient conditions on a category with finite limits for its regular completion to be a quasi-topos. As an application we observe that it is possible to iterate regular completions to obtain hierarchies of quasi-toposes. These hierarchies will appear later in the thesis giving rise to related hierarchies of toposes.

In Chapter 5 we give a characterization of the categories with finite limits whose exact completions are toposes. The key notion here is that of a *generic proof*. Most of the chapter is devoted to the proof of the characterization, but there are also a couple of related results. One of these results involves the relation of such categories with their regular completions. This relation is expressed in terms of a connection between generic monos and generic proofs. Finally, there is a discussion of the relation of the characterization with work relating set theory and type theory [2, 33].

Chapters 6 and 8 can be seen as applications of the characterization, while Chapter 7 introduces the conceptual treatment of chaotic objects and proves some technical results needed in Chapter 8. Let us discuss their contents in a bit more detail.

In [16], it is observed that the exact completion of a topos is not always a topos. Our characterization in Chapter 5 gives a characterization of the toposes for which this is the case. But for a restricted class of toposes we can give a very concrete answer. This is the content of Chapter 6 where we characterize the presheaf toposes whose exact completions are toposes. We also discuss briefly the connection of one of these toposes with Läuchli's realizability. Finally, the characterization proved in this chapter also allows us to find other examples of toposes whose exact completions are toposes.

In [59, 61] it is explained how some categories have objects that can naturally be seen as having "as much structure as possible" or as being "chaotic". Moreover, it is shown how to axiomatize such a situation. In Chapter 7 we accommodate these results to our setting, explain how they arise in some of our examples and set up the machinery needed for the results in the following chapter.

In Chapter 8 we show how the existence of chaotic objects can be used to simplify the characterization of Chapter 5. This is done by exposing a strong relation between chaotic objects, generic objects, generic monos and generic proofs. In turn, this provides a very simple way of introducing realizability toposes. Moreover, it allows us to recognize hierarchies of new examples of toposes that are exact completions. These hierarchies are related to the hierarchies of quasi-toposes discussed in Chapter 4. Finally we discuss the relation with the work in [65] on the inevitability of untypedness for realizability toposes and also show how the existence of chaotic objects simplify the sufficient conditions for regular completions to be quasi-toposes discussed in Chapter 4.

Chapters 9 and 10 deal with universal closure operators in regular and exact completions. In the case of toposes, universal closure operators coincide with Lawvere-Tierney topologies which, in turn, coincide with subtoposes of the given topos. Moreover, in the case of the topos of presheaves on a small category $\mathbf{C}$, universal closure operators coincide with Grothendieck topologies on $\mathbf{C}$.

In Chapter 9 we obtain a similar characterization of universal closure operators in the regular and exact completions of a category with finite limits $\mathbf{C}$. Indeed, we show that they coincide with certain "topologies" on $\mathbf{C}$ that look very similar to Grothendieck topologies... but not quite.

In Chapter 10 we continue our study of topologies. We review the notion of sheaf and of separated object and give an explicit description of the category of separated objects for a universal closure operator in an exact completion. We then concentrate on two "extreme" topologies. On the one hand we identify the largest topology in a category $\mathbf{C}$ for which every $\mathbf{C}$-object is separated in the exact completion. We call it the *sep-canonical* topology. We then show that under mild conditions on $\mathbf{C}$, the category of separated objects (in the exact completion of $\mathbf{C}$) for this topology is equivalent to the regular completion of $\mathbf{C}$.

The second extreme topology is the largest one that makes every $\mathbf{C}$-object a sheaf. Naturally, we call it *canonical*. We are also able to characterize the category of sheaves for this topology leading to the notion of a *complete* equivalence relation. Complete equivalence relations appear already in the seminal work of Higgs, Fourman-Scott and Hyland-Johnstone-Pitts, but not under this perspective. We will see that this perspective sheds light on the question of when is the ex/reg completion of a regular category a topos.

In Chapter 11 we study locally cartesian closed regular categories $\mathbf{D}$ with a generic mono. In particular, we study the ex/reg completion of such a category. First we show that this completion actually coincides with the category of sheaves in $\mathbf{D}_{ex}$ for the canonical topology on $\mathbf{D}$. We use this to show that the ex/reg completion of $\mathbf{D}$ is locally cartesian closed and finally show that it is also a topos. As a corollary, we obtain a characterization of the locally cartesian closed regular categories $\mathbf{D}$ whose associated category $\mathbf{Ceq}(\mathbf{D})$ of complete equivalence relations is a topos. We also discuss the relation of our work with tripos theory.

The last chapter summarizes the results and some of the problems left open.

The contents of the thesis also include many diagrams. For these, I acknowledge the use of Paul Taylor's useful diagrams and proof-tree packages.

## 1.3 Prerequisites

The basic examples use some recursion theory [97], topology [49] and the theory of locales [47]. Almost all the category theory involved can be found in Chapters I to V of [70]. As a convention, when we speak of a *category* (unqualified) we mean a *locally small* category. We may also omit to say that a category $\mathbf{C}$ *has finite limits* if the context makes this clear.

On the other hand, we refer the reader to [104] for the notion of a *bi-adjoint* and to [31] for the definition and results on *factorization systems*. For topos theory, Chapters I to VII of [71] should suffice.

# Chapter 2

# Regular, exact and lextensive categories

In this chapter we present the main examples and use them to motivate the axioms for regular, exact and lextensive categories.

## 2.1 The raw material

In this section we introduce the most basic building blocks of our examples.

### 2.1.1 Partial combinatory algebras

We introduce partial combinatory algebras and review some basic facts about them. For more details on the material in this section see [66] or [9].

**Definition 2.1.1.** A *partial applicative structure* $\mathbf{A} = (A, \cdot)$ is a set $A$ equipped with a partial binary operation $(a, a') \mapsto a \cdot a'$.

The intuition is to think of $A$ as an untyped universe of programs or of names of functions and the operation $\cdot$ as function application. Because of the lack of types it is reasonable to apply any element to any other (even to itself) and expect a value. As $\cdot$ is a partial operation the value may not exist. In calculations involving partial applicative structures we will use terms or expressions involving variables and due to the partiality of $\cdot$ these expressions may not be defined. We write $a = a'$ to mean "$a$ and $a'$ are both defined and are equal" and we write $a \simeq a'$ to mean "if either $a$ or $a'$ is defined, so is the other and then they are equal". We will omit the $\cdot$ from now on and write the operation as juxtaposition, so $a \cdot a'$ will be written $aa'$ and assumed to be left associative so that $aa_1(a_0a_1)$ is to be understood as $(aa_1)(a_0a_1)$.

**Definition 2.1.2.** A *partial combinatory algebra* (PCA) is a partial applicative structure $\mathbf{A}$ such that there exist elements $k$ and $s$ in $\mathbf{A}$ satisfying the following axioms (for every $a, a_0, a_1$ in $\mathbf{A}$).

1. $kaa_0 = a$
2. $saa_0$ is defined
3. $saa_0a_1 \simeq aa_1(a_0a_1)$

An important example of a PCA is that of the natural numbers equipped with *Kleene application*: assume some enumeration of the partial recursive functions and for any natural number $m$ let $\{m\}$ be the partial recursive function coded by $m$. Then define $aa' \simeq \{a\}a'$. We call this PCA "*Kleene's* PCA" and we denote it by $K_1$.

There are many other examples of PCAs built using the theory of the lambda calculus, domain theory and, more recently, game theory (see [92, 66, 1] for example).

It is possible to interpret a simple lambda calculus in any PCA and it is also possible to code easily data-types such as pairs or booleans. Because of this, we will write for example $\lambda\langle a, a'\rangle.fa$ and work with this expression as if it were an element of the algebra that, when applied to any element of the algebra, interprets this element as a pair, takes its first projection and applies $f$ to it. For details of these encodings and their behaviour see for example [105] or [66].

We are going to use PCAs to build categories of so called *assemblies* so we will not attempt to define a category of PCAs (although, see Section 8.1). But just to say a word about them, it is not obvious what a morphism of PCAs is and the interested reader should consult [66].

### 2.1.2 Frames

Recall that a Heyting algebra $H$ is a lattice such that for each pair of elements $a, b \in H$ there exists an element $a \to b$ such that $c \leq a \to b$ if and only if $c \wedge a \leq b$.

On the other hand, consider the notion of a *frame*.

**Definition 2.1.3.** A *frame* is a partially ordered set $H$ with finite meets, arbitrary joins and such that the following distributivity law holds for every $x$ in $H$ and $Y \subseteq H$.

$$x \wedge \bigvee Y = \bigvee\{x \wedge y | y \in Y\}$$

As explained in [47] Ch.II, frames are exactly the complete Heyting algebras. However, intuitions from topology [49] suggest maps between frames that do not preserve implication (and so, are not Heyting algebra homomorphisms). Because of this, it is useful to introduce this extra terminology.

Let **Frm** be the category whose objects are frames and whose morphisms are functions preserving finite meets and arbitrary joins. (The morphisms are not required to preserve implication.)

## 2.2 Lextensive categories

At some points in the thesis we will deal with coproducts so it is important to know what a well behaved coproduct is. For any two objects $X$ and $Y$ we denote their coproduct by $X+Y$ and the injections by $in_0 : X \longrightarrow X+Y$ and $in_1 : Y \longrightarrow X+Y$.

**Definition 2.2.1.** In a category with finite coproducts.

1. Coproducts are said to be *disjoint* if for every pair of objects $X, Y$, the injections $in_0$ and $in_1$ are mono, their pullback exists and it is the initial object.

$$\begin{tikzcd} 0 \arrow[r, "!"] \arrow[d, "!"'] & Y \arrow[d, "in_1"] \\ X \arrow[r, "in_0"'] & X+Y \end{tikzcd}$$

2. Coproducts are said to be *stable* if pullbacks of injections exist and for any coproduct diagram

$$X \xrightarrow{in_0} X+Y \xleftarrow{in_1} Y$$

pulling back along any morphism to $X+Y$ gives a coproduct diagram.

We can now introduce the notion of a category with finite limits and well behaved coproducts [18].

**Definition 2.2.2.** A category is *lextensive* if it has finite limits and stable disjoint finite coproducts.

The following two subsections introduce two important examples of lextensive categories.

### 2.2.1 Topological spaces

Topological spaces are the subject matter of *topology* [49]. Here we will only recall the most basic definitions and describe some of the "imperfections" that the category of topological spaces has.

**Definition 2.2.3.** A *topology* on a set $S$ is a collection of subsets of $S$ closed under finite intersections and arbitrary unions. In particular, the empty subset and the whole of $S$ must be in the topology.

A *topological space* $X$ is a set $|X|$ equipped with a topology. The elements of the topology of $X$ are called its *open* subsets. Also, an open subset $U$ such that $x \in U$ is called an *open neighbourhood* of $x$.

We say that a function $f : |Y| \longrightarrow |X|$ is *continuous* if for every open $U$ in $X$, $f^*U = \{y | fy = x \in U\}$ is open in $Y$.

Topological spaces and continuous functions form a category that we call **Top**. This category is lextensive, complete, cocomplete and has stable epi/regular-mono factorizations. This last property will be shared with many of the categories that we will use in the thesis.

Now recall that an object $X$ in a category with products is called *exponentiable* if the functor $X \times (\_)$ has a right adjoint $(\_)^X$. Recall also that a category is *cartesian closed* if every object is exponentiable. In this case we may also say that the category *has exponentials*.

Regular epis in **Top** are not preserved by all functors $X \times (\_)$ and hence **Top** is not cartesian closed ([44] and references therein) and regular epis are not stable under pullback [25] (recall that an epi map is called *regular* if it is the coequalizer of some pair of maps). These "imperfections" motivated research in order to find better categories of spaces (see for example [24, 70, 38, 46, 59] and more recently [102, 8, 21, 77, 78]).

There is an obvious forgetful functor $|\_| : \mathbf{Top} \longrightarrow \mathbf{Set}$ that assigns to each topological space its underlying set. This functor has both a left adjoint $\Delta$ and a right adjoint $\nabla$, both of which are full and faithful.

The functor $\Delta : \mathbf{Set} \longrightarrow \mathbf{Top}$ assigns to each set $S$ the "discrete" topological space with underlying set $S$ and such that every subset of $S$ is open.

On the other hand, the functor $\nabla : \mathbf{Set} \longrightarrow \mathbf{Top}$ assigns to each set $S$ the "chaotic" topological space with underlying set $S$ and, as open sets, only $S$ itself and the empty set.

Discrete and chaotic objects can be dealt with abstractly as explained in [59, 61]. We will discuss this in more detail in Chapter 7.

### 2.2.2 Partitioned assemblies

In this section we present the category of *partitioned assemblies* for a partial combinatory algebra $\mathbf{A}$. For historical and practical reasons they are very important examples in the thesis. Recall from Section 1.1 that realizability toposes are the exact completions of these categories [96]. Indeed, loosely speaking, these categories have as little structure as possible in order to give rise to a topos, so they motivate many of the more obscure conditions in the results that appear in the thesis.

**Definition 2.2.4.** A *partitioned assembly* is a pair $X = (|X|, \|\_\|_X)$ consisting of a set $|X|$ and a function $\|\_\|_X : |X| \to \mathbf{A}$. We usually omit subscripts.

A *morphism* $f : Y \to X$ *of partitioned assemblies* is a function $f : |Y| \to |X|$ such that there exists an $a \in \mathbf{A}$ such that for every $y \in |Y|, a\|y\|$ is defined and $a\|y\| = \|fy\|$.

In this way partitioned assemblies form a lextensive category $\mathbf{PAss}$ (or, more explicitely, $\mathbf{PAss}(\mathbf{A})$ if we want to be specific about which particular PCA is being used). The construction of finite products and coproducts is left as an easy exercise involving the coding of pairs and booleans mentioned in Section 2.1.1. The equalizer $e : E \longrightarrow Y$ of two maps $f, g : Y \longrightarrow X$ is given by $|E| = \{y | fy = gy\}$ with $\|y\|_E = \|y\|_Y$. Then, of course, the regular monos $E' \rightarrowtail Y$ are those monos such that $E'$ is isomorphic over $Y$ to some $E$ as above.

It is also easy to check that $\mathbf{PAss}$ has stable epi/regular-mono factorizations.

On the other hand, $\mathbf{PAss}$ is not cartesian closed, does not have all coequalizers and does not have arbitrary products or coproducts.

Again, there is an obvious faithful functor $|\_| : \mathbf{PAss} \longrightarrow \mathbf{Set}$ which sends every object to its underlying set. As in the case of topological spaces this "underlying set" functor has a full and faithful right adjoint $\nabla : \mathbf{Set} \longrightarrow \mathbf{PAss}$ that assigns to each set the associated "chaotic" partitioned assembly. In order to define this right adjoint, first choose some element $*$ in the underlying PCA. Then define for each set $S$ the partitioned assembly $\nabla S$ that has underlying set $S$ and such that $\|\_\|_{\nabla S}$ assigns the constant $*$ to each element of $S$.

In contrast with topological spaces, the "underlying set" functor does not have a left adjoint. Nevertheless, the functor $|\_|$ does preserve finite limits.

## 2.3 Regular categories

In this section we introduce *regular* categories [6, 15, 32, 75]. The intuition behind these categories is that a good class of quotients exists and moreover, these quotients are well behaved. Recall that the *kernel pair* of a map $f$ is the (parallel) pair of maps that form the pullback of $f$ along itself.

**Definition 2.3.1.** A category with finite limits is *regular* if

1. every kernel pair has a coequalizer
2. pullbacks of regular epis are regular epis.

Let us try to motivate these conditions by showing examples of how they can fail. The category **Top** of topological spaces has all colimits so, in particular, every kernel pair has a coequalizer. On the other hand, it is well known [25] that regular epis in **Top** are not stable under pullback.

Consider now the category $\mathbf{PAss}(K_1)$ of partitioned assemblies for Kleene's PCA. We are going to show that not every kernel pair has a coequalizer. First, let $R$ be the set of recursive functions. Then let $X$ be the partitioned assembly such that $|X| = \{(f,a) | f : \mathbb{N} \longrightarrow \mathbb{N} \in R \text{ and } a \in K_1 \text{ is a code for } f\}$ and such that $\|(f,a)\| = a$. Finally, consider the obvious map $X \longrightarrow \nabla R$ sending $(f,a)$ to $f$. It is an easy exercise in recursion theory to prove that the kernel pair of this map can not have a coequalizer in **PAss**.

So neither **Top** nor the categories of partitioned assemblies in general are regular. On the other hand, let us now discuss some good properties that regular categories have.

**Definition 2.3.2.** A diagram

$$X' \underset{e_1}{\overset{e_0}{\rightrightarrows}} X \overset{e}{\twoheadrightarrow} X''$$

is called an *exact sequence* if it is both a pullback and a coequalizer. That is, if it is a coequalizer diagram and $e_0, e_1$ is the kernel pair of $e$.

We now present as a lemma, some well known facts about regular categories whose proofs can be found in the references mentioned above. As usual, we denote by $\alpha^*$ the operation of pulling back along the map $\alpha$.

**Lemma 2.3.3.** *In a regular category,*

1. *every map $f : Y \to X$ factors as a regular epi followed by a mono. This factorization is denoted by $X \twoheadrightarrow Im(f) \rightarrowtail Y$*

2. *if $f$ factors through $g$ then $Im(f) \leq Im(g)$ as subobjects of $X$*

3. *exact sequences are stable under pullback*

4. *given a commutative diagram as below such that both rows are exact and the two left hand squares are pullbacks (so that $e_0^* f \cong e_1^* f$):*

$$
\begin{tikzcd}
Y' \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "f'"'] & Y \arrow[r, two heads, "d"] \arrow[d, "f"'] & Y'' \arrow[d, "f''"'] \\
X' \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & X \arrow[r, two heads, "e"] & X''
\end{tikzcd}
$$

*then the right hand square is a pullback.*

The existence of regular-epi/mono factorizations has a very nice implication that we now discuss. First, for any category with finite limits $\mathbf{C}$, we denote by $Sub$ the contravariant functor that assigns, to each object in $\mathbf{C}$, the class of its subobjects. The action of $Sub$ on maps is by pullback so that for any $f : Y \longrightarrow X$ in $\mathbf{C}$, $Sub(f) = f^* : Sub(X) \longrightarrow Sub(Y)$. Now, for any $f : Y \longrightarrow X$ and subobject $m : U \rightarrowtail Y$ we can define $\exists_f(U) = Im(f.m)$. This induces a map $\exists_f : Sub(Y) \longrightarrow Sub(X)$ with the following properties [6, 32].

**Lemma 2.3.4.** *Let $f : Y \longrightarrow X$.*

1. *$\exists_f : Sub(Y) \longrightarrow Sub(X)$ is monotone and left adjoint to $f^*$*

2. *for any $g : Z \longrightarrow Y$, $\exists_f.\exists_g = \exists_{(f.g)}$*

3. *if $f$ is a regular epi then the adjunction $\exists_f \dashv f^*$ is actually a reflection and so $\exists_f.f^* = id$*

We will use these adjunctions in Chapter 9.

Before introducing some examples let us define the natural notion of mapping between regular categories.

**Definition 2.3.5.** A functor between regular categories is *exact* if it preserves finite limits and exact sequences or, equivalently (see [15]) if it preserves finite limits and regular epis.

We now introduce some examples of regular categories built upon the topological and recursive theoretic notions that we have introduced in this chapter. Together with **Top** and **PAss** they are the source of examples and counterexamples that we are going to use throughout the thesis.

### 2.3.1 Sets valued on a frame

For any frame $H$ we define a category $H_+$ as follows. An object of $H_+$ is a pair $X = (|X|, \|\_\|_X)$ such that $|X|$ is a set and $\|\_\|_X : |X| \longrightarrow H$ is a function valued in the underlying set of the frame $H$.

A morphism $f : Y \longrightarrow X$ is a function $f : |Y| \longrightarrow |X|$ such that for every $y \in |Y|$, $\|y\| \leq \|fy\|$.

The category $H_+$ is regular and cartesian closed (see [83]). It also has coproducts and it is actually a *quasi-topos* as we will see in Section 2.3.6.

As before, there is a faithful "underlying set" functor $|\_| : H_+ \longrightarrow$ **Set** which, as in the case of topological spaces, has both a left and a right adjoint (both full and faithful).

The right adjoint $\nabla :$ **Set** $\longrightarrow H_+$ assigns to each set $S$ the object $\nabla S$ with underlying set $S$ and such that every element is valued in $\top$ the top element of $H$.

On the other hand $\Delta S$ has the same underlying set but, in this case, every element is valued in $\perp$ the least element of $H$.

### 2.3.2 Assemblies

For any PCA **A** we define now its associated category of *assemblies* **Ass** (or **Ass**(**A**) in case of possible confusion).

An assembly is a pair $X = (|X|, \|\_\|_X)$ where $|X|$ is a set and $\|\_\|_X$ assigns to each $x$ in $|X|$, a non empty subset of **A**.

A map $f : Y \to X$ between assemblies is a function $f : |Y| \to |X|$ for which there exists and $a \in$ **A** such that for every $y$ in $|Y|$ and $b$ in $\|y\|$, $ab$ is defined and in $\|fy\|$.

There is an adjunction $|\_| \dashv \nabla :$ **Set** $\longrightarrow$ **Ass** analogous to the case of partitioned assemblies (Section 2.2.2).

The category **Ass** is regular. As in the case of $H_+$, it is also a quasi-topos (see Section 2.3.6 and [39] where $\mathbf{Ass}(K_1)$ appears as the category of separated objects for the $\neg\neg$-topology in the effective topos **Eff**).

### 2.3.3 Subsequential spaces

In this subsection we present a notion of "space" different from the spaces defined in Section 2.2.1. The category of these new spaces embeds very interesting categories of topological spaces (see Section 7.6) and does not suffer many of the pathologies of **Top**.

**Definition 2.3.6.**

1. A *subsequential space* $X$ consists of a set $|X|$ together with a distinguished family of functions $(\mathbb{N} \cup \{\infty\}) \to |X|$, called *convergent sequences* in $X$. We say that $(x_i)$ *converges to* $x_\infty$ in $X$ if the induced function $(\mathbb{N} \cup \{\infty\}) \to X$ is one of the convergent sequences in $X$. The convergent sequences must satisfy the following axioms:

   (a) the constant sequence $(x)$ converges to $x$;

   (b) if $(x_i)$ converges to $x$, then so does every subsequence of $(x_i)$;

   (c) if $(x_i)$ is a sequence such that every subsequence of $(x_i)$ contains a subsequence converging to $x$, then $(x_i)$ converges to $x$.

2. A function between subsequential spaces is said to be *continuous* if it preserves convergent sequences.

We usually write $(x_i) \to x$ as a shorthand for $(x_i)$ converges to $x$.

Let **SSeq** be the category of subsequential spaces and continuous functions. In [46], it is shown that it arises as the full and reflective subcategory of $\neg\neg$-separated objects of the Grothendieck topos described therein. It is therefore regular, indeed a quasi-topos.

As in the case of topological spaces, the obvious "underlying set" functor has both a left and a right adjoint (both full and faithful). The "chaotic" right adjoint $\nabla : \mathbf{Set} \longrightarrow \mathbf{SSeq}$ assigns to each set $S$ the space with the same underlying set and in which every sequence converges to every point. As we mentioned before, we will discuss this in further detail in Chapter 7.

### 2.3.4 Generalized enumerated sets

The *recursive topos* was introduced in [79] as a "suitable arena for discussion of higher type recursion" [81] (see also [80]). It is the topos of sheaves for the canonical topology on the monoid of recursive functions.

We now introduce the category **GEn** which was shown in [99] to be the category of separated objects for the double negation topology of the recursive topos. For this purpose, let $\mathbf{R}$ be the set of recursive functions and for any set $S$, let $S^{\mathbb{N}}$ be the set of functions $\mathbb{N} \longrightarrow S$.

**Definition 2.3.7.**

1. A *generalized enumerated set* is a pair $X = (|X|, E)$ such that $|X|$ is a set and $E$ is a subset of $|X|^{\mathbb{N}}$ such that the following hold.

   (a) the images of the functions in $E$ cover the set $|X|$

   (b) for all $e_0, e_1 \in E$ there exists $e \in E$ and $r_0, r_1 \in \mathbf{R}$ such that $e_i = e.r_i$ for $i = 0, 1$

   (c) if $e \in E$ and $r \in \mathbf{R}$ then $e.r \in E$

2. A map $f : (|Y|, D) \longrightarrow (|X|, E)$ between generalized enumerated sets is a function $f : |Y| \longrightarrow |X|$ such that $f.D \subseteq E$ where $f.D = \{f.d | d \in D\}$.

We denote the category of generalized enumerated sets by **GEn**.

In this case, the right adjoint to the obvious "underlying set" functor assigns to each set $S$ the object $\nabla S = (S, S^{\mathbb{N}})$. We will discuss the "discrete" left adjoint in Chapter 7.

### 2.3.5 Equilogical spaces

Slightly generalizing Scott's original terminology [102, 8] let us introduce the following definition.

**Definition 2.3.8.**

1. An *equilogical space* is a pair $(X, \sim)$ where $X$ is a topological space and $\sim$ is an arbitrary equivalence relation on the underlying set of $X$.

2. An *equivariant map* $\phi : (X, \sim_X) \to (Y, \sim_Y)$ is a function $\phi$ from the quotient set $X/\sim_X$ to the quotient set $Y/\sim_Y$ that is *realized* by some continuous

$f : X \to Y$ that preserves the equivalence relations (i.e. the diagram below commutes).

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & Y \arrow[d] \\ X/{\sim_X} \arrow[r, "\phi"'] & Y/{\sim_Y} \end{tikzcd}$$

We write **Equ** for the category of equilogical spaces and equivariant maps.

In the original definition of equilogical space [102] the topological spaces involved were required to be $T_0$. One of the important facts about this category is that it is cartesian closed. In the original presentation the proof of this fact was very concrete. Soon afterwards, an abstract account of the reasons why the category is cartesian closed [21, 13] showed that the $T_0$ restriction was not necessary to achieve this. This is why we use the term equilogical space to mean the natural generalization presented in Definition 2.3.8. Also, in [8] the equivariant maps are defined as equivalence classes of equivalence-relation-preserving continuous functions, rather than as functions between quotient sets. This is essentially equivalent to the definition presented above.

There are chaotic and discrete inclusions of **Set** into **Equ** analogous to the case of **Top**.

### 2.3.6 Quasi-toposes

The categories **Ass**, $H_+$, **SSeq**, **GEn** and **Equ** have a lot more structure than being just regular. In fact, they are *quasi-toposes* [90, 106]. We briefly introduce these here.

**Definition 2.3.9.** A mono $m : U \rightarrowtail X$ is *strong* if for every commutative square with top map epi as below, there exists a (necessarily unique) diagonal map as shown making the two triangles commute.

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[d] & Y \arrow[d] \arrow[dl, "\exists"'] \\ U \arrow[r, tail, "m"'] & X \end{tikzcd}$$

It is easy to see that regular monos are strong and that strong monos are closed under pullback and composition. It is also easy to see that in a category with epi/regular-mono factorizations, every strong mono is regular. So, for example, this is the case in **PAss** (see Section 2.2.2) and in **Top** (see Section 2.2.1).

**Definition 2.3.10.** A *strong-subobject classifier* is a map $\top : 1 \rightarrowtail \Omega$ such that for every strong mono $m : U \rightarrowtail X$ there exists a unique $\chi_m : X \longrightarrow \Omega$ such that the following square is a pullback.

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

As any map from the terminal object is a (split) regular mono and regular monos are closed under pullback, it follows that in a category with a strong subobject classifier regular and strong monos coincide. In particular, we can conclude that in such a category regular monos compose.

As an example of a strong-subobject classifier let us consider again the category **PAss**. Let $\Omega = \nabla\{T, F\}$ and let the map $\top : 1 \rightarrowtail \Omega$ send the unique element of 1 to $T$.

It is an easy exercise, using the description of the regular monos in Section 2.2.2, to prove that $\top$ gives a strong-subobject classifier.

As another example consider **Top**. As in the case of **PAss** the strong-subobject classifier is given by the inclusion of $T$ into the "chaotic" topological space $\nabla\{T, F\}$.

Recall that a category **C** is *locally* cartesian closed if every slice $\mathbf{C}/X$ is cartesian closed. As we mentioned already, neither **PAss** nor **Top** are even cartesian closed so they are not quasi-toposes in the sense of the definition below.

**Definition 2.3.11.** A *quasi-topos* is a locally cartesian closed category with finite limits, finite colimits and a strong-subobject classifier.

Quasi-toposes are regular categories (see [106]) and as we mentioned in the beginning of this section all of **Ass**, $H_+$, **SSeq**, **GEn** and **Equ** are quasi-toposes. Except for **GEn**, finite limits, colimits and exponentials in these categories have been explicitely described elsewhere: for **Ass** see [39, 66], for $H_+$ see [83], for

**SSeq** see [77, 78] (beware! subsequential spaces are called *limit spaces* there) and finally, for **Equ** see [102, 8].

So let us describe the strong-subobject classifiers. In all cases, it is the inclusion of $T$ into the "chaotic" $\nabla\{T, F\}$. We leave it as an easy exercise to check that these are actually strong-subobject classifiers.

Notice that in all cases, $\Omega$ has, intuitively, as much structure as possible. Compare, for example, the object $\Omega$ in $H_+$ with the "discrete" object $\Delta\{T, F\}$. This last object can be thought of as having as little structure as possible.

In Chapter 7 we will see that there is a conceptual way of looking at "chaotic" (and "discrete") objects and that they can be used to explain important properties of our examples.

We end this section with a comment about coproducts in a quasi-topos. In general, these are stable (indeed, all colimits in a quasi-topos are stable [106]). On the other hand, coproducts need not be disjoint. So quasi-toposes are not lextensive in general (see Section 46 in [106]). Notice however, that all quasi-toposes that we have introduced *are* lextensive.

## 2.4 Relations in regular categories

In this section we review the basic facts of the theory of relations in a regular category [32, 75, 23].

**Definition 2.4.1.** A *relation* from $Y$ to $X$ is a subobject of $Y \times X$.

So a relation from $Y$ to $X$ is determined by a jointly monic pair of maps $\langle f_Y, f_X\rangle : R \rightarrowtail Y \times X$. We may sometimes abuse terminology by denoting such a relation by $R$. Let us look at some examples of relations.

For any $X$ we have the *diagonal* relation $\Delta_X = \langle id, id\rangle : X \rightarrowtail X \times X$. We usually forget the subscript. If $\langle f_Y, f_X\rangle : R \rightarrowtail Y \times X$ is a relation then its *opposite* $\langle f_X, f_Y\rangle : R \rightarrowtail X \times Y$ is also a relation and we denote it by $R^\circ$.

Any map $f : Y \longrightarrow X$ can be viewed as a relation $\langle id, f\rangle : Y \rightarrowtail Y \times X$. We call this relation the *graph* of $f$. We sometimes denote the graph of $f$ by $f$.

Another important class of relations is the following.

**Definition 2.4.2.** An *equivalence relation* is a relation $\langle d_0, d_1\rangle : X_1 \rightarrowtail X_0 \times X_0$ such that:

1. (reflexivity) there exists a map $r : X_0 \to X_1$ such that $d_0.r = d_1.r = id$
2. (symmetry) there exists a map $s : X_1 \to X_1$ such that $d_0.s = d_1$ and $d_1.s = d_0$

3. (transitivity) if $p_0 : P \to X_1$ and $p_1 : P \to X_1$ form the pullback of $d_1$ and $d_0$ as in the following diagram

$$\begin{tikzcd} & & P \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & & \\ & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ X_0 & & X_0 & & X_0 \end{tikzcd}$$

then there exists a map $t : P \to X_1$ such that $d_0.t = d_0.p_0$ and $d_1.t = d_1.p_1$.

Notice that this definition assumes finite limits. There is an alternative definition [6]: an equivalence relation on $X_0$ is a jointly monic pair of maps $d_0, d_1 : X_1 \longrightarrow X_0$ such that for every object $X$, the relation (in the usual sense) $R_X = \{(d_0.f, d_1.f) | f : X \to X_1\}$ on the set $Hom(X, X_0)$ is an equivalence relation (again, in the usual sense). One of the advantages of this definition is that it does not require products. Another advantage is that some equivalence relations may be more easily recognized in this guise. In the presence of finite limits (which is the case in this thesis), this definition and Definition 2.4.2 are equivalent.

Kernel pairs are equivalence relations but not every equivalence relation is a kernel pair in general.

In a regular category, relations can be composed as follows, if $\langle f_Y, f_X\rangle : R \rightarrowtail Y \times X$ and $\langle g_Z, g_Y\rangle : S \rightarrowtail Z \times Y$ are relations from $Y$ to $X$ and $Z$ to $Y$ respectively then their composition $\langle h_Z, h_X\rangle : SR \rightarrowtail Z \times X$ from $Z$ to $X$ is defined as the mono part of the regular-epi/mono factorization of the rightmost map below.

$$
\begin{tikzcd}
S \times_Y R \arrow[r, "\pi_R"] \arrow[d, "\pi_S"'] & R \arrow[d, "f_Y"] \\
S \arrow[r, "g_Y"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
S \times_Y R \arrow[d, "reg"] \\
SR \arrow[d, tail, "\langle h_Z, h_X \rangle"] \\
Z \times X
\end{tikzcd}
\qquad
\begin{tikzcd}
S \times_Y R \arrow[d, "\langle g_Z . \pi_S , f_X . \pi_R \rangle"] \\
Z \times X
\end{tikzcd}
$$

It is important to mention that composition of relations is associative (see [32] 1.569 where it is explained how regular categories are precisely what is needed to make composition of relations associative).

As is standard practice, the order in which we write the composition of relations is inverse to the one in which we write the composition of maps.

Notice also that relations from $Y$ to $X$ inherit a partial order $\leq$ as subobjects of $Y \times X$. Moreover, composition and $(\_)^\circ$ preserve this partial order.

Using these ideas and notation we can reformulate the notion of equivalence relation as follows. A relation $E$ is an *equivalence relation* if the following hold.

**reflexivity** $\Delta \leq E$

**symmetry** $E = E^\circ$

**transitivity** $EE \leq E$

It is not difficult to figure out the correspondence between this definition of equivalence relation and the one given in Definition 2.4.2.

### 2.4.1 Functional relations

In this section we describe two notions of morphisms between equivalence relations and explain the relation between them.

Given two equivalence relations $\langle e_0, e_1 \rangle : E \rightarrowtail X \times X$ and $\langle d_0, d_1 \rangle : D \rightarrowtail Y \times Y$ we say that a relation $\langle f_Y, f_X \rangle : F \rightarrowtail Y \times X$ is

1. *defined from* $D$ if $DF = F$

2. *defined to* $E$ if $F = FE$

3. *total* if $D \leq FF^\circ$

4. *single valued* if $F^{\circ}F \leq E$

5. *functional from $D$ to $E$* if it is defined from $D$ and to $E$ and is total and single valued.

We now state the main properties of functional relations ([32],[75]).

**Proposition 2.4.3.**

1. *any equivalence relation is functional from itself to itself*

2. *(functional relations compose) if $F$ is functional from $E$ to $E'$ and $F'$ is functional from $E'$ to $E''$ then $FF'$ is functional from $E$ to $E''$.*

3. *A relation $F$ is functional from $\Delta_Y$ to $\Delta_X$ if and only if it is the graph of a necessarily unique map $f : Y \longrightarrow X$.*

4. *if $F$ and $G$ are both functional from $D$ to $E$ and $F \leq G$ then $F = G$.*

There is another natural notion of morphism between equivalence relations. Given $D$ and $E$ equivalence relations as above we say that a map $f : Y \longrightarrow X$ *induces a map* from $D$ to $E$ if there exists a map $f' : D \longrightarrow E$ such that for $i = 0, 1$ it holds that $f.d_i = e_i.f'$. This is justified by the following result.

**Proposition 2.4.4.** *Let $f, g$ induce maps from $D$ to $E$ as below.*

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "g'"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

*Then the following hold.*

1. *$DfE$ is a functional relation from $D$ to $E$*

2. *$DfE = DgE$ if and only if there exists $h : Y \longrightarrow E$ such that $e_0.h = g$ and $e_1.h = f$.*

*Proof.* This is easy but tiresome. I have been unable to find a published statement so I included a proof in Appendix A. ☐

The next proposition is useful to recognize functional relations induced by maps. Moreover, it implies (by the implication $2 \Rightarrow 1$) that the morphisms giving rise to functional relations are exactly those that induce maps.

**Proposition 2.4.5.** *Let $D$ and $E$ be equivalence relations as above. Let $\langle h_Y, h_X\rangle : H \longrightarrow Y \times X$ be a functional relation from $D$ to $E$ and let $h : Y \longrightarrow X$. Then, the following are equivalent.*

1. *there exists an $h' : D \longrightarrow E$ such that $e_0.h' = d_0.h$ and $e_1.h' = d_1.h$ as in the square below and also $DhE = H$.*

$$\begin{tikzcd} D \arrow[r, "h'"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$

2. *$h \leq H$*

3. *there exists an $h_E : H \longrightarrow E$ such that the following square commutes.*

$$\begin{tikzcd} H \arrow[r, "h_E"] \arrow[d, "\langle h_X, h_Y\rangle"'] & E \arrow[d, "\langle e_0, e_1\rangle"] \\ X \times Y \arrow[r, "id \times h"'] & X \times X \end{tikzcd}$$

*Proof.* Idem proof of Proposition 2.4.4. □

## 2.5 Exact categories and toposes

As we have seen, regular categories form a good setting in which to develop a theory of relations. On the other hand, there is one important aspect in which this theory of relations differs from the usual one among sets: equivalence relations are not required to have a "good" quotient. At this point this is not a precise statement and this leads us to the notion of an exact category.

**Definition 2.5.1.** (Effective equivalence relations and exact categories)

1. An equivalence relation is *effective* if it is the kernel pair of some arrow.

2. A category is *exact* if it is regular and every equivalence relation is effective.

The intuition is that an effective equivalence relation is one that has a "good" quotient. Let us mention two important classes of examples: abelian categories [30] (which we are not going to touch) and toposes.

Topos theory (see [46, 71, 7, 75] and references therein) is a very rich theory. In this section we will only recall the most basic definitions.

**Definition 2.5.2.** A *subobject classifier* is a map $\top : 1 \rightarrowtail \Omega$ such that for every mono $m : U \rightarrowtail X$, there exists a unique $\chi_m : X \longrightarrow \Omega$ such that the square below is a pullback.

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

Probably, the best known example of a subobject classifier is the the inclusion $\{T\} \rightarrowtail \{T, F\}$ in **Set** which is, in turn, the best known topos.

**Definition 2.5.3.** A *topos* is a category of with finite limits, exponentials and a subobject classifier.

It is not trivial to prove that toposes have finite colimits (see [71] for example), but they do and they are also locally cartesian closed. So it is clear that every topos is a quasi-topos. On the other hand, a quasi-topos $\mathcal{E}$ is a topos if and only if every mono in $\mathcal{E}$ is strong.

We will not attempt to give an introduction to topos theory. Everything we will need can be found in any of the references above. At some point in Chapter 8 we will deal with presheaf toposes, for readers unfamiliar with these, Chapter I of [71] is an excellent introduction. In Chapters 9, 10 and 11 we are going to work with topologies and universal closure operators. Readers familiar with Grothendieck topologies (Chapters II and III of [71]) will find the material in the chapters here a lot easier to read.

On the other hand, the topic of realizability toposes does not seem to have such definitive accounts. Of course, anyone interested in the subject should take a look at [39, 94, 42] (see also [93] and the more recent [95]). For later accounts, closer to the approach of this thesis [96, 16] are also essential. Moreover, also closely related to the approach of the thesis (especially in connection with the results in Chapter 11) are the accounts of the effective topos **Eff** in [32, 75] (see also

[17]). The thesis [82] has a very good introduction to and survey of realizability (not restricted to its appearance in topos theory), numerous examples and a good bibliography (see also [87]). For more on realizability toposes and applications of these ideas to the semantics of programming languages, logics and computation see for example [40, 43, 100, 92, 66, 69, 85, 84, 88, 68, 67, 65, 11, 12].

Although we are not going to be particularly interested in the class of *pretoposes* [74, 45] as such, we are going to encounter them (apart from dealing with toposes) from time to time, so we might as well define them.

**Definition 2.5.4.** A *pretopos* is an exact lextensive category.

# Chapter 3

# Completions

The main purpose of this chapter is to introduce four free constructions that are fundamental in the thesis. We will review the constructions of the regular and exact completions of a category with finite limits [72], the coproduct completion of any category [72] and the ex/reg completion of a regular category [54, 32, 75] (see also [16] for a survey of all the constructions).

We also present important properties of these constructions that will be used in the remaining chapters. Of particular importance are the characterizations of the categories that arise as coproduct, regular and exact completions [72, 19, 16] and also the characterization of the categories with finite limits whose exact completions are locally cartesian closed [21].

Finally we present the notion of *suitable* functor [96] that will play an important role in later chapters.

## 3.1 Coproduct completions

For categories $\mathbf{C}$ and $\mathbf{C}'$ let $\mathbf{CAT}(\mathbf{C}, \mathbf{C}')$ denote the category of functors from $\mathbf{C}$ to $\mathbf{C}'$ and natural transformations between them. Also, for categories with small coproducts $\mathbf{D}$ and $\mathbf{D}'$ let $\mathbf{COP}(\mathbf{D}, \mathbf{D}')$ denote the category of coproduct preserving functors and natural transformations between them.

For any category $\mathbf{C}$ there exists a unique (up to equivalence) category $\mathbf{C}_+$ with small coproducts and a full and faithful functor $\mathbf{y} : \mathbf{C} \longrightarrow \mathbf{C}_+$ satisfying the following universal property: for every category $\mathbf{D}$ with coproducts, the functor $(\_).\mathbf{y} : \mathbf{COP}(\mathbf{C}_+, \mathbf{D}) \longrightarrow \mathbf{CAT}(\mathbf{C}, \mathbf{D})$ is an equivalence of categories.

We construct $\mathbf{C}_+$ below. Any category equivalent to $\mathbf{C}_+$ will be called the *coproduct completion* of $\mathbf{C}$.

An object of $\mathbf{C}_+$ is a family of objects $\{X_i\}_{i\in I}$ in $\mathbf{C}$ indexed by a set $I$. A map between $\{X_i\}_{i\in I}$ and $\{Y_j\}_{j\in J}$ is a family $f = \{f_i : X_i \longrightarrow Y_{\phi i}\}_{i\in I}$ with $f_i$

in $\mathbf{C}$ and $\phi$ a function from $I$ to $J$.

The coproduct completion of any category always has stable and disjoint coproducts [16, 18]. Moreover, if $\mathbf{C}$ has finite limits, then so does $\mathbf{C}_+$ and the embedding $\mathbf{C} \rightarrowtail \mathbf{C}_+$ preserves them [16].

**Definition 3.1.1.** An object $X$ is *connected* if the corresponding covariant hom-functor $Hom(X, \_)$ preserves existing coproducts.

In the presence of stable and disjoint coproducts, an object $X$ is connected if and only if $X$ is not initial and cannot be decomposed as a coproduct of non-initial objects. The following simple result is stated in [22].

**Proposition 3.1.2.** *A category* $\mathbf{E}$ *is the coproduct completion of a small category* $\mathbf{C}$ *if and only if* $\mathbf{E}$ *is locally small with small coproducts and there exists a small subcategory* $\mathbf{C}'$ *of* $\mathbf{E}$ *equivalent to* $\mathbf{C}$ *consisting of connected objects and such that every object in* $\mathbf{E}$ *is isomorphic to a coproduct of objects in* $\mathbf{C}'$.

We have already presented an example of a coproduct completion: $H_+$ as introduced in Section 2.3.1 is the coproduct completion of $H$ seen as a small category.

## 3.2 Regular completions

For categories $\mathbf{C}$ and $\mathbf{C}'$ with finite limits, let $\mathbf{LEX}(\mathbf{C}, \mathbf{C}')$ be the category whose objects are the functors from $\mathbf{C}$ to $\mathbf{C}'$ that preserve finite limits and whose maps are the natural transformations between these functors.

Similarly, for regular categories $\mathbf{D}$ and $\mathbf{D}'$, let $\mathbf{REG}(\mathbf{D}, \mathbf{D}')$ be the category of exact functors from $\mathbf{D}$ to $\mathbf{D}'$ and natural transformations between them.

