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Extension of Natural Deduction

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by

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ECS-LFCS-87-20
(Also published as CSR-225-87)

LFCS Report Series

March 1987

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10 september 1986

Our purpose here is to provide an example of how the use of Gentzen-type sequential calculus considerably simplifies a complex Natural Deduction formalism. In this case we deal with Schroeder-Heister's system of [1]. This system is important from both philosophical and practical points of view: Its philisophical importance is due to the characterization which it provides for the intuitionistic connectives, while the practical one is due to the fact that its notion of higher-order rules and its method of treating the elimination rules were incorporated into the Edinburgh LF (A general logical framework for implementing logical formalisms on a computer, which was developed in the computer science department of the university of Edinburgh). We shall show, how the notions of S.H. that are the most difficult to handle (discharge functions and subrules) become redundant in the Gentzen-type version, and that the unusual form of some of the elimination rules of S.H. corresponds to natural, standard form of antecedent rules in sequential calculi. We note also that the complex normalization proof of S.H