For any category with finite limits $\mathbf{C}$, there exists a unique (up to equivalence) regular category $\mathbf{C}_{reg}$ and a full and faithful functor $\mathbf{y} : \mathbf{C} \longrightarrow \mathbf{C}_{reg}$ preserving finite limits satisfying the following universal property: for any regular category $\mathbf{D}$, $(\_).\mathbf{y} : \mathbf{REG}(\mathbf{C}_{reg}, \mathbf{D}) \longrightarrow \mathbf{LEX}(\mathbf{C}, \mathbf{D})$ is an equivalence of categories.

We build $\mathbf{C}_{reg}$ below. Any category equivalent to $\mathbf{C}_{reg}$ will be called the *regular completion* of $\mathbf{C}$.

The category $\mathbf{C}_{reg}$ has an easy description. Its objects are maps $f : X \to Y$ in $\mathbf{C}$. A map $[l] : (f : X \to Y) \to (g : U \to V)$ in $\mathbf{C}_{reg}$ is an equivalence class of maps $l : X \to U$ such that $g.l.f_0 = g.l.f_1$ where $f_0$ and $f_1$ form the kernel pair of $f$. Two such maps $l, m$ are considered equivalent if $g.l = g.m$.

The idea is that the object $(f : X \longrightarrow Y)$ represents the image of the map $f : X \longrightarrow Y$ in $\mathbf{C}_{reg}$.

If we let $\simeq$ denote equivalence of categories then we have, for example, that $\mathbf{Ass} \simeq \mathbf{PAss}_{reg}$ (see [16]) and that $\mathbf{Equ} \simeq \mathbf{Top}_{reg}$ (see [21, 13]).

The categories that arise as regular completions can be characterized as follows.

**Definition 3.2.1.** An object $X$ is *projective* if for every regular epi $e : A \twoheadrightarrow B$ and map $g : X \to B$ there exists a map $f : X \to A$ such that $e.f = g$.

About our terminology, notice that what we call *projectives* are usually called *regular* projectives.

We say that a category *has enough projectives* if for every object $A$ there exists a projective $X$ and a regular epi $q : X \twoheadrightarrow A$. We say that $q$ is a *projective cover of* $A$.

The following result appears in [16].

**Proposition 3.2.2.** *A regular category* $\mathbf{D}$ *is a regular completion if and only if it has enough projectives, projectives are closed under finite limits and every object is a subobject of a projective. Moreover, in this case* $\mathbf{D}$ *is the regular completion of its full subcategory of projectives.*

For any category $\mathbf{D}$ we denote by $\mathbf{y} : \mathit{Proj}(\mathbf{D}) \rightarrowtail \mathbf{D}$ the embedding of the full subcategory of projectives. It follows that for any category with finite limits $\mathbf{C}$, $\mathbf{C} \simeq \mathit{Proj}(\mathbf{C}_{reg}) \rightarrowtail \mathbf{C}_{reg}$.

For more on regular completions see [22, 37] and the remark at the end of the next section.

## 3.3 Exact completions

For exact categories $\mathbf{E}$ and $\mathbf{E}'$, let $\mathbf{EX}(\mathbf{E}, \mathbf{E}')$ be the category of exact functors from $\mathbf{E}$ to $\mathbf{E}'$ and natural transformations between them.

For any category with finite limits $\mathbf{C}$, there exists a unique (up to equivalence) exact category $\mathbf{C}_{ex}$ and a full and faithful functor $\mathbf{y} : \mathbf{C} \longrightarrow \mathbf{C}_{ex}$ preserving finite limits with the following universal property: for every exact category $\mathbf{E}$, $(\_).\mathbf{y} : \mathbf{EX}(\mathbf{C}_{ex}, \mathbf{E}) \longrightarrow \mathbf{LEX}(\mathbf{C}, \mathbf{E})$ is an equivalence of categories.

We build $\mathbf{C}_{ex}$ below. Any category equivalent to $\mathbf{C}_{ex}$ is called the *exact completion* of $\mathbf{C}$.

The objects of $\mathbf{C}_{ex}$ are pseudo equivalence relations in $\mathbf{C}$ in the following sense.

**Definition 3.3.1.** A *pseudo equivalence relation* is a (not necessarily jointly monic) pair of maps $X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0$ satisfying reflexivity, symmetry and transitivity in the sense of Definition 2.4.2.

A map $[f] : (X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0) \longrightarrow (Y_1 \overset{e_0}{\underset{e_1}{\rightrightarrows}} Y_0)$ in $\mathbf{C}_{ex}$ is an equivalence class of maps $f : X_0 \to Y_0$ such that there exists an $f' : X_1 \to Y_1$ and such that the two squares $e_0.f' = f.d_0$ and $e_1.f' = f.d_1$ commute.

Two such maps $f$ and $g$ are equivalent if there exits an $h : X_0 \to Y_1$ such that $e_0.h = f$ and $e_1.h = g$.

**Proposition 3.3.2 (Joyal; Carboni-Magno).** *An exact category* $\mathbf{E}$ *is an exact completion if and only if it has enough projectives and projectives are closed under finite limits in* $\mathbf{E}$*. Moreover, in this case* $\mathbf{E}$ *is the exact completion of its full subcategory of projectives.*

This characterization was used in [96] in order to prove that the realizability topos associated to a partial combinatory algebra $\mathbf{A}$ arises as the exact completion of $\mathbf{PAss}(\mathbf{A})$, the main example being $\mathbf{Eff} \simeq (\mathbf{PAss}(K_1))_{ex}$.

Many other toposes arise as exact completions as is clear from the following proposition that relates presheaf toposes with coproduct and exact completions.

**Proposition 3.3.3.** *Let* $\mathbf{C}$ *be a small category. If* $\mathbf{C}_+$ *has finite limits then* $(\mathbf{C}_+)_{ex}$ *is equivalent to* $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$*.*

*Proof.* This is the argument used in the Corollary in p. 130 of [16]. See also Corollary 43 in [22]. □

Notice that $\mathbf{C}$ need not have finite limits (see [36] for an analysis of limits in coproduct completions).

As explained in [16], there is for any $\mathbf{C}$, an exact functor $Ker : \mathbf{C}_{reg} \rightarrowtail \mathbf{C}_{ex}$ (given by the universal property of $\mathbf{C}_{reg}$) taking an object $(f : Y \longrightarrow X)$ in $\mathbf{C}_{reg}$ to the object in $\mathbf{C}_{ex}$ given by the kernel pair of $f$.

Finally, we briefly discuss related work on regular and exact completions. In [22] it is explained how to construct regular and exact completions of categories with *weak* finite limits in a similar way as described here.

A different approach is that in [37]. For any locally small category $\mathbf{C}$ with finite limits (actually they do not restrict to the finite case) they present the regular and exact completions of $\mathbf{C}$ as certain full subcategories of the category of contravariant functors from $\mathbf{C}$ to $\mathbf{Set}$.

### 3.3.1 Local cartesian closure

In this section we review the characterization of the categories with finite limits whose exact completions are locally cartesian closed [21].

**Definition 3.3.4.** A *weak dependent product* of a map $f : X \to J$ along a map $\alpha : J \to I$ consists of maps $\zeta : Z \to I$ and $\epsilon : J \times_I Z \to X$ such that $f.\epsilon = \alpha^*\zeta$. Moreover, the pair $\epsilon, \zeta$ is weakly universal in the sense that for any other pair of maps $\zeta' : Z' \to I$ and $\epsilon' : J \times_I Z' \to X$ such that $f.\epsilon' = \alpha^*\zeta'$ there exists a (not necessarily unique) $f' : J \times_I Z' \longrightarrow J \times_I Z$ such that $\alpha^*\zeta' = (\alpha^*\zeta).f'$ and $\epsilon.f' = \epsilon'$.

$$\begin{tikzcd} J \times_I Z \arrow[rr, "\epsilon"] \arrow[dr, "\alpha^*\zeta"'] & & X \arrow[dl, "f"] \\ & J & \end{tikzcd}$$

**Proposition 3.3.5 (Carboni-Rosolini).** $\mathbf{C}_{ex}$ *is locally cartesian closed if and only if* $\mathbf{C}$ *has weak dependent products.*

As explained in [13, 21], both **Top** and **PAss** have weak dependent products and this gives short conceptual proofs that $\mathbf{Top}_{ex}$ and $(\mathbf{PAss}(K_1))_{ex} \simeq \mathbf{Eff}$ are locally cartesian closed. Moreover, an important intended application was to give a conceptual proof that $\mathbf{Top}_{reg} \simeq \mathbf{Equ}$ is locally cartesian closed. In order to do this they use the following result [13].

First, call an equivalence relation $\langle e_0, e_1\rangle : E \rightarrowtail X \times X$ *regular* if $\langle e_0, e_1\rangle$ is a regular mono. Then, for any category $\mathbf{C}$ with finite limits let $\mathbf{C}_{eq}$ be the full subcategory of $\mathbf{C}_{ex}$ induced by the regular equivalence relations.

**Proposition 3.3.6 (Birkedal,Carboni,Rosolini,Scott).** *Suppose* $\mathbf{C}$ *has finite limits and stable epi/regular-mono factorizations. Then, the inclusion* $\mathbf{C}_{eq} \to \mathbf{C}_{ex}$ *has a left adjoint which preserves products and commutes with pullbacks along maps in* $\mathbf{C}_{eq}$*. Hence, if* $\mathbf{C}_{ex}$ *is locally cartesian closed so is* $\mathbf{C}_{eq}$*.*

Actually, the result in [13] is slightly more general and involves any stable factorization system, but for our purposes the statement above suffices.

**Corollary 3.3.7.** *If* $\mathbf{C}$ *has weak dependent products, stable epi/regular-mono factorizations and is such that every regular equivalence relation is a kernel pair then* $\mathbf{C}_{reg}$ *is locally cartesian closed.*

*Proof.* As every equivalence relation is a kernel pair it follows that $\mathbf{C}_{reg}$ is equivalent to $\mathbf{C}_{eq}$ and then the result follows from Proposition 3.3.6. □

In Section 10.6 we will give an alternative proof of local cartesian closure of $\mathbf{C}_{eq}$ using the fact that $\mathbf{C}_{eq}$ is the category of separated objects for a universal closure operator in $\mathbf{C}_{ex}$.

## 3.4 The exact completion of a regular category

For any regular category $\mathbf{D}$ there exists an exact category $\mathbf{D}_{ex/reg}$ (unique up to equivalence) and a full and faithful exact functor $\mathbf{y} : \mathbf{D} \longrightarrow \mathbf{D}_{ex/reg}$ satisfying the following universal property: for any exact category $\mathbf{E}$, it holds that the functor $(\_).\mathbf{y} : \mathbf{EX}(\mathbf{D}_{ex/reg}, \mathbf{E}) \longrightarrow \mathbf{REG}(\mathbf{D}, \mathbf{E})$ is an equivalence of categories.

We build the category $\mathbf{D}_{ex/reg}$ below. Any category equivalent to $\mathbf{D}_{ex/reg}$ will be called the *ex/reg completion* of $\mathbf{D}$.

The objects of $\mathbf{D}_{ex/reg}$ are the equivalence relations in $\mathbf{D}$ and its maps are the functional relations between them as defined in Section 2.4.1. If $e_0, e_1 : E \longrightarrow X$ is an equivalence relation on $X$ we will usually denote the corresponding object in $\mathbf{D}_{ex/reg}$ by $X/E$ so as to suggest that the object must be thought of as the quotient of $X$ by $E$. Proposition 2.4.3.1 gives the identities, 2.4.3.2 gives composition and 2.4.3.3 shows that there is a full and faithful embedding $\mathbf{y} : \mathbf{D} \rightarrowtail \mathbf{D}_{ex/reg}$.

As explained in [16], the ex/reg-completion is an idempotent construction in the sense that the embedding $\mathbf{D} \rightarrowtail \mathbf{D}_{ex/reg}$ is an equivalence if and only if $\mathbf{D}$ is exact.

Notice also that for any category with finite limits $\mathbf{C}$, $\mathbf{C}_{ex} \simeq (\mathbf{C}_{reg})_{ex/reg}$. This is very easy to see if we understand the completions as inducing bi-adjunctions between 2-categories [104]. Indeed, we can see the regular completion construction as inducing a left bi-adjoint to the forgetful functor from the 2-category of regular categories and exact functors to that of categories with finite limits and functors preserving them. Also, we can see the ex/reg completion construction as a left bi-adjoint to the (full) forgetful functor from the 2-category of exact categories and exact functors to that of regular categories. So the statement $\mathbf{C}_{ex} \simeq (\mathbf{C}_{reg})_{ex/reg}$ is just the fact that bi-adjoints compose.

Any finite-limit-preserving functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ induces a natural transformation $Sub_{\mathbf{C}}(X) \longrightarrow Sub_{\mathbf{C}'}(FX)$. We say that $F$ *preserves subobjects* if this transformation is actually a natural iso.

**Lemma 3.4.1.** *The exact functor* $\mathbf{y} : \mathbf{D} \longrightarrow \mathbf{D}_{ex/reg}$ *preserves subobjects. That is, if* $X$ *is in* $\mathbf{D}$ *and* $Y/D \rightarrowtail X$ *is mono in* $\mathbf{D}_{ex/reg}$ *then* $Y/D$ *is isomorphic* *over $X$ to an object in* $\mathbf{D}$.

*Proof.* See for example [75] where $\mathbf{D}_{ex/reg}$ is denoted by $\mathbf{Map}(\mathbf{D})$. □

In other words, the embedding $\mathbf{D} \rightarrowtail \mathbf{D}_{ex/reg}$ adds no subobjects of objects in $\mathbf{D}$.

For another presentation of ex/reg completions see [52]. Any *small* regular category $\mathbf{D}$ embeds through the Yoneda embedding into the topos of sheaves for the Grothendieck topology on $\mathbf{D}$ induced by the regular epis. It is then possible to characterize $\mathbf{D}_{ex/reg}$ as the closure of $\mathbf{D}$ in the topos of sheaves under finite limits and coequalizers of equivalence relations (see Proposition 3.2 loc. cit.).

## 3.5 Covered categories and suitable functors

Propositions 3.2.2 and 3.3.2 and also the construction in Section 3.4 show that in regular, exact and ex/reg completions every object is the codomain of a regular epi whose domain is an object in the original category. In this section we explain how to profit from this.

**Definition 3.5.1.** Let $\mathbf{D}$ be a full subcategory of $\mathbf{E}$. We say that $\mathbf{D}$ *covers* $\mathbf{E}$ (or that $\mathbf{E}$ is *covered* by $\mathbf{D}$) if for every $Q$ in $\mathbf{E}$ there exists a regular epi $X \twoheadrightarrow Q$ with $X$ in $\mathbf{D}$.

For example, regular and exact completions $\mathbf{C}_{reg}$ and $\mathbf{C}_{ex}$ are covered by $\mathbf{C}$ while ex/reg completions $\mathbf{D}_{ex/reg}$ are covered by $\mathbf{D}$.

If $\mathbf{y} : \mathbf{D} \rightarrowtail \mathbf{E}$ is the embedding of a covering category, we say that a functor $F : \mathbf{E}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ is *representable over* $\mathbf{D}$ if there exists an object $R$ of $\mathbf{E}$ such that $F.\mathbf{y} \cong \mathbf{E}(\mathbf{y}(\_), R)$.

**Definition 3.5.2.** A functor $F : \mathbf{E}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ is *suitable* if it takes exact sequences to equalizer diagrams.

(It is worth noting that suitable functors are the sheaves in $\mathbf{Set}^{\mathbf{E}^{\mathbf{op}}}$ for the topology given by the regular epis.)

Consider for example a well-powered regular category $\mathbf{E}$. Then the functor $Sub : \mathbf{E}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ that takes every object to its set of subobjects and acts on arrows by pullback is suitable by item 4 of Lemma 2.3.3.

As strong subobjects are closed under pullback we can consider the sub-functor $SSub : \mathbf{E}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ of $Sub$ that takes each object to its set of strong subobjects. It follows that $SSub$ is also suitable.

**Proposition 3.5.3.** *Let* $\mathbf{y} : \mathbf{D} \rightarrowtail \mathbf{E}$ *be an embedding such that* $\mathbf{D}$ *covers* $\mathbf{E}$. *If* $F : \mathbf{E}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ *is a suitable functor that is representable over* $\mathbf{D}$ *then* $F$ *is representable.*

*Proof.* The proof in [96] of the related result involving projectives works in this case. We reproduce it here for completeness.

Assume that there exists an object $R$ such that $F.\mathbf{y} \cong \mathbf{E}(\mathbf{y}(\_), R)$. Let $Q$ be any object in $\mathbf{E}$. Because $\mathbf{D}$ covers $\mathbf{E}$ there exists a regular epi $q : X \twoheadrightarrow Q$ with $X$ in $\mathbf{D}$. Let $e_0, e_1 : E \longrightarrow X$ be the kernel pair of $q$. Again using that $\mathbf{D}$ covers $\mathbf{E}$, let $e : Y \twoheadrightarrow E$ be a regular epi with $Y$ in $\mathbf{D}$. We then have a coequalizer diagram as below.

$$\begin{tikzcd} Y \arrow[r, two heads, "e"] & E \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & X \arrow[r, two heads, "q"] & Q \end{tikzcd}$$

Let $f_0 = e_0.e$ and $f_1 = e_1.e$. It is clear that suitable functors carry regular epis to regular monos so it follows that $F$ carries the coequalizer diagram $q.f_0 = q.f_1$ to an equalizer diagram. The representable $\mathbf{E}(\_, R)$ also carries coequalizers to equalizers so we obtain the following two equalizer diagrams.

$$\begin{tikzcd} FQ \arrow[r, tail] & FX \arrow[r, shift left, "Ff_0"] \arrow[r, shift right, "Ff_1"'] \arrow[d, "\cong"'] & FY \arrow[d, "\cong"] \\ \mathbf{E}(Q, R) \arrow[r, tail] & \mathbf{E}(\mathbf{y}X, R) \arrow[r, shift left, "(\_).f_0"] \arrow[r, shift right, "(\_).f_1"'] & \mathbf{E}(\mathbf{y}Y, R) \end{tikzcd}$$

So it must be the case that $FQ \cong \mathbf{E}(Q, R)$. □

Let us put in elementary terms what does it mean for *Sub* and *SSub* to be representable over a subcategory $\mathbf{D}$ of $\mathbf{E}$.

**Definition 3.5.4.** Let $\mathbf{D}$ be a full subcategory of $\mathbf{E}$. A *classifier of (strong) subobjects of* $\mathbf{D}$*-objects* is a (strong) mono $\Omega' \rightarrowtail \Omega$ in $\mathbf{E}$ such that for every object $X$ of $\mathbf{D}$ and (strong) mono $m : U \rightarrowtail X$ in $\mathbf{E}$ there exists a unique $\chi_m : X \longrightarrow \Omega$ such that the following square is a pullback.

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "m"'] & \Omega' \arrow[d, tail, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

In the case of regular and exact completions, by Propositions 3.2.2 and 3.3.2, we can speak of a *classifier of subobjects of projectives*.

This is the formulation that we are going to use in Sections 4.2, 5.3 and 11.3 together with the following.

**Corollary 3.5.5.** *Let* $\mathbf{C}$ *be a category with finite limits and let* $\mathbf{D}$ *be a regular category.*

1. $\mathbf{C}_{reg}$ *has a strong-subobject classifier if and only if it has a classifier of strong subobjects of projectives.*

2. $\mathbf{C}_{ex}$ *has a subobject classifier if and only if it has a classifier of subobjects of projectives.*

3. $\mathbf{D}_{ex/reg}$ *has a subobject classifier if and only if it has a classifier of subobjects of* $\mathbf{D}$*-objects.*

*Proof.* These are just corollaries of Proposition 3.5.3 together with Proposition I.3.1 of [71] in order to prove that $\Omega'$ must be 1. □

Most of the thesis will address properties of regular and exact completions. In order to treat them in a concise way let us say that a category is *suitable* if it is regular, it is covered by its full subcategory of projectives and moreover, projectives are closed under finite limits.

# Chapter 4

# Regular completions, colimits and quasi-toposes

In Chapters 2 and 3 we saw that there exist categories that are both regular completions and quasi-toposes. In this chapter we analyze this phenomenon by giving sufficient conditions for regular completions to have colimits and to have a strong-subobject classifier. In this way, we obtain sufficient conditions for regular completions to be lextensive quasi-toposes. We will also see that any lextensive quasi-topos satisfies these conditions and it is then possible to iterate the regular completion construction to obtain hierarchies of quasi-toposes that have apparently not been encountered before. In Chapter 8 we will see that under certain extra conditions these hierarchies have companion hierarchies of toposes.

## 4.1 Colimits in regular completions

In this section we are going to address the problem of the existence of finite colimits in regular and exact completions. First, let us review coproducts.

**Proposition 4.1.1.** *For any category* $\mathbf{C}$ *with finite limits, the following are equivalent.*

1. $\mathbf{C}$ *is lextensive*
2. $\mathbf{C}_{ex}$ *is lextensive (and so a pretopos). Moreover, the embedding* $\mathbf{C} \rightarrowtail \mathbf{C}_{ex}$ *preserves coproducts.*
3. $\mathbf{C}_{reg}$ *is lextensive and the embeddings* $\mathbf{C} \rightarrowtail \mathbf{C}_{reg}$ *and* $\mathbf{C}_{reg} \rightarrowtail \mathbf{C}_{ex}$ *preserve coproducts.*

*Proof.* In [16] Lemma 2.2, it is stated that 1 implies 2.

We now show that 2 implies 3. In [72] it is explained that if $A, B$ in $\mathbf{C}_{ex}$ arise as the quotients of pseudo equivalence relations $p_0, p_1 : X' \longrightarrow X$ and $q_0, q_1 : Y' \longrightarrow Y$ respectively then $A + B$ is defined as the quotient of the "coproduct" pseudo equivalence relation $[p_0, q_0], [p_1, q_1] : X' + Y' \longrightarrow X + Y$. Using lextensivity, it is easy to see that if the pseudo equivalence relations are kernel pairs then so is their "coproduct". Together with the fact that the embedding $\mathbf{C}_{reg} \rightarrowtail \mathbf{C}_{ex}$ preserves finite limits this shows that if $\mathbf{C}_{ex}$ has stable and disjoint coproducts then so does $\mathbf{C}_{reg}$ and that the embedding $\mathbf{C}_{reg} \rightarrowtail \mathbf{C}_{ex}$ preserves them. Finally, it is also easy to see that the embedding $\mathbf{C} \rightarrowtail \mathbf{C}_{reg}$ preserves coproducts. This finishes the proof that 2 implies 3.

To prove that 3 implies 1, notice that as coproducts of projectives are projective and the embedding $\mathbf{C} \longrightarrow \mathbf{C}_{reg}$ preserves finite limits it follows that if $\mathbf{C}_{reg}$ has stable and disjoint coproducts then so does $\mathbf{C}$ and that the embedding preserves them. $\square$

In order to analyze coequalizers let us introduce the following definition which is inspired by Carboni's notion of a quasi-effective category (see Definition 4.1.6).

**Definition 4.1.2.** Let $f, g : Y \longrightarrow X$ be a parallel pair of maps. A *quasi-coequalizer* of $f$ and $g$ is a map $q : X \longrightarrow Q$ such that $q.f = q.g$ and such that if $q' : X \longrightarrow Q'$ is another map with $q'.f = q'.g$ then $q'.q_0 = q'.q_1$ where $q_0$ and $q_1$ form the kernel pair of $q$.

This notion seems a bit unnatural but perhaps the following two results will help in this respect.

**Proposition 4.1.3.** *If* $\mathbf{C}$ *is lextensive and has quasi-coequalizers then both* $\mathbf{C}_{reg}$ *and* $\mathbf{C}_{ex}$ *have coequalizers (and so, they are finitely cocomplete).*

*Proof.* Let $\mathbf{D}$ be either $\mathbf{C}_{reg}$ or $\mathbf{C}_{ex}$. First, it is very easy to see that any pair of maps $f, g : Y \longrightarrow X$ in $\mathbf{C}$ has a coequalizer in $\mathbf{D}$. Just take the regular epi part of the factorization of any of their quasi-coequalizers in $\mathbf{C}$.

So let $f', g' : B \longrightarrow A$ be a parallel pair in $\mathbf{D}$. Let $Y$ and $X$ be projective covers of $B$ and $A$ respectively and let $f$ and $g$ arise by projectivity as in the following diagram.

$$
\begin{tikzcd}
Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & X \arrow[d, "a"] \\
B \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & A
\end{tikzcd}
$$

Let $a$ arise as the coequalizer of the pair of maps $a_0, a_1 : X' \longrightarrow X$ in $\mathbf{C}$ (recall Section 3.5).

We can then take the coequalizer $r : X \longrightarrow R$ (in $\mathbf{D}$) of the parallel pair $[a_0, f], [a_1, g] : X' + Y \longrightarrow X$ in $\mathbf{C}$.

Clearly, $r.a_0 = r.a_1$. As $a$ is a coequalizer in $\mathbf{D}$, there exists a unique $\overline{r} : A \longrightarrow R$ such that $\overline{r}.a = r$.

Let $\overline{r}$ factor as a regular epi $r' : A \twoheadrightarrow R'$ followed by a mono $m : R' \rightarrowtail R$. We now show that $r'$ is the coequalizer of $f'$ and $g'$.

It is easily seen using that $m$ is mono and the cover $Y \twoheadrightarrow B$ epi that $r'.f' = r'.g'$.

So let $h : A \longrightarrow C$ be such that $h.f' = h.g'$. It follows that $h.a.f = h.a.g$. But then $h.a$ factors through $r = m.r'.a$ and so $h$ factors through $m.r'$ and hence, through $r'$.

$$
\begin{tikzcd}
X' + Y \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, two heads, "r"] \arrow[d, "a"'] & R \arrow[drr, "\exists!"] & & \\
B \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & A \arrow[r, two heads, "r'"'] \arrow[drrr, "h"'] & R' \arrow[u, tail, "m"'] & & \\
& & & & C
\end{tikzcd}
$$

□

For the case of $\mathbf{C}_{reg}$ we can prove a little more.

**Corollary 4.1.4.** *Let $\mathbf{C}$ be lextensive. Then $\mathbf{C}$ has quasi-coequalizers if and only if $\mathbf{C}_{reg}$ has coequalizers (and so, $\mathbf{C}_{reg}$ is finitely cocomplete).*

*Proof.* By Proposition 4.1.3 we need only consider the *if* direction. Let $f, g : Y \longrightarrow X$ be a parallel pair. Let $c : X \longrightarrow C$ be their coequalizer in $\mathbf{C}_{reg}$. By Proposition 3.2.2 there exists a mono $m : C \rightarrowtail Q$ into a projective $Q$. It is very easy to see that $m.c : X \longrightarrow Q$ (which is in $\mathbf{C}$) is a quasi-coequalizer of $f$ and $g$. □

Of course, the coequalizer of a parallel pair is trivially a quasi-coequalizer. So, for example, as **Top** is cocomplete it follows that $\mathbf{Top}_{reg} \simeq \mathbf{Equ}$ is also cocomplete. On the other hand, consider **PAss**. It has finite coproducts but, as we have already observed in Section 2.3, not every pair of maps has a coequalizer.

**Example 4.1.5 (PAss *has quasi-coequalizers*).** Recall the "chaotic" inclusion of **Set** into **PAss** (Section 2.2.2). Now, let $f, g : Y \longrightarrow X$ be a parallel pair in **PAss**. Let $q : |X| \twoheadrightarrow Q$ be the coequalizer of $|f|$ and $|g|$ in **Set**. It is easy to see that the map $X \longrightarrow \nabla Q$ in **PAss** with underlying function $q$ and realized by the constant recursive function sending everything to $*$ is a quasi-coequalizer of $f$ and $g$.

We have already mentioned that the notion of a quasi-coequalizer is inspired in that of a quasi-effective category [16]. With the terminology we have introduced, these are defined as follows.

**Definition 4.1.6.** A category is *quasi-effective* if every pseudo equivalence relation has a quasi-coequalizer.

The point about these categories is the following result taken from [16].

**Proposition 4.1.7.** *The embedding $\mathbf{C}_{reg} \rightarrowtail \mathbf{C}_{ex}$ has a left adjoint if and only if $\mathbf{C}$ is quasi-effective.*

## 4.2 Regular completions and strong-subobject classifiers

In this section we show how regular completions inherit strong-subobject classifiers.

First, we recall a nice folklore fact about regular categories. Its statement appears in [50], but being unable to find a published proof we give one for completeness.

**Lemma 4.2.1.** *In a regular category consider the following diagram with $d$ and $e$ regular epis and such that both the left hand square and the outer rectangle are pullbacks.*

$$\begin{tikzcd} {} \arrow[r, two heads, "d"] \arrow[d, "x"'] & {} \arrow[r, "g"] \arrow[d, "y"] & {} \arrow[d, "z"] \\ {} \arrow[r, two heads, "e"'] & {} \arrow[r, "f"'] & {} \end{tikzcd}$$

*Then the right hand square is a pullback.*

*Proof.* First pullback $z$ along $f$ followed by the exact sequence given by $e$ and its kernel pair.

$$\begin{tikzcd} {} \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d] & {} \arrow[r, two heads, "e'"] \arrow[d, "x"'] & {} \arrow[r, "h'"] \arrow[d, "h"] & {} \arrow[d, "z"] \\ {} \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & {} \arrow[r, two heads, "e"'] & {} \arrow[r, "f"'] & {} \end{tikzcd}$$

In order to prove the result we are going to show that $g = h'$ and $h = y$. First notice that $(f.e)^*z = x$ follows because the outer rectangle in the statement of the lemma is a pullback. Also, as the left hand square in the statement is a pullback, it must be the case that $d_0$ and $d_1$ form the kernel pair of $d$. As $e'$ is also the coequalizer of $d_0$ and $d_1$ it follows that $d$ and $e'$ are isomorphic. We can assume they are equal and it follows that $h = y$. Also, $h'.d = z^*(f.e) = g.d$ which implies (being d epi) that $h' = g$. So the right hand square in the statement is a pullback. □

We can now relate regular monos in $\mathbf{C}$ and $\mathbf{C}_{reg}$.

**Lemma 4.2.2.** *Let* $\mathbf{C}$ *have a strong-subobject classifier. Consider the following pullback square in* $\mathbf{C}_{reg}$ *with the horizontal maps being projective covers.*

$$\begin{tikzcd} Y \arrow[r, two heads] \arrow[d, "m"'] & B \arrow[d, "n"] \\ X \arrow[r, two heads, "e"'] & A \end{tikzcd}$$

*Then $m$ is a regular mono if and only if $n$ is.*

*Proof.* The *if* direction is trivial so consider the converse. As $m$ is a regular mono, we have a classifying map $\chi_m : X \longrightarrow \Omega$. Let $e_0, e_1$ be the kernel pair of $e$. As the square in the statement is a pullback, it follows that $e_0$ and $e_1$ pull $m$ back to the same subobject. Then, $\chi_m.e_0 = \chi_m.e_1$. It follows that $\chi_m$ factors uniquely through $e$ via the dashed arrow in the diagram below. It follows that the outer rectangle is a pullback.

$$\begin{tikzcd}
Y \arrow[r, two heads] \arrow[d, "m"'] & B \arrow[r, "!"] \arrow[d, "n"] & 1 \arrow[d, "\top"] \\
X \arrow[r, two heads, "e"'] & A \arrow[r, dashed] & \Omega
\end{tikzcd}$$

We can apply Lemma 4.2.1 to conclude that $n$ is a pullback of the regular mono $\top$ and so a regular mono itself. □

The following fact is the main ingredient of Proposition 4.2.4 to be proved below.

**Lemma 4.2.3.** *If* $\mathbf{C}$ *has a strong-subobject classifier then regular monos compose in* $\mathbf{C}_{reg}$.

*Proof.* Let $n' : C \rightarrowtail B$ and $n : B \rightarrowtail A$ be two regular monos in $\mathbf{C}_{reg}$. Let $e : X \twoheadrightarrow A$ be a projective cover and pull the composition $n.n'$ back along it.

$$\begin{tikzcd}
Z \arrow[r, tail] \arrow[d, two heads] & Y \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads, "e"] \\
C \arrow[r, tail, "n'"'] & B \arrow[r, tail, "n"'] & A
\end{tikzcd}$$

As regular monos compose in $\mathbf{C}$ (recall Section 2.3.6), the top composition is a regular mono. By Lemma 4.2.2, it follows that the bottom composition is also a regular mono. □

We can now state the main fact about regular completions and strong-subobject classifiers.

**Proposition 4.2.4.** *Let* $\mathbf{C}$ *be lextensive and have quasi-coequalizers. If* $\mathbf{C}$ *has a strong-subobject classifier then so does* $\mathbf{C}_{reg}$.

*Proof.* By Corollary 4.1.4, $\mathbf{C}_{reg}$ has finite colimits and by Lemma 4.2.3, regular monos compose in $\mathbf{C}_{reg}$.

Proposition 12.5 in [106] says that in any category with finite limits and such that regular monos compose, if a map has a cokernel pair then it factors as an epi followed by a regular mono. It follows that in our case, every map in $\mathbf{C}_{reg}$ has an epi/regular-mono factorization.

Recall from Section 2.3.6 that in a category with this type of factorization, strong and regular monos coincide. It follows that every strong subobject in $\mathbf{C}_{reg}$ of a projective is a regular subobject and so is itself projective (because $\mathbf{C}$ is closed under finite limits). Then, as $\mathbf{C}$ has a strong-subobject classifier, $\mathbf{C}_{reg}$ has a classifier of strong subobjects of projectives (recall Definition 3.5.4). Then, by Corollary 3.5.5, $\mathbf{C}_{reg}$ has a strong-subobject classifier. □

## 4.3 Regular completions and quasi-toposes

In this section we put together the results of the previous sections and with the help of the analysis of local cartesian closure done in Section 3.3.1 we present sufficient conditions on a category $\mathbf{C}$ for its regular completion to be a lextensive quasi-topos. An interesting observation is that every lextensive quasi-topos satisfies these conditions itself.

Recall from Section 3.3.1 that an equivalence relation is called *regular* if the corresponding mono is.

**Proposition 4.3.1.** *Let $\mathbf{C}$ be lextensive in which every regular equivalence relation is a kernel pair. Moreover, let $\mathbf{C}$ have the following properties:*

1. *stable epi/regular-mono factorizations*
2. *quasi-coequalizers*
3. *weak dependent products*
4. *a strong-subobject classifier*

*then $\mathbf{C}_{reg}$ is a lextensive quasi-topos.*

*Proof.* By Proposition 4.1.1 $\mathbf{C}_{reg}$ is lextensive and by Corollary 3.3.7 $\mathbf{C}_{reg}$ is locally cartesian closed.

As $\mathbf{C}$ is lextensive and has quasi-coequalizers we obtain by Corollary 4.1.4 that $\mathbf{C}_{reg}$ has finite colimits.

As $\mathbf{C}$ is lextensive, has quasi-coequalizers and a strong-subobject classifier, it follows by Proposition 4.2.4 that $\mathbf{C}_{reg}$ has a strong-subobject classifier. □

In the cases we are involved with, the fact that every regular equivalence relation is a kernel pair is actually a consequence of a more basic fact.

**Definition 4.3.2.** In any category call an object $X$ *exponentiating* if for every object $Y$ the exponential $X^Y$ exists.

Also, we say that a strong-subobject classifier $\top : 1 \rightarrowtail \Omega$ is exponentiating if $\Omega$ is.

**Proposition 4.3.3.** *In a category with finite limits and an exponentiating strong-subobject classifier every regular equivalence relation is a kernel pair.*

*Proof.* This is the argument to prove that in toposes every equivalence relation is effective [45, 51]. For any regular equivalence relation $e = \langle e_0, e_1 \rangle$ on $X$ consider its classifying map $\chi_e : X \times X \longrightarrow \Omega$. It is then proved that $e_0, e_1$ is the kernel pair of the transposition $\overline{\chi_e} : X \longrightarrow \Omega^X$. $\square$

So we easily obtain the following corollary.

**Corollary 4.3.4.** *Let* $\mathbf{C}$ *be a lextensive category with the following properties:*

1. *stable epi/regular-mono factorizations*

2. *quasi-coequalizers*

3. *weak dependent products*

4. *an exponentiating strong-subobject classifier*

*Then* $\mathbf{C}_{reg}$ *is a lextensive quasi-topos.*

In Section 8.4 we will review this result under a different light that will allow us to simplify it by subsuming most of the conditions in its statement under one simple conceptual property that our examples enjoy. In the meantime let us apply the result as it is.

Consider the category **PAss** of partitioned assemblies for some PCA. We mentioned in Section 2.2.2 that it has stable epi/regular-mono factorizations, we described quasi-coequalizers in Example 4.1.5 and it was shown in [13, 21] that it has weak dependent products. Finally, it is easy to show that the strong-subobject classifier described in Section 2.3.6 is exponentiable (abstractly, this follows from properties of the adjunction $|\_| \dashv \nabla$). So it follows by Corollary 4.3.4 that $\mathbf{PAss}_{reg} \simeq \mathbf{Ass}$ is a quasi-topos.

There is another conceptual explanation for this. The category **Ass** is a quasi-topos because it is a category of separated objects for a topology in the topos $\mathbf{PAss}_{ex}$ [39, 46, 16]. On the other hand, this explanation does not work in the case of **Top**.

The exact completion of **Top** is not a topos (it is not well powered). But, **Top** has epi/regular-mono factorizations, it is cocomplete and hence has quasi-coequalizers trivially, has weak dependent products as explained in [13, 21] and it is again easy to see that the strong-subobject classifier described in Section 2.3.6 is exponentiating (same argument as in the case of **PAss**). This is why $\mathbf{Top}_{reg} \simeq \mathbf{Equ}$ is a quasi-topos.

As quasi-toposes are locally cartesian closed it follows by Proposition 4.3.3 that every regular equivalence relation in them is a kernel pair. It is clear then that lextensive quasi-toposes satisfy the premises of Corollary 4.3.4 and so it is possible to iterate this result in the following sense. Let $\mathbf{C}_{reg(0)} = \mathbf{C}$ and $\mathbf{C}_{reg(n+1)} = (\mathbf{C}_{reg(n)})_{reg}$.

**Corollary 4.3.5.** *Let* $\mathbf{C}$ *be as in Corollary 4.3.4. Then for every* $n > 0$*,* $\mathbf{C}_{reg(n)}$ *is a quasi-topos.*

Finally, let us prove that these hierarchies are not trivial.

**Proposition 4.3.6.** *If* $\mathbf{C}$ *is not a regular completion then for any* $n, m$*,* $\mathbf{C}_{reg(n)} \simeq \mathbf{C}_{reg(m)}$ *implies that* $n = m$*.*

*Proof.* For the sake of contradiction let us assume that there exist an $n < m$ such that $\mathbf{C}_{reg(n)}$ is equivalent to $\mathbf{C}_{reg(m)}$. As $\mathbf{C}$ is not a regular completion it follows that $n > 0$. Now, let $m$ be the least number such that there exists an $n < m$ such that $\mathbf{C}_{reg(n)} \simeq \mathbf{C}_{reg(m)}$. It follows that $Proj(\mathbf{C}_{reg(n)})$ is equivalent to $Proj(\mathbf{C}_{reg(m)})$. That is, that $\mathbf{C}_{reg(n-1)}$ is equivalent to $\mathbf{C}_{reg(m-1)}$. But this is absurd because we have assumed that $m$ was the least one. $\square$

If we let $\mathbf{C}$ be $\mathbf{PAss}(K_1)$ then we have that $\mathbf{C}_{reg(1)} = \mathbf{C}_{reg}$ is the category **Ass**. For $n > 1$ see Section 8.2.

For $\mathbf{C} = H_+$ we have a clearer picture.

## 4.4 The hierarchy associated to a frame

Let **SLat** be the category whose objects (called semilattices) are partially ordered sets with all finite meets and whose morphisms are functions preserving finite meets. There is an obvious forgetful functor $\mathbf{Frm} \longrightarrow \mathbf{SLat}$.

A subset $A'$ of a semilattice $A$ is a *lower subset* if for every $a$ in $A'$ and $a \geq a' \in A$ it follows that $a' \in A'$. For any $a$ in $A$, we have that $\downarrow a = \{a' | a' \leq a\}$ is a lower set.

Now, for any semilattice $A$ let $DA$ denote the set of all lower subsets of $A$ ordered by inclusion. In [47] (Theorem 1.2) it is shown that this gives rise to a functor $D : \mathbf{SLat} \longrightarrow \mathbf{Frm}$ that is left adjoint to the forgetful $\mathbf{Frm} \longrightarrow \mathbf{SLat}$.

We need a slightly modified version of $D$. Indeed, let $D_+A$ denote the set of all *non-empty* lower subsets of $A$. It is not difficult to show that $D_+A$ has a least element if $A$ does and that in this case $D_+A$ is a frame.

Also, let us describe regular epis in categories of the form $H_+$. It is not difficult to show that they are characterized as the epi maps $q : Z \longrightarrow Y$ such that $\|y\|_Y = \bigvee\{\|z\|_Z | qz = y\}$.

**Proposition 4.4.1.** *For any frame $H$, $(H_+)_{reg} \simeq (D_+H)_+$.*

*Proof.* As $D_+H$ is a frame, $(D_+H)_+$ is a regular category. We are going to use Proposition 3.2.2 to show that it is the regular completion of $H_+$.

Using that $\downarrow a \subseteq \downarrow a'$ if and only if $a \leq a'$, it is easy to show that there is an embedding $\mathbf{y} : H_+ \longrightarrow (D_+H)_+$ that assigns to each $X$ in $H_+$ the object $\mathbf{y}X = (|X|, \downarrow \|\text{-}\|_X)$ in $(D_+H)_+$.

Let us now prove that the objects of the form $\mathbf{y}X$ are projective. So let $f : \mathbf{y}X \longrightarrow Y$ be any map in $(D_+H)_+$ to the codomain of a regular epi $q$ as above. Then, for every $x$ in $\mathbf{y}X$ we have that $\|x\| = \downarrow \|x\|_X \subseteq \|fx\|_Y = \bigcup\{\|z\|_Z | qz = fx\}$. As $q$ is a regular epi, there exists a $z_x$ in $Z$ such that $\downarrow \|x\|_X \subseteq \|z_x\|$. It follows that the assignment $x \longmapsto z_x$ induces a map $\mathbf{y}X \longrightarrow Z$ such that composed with $q$ gives $f$. Hence, $\mathbf{y}X$ is projective.

Now suppose that $P$ is projective in $(D_+H)_+$. We show that it is in the image of $\mathbf{y} : H_+ \longrightarrow (D_+H)_+$. For any element $p$ of $P$, let $Z_p$ be the object given by $|Z_p| = \{a \in H | a \in \|p\|\} + \{\top\}$, $\|in_l a\| = \downarrow a$ and $\|in_r \top\| = H$.

Also, let $Y_p$ be given by $|Y_p| = \{p\} + \{\top\}$, $\|in_l p\| = \|p\|_P$ and $\|in_r \top\| = H$.

There is an obvious epi $q_p : Z_p \twoheadrightarrow Y_p$ that sends $in_r\top$ to $in_r\top$ and everything else to $in_l p$. It is clearly a regular epi.

Now, consider the map $P \longrightarrow Y_p$ that sends $p$ to $in_l p$ and everything else to $\top$. The fact that $P$ is projective shows that $\|p\| \subseteq \downarrow a$ for some $a \in \|p\|$. It follows that $\|p\| = \downarrow a$. As this is for every $p$, $P$ is in the image of $\mathbf{y}$.

So we have characterized the image of $\mathbf{y}$ as the full subcategory of projectives.

Now, let us prove that every object $C$ in $(D_+H)_+$ is covered by a projective.

Let $X$ be the object in the image of $\mathbf{y}$ given by $|X| = \{(c, a) | c \in C, a \in \|c\|\}$ and $\|c, a\| = \downarrow a$. There is an obvious regular epi $X \twoheadrightarrow C$.

There only remains to prove that every $C$ is embedded in a projective. For this, let $X$ be such that $|X| = |C|$ and $\|c\|_X = \downarrow(\bigvee \|c\|_C)$. Clearly, $X$ is projective and it is easy to prove that there is a mono $C \rightarrowtail X$. $\square$

A word of acknowledgement goes to Jaap van Oosten who found a mistake in a previous version of this result.

It follows that $(H_+)_{reg(n)} \simeq ((D_+)^n H)_+$.

# Chapter 5

# The categories whose exact completions are toposes

In this chapter we give a characterization of the categories with finite limits whose exact completions are toposes. The main ingredient is the notion of a *generic proof*. These are introduced in Section 5.2 together with the statement of the characterization and some examples showing, in particular, how to use the characterization to present realizability toposes. We prove the characterization in Section 5.3 and then observe a curious relation between a strong version of generic proofs and the axiom of choice. We also describe the structure that arises in the regular completion of a category with a generic proof and finally discuss the relevance of the different classifying and generic notions that we introduce.

## 5.1 The proof-theoretic power set functor

For any category with finite limits $\mathbf{C}$, let us denote by $Prf$ the contravariant functor that for every object $X$ in $\mathbf{C}$, $Prf(X) = \widetilde{\mathbf{C}/X}$ is the poset reflection of the slice. It operates on arrows by pullback. For any $f : Y \longrightarrow X$ we denote the corresponding element in $Prf(X)$ by $\lfloor f \rfloor$. Lawvere calls this functor the *proof-theoretic power set* functor in [62]. For any $X$, the elements of $Prf(X)$ will be called *proofs*.

As explained in [62], $Prf(X)$ can be a proper class even if the category is well powered (an example given by the topos $\mathbf{Set}^{\rightrightarrows}$ of (directed) irreflexive graphs). Also, the characterization of the Grothendieck toposes for which the functor $Prf$ takes values in $\mathbf{Set}$ is posed as an open problem.

The relation of the proof-theoretic power set functor with our current problem is the following simple observation. Recall from Section 3.5 that a category $\mathbf{D}$ is called *suitable* if it is regular, has enough projectives and projectives are closed under finite limits.

**Lemma 5.1.1.** *Let* $\mathbf{D}$ *be a suitable category and let* $\mathbf{y} : \mathbf{C} \longrightarrow \mathbf{D}$ *be the embedding of the full subcategory of projectives. Then, there exists a natural isomorphism* $Sub_{\mathbf{D}}(\mathbf{y}X) \cong Prf_{\mathbf{C}}(X)$.

*Proof.* We give a sketch. For any proof $\lfloor f \rfloor \in Prf_{\mathbf{C}}(X)$ assign $Im(f) \in Sub_{\mathbf{D}}(\mathbf{y}X)$. It follows from Lemma 2.3.3 that this assignment is well defined. On the other hand, given any $m : U \rightarrowtail \mathbf{y}X$ in $Sub_{\mathbf{D}}(\mathbf{y}X)$, we can cover $U$ with a projective $\mathbf{y}Y$ giving a map $f : Y \longrightarrow X$ and thus a proof $\lfloor f \rfloor$. Because we are covering with projective objects, any two such coverings induce the same proof. So this assignment is also well defined. We leave it as an easy exercise to prove that these assignments give a natural iso. □

So the proofs in $\mathbf{C}$ give a perfect picture of the subobjects of projectives in $\mathbf{D}$. This is an essential ingredient in the proof of our characterization.

## 5.2 Generic proofs

As we will show in Section 5.4, only in very special cases it is possible to classify proofs. On the other hand, the possibility of weakly classifying them is intimately related to our characterization.

**Definition 5.2.1.** A *generic proof* is a map $\theta : \Theta \to \Lambda$ such that for every map $f : Y \to X$ there exists a (not necessarily unique) $\nu_f : X \to \Lambda$ such that $f$ factors through $\nu_f^* \theta$ and $\nu_f^* \theta$ factors through $f$.

$$\begin{tikzcd}
Y \arrow[r, shift left] \arrow[dr, "f"'] & Y' \arrow[l, shift left] \arrow[r] \arrow[d, "\nu_f^*\theta"] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d, "\theta"] \\
& X \arrow[r, "\nu_f"'] & \Lambda
\end{tikzcd}$$

With this terminology we are ready to state our characterization.

**Theorem 5.2.2.** *If* $\mathbf{C}$ *has finite limits,* $\mathbf{C}_{ex}$ *is a topos if and only if* $\mathbf{C}$ *has weak dependent products and a generic proof.*

Let us look at a couple of simple examples.

**Example 5.2.3 (PAss(A) has a generic proof).** Recall the "chaotic" inclusion of **Set** into **PAss** (Section 2.2.2). Let $\wp\mathbf{A}$ be the set of subsets of $\mathbf{A}$ and let $\Lambda = \nabla(\wp\mathbf{A})$. Moreover, let $\Theta$ be defined so that $|\Theta| = \{(U,a)|U \subseteq \mathbf{A} \text{ and } a \in U\}$ and $\|(U,a)\| = a$.

We have an obvious map $\theta : \Theta \to \Lambda$ with first projection as underlying function. We now prove that $\theta$ is a generic proof. Let $f : Y \to X$ be realized by $a_f$. Then define $\nu : X \to \Lambda$ by $\nu x = \{\|y\| \; |fy = x\}$.

Let $P$ be the pullback of $\nu$ and $\theta$. It has underlying set $|P| = \{(x, \nu x, a)|a \in \nu x\}$ and $\|(x,U,a)\| = \langle \|x\|, a\rangle$. It is easy to see that $f$ factors through $\pi : P \to X$ via the function $y \longmapsto (fy, \nu fy, \|y\|)$ which is realized by $a \longmapsto \langle a_f a, a\rangle$.

Now for each $(x,U,a) \in |P|$ there exists a $gx \in |Y|$ such that $fgx = x$ and $\|gx\| = a$. Using *choice* we obtain a function $g : |P| \to |Y|$ that is realized by the projection $(x,U,a) \longmapsto a$. It clearly holds that $f.g = \pi$.

Realizability toposes for PCAs have been shown to be exact completions of the respective categories of partitioned assemblies in [96]. So it follows by Theorem 5.2.2 that the latter categories must have generic proofs. On the other hand, the point we want to emphasize is the alternative way of presenting realizability toposes. The proof in [96] is done by constructing the toposes and then using the characterization of exact completions (Proposition 3.3.2). With Theorem 5.2.2 and Example 5.2.3 together with the fact that **PAss** has weak dependent products (see Section 3.3.1) it is possible to *introduce* realizability toposes as exact completions without having to build the toposes first in another way.

At this point it is important to discuss the use of the axiom of choice in Example 5.2.3. We claim that it is essential. In order to see this, consider a realizability topos over a PCA **A** seen as the ex/reg completion of the corresponding category of assemblies (this presentation does not require the axiom of choice). Now assume that partitioned assemblies are projective in the topos. We now show that it follows that every epi splits in **Set**.

It is not difficult to show that the embedding $\nabla : \mathbf{Set} \longrightarrow \mathbf{PAss}$ preserves regular epis. The embedding of **PAss** into **Ass** also preserves the regular epis coming from **Set** and hence, so does the embedding of **PAss** into the corresponding realizability topos. So, if partitioned assemblies are to be projective then regular epis split in **Set**. That is, if we want realizability toposes over PCAs to be exact completions of the corresponding categories of partitioned assemblies then we must accept the axiom of choice in **Set**.

**Example 5.2.4 ($H_+$ has a generic proof).** Recall Section 2.3.1 and also Section 3.1 where we mentioned that $H_+$ is the coproduct completion of $H$. It follows by Proposition 3.3.3 that the exact completion of $H_+$ is the topos of presheaves on $H$. Let us describe a generic proof in $H_+$.

Let $\wp H$ be the set of subsets of $H$ and define $\Lambda = \nabla(\wp H)$.

Also, let $|\Theta| = \{(U,a) | a \in U \in |\Lambda|\}$ and $\|(U,a)\| = a$.

Using the same idea as in Example 5.2.3, it is not difficult to prove that the first projection $\Theta \to \Lambda$ is a generic proof in $H_+$.

Now, by Proposition 3.3.3 and Theorem 5.2.2 it must be the case that every coproduct completion with finite limits has a generic proof. We now give an explicit construction of them in order to give more examples of generic proofs.

First, we need a small lemma.

**Lemma 5.2.5.** *If* $\mathbf{C}$ *be a small category then for any* $X$ *in* $\mathbf{C}_+$, $\mathit{Prf}(X)$ *is a set.*

*Proof.* Let $f : Y \longrightarrow X$ in $\mathbf{C}_+$. We can assume that $Y$ is a small coproduct $\coprod_{i\in I} C_i$ of objects in $\mathbf{C}$. It follows that $f$ is determined by a family of maps $\{f_i : C_i \longrightarrow X\}_{i\in I}$. Reordering things a little bit it is easy to see that $\lfloor f \rfloor$ is determined by a family $\{U_C\}_{C\in\mathbf{C}}$ where each $U_C$ is a subset (maybe empty) of $\mathbf{C}_+(C,X)$. That is $\lfloor f \rfloor$ is determined by a subset of the small coproduct $\coprod_{C\in\mathbf{C}} \mathbf{C}_+(C,X)$. Hence, $\mathit{Prf}(X)$ is bounded by the set $\mathit{Sub}(\coprod_{C\in\mathbf{C}} \mathbf{C}_+(C,X))$. $\square$

It is easy to check that in a category with stable and disjoint coproducts the functor $\mathit{Prf}$ carries coproducts to products. That is, there exists a natural isomorphism

$$\prod_{i\in I} \mathit{Prf}(X_i) \cong \mathit{Prf}(\coprod_{i\in I} X_i)$$

Notice also that $\mathbf{C}$ has a generic proof if and only if there exists an object $\Lambda$ and a natural epi $\mathbf{C}(X,\Lambda) \twoheadrightarrow \mathit{Prf}(X)$. We can say $\mathit{Prf}$ is *weakly representable.*

We can now describe the generic proofs.

**Proposition 5.2.6.** *If* $\mathbf{C}$ *is a small category then* $\mathbf{C}_+$ *has a generic proof.*

*Proof.* Let $\mathbf{P} = \{(p,C) | C \in \mathbf{C}, p \in \mathit{Prf}(C)\}$. It is a set because $\mathbf{C}$ is small and by Lemma 5.2.5, so is each $\mathit{Prf}(C)$. For each $(p,C) \in \mathbf{P}$ *choose* a map $f_p : X_p \longrightarrow C$ such that $\lfloor f_p \rfloor = p$. Now consider the following small coproduct of maps.

$$\coprod_{(p,C)\in\mathbf{P}} f_p : \coprod_{(p,C)\in\mathbf{P}} X_p \longrightarrow \coprod_{(p,C)\in\mathbf{P}} C$$

Denote this map by $\theta : \Theta \longrightarrow \Lambda$. We now prove that $\theta$ is a generic proof. To do this, consider first a connected object $C$. We can assume it is in $\mathbf{C}$. Let $g : Z \longrightarrow C$ be any map and consider the following diagram.

$$\begin{tikzcd}
Z \arrow[r, shift left] \arrow[drr, "g"'] & X_{\lfloor g \rfloor} \arrow[l, shift left] \arrow[r] \arrow[d, "f_{\lfloor g \rfloor}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta \arrow[d, "\theta"] \\
 & C \arrow[r, "in_{(\lfloor g \rfloor, C)}"'] & \Lambda
\end{tikzcd}$$

So $\Lambda$ is a generic "proof of connected object".

Now for an arbitrary $X$. Again, by Proposition 3.1.2 and without loss of generality we can assume that $X = \coprod_{i \in I} C_i$ with $I$ a set and for each $i \in I$, $C_i$ in $\mathbf{C}$. The following calculation shows that $\Lambda$ weakly represents $Prf$.

$$Prf(X) = Prf(\coprod_{i \in I} C_i) \cong$$

$$\cong \prod_{i \in I} Prf(C_i) \twoheadleftarrow \prod_{i \in I} \mathbf{C}_+(C_i, \Lambda) \cong \mathbf{C}_+(\coprod_{i \in I} C_i, \Lambda) = \mathbf{C}_+(X, \Lambda)$$

□

Strictly speaking, $\mathbf{C}_+$ should have finite limits in order for $Prf$ to be a functor and therefore the argument above be completely correct. But as pullbacks along injections exists in $\mathbf{C}_+$ there is really no essential problem in showing that the map $\Theta \longrightarrow \Lambda$ above is a generic proof. For more on limits in coproduct completions see [36].

## 5.3 The proof of the characterization

First, we need to introduce a technical notion related to the classification of subobjects.

**Definition 5.3.1.** A *generic subobject-of-projective* is a mono $\tau : \Upsilon \rightarrowtail \Lambda$ such that for every projective $X$ and mono $m : U \rightarrowtail X$ there exists a (not necessarily unique) $\nu_m : X \longrightarrow \Lambda$ such that the following square is a pullback.

$$\begin{tikzcd}
U \arrow[r] \arrow[d, tail, "m"'] & \Upsilon \arrow[d, tail, "\top"] \\
X \arrow[r, "\nu_m"'] & \Lambda
\end{tikzcd}$$

Also, a generic subobject-of-projective $\tau : \Upsilon \rightarrowtail \Lambda$ is *projective* if $\Lambda$ is.

We have already mentioned that $\mathbf{C}$ has a generic proof if and only if there exists an object $\Lambda$ and a natural epi $\mathbf{C}(X, \Lambda) \twoheadrightarrow \mathit{Prf}(X)$. That is, if $\mathit{Prf}$ is *weakly representable.*

Now, consider a suitable category $\mathbf{D}$ with $\mathbf{C}$ as its full subcategory of projectives. The existence of a projective generic subobject-of-projective in a suitable category $\mathbf{D}$ is equivalent to the fact that $\mathit{Sub} : \mathbf{D}^{\mathbf{op}} \to \mathbf{Set}$ is *weakly representable over projectives.* That is, that there exists a natural epimorphism $\mathbf{C}(X, \Lambda) \cong \mathbf{D}(\mathbf{y}X, \mathbf{y}\Lambda) \twoheadrightarrow \mathit{Sub}_{\mathbf{D}}(\mathbf{y}X)$ (recall that $\mathbf{y}$ denotes the embedding of the full subcategory of projectives $\mathbf{C}$ into $\mathbf{D}$).

**Lemma 5.3.2.** *The following are equivalent.*

1. $\mathbf{C}$ *has a generic proof*

2. $\mathbf{D}$ *has a projective generic subobject-of-projective*

3. $\mathbf{D}$ *has a generic subobject-of-projective*

*Proof.* The equivalence between 1 and 2 is trivial by Lemma 5.1.1. Indeed, this is just $\mathbf{C}(X, \Lambda) \twoheadrightarrow \mathit{Prf}_{\mathbf{C}}(X) \cong \mathit{Sub}_{\mathbf{D}}(\mathbf{y}X)$. To be more explicit, given a generic proof $\theta : \Theta \longrightarrow \Lambda$ in $\mathbf{C}$, the mono part of its factorization in $\mathbf{D}$ is a (projective) generic subobject-of-projective. On the other hand, given any projective generic subobject-of-projective $\tau : \Upsilon \rightarrowtail \Lambda$ in $\mathbf{D}$, the map to $\Lambda$ induced by a projective cover of $\Upsilon$ is a generic proof in $\mathbf{C}$.

Trivially, 2 implies 3. So we need only prove that 3 implies 2. Let $\tau' : \Upsilon' \rightarrowtail \Lambda'$ be a generic subobject-of-projective in $\mathbf{D}$. Let $\rho : \Lambda \twoheadrightarrow \Lambda'$ be a projective cover. Then define $\tau : \Upsilon \rightarrowtail \Lambda$ to be the pullback of $\tau'$ along $\rho$. To prove that this $\tau$ is a generic subobject-of-projective, let $X$ be projective and let $m : U \rightarrowtail X$. By hypothesis, there exists a $\chi_m : X \to \Lambda'$ such that $m$ is the pullback of $\tau'$ along $\chi_m$. Now, as $X$ is projective and $\rho$ is a regular epi, there exists a $\nu_m : X \to \Lambda$ such that $\rho.\nu_m = \chi_m$. It follows by the Pasting Lemma that $m$ is the pullback of $\tau$ along $\nu_m$ as in the following diagram.

$$\begin{tikzcd}
U \arrow[r] \arrow[d, tail, "m"] & \Upsilon \arrow[r, two heads] \arrow[d, tail, "\tau"] & \Upsilon' \arrow[d, tail, "\tau'"] \\
X \arrow[r, "\nu_m"'] & \Lambda \arrow[r, two heads, "\rho"'] & \Lambda'
\end{tikzcd}$$

□

The strategy for the proof of Theorem 5.2.2 is to build a classifier of subobjects of projectives (Definition 3.5.4) out of a generic subobject-of-projective (Definition 5.3.1). In order to do this, we are going to use the locally cartesian closed structure of the exact completion to build an equivalence relation on $\Lambda$. The quotient of this equivalence relation will classify subobjects of projectives. Then, using Corollary 3.5.5 we will be able to conclude that there exists a subobject classifier.

The following lemma explains how to build the equivalence relation.

**Lemma 5.3.3.** *Let* $\mathbf{E}$ *be locally cartesian closed and let* $m : U \rightarrowtail X$*. Then there exists an arrow* $m' : U' \rightarrowtail X \times X$ *such that* $\langle f, g\rangle : Z \to X \times X$ *factors through* $m'$ *if and only if* $f$ *and* $g$ *pull* $m$ *back to the same subobject (i.e.* $f^*m \cong g^*m$*).*

*Proof.* Consider $\gamma = m \times id_X : U \times X \rightarrowtail X \times X$ and $\delta = id_X \times m : X \times U \rightarrowtail X \times X$ as objects in the slice $\mathbf{E}/(X \times X)$. We can then build the mono $m' = \gamma^\delta \times \delta^\gamma : U' \rightarrowtail X \times X$ using the product and exponentiation in the slice.

Now, let $\langle f, g\rangle : Z \to X \times X$ factor through $m'$. That is, we have an arrow $\langle f, g\rangle \to m'$ in the slice $\mathbf{E}/(X \times X)$. This is uniquely determined by arrows $\langle f, g\rangle \to \gamma^\delta$ and $\langle f, g\rangle \to \delta^\gamma$. Let us concentrate on the arrow on the left. It is uniquely determined by an arrow $\langle f, g\rangle \times \delta \to \gamma$. Products in the slice are just pullbacks in $\mathbf{E}$, so we have an easy description of the domain of this arrow

$$\begin{tikzcd} g^*U \arrow[r, "\langle f.(g^*m), m^*g\rangle"] \arrow[d, tail, "g^*m"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times U \arrow[d, tail, "id_X \times m"] \\ Z \arrow[r, "\langle f, g\rangle"'] & X \times X \end{tikzcd}$$

So, our arrow $\langle f, g\rangle \times \delta \to \gamma$ is just an arrow $\langle h, h'\rangle : g^*U \to U \times X$ such that $(m \times id_X).\langle h, h'\rangle = \langle f, g\rangle.g^*m = (id_X \times m).\langle f.(g^*m), m^*g\rangle$. This implies $m.h = f.g^*m$ and then it follows that $g^*m \leq f^*m$.

$$
\begin{tikzcd}
g^*U \arrow[dr, dashed, "\exists"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g^*m"'] & & \\
& f^*U \arrow[r] \arrow[d, tail, "f^*m"] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, tail, "m"] \\
& Z \arrow[r, "f"'] & X
\end{tikzcd}
$$

Similarly, the arrow $\langle f, g\rangle \to \delta^\gamma$ implies that $f^*m \leq g^*m$.

On the other hand, if we start assuming that $g^*m \leq f^*m$, by following the proof above from bottom to top, it is easy to prove that there is an arrow $\langle f, g\rangle \to \gamma^\delta$. Using the same idea, starting from $f^*m \leq g^*m$ it is easy to prove the existence of $\langle f, g\rangle \to \delta^\gamma$. So, if $f$ and $g$ pullback $m$ to the same subobject, then $\langle f, g\rangle$ factors through $m'$. ☐

Clearly, "pulling back an arrow with codomain $X$ to the same thing" determines an equivalence relation on the hom-sets $\mathbf{E}(\_,X)$. It follows by the remark below Definition 2.4.2 that the $m' = \langle m_0, m_1\rangle$ built above determines an equivalence relation. Notice that $U'$ can be defined, using the internal logic, by $U' := x, x' : X \vdash (Ux \to Ux') \wedge (Ux' \to Ux)$.

**Proposition 5.3.4.** *If $\mathbf{C}_{ex}$ is locally cartesian closed then the following are equivalent:*

1. *$\mathbf{C}_{ex}$ is a topos*

2. *$\mathbf{C}$ has a generic proof*

*Proof.* To see that 1 implies 2, notice that the subobject classifier is trivially a generic subobject-of-projective. It follows by Lemma 5.3.2 that $\mathbf{C}$ has a generic proof.

To prove that 2 implies 1, we apply again Lemma 5.3.2 to obtain a generic subobject-of-projective $\tau : \Upsilon \rightarrowtail \Lambda$ in $\mathbf{C}_{ex}$. By hypothesis, the slice $\mathbf{C}_{ex}/(\Lambda \times \Lambda)$ is cartesian closed. So we can apply Lemma 5.3.3 to obtain an equivalence relation $\tau' : \Upsilon' \rightarrowtail \Lambda \times \Lambda$ with the properties specified. We can then take the quotient:

$$
\begin{tikzcd}
\Upsilon' \arrow[r, shift left, "\tau_0"] \arrow[r, shift right, "\tau_1"'] & \Lambda \arrow[r, two heads, "\rho"] & \Omega
\end{tikzcd}
$$

Trivially, $\langle \tau_0, \tau_1 \rangle = \tau'$ factors through $\tau'$. Then $\tau_0^*\tau \cong \tau_1^*\tau$ by Lemma 5.3.3. Also, $\tau$ pulls the equivalence relation $\tau_0, \tau_1$ back to another equivalence relation (recall Section 2.5). As $\mathbf{C}_{ex}$ is exact, we can take its effective quotient and obtain the top exact sequence in the diagram below. Using the universal property of coequalizers we obtain the map $\top$ making the right hand square commute. It follows by Lemma 2.3.3 that the right hand square is a pullback. That is, $\rho^*\top = \tau$.

$$\begin{tikzcd}
\tau_i^*\Upsilon \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & \Upsilon \arrow[r, two heads] \arrow[d, tail, "\tau"'] & \cdot \arrow[d, tail, "\top"] \\
\Upsilon' \arrow[r, shift left, "\tau_0"] \arrow[r, shift right, "\tau_1"'] & \Lambda \arrow[r, two heads, "\rho"] & \Omega
\end{tikzcd}$$

We now prove that $\Omega$ classifies subobjects of projectives. It will then follow by Corollary 3.5.5 that $\mathbf{C}_{ex}$ is a topos.

So let $X$ be projective and let $m : U \rightarrowtail X$ be an arbitrary subobject. Then, as $\Lambda$ is a generic subobject-of-projective, there exists a $\nu_m$ such that $m = \nu_m^*\tau = \nu_m^*(\rho^*\top) = (\rho.\nu_m)^*\top$. This means that $\top$ is also a generic subobject-of-projective.

We need to prove that there is only one arrow classifying each subobject. So let $f', g' : X \to \Omega$ pull $\top$ back to the same subobject. As $X$ is projective, it follows that $f'$ and $g'$ factor through $\rho$, say via $f$ and $g$. Then $f$ and $g$ pullback $\tau$ to the same subobject. So there exists an $h$ such that $\langle f, g \rangle = \tau'.h$ by Lemma 5.3.3. But then $f = \tau_0.h$ and $g = \tau_1.h$. So $\rho.f = \rho.\tau_0.h = \rho.\tau_1.h = \rho.g$. That is, $f' = g'$. □

This proposition is the main ingredient in our characterization.

**Corollary 5.3.5.** (Theorem 5.2.2) *$\mathbf{C}_{ex}$ is a topos if and only if $\mathbf{C}$ has weak dependent products and a generic proof.*

*Proof.* Follows from Propositions 3.3.5 and 5.3.4. □

## 5.4 Proof classifiers

It is natural to wonder if there exist cases in which the functor *Prf* is representable.

By a *proof classifier* we mean a generic proof $\theta : \Theta \to \Lambda$ such that for every $f : Y \to X$ there is a *unique* $\nu_f : X \to \Lambda$ such that the diagram in Definition 5.2.1 commutes.

Of course, by Yoneda, this is equivalent to the fact that $\mathit{Prf}$ is representable. That is, to the existence of a natural isomorphism $Hom(X,\Lambda) \to \mathit{Prf}(X)$.

Toposes in which every epi splits provide trivial examples of proof classifiers as in this case $\mathit{Prf}(X)$ is isomorphic to $\mathit{Sub}(X)$. In fact, these are the only examples.

**Proposition 5.4.1.** *If $\mathit{Prf}$ is representable then every epi splits.*

*Proof.* Suppose that there exists an object $\Lambda$ and an isomorphism $Hom(X,\Lambda) \to \mathit{Prf}(X)$ natural in $X$.

Let $e : X \twoheadrightarrow Q$ be epi. Then $(\_.e) : Hom(Q,\Lambda) \rightarrowtail Hom(X,\Lambda)$ must be an inclusion. Hence, so is $e^* : \mathit{Prf}(Q) \rightarrowtail \mathit{Prf}(X)$. This means that for every $f : Z \to Q$ and $f' : Z' \to Q$, if $e^*f$ and $e^*f'$ factor through each other then so do $f$ and $f'$.

Now, clearly, $e^*e$ and $id_X = e^*id_Q$ factor through each other. Then, by hypothesis, $e$ and $id_Q$ factor through each other. That is, $e$ splits. □

## 5.5 Generic monos

In this section we identify the structure in $\mathbf{C}_{reg}$ corresponding to a generic proof in $\mathbf{C}$ and show two examples.

**Definition 5.5.1.** A *generic mono* in a category $\mathbf{D}$ is a mono $\tau : \Upsilon \rightarrowtail \Lambda$ such that every mono $u : U \rightarrowtail A$ in $\mathbf{D}$ arises as a pullback of $\tau$ along a not necessarily unique map.

**Proposition 5.5.2.** *Let $\mathbf{C}$ be a category with finite limits. $\mathbf{C}$ has a generic proof if and only if $\mathbf{C}_{reg}$ has a generic mono.*

*Proof.* For the *if* part notice that if $\mathbf{C}_{reg}$ has a generic mono then it has a generic subobject-of-projective. So, by Lemma 5.3.2, $\mathbf{C}$ has a generic proof.

For the *only if* part, again by Lemma 5.3.2, we need only prove that the existence of a generic subobject-of-projective in $\mathbf{C}_{reg}$ implies the existence of a generic mono.

Let $\tau : \Upsilon \rightarrowtail \Lambda$ be a generic subobject-of-projective in $\mathbf{C}_{reg}$. Now, let $u : U \rightarrowtail A$ be an arbitrary mono in $\mathbf{C}_{reg}$. By Proposition 3.2.2, there is a mono $m : A \rightarrowtail X$ into a projective $X$. Then, there exists a $\nu : X \longrightarrow \Lambda$ such that $m.u : U \rightarrowtail X$ arises as the pullback of $\tau$ along $\nu$. But then the following diagram is also a pullback.

$$\begin{tikzcd} U \arrow[rr, "\tau^*\nu"] \arrow[d, "u"'] & & \Upsilon \arrow[d, "\tau"] \\ A \arrow[r, tail, "m"'] & X \arrow[r, "\nu"'] & \Lambda \end{tikzcd}$$

This proves that $\tau$ is a generic mono.

□

It follows by Example 5.2.3 that the categories of assemblies $\mathbf{Ass} \simeq \mathbf{PAss}_{reg}$ all have generic monos. Indeed, by the proof of Proposition 5.5.2 we know that they arise as the images of the generic proofs described in Example 5.2.3. So we can calculate them immediately. The object $\Lambda$ is the same and $\Upsilon$ is given by $|\Upsilon| = \{U | \emptyset \neq U \subseteq \mathbf{A}\}$ and $\|U\| = U$.

Curiously, in spite of not being regular completions, the categories of partitioned assemblies and $H_+$ also have generic monos.

**Example 5.5.3 (PAss(A) has a generic mono).** Let $\Lambda = \nabla(\mathbf{A}+1)$. On the other hand, let $|\Upsilon| = \mathbf{A}$ and let $\|\text{\_}\|_\Upsilon : |\Upsilon| \longrightarrow \mathbf{A}$ be the identity function. Finally, let $\tau : \Upsilon \rightarrowtail \Lambda$ be the obvious (left) inclusion which is realized by a constant function.

Let us prove that so defined, $\tau$ is a generic mono. So let $m : U \rightarrowtail X$ be mono. Then let $\nu_m : X \longrightarrow \Lambda$ be defined as follows.

$$\nu_m x = \begin{cases} \|x\|_U & , \ \textit{if } x \in U \\ \perp & , \ \textit{otherwise} \end{cases}$$

It is easy to check that $\nu_m$ pulls $\tau$ back to $m$.

The case of sets valued on a frame is similar.

**Example 5.5.4 ($H_+$ has a generic mono).** Indeed, first let $\Lambda = \nabla(H+1)$. Then, let $|\Upsilon| = H$ and let $\|\text{\_}\|_\Upsilon : |\Upsilon| \longrightarrow H$ be the identity function.

Using the same idea as in Example 5.5.3 it is easy to prove that the obvious inclusion $\tau : \Upsilon \rightarrowtail \Lambda$ is a generic mono in $H_+$.

## 5.6 A word about classifying and generic things

We have introduced several similar notions of objects that weakly or strongly classify subobjects or proofs of arbitrary objects or of the class of projectives. We now briefly discuss their relevance.

The notions of classifier of subobjects of projectives (Definition 3.5.4) and of generic subobject-of-projective (Definition 5.3.1) are only good ways in which to structure the proof of Theorem 5.2.2 (Corollary 5.3.5). The former highlights at which point it is used the property of having enough projectives while the latter does the same for the existence of effective quotients.

On the other hand, of undoubted importance is the notion of a subobject classifier (Definition 2.5.2).

Now, about generic monos. We have come across them because they arise in the regular-epi/mono factorization of generic proofs. We later learned of the work [64] (see also Section 8.1) and we finally came up with Theorem 11.3.3 where they play an essential role in ex/reg completions (see Section 3.4) being toposes. So it seems that generic monos are also of independent interest and not just mono parts of generic proofs. In fact, we will see in Section 8.2 that in some cases they are at least as primitive as generic proofs in the sense that the existence of one concept is equivalent to the existence of the other. A hint of this fact is given already by Examples 5.5.3 and 5.5.4.

It should also be clear, from their role in Theorem 5.2.2, that generic proofs (Definition 5.2.1) are also of intrinsic interest. The applications of Theorem 5.2.2 discussed in Chapters 6 and 8 will make this more evident. Evidence of a different kind is given below.

### 5.6.1 Russell's axiom of reducibility

Here we show the connection between generic proofs and the type theoretic versions of Russell's *axiom of reducibility* as explained in [2, 33]. In order to explain this we recall the interpretation of type theory in a locally cartesian closed category [103]. Given a locally cartesian closed category $\mathbf{C}$, the idea is to interpret contexts as objects in $\mathbf{C}$, types in context $\Gamma$ as maps in $\mathbf{C}$ with codomain (the interpretation) of $\Gamma$ and terms as sections of (the interpretation of) their types. Moreover product and exponential (arrow) types are interpreted as products and exponentials in the appropriate slice of $\mathbf{C}$.

For example, we say that two types $A$ and $B$ in context $\Gamma$ are *equivalent* if there is a term (in context $\Gamma$) of type $B^A \times A^B$. We abbreviate this type by $A \leftrightarrow B$. By interpreting this definition in a locally cartesian closed $\mathbf{C}$ we obtain that $A$ and $B$ are equivalent in $\mathbf{C}$ if the interpretation of $A$ (a map with codomain $\Gamma$) factors through the interpretation of $B$ and vice-versa.

The type theoretic formulation of the axiom of reducibility is given as follows: There exists a type $\mathbf{P}$ whose elements are codes for types such that for every type

$A$ there exists an element $\langle A\rangle$ of $\mathbf{P}$ equivalent to $A$.

We now reproduce the formal rules expressing this using the notation of [33] and then explain how they are connected to generic proofs.

$$\frac{\Gamma \vdash A \ \mathit{type}}{\Gamma \vdash \langle A\rangle : \mathbf{P}} \qquad \frac{\Gamma \vdash A \ \mathit{type}}{\Gamma \vdash \iota : \mathbf{T_P}(\langle A\rangle) \leftrightarrow A} \qquad \frac{\Gamma \vdash a : \mathbf{P}}{\Gamma \vdash \mathbf{T_P}(a) \ \mathit{type}}$$

In order to see the connection to our work think of $\mathbf{P}$ as $\Lambda$ and let us work out how this rules are interpreted in a locally cartesian closed category.

The judgement $\Gamma \vdash A$ *type* should be thought of as giving a map $A \longrightarrow \Gamma$ and the judgement $\Gamma \vdash a : \mathbf{P}$ as giving a map $a : \Gamma \longrightarrow \mathbf{P}$.

So the right hand rule says that if we have a map $a : \Gamma \longrightarrow \mathbf{P}$ then there exists a map $\mathbf{T_P}(a) \longrightarrow \Gamma$. Actually, it says something more explicit. The map $id : \mathbf{P} \longrightarrow \mathbf{P}$ gives a map that we choose to denote $\theta : \Theta \longrightarrow \mathbf{P}$. Now for every map $a : \Gamma \longrightarrow \mathbf{P}$, the map $\mathbf{T_P}(a) \longrightarrow \Gamma$ induced by the rule is just $a^*\theta$.

On the other hand, the left hand rule is saying that for every map $f : A \longrightarrow \Gamma$ there exists a map $\langle f\rangle : \Gamma \longrightarrow \mathbf{P}$. Think of $\langle f\rangle$ as $\nu_f$.

Finally, the rule in the middle is saying that for any $f : A \longrightarrow \Gamma$, $f$ is equivalent to $\mathbf{T_P}(\nu_f)$. That is, $\lfloor f\rfloor = \lfloor \nu_f^*\theta\rfloor$.

So generic proofs are the same thing as the type theoretic expression of the axiom of reducibility. This suggests an interesting connection between type theory with the axiom of reducibility and locally cartesian closed categories $\mathbf{C}$ with a generic proof. By Theorem 5.2.2, the exact completion of any such is a topos and moreover, the natural isomorphism $Sub_{\mathbf{C}_{ex}}(\mathbf{y}X) \cong Prf_{\mathbf{C}}(X)$ gives the possibility of "internalizing" the type theory of the original category in the internal logic of the topos.

# Chapter 6

# Boolean presheaf toposes

In this very short chapter we characterize the presheaf toposes that have a generic proof. We also explain briefly the connection of these toposes with Läuchli's realizability [53, 62]. Moreover, the characterization above will also let us find other examples of Grothendieck toposes whose exact completions are themselves toposes.

## 6.1 Boolean presheaf toposes

In [16] the observation that for the topos $\mathbf{Set}^{\rightrightarrows}$ of irreflexive directed graphs, $Prf(1)$ is not small is attributed to Lawvere. From this, it follows that the exact completion of $\mathbf{Set}^{\rightrightarrows}$ is not a topos as it is not well powered.

Also in [16], the characterization of the class of toposes whose exact completions are toposes is posed as an open problem. We now know that they are the ones that have a generic proof. Moreover, for the restricted class of presheaf toposes we can give a very concrete description. Indeed, we extend the characterization of boolean presheaf toposes [45, 32, 71] as follows.

**Theorem 6.1.1.** *Let* $\mathbf{C}$ *be a small category. Then the following are equivalent.*

1. $\mathbf{Set}^{\mathbf{C}^{op}}$ *is boolean*
2. $\mathbf{C}$ *is a groupoid*
3. $\mathbf{Set}^{\mathbf{C}^{op}}$ *has an essentially small class of connected objects*
4. $\mathbf{Set}^{\mathbf{C}^{op}}$ *is the coproduct completion of a small category*
5. $\mathbf{Set}^{\mathbf{C}^{op}}$ *has a generic proof*
6. $(\mathbf{Set}^{\mathbf{C}^{op}})_{ex}$ *is a topos*

*Proof.* The equivalence between 1 and 2 is well known [45, 32, 71]. That between 3 and 4 is Proposition 3.1.2. That between 5 and 6 is Theorem 5.2.2. Also, Proposition 5.2.6 gives that 4 implies 5. So we only need to prove that 2 implies 3 and that 5 implies 2.

In order to prove that 2 implies 3 consider first the case that $\mathbf{C}$ is a group. Then, $\mathbf{Set}^{\mathbf{C}^{op}}$ is the topos of sets acted on by the group $\mathbf{C}$ [45, 71, 3]. It is well known (see [3] for example) that the connected objects in such a topos are the non-empty ones with only one orbit. Moreover, every connected object is isomorphic to one given by a *coset space* (see Proposition 4 in Section 3 of Chapter 1 in [3]) and so, the class of connected objects in $\mathbf{Set}^{\mathbf{C}^{op}}$ is essentially small.

If $\mathbf{C}$ is a groupoid then it is equivalent, as a category, to a small coproduct of groups. It follows that $\mathbf{Set}^{\mathbf{C}^{op}}$ is equivalent to a small product of toposes (as categories) of the form dealt with in the previous paragraph. Hence, the class of connected objects in $\mathbf{Set}^{\mathbf{C}^{op}}$ is essentially small for this case too.

It remains to show that 5 implies 2. Suppose, for the sake of contradiction, that there exists a generic proof $\theta : \Theta \longrightarrow \Lambda$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ and that $\mathbf{C}$ is not a groupoid. Then there exists a map $a : A \longrightarrow B$ in $\mathbf{C}$ that does not have a section. That is, there is no map $s : B \longrightarrow A$ such that $a.s = id_B$. The contradiction we are aiming for is that $\Theta A$ cannot be a set.

Define $U$ as the following pushout.

$$\begin{tikzcd}
\mathbf{C}(_, A) \arrow[r, "\mathbf{C}(_, a)"] \arrow[d, "\mathbf{C}(_, a)"'] & \mathbf{C}(_, B) \arrow[d, "j_1"] \\
\mathbf{C}(_, B) \arrow[r, "j_0"'] & U
\end{tikzcd}$$

If we call $in_0, in_1 : \mathbf{C}(\_, B) \longrightarrow \mathbf{C}(\_, B) + \mathbf{C}(\_, B)$ the injections then $U$ is the quotient of $\mathbf{C}(\_, B) + \mathbf{C}(\_, B)$ by the equivalence relation generated by the pairs $(in_0 f, in_1 f)$ for $f = a.g$ for some $g$.

If we use square brackets to denote equivalence classes in $U$ then it is clear that $[in_0 a] = [in_1 a]$ in $UA$. Let us call this element $\alpha$. On the other hand, we have that $\beta_0 = [in_0 id_B] \neq [in_1 id_B] = \beta_1$ in $UB$ because $a$ does not have a section.

Now, fix some index set $I$ and let $U^I$ be the $I$-fold product of $U$. Let $\alpha_I \in U^I A$ be the $I$-tuple such that every component is $\alpha$. Also, for each $d : I \longrightarrow \{0, 1\}$ we can consider the element $I_d \in U^I B$ given by the I-uple $(..., \beta_{d(i)}, ...)_{i \in I}$. Notice that for $d, e \in \{0, 1\}^I$, $I_d = I_e$ if and only if $d = e$. Also, for each such $d$ we have $\overline{d} : \mathbf{C}(\_, B) \longrightarrow U^I$ given by the universal property of the power $U^I$ as follows.

$$\begin{tikzcd}
\mathbf{C}(_, B) \arrow[d, "\exists!\overline{d}"'] \arrow[dr, "j_{d(i)}"] & \\
U^I \arrow[r, "\pi_i"'] & U
\end{tikzcd}$$

Finally we have a unique map $f$ given by the universal property of the following coproduct.

$$\begin{tikzcd}
\mathbf{C}(_, B) \arrow[d, "in_d"'] \arrow[dr, "\overline{d}"] & \\
Y = \left( \coprod_{d \in \{0,1\}^I} \mathbf{C}(_, B) \right) \arrow[r, "\exists! f"'] & U^I
\end{tikzcd}$$

First, notice that for each $d \in \{0,1\}^I$, $f(in_d a) = \overline{d}a = \alpha_I \in U^I A$ and that $f(in_d id_B) = \overline{d}(id_B) = I_d \in U^I B$. Notice also that $(in_d id_B)$ is the unique element over $I_d$ via $f$. Indeed, if for some $k : B \longrightarrow B$ we have that $f(in_d k) = I_d$ then $\overline{d}k = I_d$ and so, for each $i \in I$, $\pi_i(\overline{d}k) = j_{di}k = [in_{di}k] = [in_{di}id] = j_{di}id = \pi_i I_d$. As $j_0$ and $j_1$ are injective, we have that $k = id$.

As $\theta$ is a generic proof we have the following situation.

$$\begin{tikzcd}
Y \arrow[r, shift left, "g"] \arrow[dr, "f"'] & Y' \arrow[l, shift left, "h"] \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "P.B."] & \Theta \arrow[d, "\theta"] \\
& U^I \arrow[r, "\nu_f"'] & \Lambda
\end{tikzcd}$$

Let $J_d = g(in_d id_B) \in Y'B$. As $f(hJ_d) = f'(g(in_d id_B)) = f(in_d id_B) = I_d$, it must be the case that $hJ_d = (in_d id_B)$ and in particular, that all the $J_d$'s are different for different $d$'s.

Also, as $h((Y'a)J_d) = (Ya)(hJ_d) = (in_d a)$ it must be the case that all the $((Y'a)J_d)$'s are different in $Y'A$ (because the $(in_d a)$'s are different for different $d$'s).

But $f'((Y'a)J_d) = f(h((Y'a)J_d)) = f(in_d a) = \alpha_I \in U^I A$.

So we have $\{0,1\}^I$ different elements in $Y'A$ over $\alpha_I \in U^I A$ via $f'_A$. As $f'$ is the pullback of $\theta$ along $\nu_f$, it follows by the construction of pullbacks in presheaf toposes that there must be $\{0,1\}^I$ different elements in the inverse image $\theta_A^{-1}(\nu_f \alpha_I)$ which is a subset of $\Theta A$. But this is for every set $I$! So $\Theta A$ cannot be a set. Absurd. $\square$

A word of acknowledgement and gratitude: the equivalence between 2 and 3 is due to Lawvere and Schanuel and it was communicated to me by the former. In their proof they use the object $U$ and its powers in order to produce a proper class of connected objects. After seeing their proof we realized that it was possible to use these objects in order to generalize our result in [76] of the fact that the exact completion of the topos $\mathbf{Set}^{\rightarrow}$ is not a topos (notice that this fact is harder than the case of $\mathbf{Set}^{\rightrightarrows}$ because the classes of proofs of $\mathbf{Set}^{\rightarrow}$ are small). This generalization is the main proof presented above.

## 6.2 A word about Läuchli's realizability

Due to its connection with Läuchli's abstract notion of realizability and completeness result [53], it may be of interest to pay special attention to the exact completion of the topos of $\mathbb{Z}$-sets. In [73] (see also [62]) the hyperdoctrine that assigns to each object $X$ of $\mathbf{Set}^{\mathbb{Z}}$ the small Heyting algebra $Prf(X) = (\widetilde{\mathbf{Set}^{\mathbb{Z}}}/X)$ is used to give an abstract account of Läuchli's completeness result.

The topos $(\mathbf{Set}^{\mathbb{Z}})_{\mathbf{ex}}$ should provide a useful tool to improve our understanding of Läuchli's work. Indeed, for every $X$ in $\mathbf{Set}^{\mathbb{Z}}$, the lattice of subobjects of $X$ in $(\mathbf{Set}^{\mathbb{Z}})_{ex}$ is isomorphic to the $Prf(X)$ above. This shows that the internal logic of $(\mathbf{Set}^{\mathbb{Z}})_{ex}$ is the logic of Läuchli's realizability (some care is required to give Läuchli's interpretation of falsity).

Moreover, in order to study this topos there is an alternative description that is perhaps simpler than the construction using pseudo-equivalence relations. Indeed, by Proposition 3.1.2, for any group $G$, the topos $(\mathbf{Set}^{G})_{ex}$ is the presheaf topos on the full subcategory of $\mathbf{Set}^{G}$ induced by the connected objects (which, as we have argued in the proof above, is essentially small).

## 6.3 Continuous group actions

If $G$ is a topological group, then it is possible to consider the topos $\mathbf{B}G$ of *continuous* $G$-sets. That is, those actions $X \times G \longrightarrow X$ that are continuous when $X$ is equipped with the discrete topology.

**Corollary 6.3.1.** *For any topological group $G$, $(\mathbf{B}G)_{ex}$ is a topos.*

*Proof.* Let $G^\delta$ denote the group $G$ but with the discrete topology. Then the topos $\mathbf{B}G^\delta$ is just the usual topos $\mathbf{Set}^G$ of $G$-sets for the underlying group $G$. It is shown in [71] that $\mathbf{B}G$ is a coreflective subcategory of $\mathbf{B}G^\delta$. It is easily seen that generic proofs are inherited by coreflective subcategories. So it follows that for any topological group $G$, $(\mathbf{B}G)_{ex}$ is a topos. □

This gives more examples of toposes whose exact completions are themselves toposes. Moreover, in [71] it is also shown that $\mathbf{B}G$ is a Grothendieck topos whose site of definition is not a groupoid in general. So even if $\mathbf{C}$ is not a groupoid, it may still be the case that there are some subtoposes of $\mathbf{Set}^{\mathbf{C}^{\mathbf{op}}}$ with non trivial generic proofs. Of course, as **Set** arises as a sheaf subtopos of many Grothendieck toposes, it was clear that there are non-groupoidal sites whose corresponding sheaf toposes have a generic proof. But the case of sets is not interesting because in this case, as every epi splits, the subobject classifier is a proof classifier.

It would be interesting to characterize the categories and the Grothendieck topologies on them whose corresponding sheaf subtoposes have generic proofs.

# Chapter 7

# Chaotic situations

The category of sets is embedded in many of our examples in such a way that the image of this embedding consists of objects that can be thought of as having *as much structure as possible*. We can call these objects *chaotic*. As we will see in this chapter and also in Chapter 8, the existence of chaotic objects can be used to conceptualize many properties of our examples and also to see more deeply into the construction of generic proofs and monos. Moreover, this deeper understanding will help us find a variety of new examples of toposes that are exact completions.

In this chapter we borrow some ideas from [59, 61], axiomatize the existence of full subcategories of chaotic objects and develop the technical machinery necessary for proving the results of Chapter 8 in a suitable abstract way.

## 7.1 Chaotic objects

Consider the categories $H_+$ for a frame $H$, **PAss** and **Ass** for some PCA, **Top**, **SSeq**, **GEn** and **Equ**.

Consider also the following two finite categories. Let $\underset{\rightarrow}{\overset{\rightarrow}{\leftarrow}}$ be the category with two objects $A$ and $N$ and non identity maps $s, t : A \longrightarrow N$ and $r : N \longrightarrow A$ such that $s.r = id_N = t.r$. Then, let $\rightrightarrows$ be the subcategory of $\underset{\rightarrow}{\overset{\rightarrow}{\leftarrow}}$ given by the parallel pair $s$ and $t$.

The presheaf topos $\mathbf{Set}^{\underset{\rightarrow}{\overset{\rightarrow}{\leftarrow}}}$ is the topos of directed reflexive graphs. On the other hand, the presheaf topos $\mathbf{Set}^{\rightrightarrows}$ is that of directed irreflexive graphs. By *irreflexive* here we mean that the graphs do not have a *distinguished* loop for each node. See [60] for more on these toposes.

There are functors $|\_|$ from each of these categories to the category of sets. As we have already mention in Chapter 2, in the cases of $H_+$, **PAss**, **Ass**, **Top** and **SSeq** the value of this functor at each object is its underlying set. The value at a map is always the corresponding function between the underlying sets. In the case of **Equ**, $|(X,\sim)|$ is the underlying set of the quotient $X/\sim$.

For the presheaf toposes, the value of $|\_|$ at a graph is its underlying set of nodes and the value at a map is the underlying function between the sets of nodes.

All the functors $|\_|$ described above have full and faithful right adjoints that we are going to denote by a $\nabla$. We have already described most of them in Chapter 2 but we recall them here for convenience.

In the case of $H_+$, for any set $S$, $\nabla S$ has $S$ as underlying set and every element is valued as $\top$ the top element of $H$. The action on functions is evident.

Let $*$ be some fixed element of the PCA $\mathbf{A}$. The functor $\nabla : \mathbf{Set} \longrightarrow \mathbf{PAss}$ takes a set $S$ to the partitioned assembly with the same underlying set and such that every element is valued as $*$. Again, the action on functions is evident. The functor to $\nabla : \mathbf{Set} \longrightarrow \mathbf{Ass}$ can be described similarly.

In the case of **Top**, $\nabla$ assigns to each set the corresponding *chaotic* or *indiscrete* space with the two trivial opens and in the case of **Equ**, it assigns the same space together with equality as associated equivalence relation.

The functor to **SSeq** assigns to each set the space with same underlying set and such that every sequence converges to every point.

In the case of **GEn**, for any set $S$, $\nabla S = S^{\mathbb{N}}$.

The functors to the toposes of graphs assign to each set the graph with the same set of nodes and exactly one arrow between any ordered pair of nodes.

Notice that in all cases, the image of the functors $\nabla$ consists of objects that have, in some sense, as much structure as possible. One way to see this is that an object of the form $\nabla S$ has so much structure that for any object $X$, any function $|X| \longrightarrow S$ between the underlying sets is continuous or realizable (preserves structure). This is very clearly expressed in the statement that $|\_|$ is left adjoint to $\nabla$. Indeed, as it is explained in [59, 61] the adjointness properties can be used to axiomatize the existence of discrete and chaotic objects in a theory of space.

In the cases of **PAss**, **Ass**, **Top**, **SSeq**, **GEn**, **Equ** and the topos of reflexive graphs, the functor $|\_|$ is nothing but the "points" functor $\Gamma = Hom(1, \_)$. This is not the case for $H_+$ or the topos of irreflexive graphs. In the case of $H_+$, the set of points of an object is the set of elements that are valued as the top element of $H$. The set of points of an irreflexive graph is its set of loops. In the case of $H_+$, the "points" functor $\Gamma$ is *right* adjoint to $\nabla$.

We will see in this chapter and the next one that many properties of our examples follow from the fact that **Set** is embedded in them in the way described above.

Actually, few properties of sets are going to be used in this chapter so in order to present the results as abstractly as possible, let us fix from now on two categories $\mathbf{S}$ and $\mathbf{C}$ with finite limits and an adjunction $|\_| \dashv \nabla$ with $\nabla : \mathbf{S} \longrightarrow \mathbf{C}$.

## 7.2 Pre-embeddings

Consider a continuous $f : Y \longrightarrow X$ between topological spaces such that $U$ is open in $Y$ if and only if there exists an open $V$ in $X$ such that $f^*V = U$. Such an $f$ need not be a regular mono but, in some sense, it is as close as it can be.

Maps of this kind will play an important role in what follows so it is good that we can characterize them in an abstract way.

**Definition 7.2.1.** A map $f : Y \longrightarrow X$ is a *pre-embedding* if the following square is a pullback.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\eta_Y"'] & X \arrow[d, "\eta_X"] \\ \nabla|Y| \arrow[r, "\nabla|f|"'] & \nabla|X| \end{tikzcd}$$

With our conventions, pre-embeddings are maps in $\mathbf{C}$. Notice that the definition of pre-embedding is relative to the adjunction $|\_| \dashv \nabla$.

**Proposition 7.2.2.** *In each of the categories below a map $f : Y \longrightarrow X$ is a pre-embedding (relative to the adjunctions presented in the previous section) if and only if the corresponding condition holds.*

**(Top)** *$U$ is open in $Y$ if and only if there exists a $V$ in $X$ such that $f^*V = U$*

**($H_+$)** *for every $y$ in $Y$, $\|y\| = \|fy\|$*

**(PAss and Ass)** *$Y$ is isomorphic over $X$ to an $f' : Y' \longrightarrow X$ such that for every $y$ in $Y'$, $\|y\| = \|f'y\|$*

**(SSeq)** *$(fy_i)$ converges to $fy$ in $X$ implies that $(y_i)$ converges to $y$ in $Y$*

**(GEn)** *if we let $Y = (|Y|, D)$ and $X = (|X|, E)$ then the condition is that $D = \{g : \mathbb{N} \longrightarrow |Y| | f.g \in E\}$*

**($\mathbf{Set}^{\overset{\rightarrow}{\leftleftarrows}}$ and $\mathbf{Set}^{\rightrightarrows}$)** *for every pair of nodes $n$, $n'$, $f$ induces an isomorphism $Arrows(n, n') \cong Arrows(fn, fn')$ (in other words, $f$ is full and faithful but not necessarily injective on nodes).*

*Proof.* See [66] (where pre-embeddings are called *cartesian maps*) for the fact in **Ass**. The fact for **SSeq** is in [77, 78]. The rest are left as easy exercises. □

The intuition behind these maps is that the structure of $Y$ is inherited from that of $X$. For another (recent) example of this phenomenon the reader is invited to characterize the pre-embeddings in **Equ**.

Let us state some easy consequences of Definition 7.2.1.

**Proposition 7.2.3.**

1. *pre-embeddings are closed under composition*
2. *if $f$ and $f.g$ are pre-embeddings then so is $g$*
3. *if $m$ is mono and $m.f$ is a pre-embedding then $f$ is a pre-embedding*

*Proof.* Easy. □

In our examples, the adjunction $|\_| \dashv \nabla$ is a *reflection* (that is, $\nabla$ is an embedding). This is why the counit $\varepsilon : |\nabla(\_)| \longrightarrow Id$ is an iso [70]. Actually, in our examples, the adjunction $|\_| \dashv \nabla$ is a *localization* (that is, a reflection such that the left adjoint preserves finite limits). This allows the following characterization of pre-embeddings.

**Proposition 7.2.4.** *Let $|\_| \dashv \nabla$ be a localization. Then $f : Y \longrightarrow X$ is a pre-embedding if and only if there exists a pullback as below.*

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[d, "h"] \\
\nabla W \arrow[r, "g"'] & \nabla V
\end{tikzcd}$$

*Proof.* The *only if* direction is trivial. So consider the converse.

The basic theory of adjunctions implies that the square above factors as follows.

$$\begin{tikzcd}
Y \arrow[r, "\eta"] \arrow[d, "f"'] & \nabla|Y| \arrow[r, "\nabla|k|"] \arrow[d, "\nabla|f|"] & \nabla W \arrow[d, "g = \nabla|g|"] \\
X \arrow[r, "\eta"'] & \nabla|X| \arrow[r, "\nabla|h|"'] & \nabla V
\end{tikzcd}$$

As both $|\_|$ and $\nabla$ preserve finite limits, the right hand square is a pullback. It follows by the Pasting Lemma that left hand square also is and hence, that $f$ is a pre-embedding. $\square$

Another useful consequence of the fact that $|\_| \dashv \nabla$ is a localization (actually, that $|\_|$ preserves products) is that the functors interact well with exponentials. So, if the exponential $S^{|A|}$ exists in $\mathbf{S}$, then $(\nabla S)^A$ exists in $\mathbf{C}$ and it is isomorphic to $\nabla(S^{|A|})$. So, if $\mathbf{S}$ is cartesian closed then every object $\nabla X$ is *exponentiating* (Definition 4.3.2) in $\mathbf{C}$. In this case, we also say that $\mathbf{S}$ is an *exponential ideal* of $\mathbf{C}$.

**Corollary 7.2.5.** *If $|\_| \dashv \nabla$ is a localization then:*

1. *pre-embeddings are closed under pullback*
2. *if $A$ is exponentiable and $f$ is a pre-embedding then $f^A$ also is*

*Proof.* The first property is trivial using Proposition 7.2.4. For the second use that $(\_)^A$ preserves pullbacks, the isomorphisms $(\nabla S)^A \cong \nabla(S^{|A|})$ for every $S$ and Proposition 7.2.4. $\square$

## 7.3 Mono-localizations

We have already mentioned that for a pre-embedding $f : Y \longrightarrow X$ we think of the structure of $Y$ being determined by that of $X$ via $f$. This is also the intuition that one has for regular monos in **Top**, $H_+$, **PAss**, **Ass**, **SSeq**, **GEn** and **Equ**. Indeed, in these categories, regular monos are pre-embeddings. It is possible to explain this in terms of the unit of the localization.

**Proposition 7.3.1.** *Let $|\_| \dashv \nabla$ be a localization. Then the following are equivalent.*

1. *$\eta$ is a natural mono*
2. *$|\_|$ is faithful*
3. *Every regular mono is a pre-embedding*
4. *if $f.g$ is a pre-embedding then so is $g$.*

*Proof.* The equivalence between the first two items is a general property of adjunctions [70]. For the fact that 1 implies 4 consider the following diagram.

$$\begin{tikzcd}
K \arrow[dr, "\exists! k"] \arrow[drr, "k_1"] \arrow[ddr, "k_0"'] & & & \\
& Z \arrow[r, "g"] \arrow[d, "\eta"] & Y \arrow[r, "f"] \arrow[d, "\eta"] & X \arrow[d, "\eta"] \\
& \nabla|Z| \arrow[r, "\nabla|g|"'] & \nabla|Y| \arrow[r, "\nabla|f|"'] & \nabla|X|
\end{tikzcd}$$

The rectangle is a pullback by hypothesis and we need to prove that the left hand square is a pullback. So we assume that $\eta.k_1 = \nabla|g|.k_0$. It follows that there exists a unique $k : K \longrightarrow Z$ such that $\eta.k = k_0$ and $f.g.k = f.k_1$. But then we have $\eta.k_1 = \nabla|g|.k_0 = \nabla|g|.\eta.k = \eta.g.k$ and as $\eta$ is mono, $k_1 = g.k$. Again because $\eta$ is mono, $k$ is unique. So the left hand square is a pullback.

We now prove that 4 implies 3. Consider for any $X$ the diagonal map $\Delta = \langle id, id\rangle : X \longrightarrow X \times X$. As $\pi.\Delta = id$ and $id$ is a pre-embedding we have, by hypothesis, that $\Delta$ is a pre-embedding. As every regular mono is a pullback of a diagonal we have that every regular mono is a pre-embedding.

Finally, we prove that 3 implies 1. As diagonal maps are (split) regular monos, they are pre-embeddings. That is, for any $X$ the following diagram is a pullback.

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\eta"'] & X \times X \arrow[d, "\eta \times \eta"] \\
\nabla|X| \arrow[r, "\Delta"'] & \nabla|X| \times \nabla|X|
\end{tikzcd}$$

But this is equivalent to the fact that $\eta$ is mono. □

Let us call a localization with a monic unit a *mono-localization.*

**Corollary 7.3.2.** *Let $|\_| \dashv \nabla$ be a mono-localization. A map $f : Y \longrightarrow X$ is a pre-embedding if and only if there exists any pullback as below.*

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[d, "h"] \\
\nabla W \arrow[r, "g"'] & Z
\end{tikzcd}$$

*Proof.* As $\eta_Z : Z \rightarrowtail \nabla|Z|$ is mono, the following square is a pullback.

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[d, "\eta_Z.h"] \\
\nabla W \arrow[r, "\eta_Z.g"'] & \nabla|Z|
\end{tikzcd}$$

Then $f$ is a pre-embedding by Proposition 7.2.4. $\square$

The next lemma is going to be essential in the proof of the main result in Chapter 8.

**Lemma 7.3.3.** *Let $|\_| \dashv \nabla$ be a mono-localization. If $f : Y \longrightarrow X$ is a pre-embedding, then $\langle f, g\rangle : Y \longrightarrow X \times Z$ is a pre-embedding for any $g : Y \longrightarrow Z$.*

*Proof.* This follows from Proposition 7.3.1 because $\pi.\langle f, g\rangle = f$ which is a pre-embedding. $\square$

In our main examples, **S** will be a topos so let us say that a category **C** has a *chaotic situation* if it has a topos as a mono-localization. Notice that at this point the theory collapses if both **C** and **S** are toposes as any mono-localization between toposes is an equivalence (see [45] Lemma 4.13, where it is proved that a geometric morphism that is both a surjection and an inclusion is an equivalence).

**Lemma 7.3.4.** *If **C** has a chaotic situation then every mono pre-embedding is a regular mono.*

*Proof.* Let $m$ be a mono pre-embedding. As $|\_|$ preserves finite limits then $|m|$ is mono. As **S** is a topos, $|m|$ is a regular mono. As $\nabla$ preserves finite limits, $\nabla|m|$ is a regular mono. As $m$ is a pre-embedding, it is a pullback of $\nabla|m|$ and hence it is also a regular mono. $\square$

# 7.4 Regular completions and mono-localizations

In this section we study some relations between a mono-localization as above and the regular completion of the category $\mathbf{C}$.

## 7.4.1 Inheriting mono-localizations

Consider a mono localization $|\_| \dashv \nabla$ with $\nabla : \mathbf{S} \longrightarrow \mathbf{C}$ and $\mathbf{S}$ regular. We now explain how $\mathbf{C}_{reg}$ inherits $\mathbf{S}$ as a mono localization.

Indeed, as the functor $|\_| : \mathbf{C} \longrightarrow \mathbf{S}$ preserves finite limits, the universal property of $\mathbf{y} : \mathbf{C} \longrightarrow \mathbf{C}_{reg}$ gives an exact functor $L : \mathbf{C}_{reg} \longrightarrow \mathbf{S}$ (that we can assume to be) such that $L.\mathbf{y} = |\_|$.

The functor $L$ has an easy description. Recall (Section 3.2) that objects in $\mathbf{C}_{reg}$ are given by maps in $\mathbf{C}$. So, for an object $(f : Y \longrightarrow X)$ in $\mathbf{C}_{reg}$, $L(f) = Im|f|$, the image of the map $|f| : |Y| \longrightarrow |X|$ in $\mathbf{S}$.

We give a concrete proof of the following simple fact, although it can be dealt with using abstract 2-categorical properties of the bi-adjoint given by regular completions.

**Lemma 7.4.1.** *The functor $L$ is part of a mono-localization $L \dashv \mathbf{y}.\nabla$.*

*Proof.* We need only check that $L$ is left adjoint to $\nabla_1 = \mathbf{y}.\nabla : \mathbf{S} \longrightarrow \mathbf{C} \longrightarrow \mathbf{C}_{reg}$ and that the unit of this adjunction is mono.

In order to check that $L$ is a left adjoint to $\nabla_1$, let $f_0, f_1$ be the kernel pair of $f : Y \longrightarrow X$ in $\mathbf{C}$ and let $S$ in $\mathbf{S}$. Then $\mathbf{S}(L(f : Y \longrightarrow X), S) \cong \mathbf{S}(Im|f|, S) \cong \{h : |Y| \longrightarrow S \quad | \quad h.|f_0| = h.|f_1|\} \cong \mathbf{C}_{reg}((f : Y \longrightarrow X), \nabla_1 S)$.

To prove that the unit of the adjunction is mono, let $f : Y \longrightarrow X$ and let $e : |Y| \twoheadrightarrow Im|f|$ be the epi part of the regular-epi/mono factorization of $|f|$. It is not difficult to see that the corresponding unit of the adjunction $L \dashv \nabla_1$ is given as follows.

$$\begin{tikzcd}
Y \arrow[r, "\eta"] \arrow[d, "f"'] & \nabla|Y| \arrow[r, "\nabla e"] & \nabla Im|f| \arrow[d, "id"] \\
X & & \nabla Im|f|
\end{tikzcd}$$

In order to prove that $[\nabla e.\eta] : (f : Y \longrightarrow X) \longrightarrow \nabla_1 L(f : Y \longrightarrow X)$ is mono let $[h], [h'] : (g : Y' \longrightarrow X') \longrightarrow (f : Y \longrightarrow X)$ be maps in $\mathbf{C}_{reg}$ such that $[\nabla e.\eta].[h] = [\nabla e.\eta].[h']$. This means that $\nabla e.\eta.h = \nabla e.\eta.h'$ and it implies that $\nabla|f|.\eta.h = \nabla|f|.\eta.h'$ and so, that $\eta.f.h = \eta.f.h'$. As $\eta$ is mono, $f.h = f.h'$ and so $[h] = [h']$. $\square$

As **Ass** and **Equ** arise as regular completions (Section 3.2), Lemma 7.4.1, together with the adjunctions from Section 7.1, explain why they have the category of sets as a mono-localization.

### 7.4.2 Local cartesian closure

Recall that Corollary 3.3.7 gives sufficient conditions for a regular completion to be locally cartesian closed. Here we show that two of the three conditions follow from the assumption of a suitable mono-localization $|\_| \dashv \nabla : \mathbf{S} \longrightarrow \mathbf{C}$.

First, notice that the fact that $|\_|$ is faithful implies that it reflects epis and monos. In turn, as $|\nabla(\_)| \cong Id_{\mathbf{S}}$, it follows that $\nabla$ preserves epis.

**Lemma 7.4.2.** *If* $\mathbf{S}$ *has stable epis then so does* $\mathbf{C}$.

*Proof.* Let $e : V \longrightarrow X$ be epi in $\mathbf{C}$ and let $d : W \longrightarrow Y$ be its pullback along some $f : Y \longrightarrow X$. As $|\_|$ is a left adjoint $|e|$ is epi in $\mathbf{S}$. As $|\_|$ is a localization and epis are stable in $\mathbf{S}$, $|d|$ is epi. As $|\_|$ is faithful it reflects epis and hence $d$ is epi. $\square$

Results in [14] (see Proposition 5.6.2) show that every localization $|\_| \dashv \nabla$ induces a factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathbf{C}$ characterized as follows.

1. $f \in \mathcal{E}$ if and only if $|f|$ is an isomorphism
2. $f \in \mathcal{M}$ if and only if $f$ is a pre-embedding

Notice that as $|\_|$ reflects epis every map in $\mathcal{E}$ is epi.

**Lemma 7.4.3.** *If* $\mathbf{S}$ *has stable epi/regular-mono factorizations then so does* $\mathbf{C}$.

*Proof.* As regular-monos and epis (by Lemma 7.4.2) are stable under pullback, we need only prove that there are epi/regular-mono factorizations. As $\nabla$ preserves epis and regular monos, every map $\nabla g$ has an epi/regular-mono factorization in $\mathbf{C}$. As pre-embeddings are pullbacks of such maps, they also factor in this way.

Now, let $f : Y \longrightarrow X$ be any map in $\mathbf{C}$. We already know that it factors as $f = m.e$ with $|e|$ and iso (and hence $e$ epi) and $m$ a pre-embedding. Then $m = n.d$ with $n$ a regular-mono and $d$ epi. So we have $f = n.(d.e)$ is an epi/regular-mono factorization of $f$. $\square$

Lemma 7.4.3 gives a conceptual explanation of why **PAss** and **Top** have this kind of factorization.

The following result should be compared with Proposition 4.3.3.

**Lemma 7.4.4.** *If every regular equivalence relation is a kernel pair in* $\mathbf{S}$ *then the same holds in* $\mathbf{C}$.

*Proof.* Let $e_0, e_1 : E \longrightarrow X$ be a regular equivalence relation. Then $|e_0|, |e_1| : |E| \longrightarrow |X|$ is also a regular equivalence relation and by hypothesis we can assume that it is the kernel pair of the map $q : |X| \longrightarrow Q$. We are going to prove that $e_0$ and $e_1$ form the kernel pair of $\nabla q.\eta : X \longrightarrow \nabla Q$. So consider the following diagram where the square is a pullback by definition.

$$\begin{tikzcd}
E \arrow[dr, "\exists ! e"] \arrow[drr, "e_1"] \arrow[ddr, "e_0"'] & & \\
& K \arrow[r, "k_1"] \arrow[d, "k_0"'] & X \arrow[d, "\nabla q.\eta"] \\
& X \arrow[r, "\nabla q.\eta"'] & \nabla Q
\end{tikzcd}$$

The outer diagram commutes because $\nabla q.\eta.e_0 = \nabla q.\nabla|e_0|.\eta = \nabla(q.|e_0|).\eta = \nabla(q.|e_1|).\eta = \nabla q.\eta.e_1$. Then, there exists an $e$ such that $\langle e_0, e_1\rangle = \langle k_0, k_1\rangle.e$. As $\langle e_0, e_1\rangle$ is a regular mono, so is $e$.

Also, as $|\_|$ preserves finite limits and $|\nabla q.\eta|$ is iso below $|X|$ to $q$, it follows that $|k_0|, |k_1| : |K| \longrightarrow |X|$ is the kernel pair of $q$. Then $|e| : |E| \longrightarrow |K|$ must be an isomorphism. As $|\_|$ reflects epis, $e$ is epi. As it is also a regular mono, it is iso. Hence $e_0, e_1 : E \longrightarrow X$ is a kernel pair. □

In order to be able to apply all the results in this section we need only assume that $\mathbf{S}$ is a regular category with stable epi/regular-mono factorizations and such that every equivalence relation is a kernel pair. Notice that any topos satisfies these conditions.

Recall our notation for Corollary 4.3.5.

**Corollary 7.4.5.** *If* $\mathbf{C}$ *has a chaotic situation and weak dependent products then for each* $n$, $\mathbf{C}_{reg(n)}$ *has weak dependent products and a chaotic situation. Actually, for every* $n > 0$, $\mathbf{C}_{reg(n)}$ *is locally cartesian closed.*

*Proof.* By induction. Assume that $\mathbf{C}_{reg(n)}$ has a chaotic situation and weak dependent products. Lemma 7.4.3 and Lemma 7.4.4 allow us to apply Corollary 3.3.7 and obtain that $\mathbf{C}_{reg(n+1)}$ is locally cartesian closed. Lemma 7.4.1 shows that $\mathbf{C}_{reg(n+1)}$ also has a chaotic situation. $\square$

## 7.5 Pre-embeddings from discrete objects

We have seen that much of the chaotic behaviour of certain classes of objects can be abstracted by the existence of an adjunction. As explained in [59, 61], *discrete* behaviour can be explained in an analogous way. Let us say that a category $\mathbf{C}$ has a *discrete situation* if there is a functor $|\_| : \mathbf{C} \longrightarrow \mathbf{S}$ to a topos $\mathbf{S}$ with a full and faithful left adjoint $\Delta : \mathbf{S} \longrightarrow \mathbf{C}$ (not to be confused with the diagonal map).

In this section we describe some discrete situations with respect to the functors $|\_|$ associated with our main examples (some of them have already been described in Chapter 2).

Let us first consider **Top**. For any set $S$, let $\Delta S$ be the space in which every subset of $S$ is open.

Now consider **SSeq**. For any set $S$, the subsequential space $\Delta S$ has underlying set $S$ and is such that a sequence converges to a point $p$ if and only if it is eventually constant with value $p$.

In the case of **GEn**, for any set $S$, $\Delta S$ has underlying set $S$ and the associated structure is given by the subset of $S^{\mathbb{N}}$ consisting of the functions with finite image.

The reader is invited to calculate what the discrete inclusions for **Equ** and the presheaf topos of reflexive graphs are.

Moreover, in the cases above, the functor $\Delta$ preserves finite limits.

The definition of a discrete situation in the first paragraph of this section does not assume that $|\_| : \mathbf{C} \longrightarrow \mathbf{S}$ is part of a chaotic situation. But this is the case in our examples.

Between toposes, the geometric morphisms $\Delta \dashv |\_| : \mathbf{C} \longrightarrow \mathbf{S}$ with $\Delta$ full and faithful and such that $|\_|$ has a right adjoint $\nabla : \mathbf{S} \longrightarrow \mathbf{C}$ are called *local* [48] (see also [61, 63] where such strings of adjoints $\Delta \dashv |\_| \dashv \nabla : \mathbf{S} \longrightarrow \mathbf{C}$ are called *unity-and-identity-of-opposites* and used to account for an abstract theory of space).

Our purpose for introducing discrete situations is to present the following concrete observation in a natural setting.

**Lemma 7.5.1.** *Let $f : \Delta S \longrightarrow X$ be a pre-embedding with discrete domain.*

1. *In* **Top** *it follows that $f$ is mono.*

2. *In* **SSeq** *it follows that $f$ is mono.*

3. *In* **GEn** *it follows that $f$ has finite fibers ($f$ is "almost mono")*

*Proof.* In all cases we use Proposition 7.2.2 describing the pre-embeddings in each case.

To prove 1, assume that $fs = fs' = x$. As $\{s\}$ is open in $\Delta S$ and $f$ is a pre-embedding, there exists an open neighbourhood $V$ of $fs$ such that $f^*V = \{s\}$. It follows that $s = s'$.

To prove 2, assume that that $fs = fs' = x$. Now consider the sequence $(t_i)$ such that $t_{2j} = s$ and $t_{2j+1} = s'$. The sequence $(ft_i)$ is constantly $x$ and so, it converges to $x$. As $f$ is a pre-embedding, it must be the case that $(t_i)$ converges to $s$ (and to $s'$). But then $s = s'$ because $f$ has discrete domain. Hence $f$ is mono.

To prove 3, let $X = (|X|, E)$ and assume there exists an infinite $U \subset S$ such that $fU = \{x\}$ for some $x$ in $|X|$. Then, for any infinite choice of elements $c : \mathbb{N} \longrightarrow U \subseteq S$, $f.c : \mathbb{N} \longrightarrow |X|$ is the function constantly $x$ which must be in $E$. As $f$ is a pre-embedding, $c$ must be in $\Delta S$ but this is absurd because $c$ has an infinite image. □

This lemma will be used in Section 8.1.

Finally, notice that we have not required $\Delta$ to preserve finite limits. May be one should, but consider the following examples.

First consider $H_+$. Recall that if for any set $S$, we let $\Delta S$ have $S$ as underlying set and be such that every element is valued in $\bot$, then we obtain a left adjoint to $|\_| : H_+ \longrightarrow \mathbf{Set}$. This left adjoint does not preserve the terminal object. Notice, though, that it preserves non-empty products and equalizers.

A similar situation arises in the case of irreflexive graphs. For any set $S$ we can let $\Delta S$ be the graph with $S$ nodes and no arrows.

Notice that in these cases, pre-embeddings from a discrete object need not be ("almost") mono.

## 7.6 Sequential spaces

Although unrelated to the main results of the thesis, we describe in this section an application of pre-embeddings to obtain an abstract characterisation of the *sequential spaces* [28, 29] which appeared in [78].

The *sequential* spaces are those topological spaces whose topologies are determined by sequence convergence. Explicitly, say that a sequence $(x_i)$ of elements of a set $X$ is *eventually in* a subset $O \subseteq X$ if there exists $l$ such that, for all $i \geq l$, $x_i \in O$. Recall that, in an arbitrary topological space $X$, a sequence $(x_i)$ is said to *converge to* a point $x$ if, for every neighbourhood of $x$, the sequence is eventually in the neighbourhood.

**Definition 7.6.1.** Let $X$ be a topological space.

1. A subset $O$ of $X$ is *sequentially open* if every sequence converging to a point in $O$ is eventually in $O$.
2. $X$ is *sequential* if every sequentially open subset is open.

Let **Seq** denote the category of sequential spaces and continuous functions. For sequential spaces, the notion of continuity has a natural reformulation. It is easily checked that a function $f : X \to Y$ between sequential spaces is continuous if and only if it preserves convergent sequences.

It is easy to see that **Seq** is a full subcategory of **SSeq**. The embedding assigns to each sequential space, the subsequential space with same underlying set and as convergent sequences those that converge topologically. This embedding has a left adjoint $F : \mathbf{SSeq} \longrightarrow \mathbf{Seq}$. In order to define $F$ notice that the definition of a sequentially open subset makes sense if $X$ is a subsequential space. Then for any subsequential space $X$ define $FX$ to be the sequential space with the same underlying set and with the sequentially open subsets of $X$ as its topology. The functor $F$ preserves finite products (see [77] or [78]). It follows [32] that **Seq** is an exponential ideal of **SSeq**.

**Lemma 7.6.2.** *If $X$ is a sequential space and $f : A \to X$ is a pre-embedding in* **SSeq** *then $A$ is also a sequential space.*

*Proof.* We are going to show that if $(a_i)$ is eventually in every sequentially open neighbourhood of $a$ then $(a_i) \to a$ in $A$. In order to do this let $U$ be an open neighbourhood of $fa$. As $f^*U$ is sequentially open, $(a_i)$ is eventually in $f^*U$. Then, $(fa_i)$ is eventually in $U$. As $X$ is topological, this means that $(fa_i) \to fa$ in $X$. As f is a pre-embedding, $(a_i) \to a$ in $A$ (recall Proposition 7.2.2). $\square$

Let $\Sigma$ be Sierpinski space (i.e. the two element space $\{\perp, \top\}$ with the singleton $\{\top\}$ as the only non-trivial open). It is easy to prove that the continuous functions from any topological space $X$ to $\Sigma$ are in one-to-one correspondence with the open subsets of $X$. Similarly, $\Sigma$ is also a subsequential space and the maps from any subsequential space $X$ to $\Sigma$ are in one-to-one correspondence with the sequentially open subsets of $X$.

By the last observation, $\Sigma^X$ in **SSeq** is an object of sequentially open subsets of a subsequential space $X$. Moreover, as **Seq** is an exponential ideal of **SSeq**, the object $\Sigma^X$ is a sequential space. For any subsequential space $X$, let $ev' : X \to \Sigma^{\Sigma^X}$ denote the transpose of the evaluation map. If $X$ is topological then it is easily checked that $ev'$ is mono if and only if $X$ is a $T_0$ space. It is useful to consider a stronger property of $ev'$.

**Definition 7.6.3.** A subsequential space $X$ is *extensional* if $ev' : X \to \Sigma^{\Sigma^X}$ is a regular mono. It is *pre-extensional* if the map is a **SSeq**-pre-embedding.

The terminology is taken from [41]. Now, recall that $F : \mathbf{SSeq} \to \mathbf{Seq}$ is the reflection functor.

**Lemma 7.6.4.** *If $(ev'x_i) \to ev'x$ in $\Sigma^{\Sigma^X}$ then $(x_i) \to x$ in $FX$.*

*Proof.* Let $O$ be sequentially open in $X$ and $x \in O$. It is clear that $(O) \to O$ in $\Sigma^X$. Then, as $(ev'x_i) \to ev'x$, $((ev'x_i)O) \to (ev'x)O$. That is, $((ev'x_i)O)$ must be eventually $\top$. In other words, $(x_i)$ must be eventually in $O$. So $(x_i) \to x$ in $FX$. □

We can now prove the characterization of the sequential spaces.

**Proposition 7.6.5.** *In* **SSeq***:*

1. *The full subcategory of pre-extensional objects is equivalent to* **Seq**.
2. *The full subcategory of extensional objects is equivalent to the category of $T_0$ sequential spaces.*

*Proof.* As $\Sigma^{\Sigma^X}$ is a sequential space, it follows, by Lemma 7.6.2, that so is any pre-extensional object. Moreover, if $X$ is extensional then $ev' : X \to \Sigma^{\Sigma^X}$ is also mono and so $X$ is $\mathrm{T}_0$. On the other hand, if $X$ is a sequential space then Lemma 7.6.4 shows that $ev' : X \to \Sigma^{\Sigma^X}$ is a pre-embedding. Moreover, if $X$ is $\mathrm{T}_0$ then $ev'$ is also mono and hence, by Lemma 7.3.4, a regular mono. □

# Chapter 8

# Generic objects, monos and proofs

We use the ideas presented in Chapter 7 in order to simplify the characterization of the categories whose exact completions are toposes for a large class of examples, especially those arising from realizability. This simplification sheds some light on the properties and construction of the generic proofs in these examples. Moreover, these ideas allow us to recognize new hierarchies of examples of toposes that arise as exact completions. They also provide a different perspective on the results about the inevitability of untypedness for realizability toposes as explained in [64, 65, 26]. Finally, these ideas can simplify the sufficient conditions for regular completions to be quasi-toposes.

Throughout this chapter we assume $\mathbf{C}$ to be a category with finite limits and a chaotic situation $\nabla : \mathbf{S} \longrightarrow \mathbf{C}$ with left adjoint $|\_| : \mathbf{C} \longrightarrow \mathbf{S}$.

## 8.1 Generic objects and generic monos

In [66] it was observed that the category of assemblies for a PCA has a generic object in the following sense.

**Definition 8.1.1.** A *generic object* is an object $\Upsilon$ such that for every $X$ there exists a pre-embedding $X \longrightarrow \Upsilon$.

Similar ideas have recently appeared in a number of works studying different typed versions of realizability [64, 26, 12]. We had been considering pre-embeddings for a while but we recognized a possible application to this area after reading [64]. What we are going to show is that if there are chaotic objects around then the existence of a generic object is equivalent to the existence of a generic mono which is equivalent (if every epi in $\mathbf{S}$ splits) to the existence of a generic proof.

Let us look at some examples.

**Example 8.1.2 (PAss(A) has a generic object).** Let $|\Upsilon| = \mathbf{A}$ and for every $a \in \mathbf{A}$, $\|a\|_\Upsilon = a$. For every object $X = (|X|, \|\_\|_X)$ the map $\|\_\|_X : X \longrightarrow \Upsilon$ is realized by the identity and is a pre-embedding by Proposition 7.2.2.

**Example 8.1.3 ($H_+$ has a generic object).** Similar to the case of **PAss**. Let $|\Upsilon| = H$ and for every $v \in H$, $\|v\|_\Upsilon = v$. The proof that $\Upsilon$ is a generic object is analogous to the case of **PAss**.

Notice that the $\Upsilon$'s in both examples are the same as the ones described in Section 5.2. Proposition 8.1.8 below explains why this is the case.

**Example 8.1.4 (Top does not have a generic object).** Assume there exists a generic object $\Upsilon$ in **Top**. For any set $S$, we can consider the *discrete* topological space $\Delta S$ as in Section 7.5. By assumption there exists a pre-embedding $f : \Delta S \longrightarrow \Upsilon$. By Lemma 7.5.1, $f$ must be mono. So the underlying set of $\Upsilon$ has cardinality greater than that of every set. Absurd.

Notice that the same arguments work for **SSeq** and **Seq**. Moreover, the same idea can be applied to a recursion theoretic example.

**Example 8.1.5 (GEn does not have a generic object).** This case is similar to the previous one by using Lemma 7.5.1 again. But notice the twist that pre-embeddings with discrete domain, in this case, need not be mono.

We now prove two results that will help us show that in the presence of a chaotic situation, the existence of a generic object is equivalent to the existence of a generic mono.

**Lemma 8.1.6.** *If $d : Y \longrightarrow Z$ is a mono such that $|d|$ is an isomorphism and $f : Y \longrightarrow X$ is a pre-embedding then there exists a pullback square as below for a unique $f'$.*

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "d"'] & X \arrow[d, "\eta"] \\
Z \arrow[r, "f'"'] & \nabla|X|
\end{tikzcd}$$

*Proof.* Just consider the diagram below.

$$\begin{tikzcd}
Y \arrow[rr, "id"] \arrow[d, "d"'] & & Y \arrow[r, "f"] \arrow[d, "\eta"] & X \arrow[d, "\eta"] \\
Z \arrow[r, "\eta"'] & \nabla|Z| \arrow[r, "\nabla(|d|^{-1})"'] & \nabla|Y| \arrow[r, "\nabla|f|"'] & \nabla|X|
\end{tikzcd}$$

The right hand square is a pullback because $f$ is a pre-embedding. The left hand rectangle is a pullback because it commutes and $\nabla(|d|^{-1}).\eta$ is mono. The long rectangle is just the pullback of the statement and $f' = \nabla(|f|.|d|^{-1}).\eta$ is unique because $d$ is epi. □

Recall [45] that every topos has partial map classifiers. For any object $S$ in $\mathbf{S}$, let $\sigma_S : S \rightarrowtail \tilde{S}$ be the corresponding partial map classifier.

**Lemma 8.1.7.** *Regular partial maps with chaotic codomain are classified in* $\mathbf{C}$. *Indeed, for every $S$ in* $\mathbf{S}$, *the map* $(\nabla\sigma_S) : \nabla S \rightarrowtail \nabla\tilde{S}$ *classifies regular partial maps with codomain* $\nabla S$.

*Proof.* Let $m : U \rightarrowtail X$ a regular mono and $f : U \longrightarrow \nabla S$. We want to prove that there exists a pullback as below for a unique $\chi$.

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, tail, "m"'] & \nabla S \arrow[d, tail, "\nabla\sigma_S"] \\
X \arrow[r, "\chi"'] & \nabla\tilde{S}
\end{tikzcd}$$

In $\mathbf{S}$, $|m|$ is a regular mono so there exists a pullback square as below.

$$\begin{tikzcd}
|U| \arrow[r, "|f|"] \arrow[d, tail, "|m|"'] & S \arrow[d, tail, "\sigma_S"] \\
|X| \arrow[r, "\chi_f"'] & \tilde{S}
\end{tikzcd}$$

As $m$ is regular, it is a pre-embedding by Proposition 7.3.1. Hence the left hand square below is a pullback. Also, as $\nabla$ preserves limits, the right hand square is a pullback. So the rectangle is a pullback.

$$\begin{tikzcd} U \arrow[r, "\eta"] \arrow[d, "m"'] & \nabla|U| \arrow[r, "\nabla|f|"] \arrow[d, "\nabla|m|"'] & \nabla S \arrow[d, "\nabla\sigma_S"] \\ X \arrow[r, "\eta"'] & \nabla|X| \arrow[r, "\nabla\chi_f"'] & \nabla\tilde{S} \end{tikzcd}$$

The top map is equal to $f$ so we have the pullback square in the statement by letting $\chi = (\nabla\chi_f).\eta$.

It is easy to see that $\chi$ is unique. $\square$

We can now show the main result of the section.

**Proposition 8.1.8.** **C** *has a generic object if and only if it has a generic mono.*

*Proof.* For the *if* direction let $\tau : \Upsilon \longrightarrow \Lambda$ be a generic mono and let $X$ be any object. As $\eta_X : X \longrightarrow \nabla|X|$ is mono, there exists a pullback square as below.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\eta"'] & \Upsilon \arrow[d, "\tau"] \\ \nabla|X| \arrow[r] & \Lambda \end{tikzcd}$$

It follows by Corollary 7.3.2 that the top map in the square is a pre-embedding.

For the *only if* direction assume that $\Upsilon$ is a generic object. Let $m : U \rightarrowtail X$ be a mono and let $m = n.e$ be its factorization induced by the localization (recall subsection 7.4.2) with $e : U \longrightarrow V$. So we have that $|e|$ is an isomorphism and $n : V \longrightarrow X$ is a pre-embedding. As $n$ is also mono (because $|n|$ is), $n$ is a regular mono by Lemma 7.3.4.

By hypothesis, there exists a pre-embedding $f : U \longrightarrow \Upsilon$ and by Lemma 8.1.6 there exists the top pullback below. Also, by Proposition 8.1.7 there exists the bottom pullback below.

$$
\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "e"'] & \Upsilon \arrow[d, "\eta"] \\
V \arrow[r, "f'"'] \arrow[d, tail, "n"'] & \nabla|\Upsilon| \arrow[d, "\nabla\sigma_{|\Upsilon|}"] \\
X \arrow[r] & \nabla\widetilde{|\Upsilon|}
\end{tikzcd}
$$

In this way we have presented any mono $m$ as a pullback of $\tau = (\nabla\sigma_{|\Upsilon|}).\eta_\Upsilon$. So $\tau : \Upsilon \rightarrowtail \nabla\widetilde{|\Upsilon|}$ is a generic mono. $\square$

Notice that in our main examples $\mathbf{S}$ is the topos of sets and so it is boolean. In this case $\nabla\widetilde{|\Upsilon|}$ is just $\nabla(|\Upsilon| + 1)$.

We close this section with some remarks on generic objects. In practice, it is easier to show the existence or non-existence of generic objects than that of generic monos (see Examples 8.1.2 to 8.1.5). Conceptually, notice that, although it is defined relative to an underlying chaotic situation, the existence of a generic object is robust. Indeed, as the notion of generic mono does not depend on a chaotic situation, Proposition 8.1.8 shows that the existence of a generic object is independent of the choice of chaotic situation.

## 8.2 Generic proofs and the axiom of choice

We say that a chaotic situation $\nabla : \mathbf{S} \longrightarrow \mathbf{C}$ is an *AC-chaotic situation* if every epi splits in $\mathbf{S}$. Notice that in this case, every map in the image of $\nabla$ factors as a split-epi followed by a mono. Hence, so does every pre-embedding.

**Theorem 8.2.1.** *Let* $\mathbf{C}$ *have an AC-chaotic situation. Then the following are equivalent.*

1. $\mathbf{C}$ *has a generic object*
2. $\mathbf{C}$ *has a generic mono*
3. $\mathbf{C}$ *has a generic proof*

*Proof.* Proposition 8.1.8 (which does not require the splitting of epis) shows the equivalence between 1 and 2.

To prove that 3 implies 1 assume that there is a generic proof $\theta : \Theta \longrightarrow \Lambda$ and let $X$ in $\mathbf{C}$. Then we have the following diagram.

$$\begin{tikzcd} X \arrow[r, shift left, "m"] \arrow[dr, "\eta"'] & X' \arrow[l, shift left, "s"] \arrow[r, "\nu'"] \arrow[d] & \Theta \arrow[d, "\theta"] \\ & \nabla|X| \arrow[r, "\nu"'] & \Lambda \end{tikzcd}$$

Using that $\eta$ is mono it is easy to prove that $s$ is a left inverse of $m$. Then $m$ is a (split) regular mono and hence a pre-embedding. As $\nu'$ is also a pre-embedding it follows that $\nu'.m$ pre-embeds $X$ in $\Theta$. Hence, $\Theta$ is a generic object.

So there only remains to prove that 2 implies 3. To do this, assume that there exists a generic mono $\tau : \Upsilon \longrightarrow \Lambda$. We can assume that $\Lambda$ is in the image of $\nabla$ (for if it were not then we could take $\eta.\tau$ as our generic mono). Then $\Lambda$ is exponentiating (recall Section 7.2).

Let $f : Y \longrightarrow X$ be any map.

Define $\xi_Y$ to be the map arising in the pullback square below. It exists because $\eta$ is mono and $\tau$ a generic mono.

$$\begin{tikzcd} Y \arrow[r, "\xi_Y"] \arrow[d, "\eta"'] & \Upsilon \arrow[d, "\tau"] \\ \nabla|Y| \arrow[r] & \Lambda \end{tikzcd}$$

By Corollary 7.3.2, $\xi_Y$ is a pre-embedding. Then, by Lemma 7.3.3, the map $\langle f, \xi_Y \rangle : Y \longrightarrow X \times \Upsilon$ also is and so it factors as a split epi $Y \twoheadrightarrow Y'$ followed by a mono $Y' \rightarrowtail X \times \Upsilon$. The generic mono gives rise to a map $\nu : X \times \Upsilon \longrightarrow \Lambda$ such that $\nu^*\Upsilon = Y'$.

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, tail] & \Upsilon \arrow[d, tail, "\tau"] \\ X \times \Upsilon \arrow[r, "\nu"'] & \Lambda \end{tikzcd}$$

We can transpose $\nu$ (using that $\Lambda$ is exponentiating) and obtain $\nu' : X \longrightarrow \Lambda^\Upsilon$. Then the diagram below shows how to build a generic proof $\Theta \longrightarrow \Lambda^\Upsilon$ and also how it classifies $f$ (all squares are pullbacks).

$$\begin{tikzcd}
Y \arrow[r, shift right, two heads] \arrow[dr, "{\langle f, \xi_Y \rangle}"'] & Y' \arrow[l, shift right, tail] \arrow[r] \arrow[d] & \Theta \arrow[r] \arrow[d] & \Upsilon \arrow[d, tail, "\tau"] \\
& X \times \Upsilon \arrow[r, "\nu' \times id"'] \arrow[d, "\pi"'] & \Lambda^\Upsilon \times \Upsilon \arrow[r, "ev"'] \arrow[d, "\pi"] & \Lambda \\
& X \arrow[r, "\nu'"'] & \Lambda^\Upsilon &
\end{tikzcd}$$

□

This result provides a very compact way to present realizability toposes: you just need to show that the categories of partitioned assemblies have weak dependent products and a generic object.

Notice also that from the proof of Theorem 8.2.1 we have clear examples of the non-uniqueness of generic monos, objects and proofs. Indeed, given a generic object $\Upsilon$, we can build a generic proof that gives us an object $\Theta$ which, as shown in the first part of the proof, is also a generic object.

By examples 8.1.4 and 8.1.5, Theorem 8.2.1 implies that the exact completions of **Top**, **SSeq** and **GEn** are not toposes. For the case of **Top** this was already known [8] as the regular completion of **Top** is not well powered. The reader is invited to figure out this argument, check if it works for the categories **SSeq** and **GEn** and compare it with the argument in terms of generic objects. For **Top** there is also an argument using the fact that the category of algebraic lattices does not have a *universal object* (see [65, 11]).

Also as a corollary of Theorem 8.2.1, we can extend the hierarchies of Corollary 4.3.5 as follows.

**Corollary 8.2.2.** *Let* $\mathbf{C}$ *have an AC-chaotic situation, weak dependent products and a generic object. Then for every* $n$, $(\mathbf{C}_{reg(n)})_{ex}$ *is a topos.*

*Proof.* By induction. Assume that $\mathbf{C}_{reg(n)}$ has a chaotic situation, weak dependent products and a generic object. Theorem 8.2.1 gives a generic proof and then we can apply Theorem 5.2.2 to obtain that $(\mathbf{C}_{reg(n)})_{ex}$ is a topos.

So we need only prove that $\mathbf{C}_{reg(n+1)}$ satisfies the same conditions. By Lemma 7.4.1, $\mathbf{C}_{reg(n+1)}$ also has an AC-chaotic situation. By Corollary 7.4.5, it is locally cartesian closed. By Proposition 5.5.2, it has a generic mono and hence by Theorem 8.2.1, a generic object. □

We also have a non triviality result.

**Corollary 8.2.3.** *If* $\mathbf{C}$ *is not a regular completion then for any* $n, m$, $(\mathbf{C}_{reg(n)})_{ex} \simeq (\mathbf{C}_{reg(m)})_{ex}$ *implies that* $n = m$.

*Proof.* Assume that $(\mathbf{C}_{reg(n)})_{ex} \simeq (\mathbf{C}_{reg(m)})_{ex}$.

Then $\mathbf{C}_{reg(n)} \simeq \mathit{Proj}((\mathbf{C}_{reg(n)})_{ex}) \simeq \mathit{Proj}((\mathbf{C}_{reg(m)})_{ex}) \simeq \mathbf{C}_{reg(m)}$.

By Proposition 4.3.6, $n = m$. □

If we let $\mathbf{C} = \mathbf{PAss}(K_1)$ then we have that $(\mathbf{C}_{reg(0)})_{ex}$ is **Eff**, the effective topos. The topos $(\mathbf{C}_{reg(1)})_{ex} = \mathbf{Ass}(K_1)_{ex}$ is the topos $\mathcal{A}$ for extensional realizability studied in [84]. This topos is defined loc. cit. using tripos theory and later shown to be the exact completion of assemblies using the characterization of exact completions (Proposition 3.3.2).

For $n > 1$ the toposes $(\mathbf{C}_{reg(n)})_{ex}$ seem not to have been studied before. In this respect, also in [84], the notion of a $\leq$*-partial combinatory algebra* ($\leq$-PCA) is introduced and it is observed that given any PCA $\mathbf{A}$, the set of non-empty subsets of $\mathbf{A}$ can be made into a $\leq$-PCA. Moreover, for any $\leq$-PCA it is possible to define its associated category of “assemblies”. These categories might provide a way of presenting the hierarchies of toposes $(\mathbf{C}_{reg(n)})_{ex}$. We will not pursue this idea here, though.

Another idea that we will not pursue here is the existence and description of the colimit of this sequence of toposes.

For the case of $\mathbf{C} = H_+$ we have a clearer picture. Indeed, by Proposition 4.4.1 it follows that $((H_+)_{reg(n)})_{ex} \simeq (((D_+)^n H)_+)_{ex} \simeq \mathbf{Set}^{((D_+)^n H)^{\mathbf{op}}}$.

## 8.3 Typed structures

The 2-category of *partial combinatory type structures* was introduced in [68] as “a natural setting for developing all the parts of the standard theory of realizability models that do not specifically exploit the untypedness of the PCA”. In particular, it is a nice setting in which to present the results in [67].

It is possible to define categories of assemblies associated to these typed structures and it is natural to wonder whether they give rise to toposes. It was shown in [65] that a typed structure gives rise to a topos if and only if it is equivalent to a PCA. Similar results have also appeared in [26].

In this section we introduce the relevant notions and present this result from the perspective of the machinery developed in this chapter.

There are slightly different versions of typed structures. We now introduce the one in [65].

**Definition 8.3.1.** A *typed partial combinatory algebra (TPCA)* is a non-empty set $\mathcal{T}$ of types together with the following data.

1. binary operations $\times$ and $\rightarrow$
2. a set $|T|$ of realizers of type $T$ for every $T \in \mathcal{T}$
3. a partial application function $\cdot_{S,T} : |S \rightarrow T| \times |S| \longrightarrow |T|$ for all $S, T \in \mathcal{T}$ (written as juxtaposition) such that for all $S, T, U \in \mathcal{T}$ there are elements

$$k_{S,T} \in |S \rightarrow T \rightarrow S|, \quad s_{S,T,U} \in |(S \rightarrow T \rightarrow U) \rightarrow (S \rightarrow T) \rightarrow (S \rightarrow U)|,$$
$$pair_{S,T} \in |S \rightarrow T \rightarrow S \times T|, \quad fst_{S,T} \in |S \times T \rightarrow S|, \quad snd_{S,T} \in |S \times T \rightarrow T|$$

satisfying

$$kab = a, \qquad sab \text{ is defined}, \qquad sabc \simeq ac(bc)$$
$$fst(pairab) = a\ , \qquad snd(pairab) = b$$

for appropiately typed $a, b, c$.

For any TPCA $\mathcal{T}$ it is possible to build the category $\mathbf{PAss}(\mathcal{T})$ as follows. An object $X = (|X|, T_X, \|\_\|_X)$ consists of a set $|X|$, a type $T_X$ and a function $\|\_\|_X : |X| \longrightarrow |T_X|$.

A map $f : Y \longrightarrow X$ is a function $f : |Y| \longrightarrow |X|$ such that there exists an element $a \in |T_Y \rightarrow T_X|$ such that for every $y \in |Y|, a\|y\|_Y = \|fy\|_X$. We say that $f$ is *realized* by $a$.

It is also possible to define categories of assemblies (which are the categories that are used in [64, 65]).

For any TPCA $\mathcal{T}, \mathbf{PAss}(\mathcal{T})$ is a lextensive category. The description of finite limits and finite coproducts is not very difficult and it is left as an exercise. The type of the pullback $Y \times_X Y'$ of $f : Y \longrightarrow X$ and $f' : Y' \longrightarrow X$ has type $T_Y \times T_{Y'}$.

There is an obvious forgetful functor $|\_| : \mathbf{PAss}(\mathcal{T}) \longrightarrow \mathbf{Set}$ that assigns to each $X$ its underlying set $|X|$. It preserves finite limits and it has a full and faithful right adjoint $\nabla : \mathbf{Set} \longrightarrow \mathbf{PAss}(\mathcal{T})$.

To define $\nabla$ choose a non-empty type $C$ in $\mathcal{T}$ (there is always one such) and some element $* \in |C|$. Then for any set $S$ let $\nabla S = (S, C, \|\_\|_{\nabla S})$ with $\|s\|_{\nabla S} = *$ for every $s$ in $S$. Also, any function $f : S' \longrightarrow S$, is realized by $k*$.

It is straighforward to show that $|\_|$ is left adjoint to $\nabla$ and so, that $\mathbf{PAss}(\mathcal{T})$ has an AC-chaotic situation. Objects in the image of $\nabla$ are called *codiscrete* in [65].

**Lemma 8.3.2.** *A map $f : Y \longrightarrow X$ is a pre-embedding in $\mathbf{PAss}(\mathcal{T})$ if and only if $f$ is isomorphic over $X$ to an $f' : Y' \longrightarrow X$ such that $T_{Y'} = T_X$ and for every $y$ in $Y'$, $\|y\|_{Y'} = \|fy\|_X$.*

*Proof.* Easy. □

A key notion on [64, 65] is the following.

**Definition 8.3.3.** Let $\mathcal{T}$ be a TPCA. A type $U$ is called a *universal* if for any type $T \in \mathcal{T}$ there exists realizers $e_T \in |T \to U|$ and $r_T \in |U \to T|$ such that for every $a \in |T|, r_T(e_T a) = a$. That is, every type is a partial ($r_T$ may be partial) retract of the type $U$.

As explained in [68, 65] it can be shown that a TPCA has a universal type if and only if it is equivalent (in a suitable sense) to a PCA.

The following result should be compared with Lemma 4.1 in [65] which is essentially the *if* direction.

**Proposition 8.3.4.** *A TPCA $\mathcal{T}$ has a universal type if and only if $\mathbf{PAss}(\mathcal{T})$ has a generic object.*

*Proof.* For any type $T$ let $\Delta_T = (|T|, T, id_{|T|})$.

Consider first the *only if* direction. So assume that there is a universal type $U$. We show that $\Delta_U$ is a generic object. Indeed, by the description of pre-embeddings in Lemma 8.3.2 it is easy to see that for any $T$, the element $e_T$ given by Definition 8.3.3 induces a pre-embedding $\Delta_T \longrightarrow \Delta_U$.

It is also easy to see that any $X$ with underlying type $T$ is pre-embedded in $\Delta_T$. As pre-embeddings compose, every $X$ is pre-embedded in $\Delta_U$.

For the *if* direction we assume that there is a generic object $\Upsilon$. We show that $U = T_\Upsilon$ is a universal type. For any type $T$, we have a pre-embedding $\Delta_T \longrightarrow \Upsilon$. By Lemma 8.3.2 this pre-embedding is iso over $\Upsilon$ to a pre-embedding $\Delta'_T \longrightarrow \Upsilon$

such that the type of $\Delta'_T$ is $U$. Realizers for the isomorphisms $\Delta_T \longrightarrow \Delta'_T$ and $\Delta'_T \longrightarrow \Delta_T$ give the elements $e_T$ and $r_T$ required by Definition 8.3.3. $\square$

So we can present the following result as a corollary of Proposition 8.3.4 and Theorem 8.2.1.

**Theorem 8.3.5 (Lietz-Streicher).** *For any TPCA $\mathcal{T}$. $\mathbf{PAss}(\mathcal{T})_{ex}$ is a topos if and only if $\mathcal{T}$ has a universal type.*

## 8.4 Quasi-toposes and chaotic situations

Here we simply put together some of the results from this chapter and the previous one in order to observe that chaotic situations can also be used to give very compact proofs that certain regular completions are quasi-toposes. The idea is that we have proved in the course of the present chapter and the previous one that a chaotic situation implies most of the sufficient conditions discussed in Section 4.3. We need only deal here with quasi-coequalizers (recall Definition 4.1.2).

**Lemma 8.4.1.** *If $\mathbf{C}$ has a chaotic situation then $\mathbf{C}$ has quasi-coequalizers.*

*Proof.* Let $f, g : Y \longrightarrow X$. By assumption, $|f|, |g| : |Y| \longrightarrow |X|$ has a coequalizer $q : X \twoheadrightarrow Q$. It is then not difficult to prove that $\nabla q.\eta : X \longrightarrow \nabla|Q|$ is a quasi-coequalizer of $f$ and $g$. $\square$

We can now state a very convenient way of showing that some regular completions are quasi-toposes.

**Corollary 8.4.2.** *Let $\mathbf{C}$ be a lextensive category with weak dependent products and a chaotic situation. Then $\mathbf{C}_{reg}$ is a quasi-topos.*

*Proof.* Lemma 8.4.1 gives quasi-coequalizers and, by Lemma 7.4.3, $\mathbf{C}$ has stable regular-epi/mono factorizations. It follows that strong and regular monos coincide in $\mathbf{C}$. As $\nabla 1$ is terminal in $\mathbf{C}$, by Proposition 8.1.7, $\nabla\sigma_1 = \nabla\top : \nabla 1 \rightarrowtail \nabla\Omega$ is a strong-subobject classifier in $\mathbf{C}$. As $\nabla\Omega$ is exponentiating, Corollary 4.3.4 implies that $\mathbf{C}_{reg}$ is a quasi-topos. $\square$

This gives very easy proofs that **Equ** and **Ass** are quasi-toposes. The reader is invited to check if the category of partitioned assemblies for a TPCA has weak dependent products.

# Chapter 9

# Topologies

It is well known [71] that universal closure operators in elementary toposes are in correspondence with the Lawvere-Tierney topologies therein. Moreover, in the case of a presheaf topos on a small category $\mathbf{C}$, the universal closure operators also correspond to the Grothendieck topologies on $\mathbf{C}$.

In this chapter we characterize, in a similar way, the universal closure operators in the regular and exact completions of a *locally* small category $\mathbf{C}$.

In order to motivate this abstract result let us mention that regular and ex/reg completions are closely related to categories of separated objects and of sheaves with respect to certain "extreme" topologies. This fact will be useful, in turn, to prove good properties of the completions. All this will be explained in detail in Chapters 10 and 11.

## 9.1 Topologies in categories with finite limits

In this section we introduce the notion of a topology in a *locally* small category $\mathbf{C}$ with finite limits. In a later section we will show that our topologies in $\mathbf{C}$ correspond to the universal closure operators in $\mathbf{C}_{reg}$ and $\mathbf{C}_{ex}$.

**Definition 9.1.1.** Let $\mathbf{C}$ be a category with finite limits. A *quasi-topology* is a function $J$ such that for every $X$ in $\mathbf{C}$, $JX$ is a class of morphisms with codomain $X$ subject to the following axioms:

**(T1)** every split epi with codomain $X$ is in $JX$

**(T2)** for $f : Y \longrightarrow X$, if $g$ is in $JX$ then $f^*g$ is in $JY$

**(T3)** for $f : Y \longrightarrow X$ in $JX$ and $g : Z \longrightarrow X$, if $f^*g$ is in $JY$ then $g$ is in $JX$

This is a compact definition. Let us introduce now an equivalent one in terms of axioms that may be easier to test in practice. Indeed, this is the case for the example we will discuss in Section 9.1.1.

**Proposition 9.1.2.** *A quasi-topology can be equivalently defined by the following axioms.*

**(T1')** *every identity* $id_X$ *is in* $JX$

**(T2)** *for* $f : Y \to X$*, if* $g$ *is in* $JX$ *then* $f^*g$ *is in* $JY$

**(T3')** *if* $g.h$ *is in* $JX$ *then* $g$ *is in* $JX$

**(T4')** *if* $f : Y \longrightarrow X$ *is in* $JX$ *and* $g$ *is in* $JY$ *then* $f.g$ *is in* $JX$

*Proof.* Let us first prove that the original set of axioms imply the second. Axiom T1' follows from T1.

Consider now T3'. If $f = g.h$ then $f^*g$ splits and so, by T1, is in $J$. As $f$ is assumed to be in $J$, by T3, $g$ is in $J$ proving T3'.

In order to prove T4' notice that $g$ factors through $f^*(f.g)$. Then, as $g$ is in $J$, we can use T3' to conclude that $f^*(f.g)$ is in $J$. But $f$ is also in $J$, so by T3 we conclude that $f.g$ is in $J$.

Conversely, assume the second set of axioms.

Axiom T1 follows from T1' and T3'.

Finally, consider axiom T3. By T4', $f.(f^*g) = g.(g^*f)$ is in $J$. Then, by T3', $g$ is in $J$. □

In order to characterize universal closure operators, we need an extra axiom which requires the following definition.

**Definition 9.1.3.** A map $h : Z \to X$ is *closed* with respect to a quasi-topology $J$ if the following holds: for every $f : Y \to X, f^*h \in JY$ implies that $f$ factors through $h$.

The next lemma presents alternative formulations of this concept.

**Lemma 9.1.4.** *Let* $J$ *be a quasi-topology. Then the following are equivalent.*

1. $h$ *is closed for* $J$

2. *for every commutative square as below,*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "g"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

*$g$ in $J$ implies that $f$ factors through $h$*

3. *$f^*h$ in $J$ implies that $f^*h$ is a split epi*

*Proof.* Easy. □

Recall from Section 5.2 that for every $f$ we denote by $\lfloor f \rfloor$ the corresponding element of $Prf(X)$. It is clear that if $h$ is closed and $\lfloor h \rfloor = \lfloor h' \rfloor$ then $h'$ is also closed.

Let us now prove two simple facts about closed maps.

**Lemma 9.1.5.** *Let $J$ be a quasi-topology on $\mathbf{C}$ and let $h : Y \to X$. Also, let $g_0, g_1$ be in $J$ and $h_0, h_1$ be closed such that $\lfloor h_0.g_0 \rfloor = \lfloor h_1.g_1 \rfloor$. Then $\lfloor h_0 \rfloor = \lfloor h_1 \rfloor$.*

*Proof.* It is enough to prove $\lfloor h_0 \rfloor \leq \lfloor h_1 \rfloor$. For this, let $k$ be such that $h_1.g_1.k = h_0.g_0$. In other words, the following square commutes.

$$\begin{tikzcd} \cdot \arrow[r, "g_1.k"] \arrow[d, "g_0"'] & \cdot \arrow[d, "h_1"] \\ \cdot \arrow[r, "h_0"'] & \cdot \end{tikzcd}$$

As $h_1$ is closed and $g_0 \in J$ it follows that $\lfloor h_0 \rfloor \leq \lfloor h_1 \rfloor$. □

Lemma 9.1.5 implies that a topology determines, for each proof $\lfloor f \rfloor$, a unique proof $\lfloor \overline{f} \rfloor$ where $\overline{f}$ is a closed map.

**Lemma 9.1.6.** *If $h$ is a closed map for a quasi-topology, then for any $f$, $f^*h$ is also closed.*

*Proof.* This is immediate from Lemma 9.1.4 item 3. □

We can now formulate the notion of topology.

**Definition 9.1.7.** A quasi-topology $J$ is a *topology* if it holds that

**(T)** for every arrow $f : Y \to X$ there exists $g : V \to W \in JW$ and a closed $h : W \to X$ such that $\lfloor f \rfloor = \lfloor h.g \rfloor$.

$$\begin{tikzcd}
Y \arrow[rr, shift right] \arrow[dr, "f"'] & & V \arrow[ll, shift right] \arrow[dl, "h.g"] \\
& X &
\end{tikzcd}$$

The most immediate examples of topologies are given by stable factorization systems $(\mathcal{E}, \mathcal{M})$ satisfying T3' (see [31], Proposition 2.1.4). Indeed, given one such, we can define $JX$ to be the maps in $\mathcal{E}$ with codomain $X$ and in this way we obtain a quasi-topology $J$. The fact that $\mathcal{E}$ is *orthogonal* to $\mathcal{M}$ (see [14]) implies that the maps in $\mathcal{M}$ are $J$-closed. The factorization property trivially implies axiom (T). So $J$ is a topology. For example, stable epi/regular-mono and stable regular-epi/mono factorizations induce topologies that we will study in more detail in Chapter 10.

On the other hand, the axiom for a topology is weaker than the usual factorization property. In the next subsection we present a class of examples that, in general, do not seem to arise from factorization systems.

### 9.1.1 Oracle topologies

In [39] (section 17) the observation that there is a connection between notions of degree and the forcing of decidability in recursive realizability is attributed to Powell. This observation finds a nice expression in the fact that the $\vee$-semilattice of Turing degrees embeds into the Heyting algebra of Lawvere-Tierney topologies in the effective topos **Eff** (see [94] p. 65).

Anticipating a bit, it is the case that our topologies in the sense of Definition 9.1.7 characterize the universal closure operators in the corresponding exact completion (Corollary 9.3.6). It follows that the Lawvere-Tierney topologies in **Eff** can be presented in this way. We now present an example of this. The main purpose of this example is to show non-trivial instances of the axiom (T) of Definition 9.1.7.

For any subset $A \subseteq \mathbb{N}$ of the natural numbers we can consider the the class of partial $A$-recursive functions [97], that is, those that in their process of computation can "ask an oracle whether a number is or not in $A$" (see also [89]).

We now build a quasi-topology in $\mathbf{PAss}(K_1)$. For each $X$, let $J_A(X)$ be the class of maps with codomain $X$ that have an $A$-recursive section. That is, $g : Z \longrightarrow X$ is in $J_A(X)$ if and only if there exists a function $s : |X| \longrightarrow |Z|$ that can be realized by a partial $A$-recursive function and such that $|g|.s = id$.

**Lemma 9.1.8.** *$J_A$ is a quasi-topology.*

*Proof.* We use Proposition 9.1.2. Axioms T1',T3' and T4' easily hold. In order to prove T2, let $g : Z \longrightarrow X$ be in $J_A$, $f : Y \longrightarrow X$ any map and $g' = f^*g : P \longrightarrow X$. We can assume that $P$ is given by $|P| = \{(y, z)|fy = gz\}$ and $\|(y, z)\| = \langle\|y\|, \|z\|\rangle$.

As $g$ is in $J_A$, there exists a section $h : |X| \longrightarrow |Z|$ that is realized by some partial $A$-recursive $a_h$. The map $y \longmapsto (y, h(fy))$ from $|X| \longrightarrow |P|$ is clearly a section of $g'$ and it is $A$-realized by $\lambda a.\langle a, a_h(a_f a)\rangle$. So $g'$ is in $J_A$. □

In order to prove that $J_A$ is a topology we are going to need a bit more work. First, we can pre-order the subsets of the natural numbers in the two following ways. For any $p, q \subseteq \mathbb{N}$ let $p \vdash q$ if and only if there exists a partial recursive $a$ such that for every $n$ in $p$, $an$ is defined and in $q$. In this case, we say that *$a$ realizes $p \vdash q$*. Similarly, we define $p \vdash_A q$ to mean that there exists a partial $A$-recursive realizer satisfying the same condition.

Also, for any set $I$ and indexed pairs $p_i, q_i \subseteq \mathbb{N}$ with $i \in I$ we say that *uniformly* $p_i \vdash q_i$ if there exists and $a$ such that for every $i \in I$, $a$ realizes $p_i \vdash q_i$. Similarly for $\vdash_A$. We say that $a$ is a *uniform* realizer.

We now borrow from [91] (see also [92]) the following fact.

**Proposition 9.1.9.** *There exists a function $\psi^*_A$ taking subsets of $\mathbb{N}$ to subsets of $\mathbb{N}$ such that $p \vdash_A q$ if and only if $p \vdash \psi^*_A(q)$.*

Although this is not mentioned in [91], the next corollary follows directly from his proof of Proposition 9.1.9.

**Corollary 9.1.10.** *Uniformly $p_i \vdash_A q_i$ if and only if uniformly $p_i \vdash \psi^*_A(q_i)$.*

It is also easy to show that uniformly $\psi^*_A\psi^*_A(q_i) \vdash \psi^*_A(q_i)$.

We can now find some closed maps with respect to $J_A$.

**Lemma 9.1.11.** *Let $h : Z \longrightarrow X$ be a map in $\mathbf{PAss}(K_1)$ such that for every $x \in X$, there exists a subset $q_x \subseteq \mathbb{N}$ such that $\|h^{-1}x\| = \{\|z\||hz = x\} = \{\langle\|x\|, a'\rangle|a' \in \psi^*_A(q_x)\}$. Then, $h$ is closed with respect to $J_A$.*

*Proof.* We use item 2 of Lemma 9.1.4. Let $f : Y \longrightarrow X$ and assume that the square below commutes for some $f'$ and $g$ in $J_A$.

$$\begin{tikzcd} |P| & P \arrow[r, "f'"] \arrow[d, "g"'] & Z \arrow[d, "h"] \\ |Y| \arrow[u, "s"] & Y \arrow[r, "f"'] & X \end{tikzcd}$$

Because we assume $g$ to be in $J_A$, there exists a partial $A$-recursive function $a_s$ realizing $s$ with $|g|.s = id$. It follows that $|h.f'|.s = |f|$ and clearly, $|f'|.s : |Y| \longrightarrow |Z|$ is $A$-realized. This means that uniformly in $y \in Y$, $\|y\| \vdash_A \|f'(sy)\|$. Let $a$ be such a uniform realizer and let $b = \lambda y.\pi_1(ay)$. By our assumptions on $\|\_\|_Z$, we have that $b$ is a uniform (in $y$) $A$-realizer for $\|y\| \vdash_A \psi_A^*(q_{h(f'(sy))}) = \psi_A^*(q_{(fy)})$. It follows that uniformly $\|y\| \vdash \psi_A^*(q_{(fy)})$. Let $c$ be a uniform realizer for this. By our assumptions again, for every $y \in Y$ there exists a $ty \in Z$ such that $h(ty) = fy$ and such that $\|ty\| = \langle \|fy\|, c\|y\| \rangle$. As $f$ has a recursive realizer, the mapping $y \longmapsto ty$ can be realized and so induces a map $t : Y \longrightarrow Z$ in $\mathbf{PAss}(K_1)$ such that $h.t = f$. So $h$ is closed with respect to $J_A$. $\square$

We can now finish this subsection by showing that the $J_A$'s are topologies.

**Proposition 9.1.12.** *For every $A \subseteq \mathbb{N}$, $J_A$ is a topology in $\mathbf{PAss}(K_1)$.*

*Proof.* Let $f : Y \longrightarrow X$ be any map in $\mathbf{PAss}(K_1)$. Let $X_0$ be the partitioned assembly such that $|X_0| = \{(x, a) | x \in X, a \in \psi_A^*(\|f^{-1}x\|)\}$ and with $\|(x, a)\| = \langle \|x\|, a \rangle$. The projection $(x, a) \longmapsto x$ is a closed map for $J_A$ by Lemma 9.1.11. Let us call it $h : X_0 \longrightarrow X$. Now let $Y_0$ be such that $|Y_0| = \{(y, a) | y \in Y, (fy, a) \in X_0\}$ and with $\|(y, a)\| = \langle \|y\|, a \rangle$. The obvious map $(y, a) \longmapsto (fy, a)$ is realized using a realizer for $f$. We now prove that the induced map $g : Y_0 \longrightarrow X_0$ is in $J_A$.

By Corollary 9.1.10, we have that, uniformly in $x \in X$, $\psi_A^*(\|f^{-1}x\|) \vdash_A \|f^{-1}x\|$. Let $b$ be a uniform realizer for this.

For every $(x, a) \in X_0$, we have that $a \in \psi_A^*(\|f^{-1}x\|)$. Then, $ba$ is defined an in $\|f^{-1}x\|$. That is, there exists a $s'x \in Y$ such that $f(s'x) = x$ and such that $\|s'x\| = ba$. Then, the map $(x, a) \longmapsto (s'x, a)$ induces a function $s : |X_0| \longrightarrow |Y_0|$ such that $g.s = id$ and such that it is realized by $\lambda\langle a', a \rangle.\langle ba, a \rangle$ which is an $A$-recursive function. So $g$ is in $J_A$.

Finally, we must prove that $\lfloor f \rfloor = \lfloor h.g \rfloor$. But this is easy because we have the map $Y \longrightarrow Y_0$ sending $y$ to $(y, \|y\|)$ and the map $Y_0 \longrightarrow Y$ sending $(y, a)$ to $y$. □

As explained in [39, 94, 91] the operation $\psi_A^*$ alone is actually enough to determine the Lawvere-Tierney topology in the effective topos **Eff**. Indeed, in toposes arising from triposes, the Lawvere-Tierney topologies are in correspondence with operations analogous to the $\psi_A^*$ above. Because of this, let us stress that our purpose in this subsection was to show a non-trivial instance of axiom (T).

On the other hand, let us also stress that universal closure operators in arbitrary exact completions (where subobject classifiers need not exist) can not be characterized in terms of operations analogous to $\psi_A^*$ (which induce Lawvere-Tierney topologies on the subobject classifier of **Eff**).

Finally, subtoposes of **Eff** are treated from yet another perspective in [86] where they are presented in terms of calculi of fractions (defined also using the ideas behind $\psi_A^*$).

## 9.2 Closure operators in suitable categories

In this section we recall the definition of universal closure operators (see for example [14]) and discuss some properties of them in suitable categories (recall Section 3.5).

**Definition 9.2.1.** A *universal closure operator* in a category with finite limits is a family of operations $\overline{(\_)}_X : Sub(X) \to Sub(X)$ indexed by the objects of the category and subject to the following axioms.

1. (monotone) $U \leq V$ implies $\overline{U} \leq \overline{V}$
2. (inflationary) $U \leq \overline{U}$
3. (idempotent) $\overline{\overline{U}} = \overline{U}$
4. (natural) for every $f : Y \to X$ and $U \rightarrowtail X$, $f^*\overline{U} = \overline{f^*U}$

A subobject $U$ of $X$ is *closed* if $\overline{U} = U$ and it is *dense* if $\overline{U} = X$. We now state some well known properties of closure operators.

**Lemma 9.2.2.** *Let* $u : U \rightarrowtail A$.

1. $\overline{U}$ *is closed*

2. *$U$ is dense as a subobject of $\overline{U}$*

3. *both closed and dense monos are stable under pullbacks*

*Proof.* See for example [14]. □

When the underlying category is regular, the regular epis have good reflection properties.

**Lemma 9.2.3.** *Consider a universal closure operator in a regular category. Then regular epis reflect both dense and closed monos. That is, if $q : Y \twoheadrightarrow X$ is a regular epi and $q^*u$ is closed (resp. dense) then $u$ is closed (resp. dense).*

*Proof.* Let us consider the case of closed monos. Assume that square (1) is a pullback with $v$ closed and $q$ a regular epi.

$$\begin{tikzcd}
 & & \overline{U} \arrow[ddl, tail] \\
V \arrow[r, two heads] \arrow[urr, two heads] \arrow[d, tail, "v"'] \arrow[dr, phantom, "(1)"] & U \arrow[ur, tail] \arrow[d, tail, "u"] & \\
Y \arrow[r, two heads, "q"'] & X &
\end{tikzcd}$$

As $q^*\overline{U} = \overline{q^*U} = \overline{V} = V$, we obtain the outer regular-epi/mono factorization of $q.v$ showing that $U$ must be iso to $\overline{U}$.

The case for dense monos is similar and can be found in [14]. □

For the rest of this chapter, let $\mathbf{D}$ be a suitable category (recall Section 3.5) and let $\mathbf{C} = \mathit{Proj}(\mathbf{D})$.

We are going to show that in such a category, universal closure operators are determined by their behaviour on projectives. For this purpose let us introduce the following definition.

**Definition 9.2.4.** A *universal closure operator on subobjects of projectives* in $\mathbf{D}$ is a family of operations $\overline{(\_)}_X : \mathit{Sub}_{\mathbf{D}}(X) \to \mathit{Sub}_{\mathbf{D}}(X)$ indexed by the projective objects of $\mathbf{D}$ and satisfying axioms 1, 2 and 3 of Definition 9.2.1 and the following.

4'. for every $f : Y \to X$ between projectives and $U \rightarrowtail X$, $f^*\overline{U} = \overline{f^*U}$

We will show that these restricted closure operators uniquely extend to honest universal closure operators.

**Lemma 9.2.5.** *Let $\overline{(\_)}$ be a universal closure operator on subobjects of projectives in $\mathbf{D}$. Let $A$ be any object and $r : R \twoheadrightarrow A$ a regular epi with $R$ projective. If square (1) is a pullback then so is square (2) where the top-right composition is the regular-epi/mono factorization of the left-bottom composition.*

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[d, "v"'] \arrow[dr, phantom, "(1)"] & U \arrow[d] \\ R \arrow[r, two heads, "r"'] & A \end{tikzcd} \qquad \begin{tikzcd} \overline{V} \arrow[r, two heads] \arrow[d, "\overline{v}"'] \arrow[dr, phantom, "(2)"] & \exists_r \overline{v} \arrow[d] \\ R \arrow[r, two heads, "r"'] & A \end{tikzcd}$$

*Proof.* Let $r_0, r_1$ be the kernel pair of $r$ and consider the following pullback diagram:

$$\begin{tikzcd} W \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "w"] & V \arrow[r, two heads] \arrow[d, "v"] \arrow[dr, phantom, "(1)"] & U \arrow[d] \\ A' \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & R \arrow[r, two heads, "r"'] & A \end{tikzcd}$$

By hypothesis we have $\overline{v} : \overline{V} \to R$. If we prove that $r_0^*\overline{v} = r_1^*\overline{v}$ then the result follows by Lemma 2.3.3.

To prove this equation let $q : Q \twoheadrightarrow A'$ be a projective cover. We then have $r_0.q, r_1.q : Q \to R$ two arrows between projectives and we can calculate: $(r_0.q)^*\overline{v} = \overline{q^*(r_0^*v)} = \overline{q^*w} = \overline{q^*(r_1^*v)} = (r_1.q)^*\overline{v}$

As $q$ is a regular epi, $q^*$ is mono so $q^*(r_0^*\overline{v}) = q^*(r_1^*\overline{v})$ implies $r_0^*\overline{v} = r_1^*\overline{v}$. □

We now show how to extend these restricted closure operators.

**Proposition 9.2.6.** *Every universal closure operator on subobjects of projectives has a unique extension to a universal closure operator.*

*Proof.* For any object $A$ in $\mathbf{D}$ we define $\overline{(\_)}_A$ as follows. Choose a projective cover $q : Q \twoheadrightarrow A$. Lemma 9.2.5 leaves only one option: for every $u : U \rightarrowtail A$ let $\overline{u} : \overline{U} \rightarrowtail A$ the mono part of the regular-epi/mono factorization of $q.\overline{q^*u}$.

What we are doing is defining $\overline{U} = \exists_q \overline{(q^*U)}$. Notice that by Lemma 9.2.5, we have that $q^*\overline{U} = \overline{q^*U}$.

Let us prove that this definition does not depend on the choice of projective cover. So let $r : R \twoheadrightarrow A$ be another projective cover. It follows that $r = q.s$ for some $s : R \to Q$. Now calculate $\exists_r \overline{r^*U} = \exists_{q.s} \overline{s^*q^*U} = \exists_q \exists_s s^* \overline{q^*U} = \exists_q \overline{q^*U}$.

Let us now prove that so defined, the operation $\overline{(\ )}$ is a universal closure operator.

As all of $\exists_q$, $f^*$ and $\overline{(\_)}$ are monotone, $\overline{(\_)}_A$ also is.

To see that it is inflationary notice that as $q^*U$ is a subobject of a projective we have that $q^*U \le \overline{q^*U}$ and so, by adjointness, $U \le \exists_q(\overline{q^*U}) = \overline{U}$.

To prove $\overline{(\_)}_A$ idempotent, notice that $\overline{\overline{U}} = \exists_q \overline{q^*\overline{U}} = \exists_q \overline{\overline{q^*U}} = \exists_q \overline{q^*U} = \overline{U}$.

We finally prove that it is natural. For this, let $U \rightarrowtail A$ and let $f : B \to A$. We need to prove that $f^*\overline{U} = \overline{f^*U}$. So let $q : Y \twoheadrightarrow B$ and $r : X \twoheadrightarrow A$ be projective covers and let $g : Y \to X$ arise by projectivity of $Y$ as in the following square.

$$
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, two heads, "q"'] & X \arrow[d, two heads, "r"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}
$$

Now calculate $q^*f^*\overline{U} = g^*r^*\overline{U} = \overline{g^*r^*U} = \overline{q^*f^*U} = q^*\overline{f^*U}$. As $q^*$ is mono, $f^*\overline{U} = \overline{f^*U}$.

This finishes the proof.

□

## 9.3 Topologies and closure operators

In this section we prove the correspondence between topologies on $\mathbf{C} = \mathit{Proj}(\mathbf{D})$ and closure operators in $\mathbf{D}$.

### 9.3.1 From closure operators to topologies

For a given universal closure operator in $\mathbf{D}$ we define a quasi-topology $J$ in $\mathbf{C}$ as follows: for every object $X$ in $\mathbf{C}$, let $JX$ to be the class of maps $h$ in $\mathbf{C}$ with codomain $X$ that factor, in $\mathbf{D}$, as regular-epi followed by a dense mono.

**Lemma 9.3.1.** *$J$ is a quasi-topology.*

*Proof.* Axiom T1'. A split epi in **C** is a regular epi in **D** so the mono part of its factorization is trivially dense.

Axiom T2 follows because dense monos are preserved by pullbacks and pullbacks are reflected by the embedding of **C** into **D**.

Axiom T3' follows because $Im(g.h) \leq Im(g)$.

Axiom T4'. Let $g = m_g.e_g : Z \to Y$ in $JY$ and $f = m_f.e_f : Y \to X$ in $JX$ with their respective regular-epi/mono factorizations.

Also, let $f.g$ factor as $e : Z \twoheadrightarrow A$ a regular epi followed by $m : A \rightarrowtail X$ a mono. Pullback $m$ along $f$ as in the following diagram.

$$\begin{tikzcd}
C \arrow[r, two heads] \arrow[d, tail, "m'"'] & E \arrow[r, tail] \arrow[d, tail, "n"'] & B \arrow[d, tail, "m"] \\
Y \arrow[r, two heads, "e_f"'] & D \arrow[r, tail, "m_f"'] & X
\end{tikzcd}$$

It is clear that $g$ factors through $f^*m = m'$. It follows that $m_g$ factors through $m'$. As $m_g$ is dense because $g$ is in $J$, $m'$ is dense. By Lemma 9.2.2, $n$ is dense and then so is $m_f.n$. This, in turn implies that $m$ is dense. So $f.g$ is in $J$. □

We now characterize the closed maps with respect to the $J$ we have defined starting from a given closure operator.

**Lemma 9.3.2.** *$h$ in **C** is closed for $J$ if and only if $h$ factors in **D** as a regular epi followed by a closed mono.*

*Proof.* Let $h = m.e : Z \to X$ with $m : U \rightarrowtail X$ mono and $e$ a regular epi.

Let us first consider the *only if* direction. So assume that $h$ is a closed map with respect to $J$.

Let $q : Y \twoheadrightarrow \overline{U}$ a projective cover and consider the following diagram.

$$\begin{tikzcd}
W \arrow[r, two heads] \arrow[d, two heads] & A \arrow[r, tail] \arrow[d, two heads] & Z \arrow[d, two heads, "e"] \\
W' \arrow[r, two heads] \arrow[d, tail] & U \arrow[r] \arrow[d, tail] & U \arrow[d, tail, "m"] \\
Y \arrow[r, two heads, "q"'] & \overline{U} \arrow[r, tail, "\overline{m}"'] & X
\end{tikzcd}$$

First notice that $W$ is in $\mathbf{C}$ because projectives are closed under finite limits in $\mathbf{D}$. Then, as $U \rightarrowtail \overline{U}$ is dense, the left hand mono is dense and so the left hand map is in $J$. By hypothesis, $h$ is closed so the bottom map factors through $h$ and hence $\overline{U} \leq U$. So $U \rightarrowtail X$ is closed.

Conversely, assume that $m$ is closed and let $f^*h$ be in $J$ with $f : Y \to X$. Then $f^*m$ is dense and as it is also closed it follows that it is an isomorphism. Hence $f$ factors through $m$. As $Y$ is projective, $f$ also factors through $h$. ☐

**Proposition 9.3.3.** *A universal closure operator in* $\mathbf{D}$ *induces a topology on* $\mathbf{C}$.

*Proof.* Given a universal closure operator on $\mathbf{D}$, we have seen how to define a $J$ that is a quasi-topology by Lemma 9.3.1.

To prove the axiom (T) for a topology, let $f : X \to Y$ and consider the following diagram in $\mathbf{D}$ where every square is a pullback.

$$\begin{tikzcd}
V \arrow[r, two heads, "reg"] \arrow[d, two heads, "reg"'] & A \arrow[r, tail, "dense"] \arrow[d, two heads, "reg"] & W \arrow[d, two heads, "reg"] \\
X \arrow[r, two heads, "reg"'] \arrow[d, "id"'] & Im(f) \arrow[r, tail, "dense"'] \arrow[d, "id"] & \overline{Im(f)} \arrow[d, tail, "closed"] \\
X \arrow[r, two heads, "reg"'] & Im(f) \arrow[r, tail] & Y
\end{tikzcd}$$

Having in mind that the bottom line is the regular-epi/mono factorization of $f$, one should look at this diagram from the bottom right corner where we have the familiar facts that the closure of an object is closed and that the embedding of an object in its closure is dense.

The regular epi $W \twoheadrightarrow \overline{Im(f)}$ is some projective cover. The remaining squares are explained by the facts that, in $\mathbf{D}$, regular epis and dense monos are closed under pullback.

Now, $V$ is a pullback of arrows between projectives so it is projective.

So let $g : V \to W$ be the top composition and let $h : W \to Y$ be the right hand composition. By our definition of $J$, $g \in JW$ and by Lemma 9.3.2, $h$ is closed for $J$. So we have the needed arrows and the fact that $\lfloor h \rfloor \geq \lfloor g.f \rfloor$.

As the map $V \twoheadrightarrow X$ is a regular epi between projectives in $\mathbf{D}$, it splits. So $\lfloor h \rfloor \leq \lfloor g.f \rfloor$. ☐

Let us concentrate now on the other side of the correspondence.

## 9.3.2 From topologies to closure operators

We first show how to build a universal closure operator from a topology and then characterize the dense monos.

**Proposition 9.3.4.** *A topology on* **C** *induces a universal closure operator in* **D**.

*Proof.* By Proposition 9.2.6 we need only prove that $J$ induces a universal closure operator on subobjects of projectives. Also, by Lemma 5.1.1, we can think of subobjects of a projective as proofs.

For any subobject $\lfloor f \rfloor$ of a projective $X$, the axiom for a topology gives a map $f_0$ in $J$ and $\overline{f}$ closed such that $\lfloor f \rfloor = \lfloor \overline{f}.f_0 \rfloor$.

Lemma 9.1.5 shows that the assignment $\overline{(\_)}$ given by $\overline{\lfloor f \rfloor} = \lfloor \overline{f} \rfloor$ is well defined. We now show that it is a universal closure operator on subobjects of projectives. It is clearly inflationary.

To prove that it is monotone, let $\lfloor f \rfloor \leq \lfloor g \rfloor$. By axiom (T), there exist $f_0, g_0$ maps in $J$ and $\overline{f}, \overline{g}$ closed such that $\lfloor f \rfloor = \lfloor \overline{f}.f_0 \rfloor$ and $\lfloor g \rfloor = \lfloor \overline{g}.g_0 \rfloor$. All this implies that we have diagrams as follows for some $p_f, p, p_g$.

$$
\begin{tikzcd}
\cdot \arrow[r, "p_f"] \arrow[d, "f_0"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "\overline{f}"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[rr, "p"] \arrow[dr, "f"'] & & \cdot \arrow[dl, "g"] \\
& \cdot &
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "p_g"] \arrow[d, "g"'] & \cdot \arrow[d, "g_0"] \\
\cdot & \cdot \arrow[l, "\overline{g}"]
\end{tikzcd}
$$

From the diagrams it is easy to see that $\lfloor f \rfloor \leq \lfloor \overline{g} \rfloor$. It follows that $f^*\overline{g}$ is a split epi and so it is in $J$. Then $p_f^* f^* \overline{g} = f_0^* \overline{f}^* \overline{g}$ is also in $J$. As $f_0$ is in $J$, so is $\overline{f}^* \overline{g}$. As $\overline{g}$ is closed, $\lfloor \overline{f} \rfloor \leq \lfloor \overline{g} \rfloor$. This finishes the proof of monotonicity.

We now prove idempotency. For $\lfloor f \rfloor$ we have that $\overline{\lfloor f \rfloor} = \lfloor \overline{f} \rfloor$ with $\overline{f}$ closed. So we have that $\lfloor \overline{f} \rfloor = \lfloor \overline{f}.id \rfloor$ with $\overline{f}$ closed and $id$ in $J$. It follows that $\overline{\lfloor \overline{f} \rfloor} = \lfloor \overline{f} \rfloor$ and so, that $\overline{\overline{\lfloor f \rfloor}} = \overline{\lfloor f \rfloor}$.

For universality notice that if $\lfloor f \rfloor = \lfloor \overline{f}.f_0 \rfloor$ with $f_0$ in $J$ and $\overline{f}$ closed then, for any $g$ in **C** with same codomain as $f$, $\lfloor g^* f \rfloor = \lfloor (g^*\overline{f}).(g^* f_0) \rfloor$. But $f_0$ pulls back to a dense map by (T2) and $\overline{f}$ pulls back to a closed map by Lemma 9.1.6. It follows that $\overline{\lfloor g^* f \rfloor} = \lfloor g^* \overline{f} \rfloor$ and so, that $\overline{g^* \lfloor f \rfloor} = g^* \overline{\lfloor f \rfloor}$. This finishes the proof of universality and so that we have a universal closure operator over projectives. □

**Lemma 9.3.5.** $\lfloor f \rfloor$ *in* $Sub(X)$ *is dense if and only if* $f : Y \longrightarrow X$ *is in* $J$.

*Proof.* $\lfloor \overline{f} \rfloor = \lfloor id \rfloor$ iff $\lfloor f \rfloor = \lfloor id.g \rfloor$ with $g$ in $J$ iff $f$ is in $J$. □

Proposition 9.2.6 shows how the behaviour of a universal closure operator in a suitable category is determined by its behaviour on subobjects of projectives.

Lemmas 9.3.2 and 9.3.5 help to show that the constructions in Propositions 9.3.3 and 9.3.4 induce a bijective correspondence between universal closure operators in a suitable category $\mathbf{D}$ and topologies in its full subcategory of projectives $\mathbf{C}$.

Our main examples of suitable categories are regular and exact completions, so let us state the following result.

**Corollary 9.3.6.** *There is a bijective correspondence between topologies on* $\mathbf{C}$ *and universal closure operators in* $\mathbf{C}_{ex}$ *and in* $\mathbf{C}_{reg}$.

Finally, it should be mentioned that Grothendieck topologies also make sense on non-small categories $\mathbf{C}$ and one can then consider separated objects and sheaves in $\mathbf{Set}^{\mathbf{C}^{\mathbf{op}}}$. Moreover, for any quasi-topology on $\mathbf{C}$ it should be possible to define the associated Grothendieck topology on $\mathbf{C}$ and then relate sheaves and separated objects in $\mathbf{C}_{ex}$ and $\mathbf{Set}^{\mathbf{C}^{\mathbf{op}}}$ using the results in Chapter 10 (see Proposition 10.1.5). On the other hand, $\mathbf{Set}^{\mathbf{C}^{\mathbf{op}}}$ lacks good properties if $\mathbf{C}$ is not small and arbitrary Grothendieck topologies on $\mathbf{C}$ do not make sense with respect to $\mathbf{C}_{reg}$ or $\mathbf{C}_{ex}$. So we may say that we have found the right workable notion of topology for locally small categories with finite limits.

# Chapter 10

# Separated objects, sheaves and canonical topologies

In this chapter we carry on the study of universal closure operators in exact completions using the description in terms of topologies explained in Chapter 9.

We study the categories of separated objects and of sheaves. First, we characterize the categories of separated objects and observe that in the case that $\mathbf{C}_{ex}$ is a pretopos, these categories have very good properties.

Then we study stable epi/regular-mono and regular-epi/mono factorizations. These factorization systems induce topologies that turn out to be extreme in a sense that we explain. We also study the associated categories of separated objects and of sheaves of these extreme topologies and relate them with regular and ex/reg completions. This provides a conceptual explanation of why the completions we have been working with have so many good properties.

The relation between ex/reg completions and categories of sheaves will also be exploited in Chapter 11 where we study when is such a completion a topos.

## 10.1 Separated objects and sheaves

Intimately related to the notion of a universal closure operator are the notions of sheaf and of separated object. These classes of objects induce full subcategories with good properties. For more on categories of separated objects and of sheaves see for example [45, 71, 106, 5, 20].

**Definition 10.1.1.** In a category with a universal closure operator we say that an object $X$ is a *sheaf* if for every dense mono $m : U \rightarrowtail Z$ and map $f : U \longrightarrow X$, there exists a unique $f' : Z \longrightarrow X$ such that $f'.m = f$.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, tail, "m"'] & X \\ Z \arrow[ur, "\exists! f'"'] & \end{tikzcd}$$

Also, we say that $X$ is *separated* if for every dense $m : U \rightarrowtail Z$ and maps $g, g' : Z \longrightarrow X$ such that $g.m = g'.m$ then $g = g'$.

In other words, $X$ is separated if in the diagram defining sheaves, the map $f'$ need not exist, but if it does then it is unique. Another useful characterization of separated objects is the following.

**Lemma 10.1.2.** *$X$ is separated if and only if the diagonal $\langle id, id\rangle : X \rightarrowtail X \times X$ is closed.*

*Proof.* Easy, using the fact that dense and closed monos are orthogonal. See also [45], for example. □

With this characterization it is easy to prove the following.

**Lemma 10.1.3.** *Let $\langle e_0, e_1\rangle : E \rightarrowtail X \times X$ be a closed equivalence relation and assume that it has an effective quotient $q : X \twoheadrightarrow X/E$. Then $X/E$ is separated.*

*Proof.* As $q$ is effective, the following square is a pullback.

$$\begin{tikzcd} E \arrow[r, two heads] \arrow[d, tail] & X/E \arrow[d, tail, "\langle id, id\rangle"] \\ X \times X \arrow[r, two heads, "q \times q"'] & X/E \times X/E \end{tikzcd}$$

The result then follows by Lemmas 9.2.3 and 10.1.2. □

Also, we say that an object is separated or a sheaf *over projectives* if the object satisfies the respective condition for subobjects of projectives.

Recall from Section 3.5 that we call a category $\mathbf{D}$ *suitable* if it is regular, it is covered by its full subcategory of projectives and moreover, projectives are closed under finite limits. Also, given a suitable category $\mathbf{D}$ we denote its full subcategory of projectives by $\mathbf{C}$.

**Lemma 10.1.4.** *Let $\overline{(\_)}$ be a universal closure operator in a suitable category $\mathbf{D}$. Then:*

1. *$A$ is separated iff $A$ is separated over projectives.*
2. *$A$ is a sheaf iff $A$ is a sheaf over projectives.*

*Proof.* Let us prove 1. The *only if* direction is trivial. For the other direction let $f_0, f_1 : B \to A$ and let $u : U \to B$ be a dense mono such that $f_0.u = f_1.u$.

Let $q : Y \twoheadrightarrow B$ be a projective cover. We then have the following diagram.

$$\begin{tikzcd}
V \arrow[r, two heads, "r"] \arrow[d, tail, "v"'] \arrow[dr, phantom, "P.B."] & U \arrow[r] \arrow[d, tail, "u"] & A \\
Y \arrow[r, two heads, "q"'] & B \arrow[ur, "f_0", "f_1"'] &
\end{tikzcd}$$

As $v$ is dense, and $A$ separated with respect to projectives, $f_0.q = f_1.q$. As $q$ is epi, $f_0 = f_1$ and hence, $A$ is separated.

To prove 2, the *only if* direction is also trivial. For the converse, we already know that $A$ is separated. So for, $u$ as above, we need only prove that for any $f : U \to A$ there exists a $f' : B \to A$ such that $f'.u = f$.

As before, consider a projective cover $q : Y \twoheadrightarrow B$ and consider the pullback square as in the diagram above. As $A$ is a sheaf with respect to projectives, there exists a unique $h : Y \to A$ such that $h.v = f.r$.

Now, take the kernel pair of $q$ and cover it with a projective obtaining a coequalizer diagram as in the bottom line of the following diagram.

$$\begin{tikzcd}
W \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail, "w"'] \arrow[dr, phantom, "P.B."] & V \arrow[r, two heads, "r"] \arrow[d, tail, "v"] & U \arrow[r, "f"] & A \\
X \arrow[r, shift left, "q_0"] \arrow[r, shift right, "q_1"'] & Y \arrow[r, two heads, "q"'] \arrow[urr, "h"] & B &
\end{tikzcd}$$

As $w$ is a pullback of u, it is dense by Lemma 9.2.2. As $A$ is separated, $h.q_0 = h.q_1$. As $q$ is the coequalizer of $q_0$ and $q_1$, there exists a unique $f' : B \to A$ such that $f'.q = f$. We need only check that $f'.u = g$. For this, use that $r$ is epi and calculate, $f'.u.r = f'.q.v = h.v = f.r$. This finishes the proof. ☐

Using Lemma 10.1.4 it is possible to formulate the conditions for a separated object or a sheaf in terms of $J$.

**Proposition 10.1.5.** *Let $J$ be a topology in $\mathbf{C}$ and consider the induced universal closure operator in $\mathbf{D}$.*

1. *$A$ is separated if and only if for every map $f : Y \longrightarrow X$ in $J$ and maps $g, h : X \longrightarrow A$, $g.f = h.f$ implies that $g = h$ (that is, $A$ believes that maps in $J$ are epi in $\mathbf{D}$).*

2. *$A$ is a sheaf if and only if for every map $f : Y \longrightarrow X$ in $J$ with kernel pair $f_0, f_1 : K \longrightarrow Y$ and map $g : Y \longrightarrow A$ such that $g.f_0 = g.f_1$ there exists a unique map $g' : X \longrightarrow A$ such that $g'.f = g$ (that is, $A$ believes that maps in $J$ are regular epis in $\mathbf{D}$).*

*Proof.* Let us prove 1. For the *only if* direction, let $m.e$ be the regular-epi/mono factorization in $\mathbf{D}$ of $f$ in $J$. By Lemma 9.3.5, $m$ is dense. As $e$ is epi, we have that $g.m = h.m$. As $A$ is separated, $g = h$.

For the *if* direction notice that by Lemma 10.1.4 we need only prove that $A$ is separated over projectives. To do this, let $m : U \rightarrowtail X$ be a dense subobject of a projective and let $g.m = h.m$ with $g, h : X \longrightarrow A$. Let $Y \twoheadrightarrow U$ be a projective cover and let $f : Y \longrightarrow X$ be the resulting map in $\mathbf{C}$, which is in $J$ by Lemma 9.3.5. It follows that $g.f = h.f$ and so, by hypothesis, that $g = h$. Hence, $A$ is separated over projectives.

Let us now prove 2. For the *only if* direction, let $m.e$ be as in the proof of item 1 with $m : U \rightarrowtail X$. As $e : Y \twoheadrightarrow U$ is the coequalizer of $f_0$ and $f_1$ in $\mathbf{D}$, it follows that there exists a unique $h : U \longrightarrow A$ such that $h.e = g$. As $A$ is a sheaf, there exists a unique $g' : X \longrightarrow A$ such that $g'.m = h$. It follows that $g'.f = g$. As $A$ is separated, $g'$ is unique.

For the *if* direction, as in case 1, we need only prove that $A$ is a sheaf over projectives. So let $m : U \rightarrowtail X$ be a dense subobject of a projective and let $g : U \longrightarrow A$. Again, let $e : Y \twoheadrightarrow U$ be a projective cover and let $m.e = f : Y \longrightarrow X$ be the resulting map in $\mathbf{C}$, which is in $J$ by Lemma 9.3.5. Let $f_0, f_1$ be the kernel pair of $f$ which is also the kernel pair of $e$. We then have that $(g.e).f_0 = (g.e).f_1$ and so, by hypothesis, we have a unique $g' : X \longrightarrow A$ such that $g'.f = g'.m.e = g.e$. As $e$ is epi in $\mathbf{D}$ we have that $g'.m = g$. So $A$ is a sheaf. $\square$

Universal closure operators are sometimes denoted by the letter $j$. For any category $\mathbf{E}$ equipped with such a $j$, we denote by $\mathbf{Sep}_j(\mathbf{E})$ and $\mathbf{Sh}_j(\mathbf{E})$ the full subcategories of $\mathbf{E}$ given by separated objects and sheaves respectively.

## 10.2 Categories of separated objects

Let us now give a concrete description of the categories of separated objects. Given a topology $J$ on a category $\mathbf{C}$, we say that a pseudo equivalence relation $P = (X_1 \overset{p_0}{\underset{p_1}{\rightrightarrows}} X_0)$ is *$J$-closed* if the map $\langle p_0, p_1\rangle : X_1 \longrightarrow X_0 \times X_0$ is closed with respect to $J$ (Definition 9.1.3).

**Proposition 10.2.1.** *Let $J$ be a topology on a category $\mathbf{C}$ and let $j$ be the induced universal closure operator in $\mathbf{C}_{ex}$. Then $\mathbf{Sep}_j(\mathbf{C}_{ex})$ is equivalent to the full subcategory of $\mathbf{C}_{ex}$ given by the $J$-closed pseudo equivalence relations.*

*Proof.* Let $P = (X_1 \overset{p_0}{\underset{p_1}{\rightrightarrows}} X_0)$ be a pseudo equivalence relation.

First we show that if $P$ is $J$-closed then it is separated. We use Proposition 10.1.5. So let $f : Y \longrightarrow X$ be in $J$ and let $g, h : X \longrightarrow X_0$ induce maps $[g], [h] : X \longrightarrow P$ in $\mathbf{C}_{ex}$ such that $[g].f = [h].f$. This means that there exists a commutative square as below.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"'] & X_1 \arrow[d, "\langle p_0, p_1\rangle"] \\
X \arrow[r, "\langle g, h\rangle"'] & X_0 \times X_0
\end{tikzcd}$$

As $\langle p_0, p_1\rangle$ is closed it follows by Lemma 9.1.4 that $\langle g, h\rangle$ factors through it. But this means that $[g] = [h]$ in $\mathbf{C}_{ex}$ so, indeed, $P$ is separated.

Now assume that $P$ is separated as an object in $\mathbf{C}_{ex}$. As $J$ is a topology, there exists $e : X_1 \longrightarrow E$ in $J$ and $\langle e_0, e_1\rangle : E \longrightarrow X_0 \times X_0$ closed such that $\lfloor\langle p_0, p_1\rangle\rfloor = \lfloor\langle e_0, e_1\rangle.e\rfloor$. We now show that $P$ is isomorphic in $\mathbf{C}_{ex}$ to the object given by the $J$-closed pseudo equivalence relation $e_0, e_1$. We already have that $\langle p_0, p_1\rangle$ factors through $\langle e_0, e_1\rangle$ so it is enough to show that $\langle e_0, e_1\rangle$ factors through $\langle p_0, p_1\rangle$. In order to do this, notice that $e_0$ and $e_1$ induce maps to $P$ such that $[e_0].e = [e_1].e$. As by hypothesis $e$ is in $J$ and $P$ is separated, $[e_0] = [e_1]$ by Proposition 10.1.5. But this means that there exists a map $h : E \longrightarrow X_1$ such that $\langle p_0, p_1\rangle.h = \langle e_0, e_1\rangle$. The result follows. □

The full subcategory of separated objects is actually reflective. Indeed, let us recall a result from [5].

**Proposition 10.2.2 (Barr).** *Let $j$ be a universal closure operator in an exact category $\mathbf{E}$. The inclusion functor $i : \mathbf{Sep}_j(\mathbf{E}) \longrightarrow \mathbf{E}$ has a left adjoint $s$ with the following properties:*

1. *$s$ preserves monos and finite products,*

2. *the units of the adjunction $s \dashv i$ are regular epimorphisms.*

*(For any $X$ in $\mathbf{E}$, $sX$ is the quotient of $X$ by the closure of its diagonal.)*

Moreover, results in [20] imply that, in many cases in practice, the categories of $J$-closed pseudo equivalence relations have very good properties. We now briefly explain this.

**Definition 10.2.3.** A *quasi-pretopos* is a regular category with the following properties:

1. strong equivalence relations are effective

2. admits stable epi/regular-mono factorizations

3. has stable (finite) sums

Notice that a quasi-pretopos need not be lextensive as coproducts need not be disjoint.

In [20] it was shown that quasi-pretoposes have the following natural characterization in terms of pretoposes (recall Definition 2.5.4).

**Proposition 10.2.4 (Carboni-Mantovani).** *A category $\mathbf{D}$ is equivalent to the category of separated objects for a universal closure operator in a pretopos if and only if $\mathbf{D}$ is a quasi-pretopos.*

Now, recall (Proposition 4.1.1) that if $\mathbf{C}$ is lextensive then $\mathbf{C}_{ex}$ is a pretopos and so, any category $J$-closed pseudo equivalence relations of $\mathbf{C}$ will be a quasi-pretopos. This does not seem to be the case for $\mathbf{C}_{reg}$ in general.

## 10.3 Sep-canonical topologies

Notice that in order to define sheaves and separated objects, we really did not need the monos involved to be dense with respect to a universal closure operator. In fact, any class of monos would do.

For example, given any quasi-topology $J$ on $\mathbf{C}$, the class of monos in $\mathbf{C}_{ex}$ that are the images in $\mathbf{C}_{ex}$ of the maps in $J$ may still be of interest.

Given a suitable category $\mathbf{D}$, we say that a quasi-topology $J$ on its full subcategory of projectives $\mathbf{C}$ is *sep-subcanonical* if every projective is separated with respect to the class of monos in $\mathbf{D}$ induced by $J$ as in the paragraph above.

**Corollary 10.3.1.** *A quasi-topology $J$ on* $\mathbf{C}$ *is sep-subcanonical if and only if every map in $J$ is epi.*

*Proof.* Use Proposition 10.1.5. □

This fact has two immediate consequences.

**Corollary 10.3.2.** *The stable epis form the largest quasi-topology for which every projective is separated.*

*Proof.* Using Proposition 9.1.2 it is easy to see that stable epis form a quasi-topology. □

Call this quasi-topology the *sep-canonical* quasi-topology.

**Corollary 10.3.3.** *If* $\mathbf{C}$ *has stable epi/regular-mono factorizations then the epis form the largest topology for which every projective is separated in* $\mathbf{C}_{ex}$.

We call this topology the *sep-canonical* topology.

Recall (Section 3.3.1) that for any category with finite limits $\mathbf{C}$ we denote by $\mathbf{C}_{eq}$ the full subcategory of $\mathbf{C}_{ex}$ induced by the regular equivalence relations.

**Corollary 10.3.4.** *If* $\mathbf{C}$ *has stable epi/regular-mono factorizations then the category of separated objects in* $\mathbf{C}_{ex}$ *for the sep-canonical topology is equivalent to* $\mathbf{C}_{eq}$.

*Proof.* This follows by Proposition 10.2.1. □

We have now a conceptual explanation of why, for example, $\mathbf{Top}_{eq} \simeq \mathbf{Equ} \simeq \mathbf{Top}_{reg}$ and $\mathbf{PAss}_{eq} \simeq \mathbf{Ass} \simeq \mathbf{PAss}_{reg}$ are categories of separated objects. The result below explains why they are, at the same time, regular completions.

**Corollary 10.3.5.** *Let* $\mathbf{C}$ *have stable epi/regular-mono factorizations. Then* $\mathbf{C}_{reg}$ *is the category of separated objects in* $\mathbf{C}_{ex}$ *for the sep-canonical topology if and only if every regular equivalence relation in* $\mathbf{C}$ *is a kernel pair.*

We end this section with a small result concerning quasi-pretoposes.

**Corollary 10.3.6.** *Lextensive quasi-pretoposes are closed under regular completions.*

*Proof.* If $\mathbf{C}$ is a lextensive quasi-pretopos then we can apply Corollary 10.3.5 to obtain $\mathbf{C}_{reg}$ as a category of separated objects for a universal closure operator in $\mathbf{C}_{ex}$. As $\mathbf{C}$ is lextensive, $\mathbf{C}_{ex}$ is a pretopos and then, by the results in [20], $\mathbf{C}_{reg}$ is a quasi-pretopos. But also by Proposition 4.1.1, $\mathbf{C}_{reg}$ is lextensive and so the result follows. □

## 10.4 Canonical topologies

In [6] it is observed that any small regular category $\mathbf{D}$ has a regular embedding into a topos. In particular, into the topos of sheaves for the (subcanonical) Grothendieck topology on $\mathbf{D}$ induced by the regular epis therein.

Regular epis on any (not necessarily small) regular category also induce a topology in our sense. In this section we study this topology and also characterize the sheaves in $\mathbf{D}_{ex}$ for it. This category of sheaves will be related to ex/reg completions in Corollary 11.2.4.

Given a suitable category $\mathbf{D}$ (recall Section 3.5), we say that a quasi-topology $J$ on its full subcategory of projectives $\mathbf{C}$ is *subcanonical* if every projective is a sheaf (as an object in $\mathbf{D}$) for $J$.

**Corollary 10.4.1.** *A quasi-topology $J$ on $\mathbf{C}$ is subcanonical if and only if every map in $J$ is a regular epi.*

*Proof.* Use Proposition 10.1.5. □

We have already mentioned that stable regular-epi/mono factorizations induce topologies. It follows by Corollary 10.4.1 that on *any* regular category $\mathbf{D}$, this topology is the largest one for which every object (as an object in $\mathbf{D}_{ex}$) is a sheaf. We call this topology, the *canonical topology* and we denote it by *can*.

We can already consider the associated category of separated objects.

**Corollary 10.4.2.** *Let $\mathbf{D}$ be a regular category. Then $\mathbf{Sep}_{can}(\mathbf{D}_{ex})$ is equivalent to the full subcategory of $\mathbf{D}_{ex}$ given by the equivalence relations.*

*Proof.* Use Proposition 10.2.1. □

We claim that the sheaves for the canonical topology on a regular category are the Higgs-complete relations in the following sense (see Section 11.1 for an explantion of the terminology). At this point the reader should be familiar with the definition of a functional relation discussed in Section 2.4.1.

**Definition 10.4.3.** An equivalence relation $\langle e_0, e_1 \rangle : E \rightarrowtail X \times X$ is *Higgs-complete* if for every equivalence relation $\langle d_0, d_1 \rangle : D \rightarrowtail Y \times Y$ and functional relation $\langle f_Y, f_X \rangle : F \rightarrowtail Y \times X$ from $D$ to $E$ there exists a map $f : Y \longrightarrow X$ such that $f \leq F$ (i.e. $f$ induces $F$).

By Proposition 2.4.5, for $F$ and $f$ as above, there exists $f' : D \longrightarrow E$ such that the following diagrams commute and such that $DfE = F$.

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

In other words, $E$ on $X$ is Higgs-complete if for every equivalence relation $D$, the inclusion $\mathbf{D}_{ex}(Y/D, X/E) \longrightarrow \mathbf{D}_{ex/reg}(Y/D, X/E)$ is actually an isomorphism.

Compare also with the proposition in page 163 of [58] where it is shown that Cauchy-complete metric spaces enjoy, with respect to bimodules, a property similar to Higgs-completeness as in Definition 10.4.3.

In order to relate Higgs-complete equivalence relations and sheaves for the canonical topology let us notice first that, in some cases, the uniqueness condition in the definition of a sheaf is unnecessary.

**Definition 10.4.4.** An object $X$ is a *quasi-sheaf* if for every dense mono $m : U \rightarrowtail Y$ and map $f : U \longrightarrow X$ there exists a (not necessarily unique) $f' : Y \longrightarrow X$ such that $f'.m = f$.

Compare with Definition 10.1.1. A quasi-sheaf is separated if and only if it is a sheaf.

**Lemma 10.4.5.** *Let* $\mathbf{D}$ *be a suitable category. Let* $e_0, e_1 : E \longrightarrow X$ *be an equivalence relation in its full subcategory of projectives* $\mathbf{C}$ *such that it has an effective quotient* $e : X \twoheadrightarrow X/E$ *in* $\mathbf{D}$*. Moreover, let* $J$ *be a subcanonical topology on* $\mathbf{C}$*. Then the following hold:*

1. *$X/E$ is separated*

2. *$X/E$ is a sheaf for $J$ in $\mathbf{D}$ if and only if it is a quasi-sheaf.*

*Proof.* Consider the first item. As $J$ is subcanonical, $\langle e_0, e_1\rangle : E \rightarrowtail X \times X$ is closed. Then $X/E$ is separated by Lemma 10.1.3.

For the second part of the result notice that the *only if* direction is trivial because a sheaf is always a quasi-sheaf. For the *if* direction we need only prove that $X/E$ is separated, but this is what we just did in the first part of the proof. $\square$

The first part of the lemma explains, for example, why **Ass** is closed under quotients of $\neg\neg$-stable equivalence relations in **Eff**. But let us concentrate on the characterization of sheaves.

We now express in more concrete terms what does it mean for an equivalence relation to be a quasi-sheaf for the canonical topology in a regular category.

First, we explain how to factor maps between projectives in an exact completion. If $f : Y \longrightarrow X$ is such a map, we can see its kernel pair $k_0, k_1 : K \longrightarrow Y$ as an object $Im(f) = (K \underset{k_1}{\overset{k_0}{\rightrightarrows}} Y)$ in $\mathbf{C}_{ex}$. The factorization of $f$ in $\mathbf{C}_{ex}$ is given by the maps $[id] : Y \twoheadrightarrow Im(f)$ and $[f] : Im(f) \rightarrowtail X$.

It follows from Lemma 9.3.5 that dense monos for the canonical topology in $\mathbf{C}_{ex}$ are given by regular epis (in $\mathbf{C}$) $e : Y \twoheadrightarrow Z$ viewed as monomorphisms $[e] : Im(e) \rightarrowtail Z$ from the kernel pair of $e$ (viewed as an object in $\mathbf{C}_{ex}$) to $Z$.

It should be clear then that a map from a dense subobject in $\mathbf{C}_{ex}$ is induced by a pair of maps as in Definition 10.4.6 below. It should also be clear that an equivalence relation $E$ on $X$ is a quasi-sheaf in $\mathbf{C}_{ex}$ for the canonical topology if and only if it is complete in the sense below.

**Definition 10.4.6.** An equivalence relation $\langle e_0, e_1 \rangle : E \rightarrowtail X \times X$ is *complete* if for every exact sequence $d.d_0 = d.d_1$ and maps $f, \overline{f}$ such that $f.d_0 = e_0.\overline{f}$ and $f.d_1 = e_1.\overline{f}$ as below

$$\begin{tikzcd}
D \arrow[rr, "\overline{f}"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\
Y \arrow[rr, "f"'] \arrow[d, two heads, "d"'] & & X \\
Z & &
\end{tikzcd}$$

there exist maps $f' : Z \longrightarrow X$ and $k : Y \longrightarrow E$ such that $e_0.k = f$ and $e_1.k = f'.d$. In other words, $[f] = [f'.d]$ as morphisms from $D$ to $E$ in the exact completion of the underlying category.

The notion of Higgs-completeness seems more intuitively appealing and it is the one that we are going to use in Section 11.2 which is essential to our proof of Theorem 11.3.3. On the other hand, the notion of completeness may be easier to test in practice due to the fact that there are no functional relations involved. Moreover, it is the natural class of objects induced by Lemma 10.4.5.

The main observation is, of course, that the two notions are equivalent.

**Proposition 10.4.7.** *An equivalence relation is complete if and only if it is Higgs-complete.*

*Proof.* Consider the *only if* direction. So let $\langle e_0, e_1\rangle : E \longrightarrow X \times X$ be complete. Let $F$ be as in the definition of Higgs-completeness. As $F$ is total, $f_Y$ is a regular epi by Lemma A.4.2. Let $k_0, k_1 : F \times_Y F \longrightarrow F$ be its kernel pair. As $F$ is single valued, there exists a map $s : F \times_Y F \longrightarrow E$ such that $\langle e_0, e_1\rangle.s = (f_X \times f_X).\langle k_0, k_1\rangle$. That is, we have the following diagram.

$$\begin{tikzcd}
F \times_Y F \arrow[r, "s"] \arrow[d, shift right, "k_0"'] \arrow[d, shift left, "k_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\
F \arrow[r, "f_X"'] \arrow[d, "f_Y"'] & X \\
Y &
\end{tikzcd}$$

By hypothesis there exists a map $f : Y \longrightarrow X$ and a $g : F \longrightarrow E$ such that $e_0.g = f_X$ and $e_1.g = f.f_Y$. The rest follows from Proposition 2.4.5 (item 3).

Consider now the *if* part. Let $E$ be Higgs-complete and let $d.d_0 = d.d_1$ be an exact sequence and $f'$ and $f$ as in the definition of complete equivalence relation (Definition 10.4.6). Let $F = DfE : (D \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y) \longrightarrow (E \overset{e_0}{\underset{e_1}{\rightrightarrows}} X)$ be the functional relation induced by $f$. Also, notice that $\langle d, id\rangle : Y \rightarrowtail Z \times Y$ induces a functional relation $I : \Delta_Z \longrightarrow (D \overset{d_0}{\underset{d_1}{\rightrightarrows}} Y)$ which has an inverse induced by $I^{-1} = \langle id, d\rangle$.

Then $IF : \Delta_Z \longrightarrow E$ is a functional relation. By hypothesis, there exists a map $\overline{f} : Z \longrightarrow X$ inducing $IF$, that is $\overline{f}E = IF$. But then, $I^{-1}\overline{f}E = I^{-1}IF = F$. Then $\overline{f}.d$ induces $F$ which is induced by $f$. By Proposition 2.4.4 there exists $g : Y \longrightarrow E$ such that $e_0.g = f$ and $e_1.g = \overline{f}.d$. So $E$ is complete. □

We can now prove our characterization.

**Corollary 10.4.8.** *Let* **D** *be a regular category. The following categories are equivalent.*

*1.* $\mathbf{Sh}_{can}(\mathbf{D}_{ex})$

*2. the full subcategory of* $\mathbf{D}_{ex}$ *given by the complete equivalence relations*

*3. the full subcategory of* $\mathbf{D}_{ex}$ *given by the Higgs-complete equivalence relations*

*Proof.* By Proposition 10.4.7 it is enough to show that the category of sheaves is equivalent to the category of complete equivalence relations in the sense of Definition 10.4.6.

As $\mathbf{Sh}_{can}(\mathbf{D}_{ex})$ is a full subcategory of $\mathbf{Sep}_{can}(\mathbf{D}_{ex})$ we already know, by Corollary 10.4.2, that every sheaf is isomorphic to some equivalence relation.

As the complete equivalence relations are exactly the ones that are quasi-sheaves, the result follows by Lemma 10.4.5. □

The definition of Higgs-completeness (Definition 10.4.3) implies that the full subcategories of $\mathbf{D}_{ex}$ and of $\mathbf{D}_{ex/reg}$ induced by the Higgs-complete equivalence relations are equivalent. Let us denote any of these equivalent categories by $\mathbf{Ceq}(\mathbf{D})$. So that we have embeddings $\mathbf{Ceq}(\mathbf{D}) \longrightarrow \mathbf{D}_{ex/reg}$ and $\mathbf{Ceq}(\mathbf{D}) \simeq \mathbf{Sh}_{can}(\mathbf{D}_{ex}) \longrightarrow \mathbf{D}_{ex}$.

## 10.5 Sheaves and the ex/reg completion

In this section we briefly discuss the category $\mathbf{Ceq}(\mathbf{D})$ from the perspective of its embedding into $\mathbf{D}_{ex/reg}$.

**Definition 10.5.1.** Let $\mathbf{D}$ be a subcategory of $\mathbf{D}'$. A map $q : Y \longrightarrow Q$ in $\mathbf{D}'$ is $\mathbf{D}$-*projecting* if for every $X$ in $\mathbf{D}$ and map $g : X \longrightarrow Q$ there exists a map $f : X \longrightarrow Y$ such that $q.f = g$.

It is very easy to see that with $\mathbf{D}$ and $\mathbf{D}'$ as in Definition 10.5.1, $\mathbf{D}$-projecting maps in $\mathbf{D}'$ are closed under pullback.

Let us relate this notion with the sheaves for the canonical topology.

**Lemma 10.5.2.** *Let* $\mathbf{D}$ *be a regular category,* $X$ *an object of* $\mathbf{D}$ *and* $q : X \twoheadrightarrow Q$ *a regular epi in* $\mathbf{D}_{ex/reg}$. *Then* $q$ *is* $\mathbf{D}$*-projecting if and only if the kernel pair of* $q$ *(an equivalence relation in* $\mathbf{D}$*) is complete.*

*Proof.* Let $\langle e_0, e_1\rangle : E \rightarrowtail X \times X$ be the kernel pair of $q$. By Lemma 3.4.1, $\langle e_0, e_1\rangle$ is an equivalence relation in $\mathbf{D}$ and we can assume that $Q$ is the object $X/E$ in $\mathbf{D}_{ex/reg}$.

The *if* direction is easy as Higgs-completeness of $E$ clearly implies that the quotient $X \twoheadrightarrow X/E$ is $\mathbf{D}$-projecting.

For the *only if* direction assume that the quotient $X \twoheadrightarrow X/E$ is $\mathbf{D}$-projecting. To prove completeness of $E$ let $d.d_0 = d.d_1$ be an exact sequence in $\mathbf{D}$ with $d_0, d_1 : D \longrightarrow Y$ and $d : Y \twoheadrightarrow Z$. Moreover, let $f : Y \longrightarrow X$ and $\overline{f} : D \longrightarrow E$ be as in the diagram in the definition of complete equivalence relation (Definition 10.4.6).

As the embedding $\mathbf{D} \longrightarrow \mathbf{D}_{ex/reg}$ is exact, $d.d_0 = d.d_1$ is an exact sequence in $\mathbf{D}_{ex/reg}$. Then, $q.f$ induces, through the universal property of $d$, a unique map $g : Z \longrightarrow Q$ such that $q.f = g.d$. As, $q$ is $\mathbf{D}$-projecting, there exists a $f' : Z \longrightarrow X$ such that $q.f' = g$. This easily implies that $e_0, e_1$ is complete. $\square$

We can immediately state the following.

**Lemma 10.5.3.** *An object $Q$ in $\mathbf{D}_{ex/reg}$ is in $\mathbf{Ceq}(\mathbf{D})$ if and only if there exists an $X$ in $\mathbf{D}$ and a $\mathbf{D}$-projecting quotient $X \twoheadrightarrow Q$.*

It follows that for every object $Q$ in $\mathbf{Ceq}(\mathbf{D})$ there exists an object $X$ of $\mathbf{D}$ and a $\mathbf{D}$-projecting quotient $X \twoheadrightarrow Q$.

Now recall Lemma 3.4.1 and consider the following result.

**Lemma 10.5.4.** *The embedding $\mathbf{Ceq}(\mathbf{D}) \longrightarrow \mathbf{D}_{ex/reg}$ preserves subobjects.*

*Proof.* Let $Q$ be a complete equivalence relation and let $u : U \rightarrowtail Q$ be a subobject of $Q$ in $\mathbf{D}_{ex/reg}$. By Lemma 10.5.3 there exists $X$ in $\mathbf{D}$ and a $\mathbf{D}$-projecting quotient $X \twoheadrightarrow Q$. Also, by the construction of $\mathbf{D}_{ex/reg}$ it must be the case that there exists an object $Y$ in $\mathbf{D}$ and a regular epi $u : Y \twoheadrightarrow U$. We can then take the following pullback diagram.

$$\begin{tikzcd}
Z \arrow[r, two heads] \arrow[d, two heads] & Im(Z) \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads, "q"] \\
Y \arrow[r, two heads, "r"'] & U \arrow[r, tail, "u"'] & Q
\end{tikzcd}$$

As the embedding of $\mathbf{D}$ into $\mathbf{D}_{ex/reg}$ is regular and preserves subobjects, it must be the case that both $Z$ and $Im(Z)$ are in $\mathbf{D}$. As $\mathbf{D}$-projecting quotients are closed under pullback, the quotient $Im(Z) \twoheadrightarrow U$ is $\mathbf{D}$-projecting. By Lemma 10.5.3, $U$ is in $\mathbf{Ceq}(\mathbf{D})$. $\square$

The following sums up some good properties of the category $\mathbf{Ceq}(\mathbf{D})$.

**Proposition 10.5.5.** *If $\mathbf{D}$ is a regular category then $\mathbf{Ceq}(\mathbf{D})$ is regular and the embedding $\mathbf{Ceq}(\mathbf{D}) \rightarrowtail \mathbf{D}_{ex/reg}$ is exact and preserves subobjects.*

*Proof.* We already know that $\mathbf{Ceq}(\mathbf{D})$ has finite limits because it is a category of sheaves. In order to prove that the embedding $\mathbf{Ceq}(\mathbf{D}) \rightarrowtail \mathbf{D}_{ex/reg}$ preserves them it is better to describe the limits as reflected from $\mathbf{D}_{ex/reg}$. As objects from $\mathbf{D}$ and $\mathbf{D}$-projecting quotients are closed under product, $\mathbf{Ceq}(\mathbf{D})$ has products and the embedding into $\mathbf{D}_{ex/reg}$ preserves them. By Lemma 10.5.4 the same follows for finite limits.

To prove that $\mathbf{Ceq}(\mathbf{D})$ has regular-epi/mono factorizations let $f : R \longrightarrow Q$ in $\mathbf{Ceq}(\mathbf{D})$ and consider its regular-epi/mono factorization $R \twoheadrightarrow T \rightarrowtail Q$ in $\mathbf{D}_{ex/reg}$.

By Lemma 10.5.4, $T$ is in $\mathbf{Ceq}(\mathbf{D})$ so we need only prove that the map $t : R \twoheadrightarrow T$ is a regular epi in $\mathbf{Ceq}(\mathbf{D})$. But as $\mathbf{D}$ covers $\mathbf{D}_{ex/reg}$, we can cover the kernel pair of $t$ with an object in $\mathbf{D}$. We then have a coequalizer diagram in $\mathbf{D}_{ex/reg}$ that is reflected by the embedding $\mathbf{Ceq}(\mathbf{D}) \longrightarrow \mathbf{D}_{ex/reg}$ showing that $t$ is a regular epi in $\mathbf{Ceq}(\mathbf{D})$.

Stability is inherited from $\mathbf{D}_{ex/reg}$ so indeed $\mathbf{Ceq}(\mathbf{D})$ is regular and its embedding into $\mathbf{D}_{ex/reg}$ is exact. □

Notice that as the embedding $\mathbf{D} \longrightarrow \mathbf{D}_{ex/reg}$ also preserves subobjects (Lemma 3.4.1) we obtain that the embedding $\mathbf{D} \longrightarrow \mathbf{Ceq}(\mathbf{D})$ does too.

The strong properties of the embedding $\mathbf{Ceq}(\mathbf{D}) \longrightarrow \mathbf{D}_{ex/reg}$ have the following consequence.

**Corollary 10.5.6.** $\mathbf{Ceq}(\mathbf{D})$ *is exact if and only if it is equivalent to* $\mathbf{D}_{ex/reg}$.

*Proof.* The *if* direction is trivial so consider the converse. For this, we need only show that the embedding $\mathbf{Ceq}(\mathbf{D}) \longrightarrow \mathbf{D}_{ex/reg}$ is essentially surjective. So let $\langle e_0, e_1 \rangle : E \rightarrowtail X \times X$ be an equivalence relation in $\mathbf{D}$ (that is, an object in $\mathbf{D}_{ex/reg}$). As the embedding $\mathbf{D} \longrightarrow \mathbf{Ceq}(\mathbf{D})$ preserves subobjects, $\langle e_0, e_1 \rangle : E \rightarrowtail X \times X$ is an equivalence relation in $\mathbf{Ceq}(\mathbf{D})$. As we are assuming that this category is exact, the equivalence relation has an effective quotient which the exact embedding $\mathbf{Ceq}(\mathbf{D}) \longrightarrow \mathbf{D}_{ex/reg}$ must preserve. This shows that this embedding is essentially surjective and hence, an equivalence. □

## 10.6 Local cartesian closure

The purpose of this section is to show that certain categories of separated objects and sheaves are locally cartesian closed. This follows from a folklore fact given our results in the previous sections.

First, let us discuss the folklore fact.

Let $j$ be a universal closure operator in a category with finite limits $\mathbf{C}$. Then, for any $X$ in $\mathbf{C}$, consider the slice category $\mathbf{C}/X$. If $f : Y \longrightarrow X$ is a map in $\mathbf{C}$ we denote the corresponding object of $\mathbf{C}/X$ by $(Y, f)$.

A map $m : (Y', g) \longrightarrow (Y, f)$ in $\mathbf{C}/X$ is mono if and only if $m : Y' \longrightarrow Y$ is mono in $\mathbf{C}$. It can be easily shown then that $j$ gives rise to universal closure operators in the slices. Indeed, let $m : (Y', g) \longrightarrow (Y, f)$ be mono in $\mathbf{C}/X$. Then we can define $(j/X)m = jm : (jY', f.(jm)) \longrightarrow (Y, f)$. It can be easily checked that this defines a universal closure operator $j/X$ in $\mathbf{C}/X$. In particular, a mono $m : (Y', g) \rightarrowtail (Y, f)$ in $\mathbf{C}/X$ is dense (for $j/X$) if and only if $m : Y' \longrightarrow Y$ (in $\mathbf{C}$) is dense for $j$.

The following is folklore but we include a proof for completeness.

**Lemma 10.6.1.** *If $X$ is separated for $j$ in $\mathbf{C}$ then $\mathbf{Sep}_{(j/X)}(\mathbf{C}/X) \cong (\mathbf{Sep}_j\mathbf{C})/X$. Moreover, if $X$ is a sheaf then $\mathbf{Sh}_{(j/X)}(\mathbf{C}/X) \cong (\mathbf{Sh}_j\mathbf{C})/X$.*

*Proof.* Let $X$ be separated for $j$. First, let us prove that $\mathbf{Sep}_{(j/X)}(\mathbf{C}/X)$ is embedded in $(\mathbf{Sep}_j\mathbf{C})/X$. For this, we assume that $a : A \longrightarrow X$ is an object in $\mathbf{Sep}_{(j/X)}(\mathbf{C}/X)$ and show that $A$ is a separated for $j$ in $\mathbf{C}$. So let $m : Y' \rightarrowtail Y$ be dense and let $h, h' : Y \longrightarrow A$ be such that $h.m = h'.m$. Then we have that $a.h.m = a.h'.m$ and as $X$ is separated, it follows that $a.h = a.h'$. But then we have that $m : (Y', a.h.m = a.h'.m) \longrightarrow (Y, a.h = a.h')$ is a dense mono in $\mathbf{C}/X$. Moreover we have that $h, h' : (Y, a.h = a.h') \longrightarrow (A, a)$ and that $h.m = h'.m$ in $\mathbf{C}/X$. As we are assuming $(A, a)$ is separated, $h = h'$. So $A$ is separated.

To prove that $(\mathbf{Sep}_j\mathbf{C})/X$ is embedded in $\mathbf{Sep}_{(j/X)}(\mathbf{C}/X)$ we are going to show that if $A$ is separated in $\mathbf{C}$ then for every $a : A \longrightarrow X$, $(A, a)$ is separated in $\mathbf{C}/X$. So let $m : (Y', f') \longrightarrow (Y, f)$ be dense in $\mathbf{C}/X$ and let $h, h' : (Y, f) \longrightarrow (A, a)$ be such that $h.m = h'.m$. But then $h.m = h'.m$ also holds as maps $Y' \longrightarrow A$ in $\mathbf{C}$. As $m$ is also dense in $\mathbf{C}$ and $A$ is separated then $h = h'$ and hence $(A, a)$ is separated in $\mathbf{C}/X$.

It is trivial to see that the embeddings are inverse to each other so indeed $\mathbf{Sep}_{(j/X)}(\mathbf{C}/X) \cong (\mathbf{Sep}_j\mathbf{C})/X$.

Now assume that $X$ is a sheaf for $j$ in $\mathbf{C}$. First, let us prove that $\mathbf{Sh}_{(j/X)}(\mathbf{C}/X)$ is embedded in $(\mathbf{Sh}_j\mathbf{C})/X$. For this, we assume that $a : A \longrightarrow X$ is an object in $\mathbf{Sh}_{(j/X)}(\mathbf{C}/X)$ and show that $A$ is a sheaf for $j$ in $\mathbf{C}$. So let $m : Y' \rightarrowtail Y$ be dense and let $g : Y' \longrightarrow A$. Then, as $X$ is a sheaf, we have the a unique $f : Y \longrightarrow X$ such that $m.f = a.g$ as below.

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, tail, "m"'] & A \arrow[r, "a"] & X \\ Y \arrow[urr, "\exists! f"'] & & \end{tikzcd}$$

But then $m : (Y', a.g) \longrightarrow (Y, f)$ is a dense mono in $\mathbf{C}/X$. Also, we have that $g : (Y', a.g) \longrightarrow (A, a)$ and so, as $a$ is a sheaf in $\mathbf{C}/X$, there exists a unique $h : (Y, f) \longrightarrow (A, a)$ as in the diagram below.

$$\begin{tikzcd} (Y', a.g) \arrow[r, "g"] \arrow[d, tail, "m"'] & (A, a) \\ (Y, f) \arrow[ur, "\exists! h"'] & \end{tikzcd}$$

So we have an $h : Y \longrightarrow A$ such that $h.m = g$. We already know that $A$ is separated by the first part of this result so $A$ is indeed a sheaf.

To prove that $(\mathbf{Sh}_j\mathbf{C})/X$ is embedded in $\mathbf{Sh}_{(j/X)}(\mathbf{C}/X)$ we are going to show that if $A$ is a sheaf in $\mathbf{C}$ then for every $a : A \longrightarrow X$, $(A, a)$ is a sheaf in $\mathbf{C}/X$. So let $m : (Y', f') \longrightarrow (Y, f)$ be dense in $\mathbf{C}/X$ and let $g : (Y', f') \longrightarrow (A, a)$. Then $m : Y' \longrightarrow Y$ is dense in $\mathbf{C}$ and as $A$ is a sheaf, there exists a unique $h : Y \longrightarrow A$ such that $h.m = g$ as in the diagram below.

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, tail, "m"'] & A \\ Y \arrow[ur, "\exists! h"'] & \end{tikzcd}$$

We also have that $a.h.m = a.g = f' = f.m : Y' \longrightarrow X$. As $m$ is dense and $X$ is a sheaf, $a.h = f$. So $h$ is a map $(Y, f) \longrightarrow (A, a)$ such that $h.m = g$. Again, we already know that $(A, a)$ is separated so $(A, a)$ is a sheaf.

It is trivial to see that the embeddings are inverse to each other so indeed $\mathbf{Sh}_{(j/X)}(\mathbf{C}/X) \cong (\mathbf{Sh}_j\mathbf{C})/X$ as stated. $\square$

It is well known (see for example the proof of Lemma V.2.1 in [71]) that if a category is cartesian closed then the categories of sheaves and of separated objects for any universal closure operator are also cartesian closed. In the case of separated objects for a universal closure operator this also follows from the preservation of products of the reflection functor (Proposition 10.2.2).

We now have an easy proof of the following.

**Corollary 10.6.2.** *Let* $\mathbf{C}$ *have weak dependent products. Then for any topology* $J$ *on* $\mathbf{C}$*, the category of* $J$*-closed pseudo equivalence relations is locally cartesian closed.*

*Proof.* By Proposition 10.2.1 the $J$-closed pseudo equivalence relations form a subcategory of separated objects of $\mathbf{C}_{ex}$ which is, by Proposition 3.3.5, locally cartesian closed. □

Moreover, these ideas give an alternative proof of Corollary 3.3.7 which we restate here for convenience.

**Corollary 10.6.3.** *Let* $\mathbf{C}$ *have stable epi/regular-mono factorizations and be such that every regular equivalence relation is a kernel pair (as is the case, by Lemma 7.4.4, if* $\mathbf{C}$ *has a chaotic situation). If* $\mathbf{C}$ *has weak dependent products then* $\mathbf{C}_{reg}$ *is locally cartesian closed.*

*Proof.* By Corollary 10.3.5, $\mathbf{C}_{reg}$ is a category of separated objects in this case. So the proof of Corollary 10.6.2 applies in this case too. □

Let us compare these results with the approach in [13] to prove local cartesian closure of certain categories of pseudo equivalence relations. In their work, topologies in the sense of Definition 9.1.7 are not considered and results on categories of separated objects are not exploited. For any stable factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathbf{C}$ they introduce the full subcategory $\mathbf{PER}(\mathbf{C}, \mathcal{M})$ of $\mathbf{C}_{ex}$ given by the pseudo equivalence relations $r_1, r_2 : X_1 \longrightarrow X_0$ such that $\langle r_1, r_2 \rangle : X_1 \longrightarrow X_0 \times X_0$ is in $\mathcal{M}$. Then they show that the embedding of $\mathbf{PER}(\mathbf{C}, \mathcal{M})$ into $\mathbf{C}_{ex}$ has a left adjoint which preserves products and commutes with pullbacks along maps in the subcategory. From this, it follows that if $\mathbf{C}_{ex}$ is locally cartesian closed then $\mathbf{PER}(\mathbf{C}, \mathcal{M})$ also is.

Notice that when $(\mathcal{E}, \mathcal{M})$ satisfies T3' then the factorization system is an example of our topologies and $\mathbf{PER}(\mathbf{C}, \mathcal{M})$ is the associated category of separated objects by Proposition 10.2.1. Then the existence of the left adjoint satisfying the properties mentioned above follows from Proposition 10.2.2 and Lemma 10.6.1.

On the other hand, left adjoints to embeddings of categories of sheaves are not as easy to construct as in the case of separated objects. Indeed, *enough injectives* are usually required [5]. But we can still use our argument to prove the following.

**Corollary 10.6.4.** *If* $\mathbf{D}$ *is a regular category with weak dependent products then* $\mathbf{Ceq}(\mathbf{D})$ *is locally cartesian closed.*

*Proof.* Recall from Section 10.4 that $\mathbf{Ceq}(\mathbf{D})$ is equivalent, by definition, to $\mathbf{Sh}_{can}(\mathbf{D}_{ex})$. So the result follows by the remark below Lemma 10.6.1. □

This result will let us show the local cartesian closure of certain ex/reg completions in Section 11.3.

## 10.7 Continuous functors?

As explained in [71], given a cocomplete topos $\mathbf{E}$ and a Grothendieck topology $J$ on a small category $\mathbf{C}$ with finite limits, the geometric morphisms $\mathbf{E} \longrightarrow \mathbf{Sh}(\mathbf{C}, J)$ from $\mathbf{E}$ to the topos of sheaves on $\mathbf{C}$ correspond to functors $\mathbf{C} \longrightarrow \mathbf{E}$ that preserve finite limits and are *continuous* in certain sense.

It is natural then to wonder what is the right notion of *continuous* functor with respect to topologies in the sense of Definition 9.1.7. Continuous functors in this sense should help to describe morphisms to or from categories of sheaves and of separated objects of exact and regular completions. Although tempted to follow it, we will not pursue this research direction here.

# Chapter 11

# Ex/reg completions that are toposes

We show that the ex/reg completion $\mathbf{D}_{ex/reg}$ of a locally cartesian closed regular category $\mathbf{D}$ with a generic mono is a topos. In the process we observe that in this case, $\mathbf{D}_{ex/reg}$ is equivalent to the category of sheaves in $\mathbf{D}_{ex}$ for the canonical topology on $\mathbf{D}$, that is, to the category $\mathbf{Ceq}(\mathbf{D})$. Moreover, this result will let us derive a characterization of the locally cartesian closed regular categories whose associated category $\mathbf{Ceq}(\mathbf{D})$ is a topos. We also discuss briefly the relation of this result with other presentations of toposes as ex/reg completions and also with tripos theory.

## 11.1 Toposes and ex/reg completions

We have already observed that $(\mathbf{Ass}(K_1))_{ex/reg} \simeq \mathbf{Eff}$. This naturally leads to the question of what are the properties of $\mathbf{Ass}$ that makes its ex/reg completion a topos. As $\mathbf{Ass}$ is a regular completion (recall that $\mathbf{Ass} \simeq \mathbf{PAss}_{reg}$), we have already answered this question in Proposition 5.5.2: it is the generic mono. But there are regular categories with generic monos that do not arise as regular completions (for example the categories $H_+$, recall Example 5.5.4). So it is natural to wonder about the effect of generic monos in ex/reg completions. We will prove in this chapter that the ex/reg completion of a locally cartesian closed regular category with a generic mono is a topos.

Let us now compare this statement with other presentations of toposes that are ex/reg completions. The first one to mention is the presentation of the effective topos in [17] and [32]. They introduce the category of assemblies, build its ex/reg completion and show that it has power objects (which is an alternative way of defining a topos: a category with finite limits and power objects [71]). No attempt is made neither in [17] nor in [32] to give sufficient conditions on a regular category for its ex/reg completion to be a topos.

In [75] there is an attempt to simplify the presentation. McLarty shows that in order to prove that the ex/reg completion of a regular category $\mathbf{D}$ is a topos it is enough to show that every object in $\mathbf{D}$ has a power object in $\mathbf{D}_{ex/reg}$ (actually, something slightly weaker: the "power" version of a classifier of subobjects of $\mathbf{D}$-objects as in Definition 3.5.4). That is, in order to prove that $\mathbf{D}_{ex/reg}$ is a topos, you do not need to build all power objects, just a good class of them. This is a good simplification, it is essentially item 3 of Corollary 3.5.5. Yet, to use this fact, you still have to build the ex/reg completion and construct objects in it.

Our result can be used to present toposes that are ex/reg completions in a way that avoids completely the actual construction of the completion. That is, they allow you to verify that $\mathbf{D}_{ex/reg}$ is a topos merely by looking at the category $\mathbf{D}$ itself.

A different approach is that of tripos theory and we will discuss its relation with our results in more detail in Section 11.4.

Let us now discuss the proof of our result. In order to motivate it, let us first briefly review Higgs' construction of the category of sheaves on a locale [34] (see also [35, 27, 10, 106]).

Let $H$ be a frame and consider the category $\mathcal{S}(H)$ defined as follows. Its objects are pairs $X = (|X|, \delta_X)$ with $|X|$ a set and $\delta_X$ a function from $|X| \times |X|$ to $H$ such that $\delta_X(x, x') = \delta_X(x', x)$ and $\delta_X(x_0, x_1) \wedge \delta_X(x_1, x_2) \leq \delta_X(x_0, x_2)$.

A map $Y \longrightarrow X$ between two such objects is a function $f : |Y| \times |X| \longrightarrow H$ such that the following hold.

1. $f(y, x) \wedge \delta_Y(y, y') \leq f(y', x)$ and $f(y, x) \wedge \delta_X(x, x') \leq f(y, x')$
2. $f(y, x) \wedge f(y, x') \leq \delta_X(x, x')$
3. $\bigvee_{x \in X} f(y, x) = \delta_Y(y, y)$

It turns out that this category is equivalent to the category of sheaves on the frame $H$. Let us outline a sketch of the proof. We use the terminology of [35]. We say that a map $f : Y \longrightarrow X$ is represented by a function $f_0 : |Y| \longrightarrow |X|$ if $f(y, x) \leq \delta_X(f_0 y, x)$ for all $y \in Y$ and $x \in X$.

Now define an object $X$ to be *ample* if every map to $X$ is represented by a function.

There is functor from the category of sheaves on $H$ to the category $\mathcal{S}(H)$ that assigns to each sheaf an ample object. This property is used to prove that the functor is full and faithful. Then it is proved that every object in $\mathcal{S}(H)$ is isomorphic to one in the image of the embedding of sheaves.

Let us stress this fact. Every object in $\mathcal{S}(H)$ is isomorphic to an ample object.

In [42], this presentation of sheaves is used to motivate the definition of a tripos. In their treatment of geometric morphisms they introduce the notion of a *weakly complete* object (see also [94]) which is very similar to the notion of an ample object. The main fact they prove is that every object in the topos induced by a tripos is isomorphic to a weakly complete one.

The resemblance of Higgs-completeness with ampleness (and also with weak completeness [42]) is evident.

We are going to borrow this idea from Higgs and then use an argument similar to that in the proof of Theorem 5.2.2 in order to prove our result.

## 11.2 Generic monos and complete equivalence relations

In this section we outline the proof that in a locally cartesian closed regular category $\mathbf{D}$ with a generic mono, every equivalence relation is isomorphic (as an object in $\mathbf{D}_{ex/reg}$) to a complete one (we sometimes say that the two equivalence relations are *relationally isomorphic*). That is, in view of Corollary 10.4.8, we are going to show that $\mathbf{Ceq}(\mathbf{D}) \simeq \mathbf{Sh}_{can}(\mathbf{D}_{ex})$ is equivalent to $\mathbf{D}_{ex/reg}$.

As we have already explained, the idea that every equivalence relation is relationally isomorphic to a complete one is essentially due to Higgs and the proof, in the case of the topos of sheaves on a frame, is in [27]. In the more abstract setting of tripos theory, the result is essentially proved in Proposition 3.3 of [42] making heavy use of internal logic. Here we recast this proof in diagrammatic terms.

If the reader is familiar with arguments using internal logic, looking at Proposition 3.3 of [42] will be enough to complete the proofs missing in this section. On the other hand, our setting is quite different from that of tripos theory so I have placed the complete details in diagrammatic terms in Appendix B.

Let $\tau : \Upsilon \rightarrowtail \Lambda$ be a generic mono in $\mathbf{D}$.

By Lemma 5.3.3, there exists an equivalence relation $\langle \tau_0, \tau_1 \rangle : \Xi \rightarrowtail \Lambda \times \Lambda$ such that $\langle f, g \rangle : Y \to \Lambda \times \Lambda$ factors through $\Xi$ if and only if $f$ and $g$ pull $\tau$ back to the same subobject of $Y$.

As $\tau$ is a generic mono, $\Xi$ arises as in the following pullback.

$$\begin{tikzcd}
\Xi \arrow[r] \arrow[d, "\langle \tau_0, \tau_1 \rangle"'] & \Upsilon \arrow[d, "\tau"] \\
\Lambda \times \Lambda \arrow[r] & \Lambda
\end{tikzcd}$$

The idea is that $\Xi$ will give rise to a subobject classifier.

For any $X$ we can consider the following pullback.

$$\begin{tikzcd}
\Xi^X \arrow[r, phantom, "="] \arrow[d, "\langle \tau_0^X, \tau_1^X \rangle"'] & \Xi^X \arrow[r] \arrow[d, "\langle \tau_0, \tau_1 \rangle^X"] & \Upsilon^X \arrow[d, "\tau^X"] \\
\Lambda^X \times \Lambda^X \arrow[r, phantom, "\cong"] & (\Lambda \times \Lambda)^X \arrow[r] & \Lambda^X
\end{tikzcd}$$

Indeed, think of $\Lambda^X$ as the object of "intensional" predicates on $X$ and of $\Xi^X$ as the equivalence relation given by bi-implication between predicates.

As $\tau$ is a generic mono, any equivalence relation $\langle e_0, e_1 \rangle : E \rightarrowtail X \times X$ gives rise to the following maps.

$$\begin{tikzcd}
E \arrow[r, "\nu_{E,\Upsilon}"] \arrow[d, tail, "\langle e_0, e_1 \rangle"'] & \Upsilon \arrow[d, "\tau"] \\
X \times X \arrow[r, "\nu_{E,\Lambda}"'] & \Lambda
\end{tikzcd}$$

---

$$X \xrightarrow{\nu_E} \Lambda^X$$

Think of $\nu_E$ as the assignment $x \longmapsto \{x' | x' E x\} = [x]$.

**Proposition 11.2.1.** *Every equivalence relation $E$ on $X$ appears in a pullback square as below.*

$$\begin{tikzcd}
E \arrow[r] \arrow[d, tail, "\langle e_0, e_1 \rangle"'] & \Xi^X \arrow[d, tail, "\langle \tau_0^X, \tau_1^X \rangle"] \\
X \times X \arrow[r, "\nu_E \times \nu_E"'] & \Lambda^X \times \Lambda^X
\end{tikzcd}$$

*Proof.* See Appendix B. □

The intuition is that there may be different "elements" in $\Lambda^X$ denoting the same "subset of" or "predicate on" $X$. Also, we think of $\Xi^X$ as containing the information of which elements of $\Lambda^X$ denote the same subset. In discussing intuition we may sometimes write $c \leftrightarrow c'$ to express that the elements $c$ and $c'$ of $\Lambda^X$ denote the same subset of $X$.

We now start the construction of a complete equivalence relation relationally isomorphic to $E$. First consider the following pullback square and induced map.

$$\begin{tikzcd}
X \arrow[r, "\nu_E"] \arrow[dr, "\exists ! s"] \arrow[ddr, "\langle \nu_E, id \rangle"'] & \Lambda^X \arrow[dr, "refl"] & \\
& \Upsilon_E \arrow[r] \arrow[d, tail, "\langle e'_0, e'_1 \rangle"] & \Xi^X \arrow[d, tail] \\
& \Lambda^X \times X \arrow[r, "id \times \nu_E"'] & \Lambda^X \times \Lambda^X
\end{tikzcd}$$

Think of $\Upsilon_E$ as $\{(c, x) | c \leftrightarrow [x]\}$.

First notice that $s$ is a split mono because $e'_1.s = id$. Let $\Upsilon_E \xrightarrow{e} \Lambda_E \xrightarrow{e'} \Lambda^X$ be the regular-epi/mono factorization of $e'_0$.

Of course, think of $\Lambda_E$ as $\{c \in \Lambda^X | c \leftrightarrow [x] \text{ for some } x \in X\}$.

Also, let $\nu'_E = e.s : X \longrightarrow \Lambda_E$ so that $\nu_E = e'.\nu'_E : X \longrightarrow \Lambda^X$.

We can now split the pullback of Proposition 11.2.1 as follows.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, tail, "\langle e_0, e_1 \rangle"'] & J \arrow[r] \arrow[d, tail, "\langle j_0, j_1 \rangle"'] & \Xi_E \arrow[r] \arrow[d, tail, "\langle c_0, c_1 \rangle"'] & \Xi^X \arrow[d, tail] \\
X \times X \arrow[r, "\nu'_E \times id"'] & \Lambda_E \times X \arrow[r, "id \times \nu'_E"'] & \Lambda_E \times \Lambda_E \arrow[r, "e' \times e'"'] & \Lambda^X \times \Lambda^X
\end{tikzcd}$$

So pulling back in this way gives rise to an equivalence relation $\Xi_E$ on an object $\Lambda_E$ and to a relation $J$ from $E$ to $\Xi_E$.

**Proposition 11.2.2.** *The relation $J$ is an isomorphism $X/E \longrightarrow \Lambda_E/\Xi_E$ between the equivalence relations $E$ and $\Xi_E$ as objects in $\mathbf{D}_{ex/reg}$.*

*Proof.* See Appendix B. $\square$

We now state the key fact about $\Xi_E$.

**Proposition 11.2.3.** *$\Xi_E$ is complete.*

*Proof.* Let $\langle d_0, d_1 \rangle : D \rightarrowtail Y \times Y$ be an equivalence relation on $Y$ and let $\langle f_Y, f_E \rangle : F \rightarrowtail Y \times \Lambda_E$ be a functional relation from $D$ to $\Xi_E$. We can form the following pullback and transposition.

$$
\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "\langle g_X, g_Y \rangle"'] & F \arrow[r, "\nu'_F"] \arrow[d, "\langle f_E, f_Y \rangle"'] & \Upsilon \arrow[d, "\tau"] \\
X \times Y \arrow[r, "\nu'_E \times id"'] & \Lambda_E \times Y \arrow[r, "\nu_F"'] & \Lambda
\end{tikzcd}
$$

$$
\overline{\hspace{12em}}
$$

$$
Y \xrightarrow{f'} \Lambda^X
$$

One then proves that $f'$ factors through a map $f : Y \longrightarrow \Lambda_E$ that induces $F$. Full details are given in Appendix B. $\square$

In the view of Section 10.5, we can formulate the results in this section as follows.

**Corollary 11.2.4.** *If $\mathbf{D}$ is a locally cartesian closed regular category with a generic mono then $\mathbf{D}_{ex/reg}$ is equivalent to $\mathbf{Ceq}(\mathbf{D}) \simeq \mathbf{Sh}_{can}(\mathbf{D}_{ex})$ the category of sheaves in $\mathbf{D}_{ex}$ for the canonical topology on $\mathbf{D}$.*

## 11.3 Sheaves, ex/reg completions and toposes

In this section we prove the main result of the chapter: the ex/reg completion of a locally cartesian closed regular category with a generic mono is a topos.

**Corollary 11.3.1.** *If $\mathbf{D}$ is a locally cartesian closed regular category with a generic mono then $\mathbf{D}_{ex/reg}$ is locally cartesian closed.*

*Proof.* By Corollary 10.6.4, $\mathbf{Sh}_{can}(\mathbf{D}_{ex})$ is locally cartesian closed. By Corollary 11.2.4 the result follows. $\square$

Moreover, as the canonical embedding $\mathbf{D} \longrightarrow \mathbf{D}_{ex/reg}$ preserves finite limits, then the universal property of $\mathbf{D}_{ex}$ induces an exact functor $a : \mathbf{D}_{ex} \longrightarrow \mathbf{D}_{ex/reg} \simeq \mathbf{Sh}_{can}(\mathbf{D}_{ex})$. This functor is easily seen to be left adjoint to the embedding $\mathbf{Sh}_{can}(\mathbf{D}_{ex}) \longrightarrow \mathbf{D}_{ex}$. So we have easily obtained an *associated sheaf* functor. Notice that as this functor preserves finite limits, we have another proof of local cartesian closure of $\mathbf{D}_{ex/reg} \simeq \mathbf{Sh}_{can}(\mathbf{D}_{ex})$.

Also, Corollary 11.2.4 gives a clearer picture of the hierarchies of toposes that we found in Chapter 8 (Corollary 8.2.2).

**Corollary 11.3.2.** *Let* $\mathbf{C}$ *have an AC-chaotic situation, weak dependent products and a generic object. Then for every* $n$*,* $(\mathbf{C}_{reg(n)})_{ex}$ *is a topos. It is the topos of sheaves in* $(\mathbf{C}_{reg(n+1)})_{ex}$ *for the canonical topology on* $\mathbf{C}_{reg(n+1)}$*.*

*Proof.* This follows because $(\mathbf{C}_{reg(n)})_{ex} \simeq (\mathbf{C}_{reg(n+1)})_{ex/reg} \simeq \mathbf{Sh}_{can}((\mathbf{C}_{reg(n+1)})_{ex})$. □

That is, each topos in the hierarchy is the topos of sheaves for the canonical topology of the next topos in the hierarchy.

But let us go back to our main purpose. With Corollary 11.3.1 we are now ready to prove the $\mathbf{D}_{ex/reg}$ is a topos. Notice that we cannot directly apply the fact that $\mathbf{D}_{ex/reg}$ is equivalent to $\mathbf{Sh}_{can}(\mathbf{D}_{ex})$ (Corollary 11.2.4) because $\mathbf{D}_{ex}$ need not be a topos (trivial examples are given by toposes without generic proofs).

In order to prove in Theorem 5.2.2 that certain exact completions are toposes, we reduced the problem of the existence of a subobject classifier to that of the existence of a classifier of subobjects of projectives.

It is possible to use the same idea in the slightly different setting of ex/reg completions. But with a twist. In the proof of Theorem 5.2.2 we used the fact that in $\mathbf{C}_{ex}$ the objects from $\mathbf{C}$ are *projective* in order to prove that $\mathbf{C}_{ex}$ has a classifier of subobjects of projectives and then immediately apply Corollary 3.5.5. In the present case we will use the fact that every object of $\mathbf{D}_{ex/reg}$ is covered by a $\mathbf{D}$-*projecting* quotient from an object in $\mathbf{D}$ in order to prove that $\mathbf{D}_{ex/reg}$ has a classifier of subobjects of $\mathbf{D}$-objects.

**Theorem 11.3.3.** *If* $\mathbf{D}$ *is a locally cartesian closed regular category with a generic mono then* $\mathbf{D}_{ex/reg}$ *is a topos.*

*Proof.* By Corollary 11.3.1, $\mathbf{D}_{ex/reg}$ is locally cartesian closed. So we need only prove that $\mathbf{D}_{ex/reg}$ has a subobject classifier. By Corollary 3.5.5 we only need a classifier of subobjects of objects from $\mathbf{D}$.

Let $\tau : \Upsilon \longrightarrow \Lambda$ be a generic mono in $\mathbf{D}$. Because $\mathbf{D}$ is locally cartesian closed we can use Lemma 5.3.3 in order to obtain an equivalence relation on $\Lambda$ and its quotient (in $\mathbf{D}_{ex/reg}$) $\rho : \Lambda \twoheadrightarrow \Omega$ such that for any pair of maps $f, g : X \longrightarrow \Lambda$ in $\mathbf{D}$, $\rho.f = \rho.g$ if and only if $f$ and $g$ pull $\tau$ back to the same subobject.

Also, as in Proposition 5.3.4, we can build a pullback square as below.

$$\begin{tikzcd}
\Upsilon \arrow[r, two heads] \arrow[d, "\tau"'] & \Omega' \arrow[d, "\top"] \\
\Lambda \arrow[r, two heads, "\rho"'] & \Omega
\end{tikzcd}$$

We now prove that $\top$ is a classifier of subobjects of objects in $\mathbf{D}$. The idea is to follow the proof of Theorem 5.2.2. The problem is that the objects of $\mathbf{D}$ are not projective in this case.

We can nevertheless use a similar idea thanks to the existence of $\mathbf{D}$-projecting covers. Indeed, as $\mathbf{Ceq}(\mathbf{D}) \simeq \mathbf{D}_{ex/reg}$ by Corollary 11.2.4, there exists a $\mathbf{D}$-projecting regular epi $\rho_0 : \Lambda_0 \longrightarrow \Omega$ by Lemma 10.5.3.

Let $\tau_0$ be be the pullback of $\top$ along $\rho_0$ as in the square below.

$$\begin{tikzcd}
\Upsilon_0 \arrow[r, two heads] \arrow[d, "\tau_0"'] & \Omega' \arrow[d, "\top"] \\
\Lambda_0 \arrow[r, two heads, "\rho_0"'] & \Omega
\end{tikzcd}$$

By Lemma 3.4.1, we can assume that $\tau_0$ is in $\mathbf{D}$.

First, we will show that $\tau_0$ is a generic mono in $\mathbf{D}$. Second, we will show that if $f_0, g_0 : X \longrightarrow \Lambda_0$ in $\mathbf{D}$ pull $\tau_0$ back to the same mono, then $\rho_0.f_0 = \rho_0.g_0$. From this, it will follow that $\top$ is a classifier of subobjects of objects in $\mathbf{D}$.

The fact that $\tau_0$ is a generic mono can be proved easily because as $\rho_0$ is $\mathbf{D}$-projecting, $\rho$ factors through $\rho_0$. It follows that $\tau$ is a pullback of $\tau_0$ and hence that $\tau_0$ is a generic mono.

Our second task takes a bit more effort. Indeed, in order to deal with it we need to show first that $\rho_0$ factors through $\rho$.

As $\tau$ is a generic mono and $\tau_0$ is in $\mathbf{D}$, there exists a pullback as below.

$$\begin{tikzcd} \Upsilon_0 \arrow[r, "\nu_0'"] \arrow[d, "\tau_0"'] & \Upsilon \arrow[d, "\tau"] \\ \Lambda_0 \arrow[r, "\nu_0"'] & \Lambda \end{tikzcd}$$

In order to show that $\rho_0 = \rho.\nu_0$, consider the pullback of $\rho$ and $\rho_0$ as below.

$$\begin{tikzcd} P \arrow[r, two heads, "e"] \arrow[d, two heads, "r"'] & \Lambda_0 \arrow[d, two heads, "\rho_0"] \\ \Lambda \arrow[r, two heads, "\rho"'] & \Omega \end{tikzcd}$$

As $r^*\tau = r^*(\rho^*\top) = e^*(\rho_0^*\top) = e^*\tau_0 = e^*(\nu_0^*\tau)$, it follows by the defining property of $\rho$ that $\rho.r = \rho.\nu_0.e$. But we also have that $\rho.r = \rho_0.e$. As $e$ is epi, $\rho_0 = \rho.\nu_0$.

We can finally deal with our second task. So let $f_0, g_0 : X \longrightarrow \Lambda_0$ pull $\tau_0$ back to the same subobject and consider the following chain of implications.

$$\begin{aligned} f_0^*\tau_0 = g_0^*\tau_0 \ &\Rightarrow\ f_0^*(\nu_0^*\tau) = g_0^*(\nu_0^*\tau) \\ &\Rightarrow\ \rho.\nu_0.f_0 = \rho.\nu_0.g_0 \\ &\Rightarrow\ \rho_0.f_0 = \rho_0.g_0 \end{aligned}$$

So our second task is finished.

To finish the proof that $\top$ is a subobject classifier, let $f, g : X \longrightarrow \Omega$ pull $\top$ back to the same subobject.

As, $\rho_0$ is **D**-projecting, there exist $f_0, g_0 : X \longrightarrow \Lambda_0$ such that $\rho_0.f_0 = f$ and $\rho_0.g_0 = g$. But this implies that $f_0^*\tau_0 = g_0^*\tau_0$ and hence that $f = \rho_0.f_0 = \rho_0.g_0 = g$. So, indeed, $\top$ is a subobject classifier and hence $\mathbf{D}_{ex/reg}$ is a topos. □

We can slightly change perspective in order to have an idea of how necessary the generic monos are.

**Corollary 11.3.4.** *Let* **D** *be a locally cartesian closed regular category. Then* **Ceq(D)** *is a topos if and only if* **D** *has a generic mono. Moreover, in this case,* **Ceq(D)** *is equivalent to* $\mathbf{D}_{ex/reg}$.

*Proof.* The *if* direction is just Theorem 11.3.3 together with Corollary 11.2.4.

For the *only if* direction let $\top : 1 \longrightarrow \Omega$ be the subobject classifier in $\mathbf{Ceq}(\mathbf{D})$. By Lemma 10.5.3 there exists a $\mathbf{D}$-projecting quotient $\rho : \Lambda \twoheadrightarrow \Omega$ with $\Lambda$ in $\mathbf{D}$. Let $\tau = \rho^*\top : \Upsilon \rightarrowtail \Lambda$. As $\mathbf{D} \longrightarrow \mathbf{Ceq}(\mathbf{D})$ preserves subobjects, $\Upsilon$ is in $\mathbf{D}$. As $\rho$ is $\mathbf{D}$-projecting it is easy to see that $\tau$ is a generic mono in $\mathbf{D}$.

By Corollary 10.5.6, $\mathbf{Ceq}(\mathbf{D})$ is equivalent to $\mathbf{D}_{ex/reg}$. ☐

Recall that we needed the axiom of choice to prove that **PAss** has a generic proof (Example 5.2.3). On the other hand, the proof that **Ass** satisfies the premises of Theorem 11.3.3 (or Corollary 11.3.4) is choice-free (see Example 5.5.3 for the generic mono). So we we have a choice-free presentation of realizability toposes as the categories $\mathbf{Ass}_{ex/reg}$ which, as explained in Section 11.1 is much simpler than those in [17, 32, 75].

The proof that $H_+$ has a generic mono (Example 5.5.4) is also choice free and we have that $(H_+)_{ex/reg}$ is the topos of sheaves on $H$.

More generally, although we do not know how to do this in general, we expect that many of the examples dealt with tripos theory [94, 42] can be dealt with using Theorem 11.3.3. We briefly explain this in the next section.

## 11.4 On the relation with tripos theory

In this section we outline some connections of tripos theory [94, 42] with our results. We will do this from the perspective of the more recent [95].

**Definition 11.4.1.** Let $\mathbf{C}$ be a category with finite products. A *first order hyperdoctrine* $\mathbf{P}$ over $\mathbf{C}$ is specified by a contravariant functor from $\mathbf{C}$ into the category of partially ordered sets and monotone functions, with the following properties.

1. For each $X$ in $\mathbf{C}$, $\mathbf{P}X$ is a Heyting algebra.

2. For each $f : X \longrightarrow Y$ in $\mathbf{C}$, $\mathbf{P}f : \mathbf{P}Y \longrightarrow \mathbf{P}X$ is a homomorphism of Heyting algebras.

3. For each diagonal morphism $\Delta_X : X \longrightarrow X \times X$ in $\mathbf{C}$, the left adjoint to $\mathbf{P}\Delta_X$ at the top element $\top \in \mathbf{P}X$ exists, in other words there is an element $=_X$ of $\mathbf{P}(X \times X)$ such that

$$\top \leq (\mathbf{P}\Delta_X)A \text{ if and only if } =_X \leq A.$$

4. For each product projection $\pi : \Gamma \times X \longrightarrow \Gamma$ in $\mathbf{C}$, the monotone function $\mathbf{P}\pi : \mathbf{P}\Gamma \longrightarrow \mathbf{P}(\Gamma \times X)$ has both a left adjoint and a right adjoint both natural in $\Gamma$.

The elements of $\mathbf{P}X$ will be referred to as $\mathbf{P}$-*predicates*.

First order hyperdoctrines can interpret first order logic with equality and one can associate to each first order hyperdoctrine ($\mathbf{C}$,$\mathbf{P}$) its *internal language* which is the signature having a sort for each object of $\mathbf{C}$, a function symbol (of the appropriate arity) for each map in $\mathbf{C}$ and a relation symbol for each $\mathbf{P}$-predicate.

The internal language can then be used to express conditions on the hyperdoctrine. We usually express this by writing that some formula in the internal language of ($\mathbf{C}$,$\mathbf{P}$) *holds* in $\mathbf{P}$ or that $\mathbf{P}$ *satisfies* the formula.

It is possible then to associate to a first order hyperdoctrine ($\mathbf{C}$,$\mathbf{P}$) its category $\mathbf{C}[\mathbf{P}]$ of partial equivalence relations. The objects of $\mathbf{C}[\mathbf{P}]$ are pairs $(X, E)$ with $X$ a $\mathbf{C}$-object and $E \in \mathbf{P}(X \times X)$ a predicate such that the sentences in the internal language expressing symmetry and transitivity of $E$ hold in $\mathbf{P}$. A map $F : (X_1, E_1) \longrightarrow (X_2, E_2)$ is a predicate $F \in \mathbf{P}(X_1 \times X_2)$ such that the sentences in the internal language expressing that $F$ respects $E_1$ and $E_2$ and that it is single valued and total from $E_1$ to $E_2$ hold.

The category $\mathbf{C}[\mathbf{P}]$ has finite limits, pullback-stable images and dual images of subobjects along morphisms, and pullback-stable finite joins of subobjects. Following [74], Pitts calls a category with this structure a *logos*. Moreover, all equivalence relations in $\mathbf{C}[\mathbf{P}]$ have quotients.

There exists an embedding $\Delta_{\mathbf{P}} : \mathbf{C} \longrightarrow \mathbf{C}[\mathbf{P}]$ that, on objects, maps $X$ to $(X, =_X)$ and, on morphisms, maps $f : X_1 \longrightarrow X_2$ to the morphism given by the formula $f(x_1) =_{X_2} x_2$ in the internal language of ($\mathbf{C}$,$\mathbf{P}$).

The functor $\Delta_{\mathbf{P}}$ is called the *constant objects* functor, preserves finite products and moreover, $\mathbf{P}X$ is naturally isomorphic to $Sub_{\mathbf{C}[\mathbf{P}]}(\Delta_{\mathbf{P}}X)$.

**Proposition 11.4.2 (Pitts).** *Let $\mathbf{C}$ be a category with finite products and $\mathbf{E}$ a logos such that every equivalence relation in it has a quotient. Let $F : \mathbf{C} \longrightarrow \mathbf{E}$ be a functor preserving finite products. Then $Sub_{\mathbf{E}}(F(\_))$ is a first order hyperdoctrine. Moreover, $\mathbf{C}[Sub_{\mathbf{E}}(F(\_))]$ is equivalent to $\mathbf{E}$ and $F$ is naturally isomorphic to the constant objects functor $\mathbf{C} \longrightarrow \mathbf{C}[Sub_{\mathbf{E}}(F(\_))]$ if and only if every object of $\mathbf{E}$ is a quotient of a subobject of some object in the image of $F$.*

If we assume that $\mathbf{C}$ has finite limits and enough structure to ensure that $\mathbf{C}_{ex}$ is a logos then Proposition 11.4.2 can be applied to the embedding $\mathbf{C} \longrightarrow \mathbf{C}_{ex}$. Something analogous happens with ex/reg completions. So, in these cases, Proposition 11.4.2 gives an alternative description of exact and ex/reg completions as categories of partial equivalence relations of first order hyperdoctrines. In the case of exact completions, the hyperdoctrine is just the proof-theoretic power set functor *Prf*.

Pitts characterized when $\mathbf{C}[\mathbf{P}]$ is a topos as follows.

**Proposition 11.4.3 (Pitts).** *Suppose* $\mathbf{C}$ *is a category with finite products and* $\mathbf{P}$ *is a first order hyperdoctrine over* $\mathbf{C}$*. Then* $\mathbf{C}[\mathbf{P}]$ *is a topos if and only if* $(\mathbf{C}, \mathbf{P})$ *satisfies the following comprehension axiom (CA): for all* $\mathbf{C}$*-objects* $X$ *there is a* $\mathbf{C}$*-object* $PX$ *and a* $\mathbf{P}$*-predicate* $In_X \in \mathbf{P}(X \times PX)$ *such that for every* $\mathbf{C}$*-object* $\Gamma$ *and* $\mathbf{P}$*-predicate* $R \in \mathbf{P}(X \times \Gamma)$*,* $\mathbf{P}$ *satisfies the following sentence of its internal language.*

$$\forall i : \Gamma . \exists s : PX . \forall x : X . In_X(x, s) \Leftrightarrow R(x, i)$$

Starting with a category with finite limits $\mathbf{C}$ with enough structure to ensure that the proof-theoretic power set functor is a first order hyperdoctrine, Proposition 11.4.3 should provide a variant of our Theorem 5.2.2 by working out what the axiom (CA) amounts to in $\mathbf{C}$.

For Theorem 11.3.3, one could check that for a locally cartesian closed regular category the subobjects functor $Sub_{\mathbf{D}}$ induces a first order hyperdoctrine and then show that the generic mono (with the help of the local cartesian closed structure) implies that (CA) holds.

Similarly, there probably exists a treatment of weak dependent products along these lines.

This strategy to deal with the question of exact completions and ex/reg completions that are locally cartesian closed or that are toposes takes a bit of a roundabout route and hides much of the simple categorical structure and arguments. On the other hand, this strategy may provide some insight into the problem of characterizing the ex/reg completions that are toposes. Although in the end, it would be nice to obtain an indexing-free formulation of the statement and proof of such a characterization.

Another question on the relation between tripos theory and the work reported in this thesis is whether every topos arising from a first order hyperdoctrine can also be generated by an ex/reg completion. Of course, as the ex/reg completion is an idempotent construction, this is trivially the case. The point of the question is whether given a first order hyperdoctrine $\mathbf{P}$ over $\mathbf{C}$ satisfying (CA) there exists a regular category $\mathbf{D}$, simpler to understand than the topos $\mathbf{C}[\mathbf{P}]$ and such that $\mathbf{D}_{ex/reg} \simeq \mathbf{C}[\mathbf{P}]$. Clear examples of this are realizability toposes as ex/reg completions of categories of assemblies. Moreover, it would be good if the category $\mathbf{D}$, on top of being easy to understand, would also have good structure, such as cartesian closed slices and a strong-subobject classifier.

There has not been much work on general first order hyperdoctrines satisfying (CA). On the other hand, in the case of triposes (which we define below) we can point out some analogies that suggest that such $\mathbf{D}$'s may exist.

**Definition 11.4.4.** Generic predicates and triposes.

1. A first order hyperdoctrine $\mathbf{P}$ over $\mathbf{C}$ has a *generic predicate* if there exists a $\mathbf{C}$-object *Prop* and a $\mathbf{P}$-predicate $prf \in \mathbf{P}Prop$ such that for any $\Gamma$ and $A \in \mathbf{P}\Gamma$ there exists a $\mathbf{C}$-morphism $\nu_A : \Gamma \longrightarrow Prop$ with $A = (\mathbf{P}\nu_A)prf$.

2. If $\mathbf{C}$ is a cartesian closed category, a $\mathbf{C}$-*tripos* is a first order hyperdoctrine over $\mathbf{C}$ with a generic predicate.

Moreover, we have already mentioned in Section 11.1 that for a $\mathbf{C}$-tripos $\mathbf{P}$, the topos $\mathbf{C}[\mathbf{P}]$ of partial equivalence relations associated with $\mathbf{P}$ has the property that every object is isomorphic to a weakly complete one. In fact, as mentioned in Section 11.2, we have actually borrowed the proof of this fact in order to show that, in a locally cartesian closed regular category $\mathbf{D}$ with a generic mono, every equivalence relation is relationally isomorphic to a complete one.

This may be suggesting that for a topos $\mathbf{E}$ induced by a tripos there exists a simpler regular subcategory $\mathbf{D}$ of $\mathbf{E}$ such that for every object $Q$ in $\mathbf{E}$ there exists an object $X$ in $\mathbf{D}$ and a $\mathbf{D}$-projecting quotient $X \twoheadrightarrow Q$.

Moreover, the generic predicate should induce a generic mono in $\mathbf{D}$ and the embedding $\mathbf{D} \longrightarrow \mathbf{E}$ should preserve subobjects. That is, $\mathbf{E}$ should be $\mathbf{Ceq}(\mathbf{D})$ (recall Section 10.5).

But we do not know if these analogies can be turned into actual constructions.

In any case, it seems that since the invention of realizability toposes, triposes were the only general abstract tool to deal with this kind of example. The results in this thesis provide an extra general tool that avoids the complications of indexed structures.

# Chapter 12

# Conclusions

We summarize the contributions of the thesis and indicate some of the problems left open.

## 12.1 Summary of the main results

Our main objective was to achieve a good understanding of the relation between quasi-toposes and toposes on one hand and the universal problems of building regular, exact and ex/reg completions on the other. The main contributions of this thesis are the following:

1. A characterization of the categories with finite limits whose exact completions are toposes (Theorem 5.2.2).

   We proved the usefulness of the characterization by applying it to obtain results on the following two problems.

   (a) A characterization of the presheaf toposes whose exact completions are toposes (Theorem 6.1.1).

   To illustrate the interest of this result it must be said that, before our characterization, there was no grasp to the problem even in the simplest cases. For example, it was not known whether the exact completion of $\mathbf{Set}^{\longrightarrow}$ was a topos or not.

   (b) Finding many new examples of toposes that are exact completions. Namely, the toposes of continuous actions for topological groups (Corollary 6.3.1) and the hierarchies over the categories of partitioned assemblies (Corollaries 8.2.2 and 11.3.2).

   In this respect, it must be said that generic proofs were an essential tool in finding these examples. It struck me as a surprise to find out that **Ass** has a generic proof, but it was not a difficult observation as the essential idea was the same as in the case of partitioned assemblies. I later learned that van Oosten had already observed that the exact completion of the category of assemblies is a topos. His observation consisted of first building the topos in question and then identifying its subcategory of projectives with assemblies. Generic proofs provided a very simple way of iterating this observation (giving rise to the hierarchies) without having to calculate with any of the induced toposes.

2. The conceptual use of "chaotic" objects to explain many relevant properties of our examples (Chapters 7 and 8). In particular, the equivalence (in this context) of generic objects, generic monos and generic proofs (Theorem 8.2.1). Indeed, generic objects should be highlighted for providing a very efficient way to present toposes that are exact completions and also to recognize new examples and counterexamples. In particular, they provide a nice perspective on the results in [65] (see Section 8.3).

3. A characterization of universal closure operators in regular and exact completions of a category $\mathbf{C}$ in terms of the notion of a *topology* on $\mathbf{C}$ (Corollary 9.3.6).

   This was an evident problem suggested by the construction of presheaf toposes as colimit completions and by the description of the universal closure operators therein as Grothendieck topologies.

   Moreover, this characterization allows us to state and prove the following nice conceptual facts.

   (a) The characterization of categories of closed pseudo equivalence relations as the categories of separated objects for universal closure operators in exact completions (Proposition 10.2.1). This allows us to conclude that the former categories are quasi-pretoposes in the presence of stable and disjoint coproducts (Section 10.2) and to obtain sufficient conditions for these categories to be locally cartesian closed (Corollary 10.6.2).

   (b) The characterization of stable epi/regular-mono factorizations as sep-canonical topologies (Corollary 10.3.3) and their relation with regular completions (Corollary 10.3.5). Sufficient conditions for regular completions to be locally cartesian closed follow (Corollary 10.6.3).

(c) The description of stable regular-epi/mono factorizations as canonical topologies (Section 10.4), the characterization of the sheaves for this topology as the (Higgs-)complete equivalence relations (Corollary 10.4.8) and the relation of the category of sheaves with ex/reg completions (Section 10.5). As in the case of separated objects, we can easily infer sufficient conditions for local cartesian closure (see Corollaries 10.6.4 and 11.3.1).

4. Sufficient conditions on a regular category **D** for its ex/reg completion to be a topos (Theorem 11.3.3) and the characterization of the locally cartesian closed regular categories whose category of complete equivalence relations is a topos (Corollary 11.3.4).

   These results improve in generality and weaken the assumptions over the related results by Higgs [34], Fourman-Scott [27], Carboni-Freyd-Scedrov [17] and McLarty [75].

5. Simple conditions for regular completions to be quasi-toposes (Corollaries 4.3.4 and 8.4.2) and in particular, we have characterized the lextensive categories whose regular completions have coequalizers (Corollary 4.1.4).

Finally, there is a healthy set of topological and recursion theoretic examples (and counterexamples) showing how the abstract concepts and results arise (or do not arise) in practice.

## 12.2 Loose ends

Regular and exact completions can also be taken over categories with weak limits [22]. It seems plausible to generalize our results in this direction. One of the toposes described in [84] already motivates such a generalization.

It may be of interest to find sharper results relating local cartesian closure and quasi-toposes with regular completions.

Our characterization of boolean presheaf toposes shows that for a restricted class of categories it is possible to simplify the conditions that ensure its exact completions to be toposes. There may be many other interesting classes. For example, it is natural to hope for a good description of the sites whose associated Grothendieck toposes have generic proofs. Also, we do not know if the exact completion of the effective topos is a topos. In fact, we do not know an example of a non-boolean topos whose exact completion is itself a topos.

The new hierarchies of toposes presented in Chapter 8 are awaiting a serious analysis. We have only proved their existence as an application of our characterization.

Our characterization in Chapter 9 of universal closure operators in regular and exact completions seems pretty tight and definitive. As well as explaining the relationship between factorization systems on a category $\mathbf{C}$ and the universal closure operators in $\mathbf{C}_{ex}$, it suggests that it one should not restrict to the former. For example, it would be nice to know if **Top** has a canonical topology. Also, we have not touched the important question of what is a good notion of a continuous functor in this setting. It would be nice if there existed a theory as rich as that for Grothendieck toposes.

The results in Chapter 10 show that categories of closed pseudo equivalence relations or of complete equivalence relations have, in general, better properties than regular and ex/reg completions. Because of this, it may be interesting to study more deeply the former categories.

Our sufficient conditions for ex/reg completions to be toposes (Theorem 11.3.3) seem not to be necessary. Concerning the search for a tighter result, one may start to attempt to characterize the regular categories whose ex/reg completions are locally cartesian closed and then combine this result with some version of Pitts' axiom characterizing the hyperdoctrines giving rise to toposes [95].

# Appendix A

# Some technical facts about relations

In this appendix we collect some folklore facts about relations that are used in the thesis (mainly in Sections 2.4.1 and 11.2). None of the results is difficult but some are a bit tiresome. It should be said that many of the proofs could be simplified if a treatment of internal logic as in [75] or of representations of regular categories as in [6] was assumed. Nevertheless I decided to include the most unassuming proofs.

In order to ease the statements let $\langle d_0, d_1\rangle : D \rightarrowtail Y \times Y$ and $\langle e_0, e_1\rangle : E \rightarrowtail X \times X$ be two arbitrary equivalence relations.

## A.1 Proposition 2.4.4

**Proposition 2.4.4.** *Let $f, g$ induce maps from $D$ to $E$ as below.*

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "g'"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

*Then the following hold.*

1. *$DfE$ is a functional relation from $D$ to $E$*

2. *$DfE = DgE$ if and only if there exists $h : Y \longrightarrow E$ such that $e_0.h = g$ and $e_1.h = f$.*

*Proof.* We prove 1 first. That $F = DfE$ is defined from $D$ to $E$ is trivial because $DD = D$ and $EE = E$. To prove that $F$ is total notice that by Lemma A.1.1,

$D \leq fEf^\circ$ and then $D = DDD \leq DfEf^\circ D = FF^\circ$. To prove that $F$ is single valued use Lemma A.1.1 again and notice that $f^\circ Df \leq E$ implies that $F^\circ F = Ef^\circ DfE \leq EEE = E$.

Let us prove 2. For the *if* direction notice that it is enough to prove that $fE \leq gE$ because then $DfE \leq DgE$ and as both relations are functional by item 1, they are equal by Proposition 2.4.3. So consider the following diagrams.

$$\begin{tikzcd}
Y \times_f E \arrow[r, "\pi_E"] \arrow[d, "\pi_Y"'] & E \arrow[d, "e_0"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times_f E \arrow[d, "\pi_Y"'] \arrow[dr, "\exists ! h_0"] \arrow[drr, "\pi_E"] & & \\
Y \arrow[dr, "h"'] & E \times_X E \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & E \arrow[d, "e_0"] \\
& E \arrow[r, "e_1"'] & X
\end{tikzcd}$$

We then have the following.

$$\begin{tikzcd}
Y \times_f E \arrow[r, "h_0"] \arrow[dr, "\exists ! h'"] \arrow[ddr, "\pi_Y"'] & E \times_X E \arrow[dr, "trans"] & \\
& Y \times_g E \arrow[r, "\pi'_E"] \arrow[d, "\pi'_Y"] & E \arrow[d, "e_0"] \\
& Y \arrow[r, "g"'] & X
\end{tikzcd}$$

It is very easy to prove that $\langle \pi'_Y, e_1.\pi'_E \rangle.h' = \langle \pi_Y, e_1.\pi_E \rangle : Y \times_f E \longrightarrow Y \times X$ which shows that $fE \leq gE$.

To prove the *only if* direction we are going to build a regular cover $M$ of $D$ and a map $m_E : M \longrightarrow E$ such that the square below commutes, inducing the inner arrow which proves the statement.

$$\begin{tikzcd}
M \arrow[r, two heads, "m_Y"] \arrow[d, "m_E"'] & Y \arrow[d, "\langle g, f \rangle"] \arrow[dl, "\exists !"'] \\
E \arrow[r, tail, "\langle e_0, e_1 \rangle"'] & X \times X
\end{tikzcd}$$

In order to build this square let us work out what does the equality $H = DfE = DgE$ mean. We first calculate the two sides of the isomorphism separately.

$$
\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "p_D"'] & Y \times_f E \arrow[r, "\pi_E"] \arrow[d, "\pi_Y"'] & E \arrow[d, "e_0"] \\
D \arrow[r, "d_1"'] & Y \arrow[r, "f"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "q"] \arrow[d, "q_D"'] & Y \times_g E \arrow[r, "\pi'_E"] \arrow[d, "\pi'_Y"'] & E \arrow[d, "e_0"] \\
D \arrow[r, "d_1"'] & Y \arrow[r, "g"'] & X
\end{tikzcd}
$$

The equality means then that, as shown in the diagram below, the maps from $P$ and $Q$ factor through the same subobject. The square in the diagram is a pullback defining $M$.

$$
\begin{tikzcd}
M \arrow[r, two heads, "\pi_P"] \arrow[d, two heads, "\pi_Q"'] & P \arrow[d, two heads, "e'"] \arrow[dr, "\langle d_0.p_D, e_1.\pi_E.p \rangle"] & \\
Q \arrow[r, two heads, "e"'] \arrow[rr, bend right, "\langle d_0.q_D, e_1.\pi'_E.q \rangle"'] & H \arrow[r, tail, "\langle h_Y, h_X \rangle"'] & Y \times X
\end{tikzcd}
$$

It is fair to think

1. of $Q$ as $\{(y_0, y, x_0) | y_0 D y \wedge (gy) E x_0\}$,

2. of $P$ as $\{(y_0, y', x_0) | y_0 D y' \wedge (fy') E x_0\}$,

3. of $H$ as $\{(y_0, x_0) | (\exists y)(y_0, y, x_0) \in Q\} = \{(y_0, x_0) | (\exists y')(y_0, y', x_0) \in P\}$,

4. and so, of $M$ as $\{(y_0, y, y', x_0) | (y_0, y, x_0) \in Q \wedge (y_0, y', x_0) \in P\}$

Using $f'$ and $g'$ it is easy to build maps $m_g, m_f : M \longrightarrow E$ such that intuitively $m_g(y_0, y, y', x_0) = (gy_0) E x_0$ and $m_f(y_0, y, y', x_0) = (fy_0) E x_0$.

Also, using that $k = e_1.\pi'_E.q.\pi_Q = e_1.\pi_E.p.\pi_Q$ (which follows from the diagram above) together with $m_g$ and $m_f$ it is easy to build a map $m_E : M \longrightarrow E$ such that intuitively $m_E(y_0, y, y', x_0) = (gy_0) E (fy_0)$. That is, letting $k' = e.\pi_Q = e'.\pi_P$, $\langle e_0, e_1 \rangle.m_E = \langle g, f \rangle.h_Y.k'$.

Clearly $k'$ is a regular epi because $e$ and $e'$ are. Also, by the first item of this proposition we know that $\langle h_Y, h_X \rangle$ is a functional relation, hence total by definition and so $h_Y$ is also a regular epi by Lemma A.4.2.

So, by letting $m_Y = h_Y.k'$ we have the square $\langle e_0, e_1 \rangle.m_E = \langle g, f \rangle.m_Y$ that we set up to build and so the proof is finished. $\square$

**Lemma A.1.1.** *$D \leq fEf^\circ$ and $f^\circ Df \leq E$*

*Proof.* Easy. $\square$

## A.2 Proposition 2.4.5

**Proposition 2.4.5.** *Let $D$ and $E$ be equivalence relations as above. Let $\langle h_Y, h_X\rangle : H \longrightarrow Y \times X$ be a functional relation from $D$ to $E$ and let $h : Y \longrightarrow X$. Then, the following are equivalent.*

1. *there exists an $h' : D \longrightarrow E$ such that $e_0.h' = d_0.h$ and $e_1.h' = d_1.h$ as in the square below and also $DhE = H$.*

$$\begin{tikzcd} D \arrow[r, "h'"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & E \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \\ Y \arrow[r, "h"'] & X \end{tikzcd}$$

2. *$\langle id, h\rangle \leq H$*

3. *there exists an $h_E : H \longrightarrow E$ such that the following square commutes.*

$$\begin{tikzcd} H \arrow[r, "h_E"] \arrow[d, "{\langle h_X, h_Y\rangle}"'] & E \arrow[d, "{\langle e_0, e_1\rangle}"] \\ X \times Y \arrow[r, "id \times h"'] & X \times X \end{tikzcd}$$

*Proof.* We first prove that 1 implies 3. Consider the following pullback diagram.

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "\pi_D"'] & Y \times_X E \arrow[r, "\pi_E"] \arrow[d, "\pi_Y"] & E \arrow[d, "e_0"'] \\ D \arrow[r, "d_1"'] & Y \arrow[r, "h"'] & X \end{tikzcd}$$

The equality $DhE = H$ means that there exists a regular epi $p : P \twoheadrightarrow H$ such that $\langle d_0.\pi_D, e_1.\pi_E.\pi\rangle = \langle h_Y, h_X\rangle.p$.

Intuitively, the map $p$ takes a 3-tuple $(y, y', x)$ such that $yDy' \wedge (hy')Ex$ and returns $yHx$.

On the other hand, consider the construction of the map $l$ below.

$$\begin{tikzcd}
P \arrow[r, "\pi"] \arrow[d, "\pi_D"'] \arrow[dr, "\exists! l"] & Y \times_X E \arrow[dr, "\pi_E"] & \\
D \arrow[dr, "h'"'] & E \times_X E \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & E \arrow[d, "e_0"] \\
 & E \arrow[r, "e_1"'] & X
\end{tikzcd}$$

Intuitively, $l$ takes a 3-uple $(y, y', x)$ as above to $(hy)E(hy')$ and $(hy')Ex$.

We then have the following outer diagram inducing the inner dotted arrow.

$$\begin{tikzcd}
P \arrow[r, two heads, "p"] \arrow[d, "l"'] & H \arrow[r, tail, "\langle h_X, h_Y \rangle"] \arrow[d, dashed, "\exists! h_E"'] & X \times Y \arrow[d, "id \times h"] \\
E \times_X E \arrow[r, "sym.trans"'] & E \arrow[r, tail, "\langle e_0, e_1 \rangle"'] & X \times X
\end{tikzcd}$$

To prove that 3 implies 2, consider the following diagrams.

$$\begin{tikzcd}
D \times_Y H \arrow[r, "\pi_H"] \arrow[d, "\pi_D"'] & H \arrow[d, "h_Y"] \\
D \arrow[r, "d_1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[dr, "\exists! h_0"] \arrow[drr, "h_E"] \arrow[ddr, "id"'] & & \\
 & H \times_X E \arrow[r, "\pi_E"] \arrow[d, "\pi_H"] & E \arrow[d, "e_0"] \\
 & H \arrow[r, "h_X"'] & X
\end{tikzcd}$$

Notice that $\pi_D$ is a regular epi because $h_Y$ is by Lemma A.4.2. As $H$ is defined from $D$ to $E$, there exist regular epis $d : D \times_Y H \longrightarrow H$ and $e : H \times_X E \longrightarrow H$ satisfying certain equations. It follows that $\langle h_Y, h_X \rangle.(e.h_0.d) = \langle id, h \rangle.d_0.\pi_D$. As $d_0.\pi_D$ is a regular epi, $\langle id, h \rangle \leq \langle h_Y, h_X \rangle$.

To prove that 2 implies 1, notice that $\langle h, id\rangle D\langle id, h\rangle \leq H^\circ DH$.

As $H$ is defined from $D$, $\langle h, id\rangle D\langle id, h\rangle \leq H^\circ H$.

As $H$ is single valued, $\langle h, id\rangle D\langle id, h\rangle \leq E$.

But this easily implies 1. ☐

## A.3 Definedness

In this section we show two simple lemmas involving the notion of definedness.

**Lemma A.3.1.** *Let $\langle f_Y, f_X\rangle : F \rightarrowtail Y \times X$, $h : Z \to X$ and $G$ be as in the following pullback.*

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\langle g_Z, g_Y\rangle"'] & F \arrow[d, "\langle f_X, f_Y\rangle"] \\ Z \times Y \arrow[r, "h \times id"'] & X \times Y \end{tikzcd}$$

*If $\langle f_Y, f_X\rangle$ is defined from $D$ then so is $\langle g_Y, g_X\rangle$.*

*Proof.* By Lemma A.3.2 we need only check that $(id_Z \times d_0)$ and $(id_Z \times d_1)$ pull $G$ back to the same subobject. But this is equivalent to $(id_X \times d_0).(h \times id)$ and $(id_X \times d_1).(h \times id)$ pulling $F$ back to the same subobject. This holds because $F$ is defined from $D$ (Lemma A.3.2 again). ☐

Intuitively, the next lemma is saying that a relation $F$ is defined from $D$ is total if and only if $\{x, y, y' | yDy' \wedge xFy\} = \{x, y, y' | yDy' \wedge xFy'\}$.

**Lemma A.3.2.** *Let $\langle f_Y, f_X\rangle : F \longrightarrow Y \times X$ a relation. Then $F$ is defined from $D$ if and only if $(id \times d_0)$ and $(id \times d_1)$ pull $\langle f_X, f_Y\rangle$ back to the same subobject.*

$$\begin{tikzcd} U \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F \arrow[d, "\langle f_X, f_Y\rangle"] \\ X \times D \arrow[r, shift left, "id \times d_0"] \arrow[r, shift right, "id \times d_1"'] & X \times Y \end{tikzcd}$$

*Proof.* For the purpose of the proof let us split the diagram above in the pullbacks below.

$$
\begin{tikzcd}
D \times_Y F \arrow[r, "\pi_F"] \arrow[d, "\pi_D"'] & F \arrow[d, "f_Y"] \\
D \arrow[r, "d_1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
U_0 \arrow[r, "u_F"] \arrow[d, "u_D"'] & F \arrow[d, "f_Y"] \\
D \arrow[r, "d_0"'] & Y
\end{tikzcd}
$$

Let us start with the *only if* direction. We must prove that $D \times_Y F$ and $U_0$ are isomorphic over $X \times D$. The fact $F$ is defined from $D$ means that $\langle d_0.\pi_D, f_X.\pi_F \rangle = \langle f_Y, f_X \rangle.e$ for some regular epi $e : D \times_Y F \twoheadrightarrow F$.

It follows that $\langle f_X, f_Y \rangle.e = (id \times d_0).\langle f_X.\pi_F, \pi_D \rangle$ and hence that $D \times_Y F \leq U_0$ over $X \times D$.

In order to prove that $U_0 \leq D \times_Y F$, first take the map $s : U_0 \longrightarrow D \times_Y F$ induced by the universal property of $D \times_Y F$ and the equality $f_Y.u_F = d_1.sym.u_D$. We then have that $f_Y.e.s = d_0.\pi_D.s = d_0.sym.u_D = d_1.u_D$. The universal property of $D \times_Y F$ shows that $U_0 \leq D \times_Y F$.

For the *if* direction we need to prove that $DF = F$ (that is, the image of $\langle d_0.\pi_D, f_X.\pi_F \rangle$ is isomorphic to $\langle f_Y, f_X \rangle$ over $Y \times X$).

First, notice that as $D$ is reflexive we have $F = F\Delta \leq FD$.

To show $F \leq DF$, use the map $F \longrightarrow D \times_Y F$ given by $d_1.\mathit{refl}.f_Y = f_Y.id$.

To prove that $DF \leq F$ just use the hypothesis, that is, that there exists an isomorphism $i : D \times_Y F \longrightarrow U_0$ such that $\langle f_X.u_F, u_D \rangle.i = \langle f_X.\pi_F, \pi_D \rangle$.

It follows that $\langle f_Y, f_X \rangle.u_F.i = \langle d_0.\pi_D, f_X.\pi_F \rangle$ and hence, that $DF \leq F$. □

## A.4 Totality

Intuitively, the following lemma shows that in a situation similar to that in Lemma A.3.1, if $h$ hits representatives of all $E$-equivalence classes then $G$ is total.

**Lemma A.4.1.** *Let $F$ be defined from $D$ to $E$ and total and let $G$ and $h$ be as in Lemma A.3.1. Consider the following pullback diagram.*

$$
\begin{tikzcd}
H \arrow[r, "h'"'] \arrow[d, "\langle h_X, h_Z \rangle"'] & E \arrow[d, "\langle e_0, e_1 \rangle"] \\
X \times Z \arrow[r, "id \times h"] & X \times X
\end{tikzcd}
$$

*If $h_X = e_0.h'$ is a regular epi then $\langle g_Y, g_Z \rangle$ is total.*

*Proof.* Consider the following diagrams.

$$\begin{tikzcd}
F \times_X H \arrow[r, "h_H"] \arrow[d, "h_F"'] & H \arrow[d, "h_X"] \\
F \arrow[r, "f_X"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
F \times_X H \arrow[r, "h_H"] \arrow[dr, "\exists ! h_0"] \arrow[ddr, "h_F"'] & H \arrow[dr, "h'"] & \\
& F \times_X E \arrow[r, "\pi_E"] \arrow[d, "\pi_F"'] & E \arrow[d, "e_0"] \\
& F \arrow[r, "f_X"'] & X
\end{tikzcd}$$

We are assuming $h_X$ is a regular epi, so $h_F$ also is. Moreover, because $F$ is defined to $E$, there exists a regular epi $e : F \times_X E \twoheadrightarrow F$ such that $\langle f_Y, f_X\rangle.e = \langle f_Y.\pi_F, e_1.\pi_E\rangle$.

We can then build the following.

$$\begin{tikzcd}
F \times_X H \arrow[r, "h_0"] \arrow[d, "\langle h_F, h_H\rangle"'] \arrow[dr, "\exists !"] & F \times_X E \arrow[dr, "e"] & \\
F \times H \arrow[dr, "f_Y \times h_Z"'] & G \arrow[r] \arrow[d, "\langle g_Y, g_Z\rangle"] & F \arrow[d, "\langle f_Y, f_X\rangle"] \\
& Y \times Z \arrow[r, "id \times h"'] & Y \times X
\end{tikzcd}$$

As $F$ is total, $f_Y$ is a regular epi, so $f_Y.h_F$ also is and hence $g_Y$ is a regular epi. By Lemma A.4.2, $G$ is total. $\square$

Below we show that a relation $\langle f_Y, f_X\rangle : F \rightarrowtail Y \times X$ is total from $D$ if and only if $(\forall y \in Y)(\exists z \in F)(f_Y z = y)$.

**Lemma A.4.2.** *Let $\langle f_Y, f_X\rangle : F \rightarrowtail Y \times X$ a relation defined from $D$. Then $F$ is total from $D$ if and only if $f_Y$ is a regular epi.*

*Proof.* Let $k_0, k_1 : F \times_X F \longrightarrow F$ the kernel pair of $f_X$.

Let us first prove the *only if* part. Totality means that we have a map $t : D \rightarrowtail FF^\circ$ as below where $\langle t_0, t_1\rangle.e = \langle f_Y.k_0, f_Y.k_1\rangle$.

$$
\begin{tikzcd}
Y \arrow[r, tail, "refl"] \arrow[drr, "\Delta"'] & D \arrow[r, tail, "t"] \arrow[dr] & FF^\circ \arrow[d, tail, "\langle t_0, t_1\rangle"] & Y \times_X Y \arrow[l, two heads, "e"'] \\
& & Y \times Y &
\end{tikzcd}
$$

Using reflexivity it follows that $t_0$ is a (split) regular epi. Then $t_0.e = f_Y.k_0$ also is and hence $f_Y$ also is.

We now prove the *if* part. Consider the left-hand pullback below. As $f_Y$ is a regular epi, $\pi_D$ also is. Now, as $F$ is defined from $D$, there exists a regular epi $e : D \times_Y F \twoheadrightarrow F$ such that $\langle f_Y, f_X\rangle.e = \langle d_0.\pi_D, f_X.\pi_F\rangle$ and so we can obtain a map $d : D \times_Y F \longrightarrow F \times_X F$ as in the right-hand diagram below.

$$
\begin{tikzcd}
D \times_Y F \arrow[r, "\pi_F"] \arrow[d, "\pi_D"'] & F \arrow[d, "f_Y"] \\
D \arrow[r, "d_1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
D \times_Y F \arrow[dr, "\exists ! d"] \arrow[drr, "\pi_F"] \arrow[ddr, "e"'] & & \\
& F \times_X F \arrow[r, "k_1"] \arrow[d, "k_0"] & F \arrow[d, "f_X"] \\
& F \arrow[r, "f_X"'] & X
\end{tikzcd}
$$

Together with $\pi_D$ being a regular epi, the following diagram shows that $F$ is total (the vertical right hand map is $\langle f_Y.k_0, f_Y.k_1\rangle$).

$$
\begin{tikzcd}
D \times_Y F \arrow[r, two heads, "\pi_D"] \arrow[d, "d"'] & D \arrow[d, dashed, "\exists !"'] \arrow[dr, "\langle d_0, d_1\rangle"] & \\
F \times_X F \arrow[r, two heads] & FF^\circ \arrow[r, tail] & Z \times Z
\end{tikzcd}
$$

□

## A.5 Pulling back relations

We now state one of the main observations that we need in Section 11.2.

It is not difficult to show that if $\langle d_0, d_1\rangle$ is an equivalence relation on $X$ then for any $j : Y \to X$, $\langle j^*d_0, j^*d_1\rangle$ is an equivalence relation on $Y$. That is, equivalence relations are closed under pullback.

**Proposition A.5.1.** *Assume that there exists a $j : X \longrightarrow Y$ such that the following two squares below are pullbacks.*

$$\begin{tikzcd}
E \arrow[r, "g"] \arrow[d, "{\langle e_0, e_1 \rangle}"'] & J \arrow[r, "h"] \arrow[d, "{\langle j_Y, j_X \rangle}"] & D \arrow[d, "{\langle d_0, d_1 \rangle}"] \\
X \times X \arrow[r, "j \times id"'] & Y \times X \arrow[r, "id \times j"'] & Y \times Y
\end{tikzcd}$$

*Then the following hold.*

1. *$\langle j_X, j_Y \rangle$ is a functional relation from $E$ to $D$.*

2. *$\langle j_Y, j_X \rangle$ is a single valued relation from $D$ to $E$.*

*Proof.* Concerning the first item. Let us prove that $\langle j_X, j_Y \rangle$ is defined from $E$. The following diagram shows that $J \leq EJ$.

$$\begin{tikzcd}
J \arrow[d, "j_X"'] \arrow[dr, "\exists!"'] \arrow[drr, "id"] & & \\
X \arrow[dr, "refl"'] & E \times_X J \arrow[r, "\pi_J"] \arrow[d, "\pi_E"'] & J \arrow[d, "j_X"] \\
& E \arrow[r, "e_1"'] & X
\end{tikzcd}$$

To show that $EJ \leq J$, it is enough to find a map $E \times_X J \longrightarrow J$ with the right properties.

$$\begin{tikzcd}
E \times_X J \arrow[r, "\pi_J"] \arrow[d, "\pi_E"'] \arrow[dr, "\exists! g'"'] & J \arrow[r, "h"] & D \arrow[d, "sym"] \\
E \arrow[d, "g"'] & D \times_Y D \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & D \arrow[d, "d_0"] \\
J \arrow[r, "h"'] & D \arrow[r, "d_1"'] & Y
\end{tikzcd}$$

Then, we build the following.

$$
\begin{tikzcd}
E \times_X J \arrow[r, "g'"] \arrow[d, "{\langle \pi_J, \pi_E \rangle}"'] \arrow[dr, "\exists!"] & D \times_Y D \arrow[r, "trans"] & D \arrow[d, "sym"] \\
J \times E \arrow[dr, "j_Y \times e_0"'] & J \arrow[r, "h"] \arrow[d, tail, "{\langle j_Y, j_X \rangle}"] & D \arrow[d, tail, "{\langle d_0, d_1 \rangle}"] \\
 & Y \times X \arrow[r, "id \times j"'] & Y \times Y
\end{tikzcd}
$$

This finishes the proof that $\langle j_X, j_Y \rangle$ is defined from $E$.

The fact that it is defined to $D$ is equivalent to the fact that $\langle j_Y, j_X \rangle$ is defined from $D$ which we will show easily in the proof of 2.

That $\langle j_X, j_Y \rangle$ is total with respect to $E$ is easily proved using Lemma A.4.2. Just notice that, as $j_X.g = e_1$ is a regular epi, so is $j_X$.

Let us prove then that it is single valued with respect to $D$. For this, let $\pi_0, \pi_1 : J \times_X J \longrightarrow J$ be the kernel pair of $j_X$.

We will find a map $J \times_X J \longrightarrow D$ satisfying the right properties.

$$
\begin{tikzcd}
J \times_X J \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, "\exists! h'"'] & J \arrow[r, "h"] & D \arrow[d, "sym"] \\
J \arrow[dr, "h"'] & D \times_Y D \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & D \arrow[d, "d_0"] \\
 & D \arrow[r, "d_1"'] & Y
\end{tikzcd}
$$

Then we have $\langle d_0, d_1 \rangle.trans.h' = \langle d_0.\pi_0.h', d_1.\pi_1.h' \rangle = \langle d_0.h.\pi_0, d_1.sym.h.\pi_1 \rangle = \langle j_Y.\pi_0, j_Y.\pi_1 \rangle$.

Now let us consider item 2. The fact that $\langle j_Y, j_X \rangle$ is defined from $D$ follows from the fact that $D$ is defined from itself to itself and Lemma A.3.1.

The fact that it is defined to $E$ is equivalent to the fact that $\langle j_X, j_Y \rangle$ is defined from $E$ which we have proved above.

Let us prove that $\langle j_Y, j_X \rangle$ is single valued with respect to $E$. So let $k_0, k_1 : J \times_Y J \longrightarrow J$ be the kernel pair of $j_Y$ and let the following diagram define $k_2$.

$$\begin{tikzcd}
J \times_Y J \arrow[r, "k_1"] \arrow[d, "k_0"'] \arrow[dr, "\exists ! k_2"] & J \arrow[dr, "h"] & \\
J \arrow[d, "h"'] & D \times_Y D \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & D \arrow[d, "d_0"] \\
D \arrow[r, "sym"'] & D \arrow[r, "d_1"'] & Y
\end{tikzcd}$$

But then the outer diagram below commutes giving rise to the inside diagonal arrow proving that $\langle j_Y, j_X \rangle$ is single valued.

$$\begin{tikzcd}
J \times_Y J \arrow[r, "k_2"] \arrow[d, "\langle k_0, k_1 \rangle"'] \arrow[dr, "\exists !"] & D \times_Y D \arrow[dr, "trans"] & \\
J \times J \arrow[dr, "j_X \times j_X"'] & E \arrow[r, "h.g"] \arrow[d, tail, "\langle e_0, e_1 \rangle"] & D \arrow[d, tail, "\langle d_0, d_1 \rangle"] \\
& X \times X \arrow[r, "j \times j"'] & Y \times Y
\end{tikzcd}$$

$\square$

# Appendix B

# Every equivalence relation is relationally isomorphic to a complete one

We give diagrammatic proofs of the results in Section 11.2 which, as explained there, have essentially appeared in [27, 42].

## B.1 Proposition 11.2.1

**Proposition 11.2.1.** *Every equivalence relation $E$ on $X$ appears in a pullback square as below.*

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "\langle e_0, e_1 \rangle"'] & \Xi^X \arrow[d, "\langle \tau_0^X, \tau_1^X \rangle"] \\
X \times X \arrow[r, "\nu_E \times \nu_E"'] & \Lambda^X \times \Lambda^X
\end{tikzcd}$$

*Proof.* First, let us prove that $(\nu_E \times \nu_E).\langle e_0, e_1 \rangle$ factors through $\Xi^X$. We use Lemma B.1.1. So we need to check that the two pullbacks below coincide.

$$\begin{tikzcd}
V \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E \arrow[r, "\nu_{E,\Upsilon}"] \arrow[d] & \Upsilon \arrow[d, "\tau"] \\
X \times E \arrow[r, shift left, "id \times e_0"] \arrow[r, shift right, "id \times e_1"'] & X \times X \arrow[r, "\nu_{E,\Lambda}"'] & \Lambda
\end{tikzcd}$$

But this is trivial because $E$ is defined from itself to itself and hence by Lemma A.3.2 the pullbacks coincide.

We now prove that the square is a pullback. Let $f_0, f_1 : Y \longrightarrow X$ and $f : Y \longrightarrow \Xi^X$ be such that $\langle \tau_0^X, \tau_1^X \rangle.f = (\nu_E \times \nu_E).\langle f_0, f_1 \rangle$. By Lemma B.1.1 we have that the following two pullbacks coincide.

$$\begin{tikzcd}
U \arrow[r, shift left, "f_0'"] \arrow[r, shift right, "f_1'"'] \arrow[d, "\langle u_0, u_1 \rangle"'] & E \arrow[r] \arrow[d, "\langle e_0, e_1 \rangle"] & \Upsilon \arrow[d] \\
X \times Z \arrow[r, shift left, "id \times f_0"] \arrow[r, shift right, "id \times f_1"'] & X \times X \arrow[r, "\nu_{E,\Lambda}"'] & \Lambda
\end{tikzcd}$$

This implies first that $u_0 = e_0.f_0' = e_0.f_1'$ and that the following squares are pullbacks.

$$\begin{tikzcd}
U \arrow[r, shift left, "f_0'"] \arrow[r, shift right, "f_1'"'] \arrow[d, "u_1"'] & E \arrow[d, "e_1"] \\
Y \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & X
\end{tikzcd}$$

As $e_1$ is a split epi, so is $u_1$. Let $u_2 : Y \longrightarrow U$ be a section of $u_1$. In order to prove the proposition it is enough to find a map $u' : Y \longrightarrow E$ such that $\langle e_0, e_1 \rangle.u' = \langle f_0, f_1 \rangle$. First, consider the following diagram.

$$\begin{tikzcd}
U \arrow[d, "f_0'"'] \arrow[dr, "\exists! u"'] \arrow[drr, "f_1'"] & & \\
E \arrow[dr, "sym"'] & E \times_X E \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & E \arrow[d, "e_0"] \\
& E \arrow[r, "e_1"] & X
\end{tikzcd}$$

Then, we use the fact that $u_1$ is a split epi to show the existence of $u'$ as follows.

$$\begin{tikzcd}
U \arrow[r, shift left, "id"] \arrow[r, shift right, "u_2.u_1"'] & U \arrow[r, "u_1"] \arrow[d, "u"'] & Y \arrow[d, "\exists! u'"'] \arrow[dr, "\langle f_0, f_1 \rangle"] & \\
& E \times_X E \arrow[r, "trans"'] & E \arrow[r, "\langle e_0, e_1 \rangle"'] & X \times X
\end{tikzcd}$$

$\square$

**Lemma B.1.1.** $\langle f, g\rangle : Z \longrightarrow \Lambda^X \times \Lambda^X$ *factors through* $\Xi^X$ *if and only if* $ev.(id_X \times f)$ *and* $ev.(id_X \times g)$ *pull* $\tau$ *back to the same subobject.*

$$
\begin{tikzcd}
U \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & \in_X \arrow[r] \arrow[d, tail] & \Upsilon \arrow[d, tail, "\tau"] \\
X \times Z \arrow[r, shift left, "id \times f"] \arrow[r, shift right, "id \times g"'] & X \times \Lambda^X \arrow[r, "ev"'] & \Lambda
\end{tikzcd}
$$

*Proof.* Just calculate the transpositions and use the defining property of $\Xi$. $\square$

## B.2 Proposition 11.2.2

**Proposition 11.2.2.** *The relation* $J$ *is an isomorphism* $X/E \longrightarrow \Lambda_E/\Xi_E$ *between the equivalence relations* $E$ *and* $\Xi_E$ *as objects in* $\mathbf{D}_{ex/reg}$.

*Proof.* By Proposition A.5.1, we need only check that $J$ is total as a relation from $\Xi_E$ to $E$. In turn, by Lemma A.4.2 this reduces to prove $j_0$ is a regular epi. It is not difficult to show that the map $\Upsilon_E \longrightarrow \Xi^X$ factors through $\Xi_E$ in a way that makes the outer diagram below commute.

$$
\begin{tikzcd}
\Upsilon_E \arrow[dr, "\exists!"] \arrow[drr] \arrow[ddr, "{\langle e, e'_1\rangle}"'] & & \\
& J \arrow[r] \arrow[d, "{\langle j_0, j_1\rangle}"] & \Xi_E \arrow[d, tail, "{\langle c_0, c_1\rangle}"] \\
& \Lambda_E \times X \arrow[r, "id \times \nu'_E"'] & \Lambda_E \times \Lambda_E
\end{tikzcd}
$$

As $e$ is a regular epi, $j_0$ has to be one too. $\square$

## B.3 Proposition 11.2.3

**Proposition 11.2.3.** $\Xi_E$ *is complete.*

*Proof.* Let $\langle d_0, d_1\rangle : D \longrightarrow Y \times Y$ be an equivalence relation on $Y$ and let $\langle f_Y, f_E\rangle : F \rightarrowtail Y \times \Lambda_E$ be a functional relation from $D$ to $\Xi_E$. We can form the following pullback and transposition.

$$
\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "\langle g_X, g_Y \rangle"'] & F \arrow[r, "\nu'_F"] \arrow[d, "\langle f_E, f_Y \rangle"'] & \Upsilon \arrow[d, "\tau"] \\
X \times Y \arrow[r, "\nu'_E \times id"'] & \Lambda_E \times Y \arrow[r, "\nu_F"'] & \Lambda
\end{tikzcd}
$$

---

$$
\begin{tikzcd}
Y \arrow[r, "f'"] & \Lambda^X
\end{tikzcd}
$$

We first prove that $f'$ factors through $\Lambda_E$. In order to do this we will first show that $G$ is total and hence that $g_Y$ is a regular epi. We will then prove that there exists a map $G \longrightarrow \Upsilon_E$ making the outer diagram below commute. These imply the existence of the inner arrow showing that indeed $f'$ factors through $\Lambda_E$.

$$
\begin{tikzcd}
G \arrow[r, two heads, "g_Y"] \arrow[dr] & Y \arrow[r, two heads] \arrow[rr, bend left, "f'"] & Im(f') \arrow[r, tail] \arrow[d, dashed] & \Lambda^X \\
 & \Upsilon_E \arrow[r, two heads, "e"'] & \Lambda_E \arrow[ur, tail, "e'"'] &
\end{tikzcd}
$$

In order to prove that $G$ is total we use Lemma A.4.1 whose sufficient condition is nothing but the content of Proposition 11.2.2.

In order to prove the existence of the map $G \longrightarrow \Upsilon_E$ we will show that the map $(f' \times \nu_E).\langle g_Y, g_X \rangle$ factors through $\Xi^X$ as in the diagram below.

$$
\begin{tikzcd}
G \arrow[d, "\langle g_Y, g_X \rangle"'] \arrow[dr, "\exists!"] \arrow[drr] & & \\
Y \times X \arrow[dr, "f' \times id"'] & \Upsilon_E \arrow[r] \arrow[d, "\langle e'_0, e'_1 \rangle"] & \Xi^X \arrow[d, tail] \\
 & \Lambda^X \times X \arrow[r, "id \times \nu_E"'] & \Lambda^X \times \Lambda^X
\end{tikzcd}
$$

By Lemma B.1.1 we need to show that $G_0$ and $G_1$ below are isomorphic over $X \times G$.

$$
\begin{tikzcd}
G_0 \arrow[r, "g_F"] \arrow[d, "\langle g_0, g_0' \rangle"] & F \arrow[r] \arrow[d, "\langle f_E, f_Y \rangle"] & \Upsilon \arrow[d, "\tau"] & G_1 \arrow[r, "g_E"] \arrow[d, "\langle g_1, g_1' \rangle"] & E \arrow[r] \arrow[d, "\langle e_0, e_1 \rangle"] & \Upsilon \arrow[d, "\tau"] \\
X \times G \arrow[r, "\nu_E' \times g_Y"'] & \Lambda_E \times Y \arrow[r, "\nu_F"'] & \Lambda & X \times G \arrow[r, "id \times g_X"'] & X \times X \arrow[r, "\nu_{E,\Lambda}"'] & \Lambda
\end{tikzcd}
$$

Let us prove first that $G_0 \leq G_1$.

$$
\begin{tikzcd}
G_0 \arrow[d, "g_0'"'] \arrow[dr, "\exists ! g_2"] \arrow[drr, "g_F"] & & \\
G \arrow[dr, "g"'] & F \times_Y F \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & F \arrow[d, "f_Y"] \\
& F \arrow[r, "f_Y"'] & Y
\end{tikzcd}
$$

As $F$ is total, there exists a map $t : F \times_Y F \longrightarrow \Xi_E$ such that $\langle c_0, c_1 \rangle.t = \langle f_E.\pi_0, f_E.\pi_1 \rangle$.

The arrow $G_0 \longrightarrow E$ below implies that $G_0 \leq G_1$.

$$
\begin{tikzcd}
G_0 \arrow[r, "g_2"] \arrow[d, "\langle g_0, g_0' \rangle"'] \arrow[dr, "\exists !"] & F \times_Y F \arrow[r, "t"] & \Xi_E \arrow[d, "sym"] \\
X \times G \arrow[dr, "id \times g_X"'] & E \arrow[r] \arrow[d, tail, "\langle e_0, e_1 \rangle"] & \Xi_E \arrow[d, tail, "\langle c_0, c_1 \rangle"] \\
& X \times X \arrow[r, "\nu_E' \times \nu_E'"'] & \Lambda_E \times \Lambda_E
\end{tikzcd}
$$

Let us prove now that $G_1 \leq G_0$.

$$\begin{tikzcd}
G_1 \arrow[r, "g_E"] \arrow[d, "g_1'"'] \arrow[dr, "\exists! g_3"] & E \arrow[r] & \Xi_E \arrow[d, "sym"] \\
G \arrow[dr, "g"'] & F \times_{\Lambda_E} \Xi_E \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \Xi_E \arrow[d, "c_0"] \\
& F \arrow[r, "f_E"'] & \Lambda_E
\end{tikzcd}$$

As $F$ is defined to $\Xi_E$, there exists a map $d : F \times_{\Lambda_E} \Xi_E \twoheadrightarrow F$ such that $\langle f_Y, f_E \rangle.d = \langle f_Y.\pi_0, c_1.\pi_1 \rangle$. The map $d.g_3 : G_1 \longrightarrow F$ can be used in the definition of $G_0$ in order to prove that $G_1 \leq G_0$.

This finishes the proof that $f' : Y \longrightarrow \Lambda^X$ factors as $f' = e'.f$ for some $f : Y \longrightarrow \Lambda_E$.

To finish the proof of completeness we need to find a map $D \longrightarrow \Xi_E$ witnessing that $f$ respects the equivalence relations as below.

$$\begin{tikzcd}
D \arrow[r, dashed] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & \Xi_E \arrow[d, shift right, "c_0"'] \arrow[d, shift left, "c_1"] \\
Y \arrow[r, "f"'] & \Lambda_E
\end{tikzcd}$$

By the definition of $\Xi_E$ it is enough to show that $\langle f'.d_0, f'.d_1 \rangle = (e' \times e').(f \times f).\langle d_0, d_1 \rangle$ factors through $\Xi^X$.

By Lemma B.1.1, this reduces to show that $ev.(id \times f').(id \times d_0) = \nu_F.(\nu_E' \times id).(id \times d_0)$ and $ev.(id \times f').(id \times d_1) = \nu_F.(\nu_E' \times id).(id \times d_1)$ pull $\tau$ to the same subobject.

In turn, this reduces to check that $id \times d_0$ and $id \times d_1$ pull $G$ back to the same subobject. But as $G$ is defined from $D$ (because $F$ is, see Lemma A.3.1) the above holds (see Lemma A.3.2). $\square$

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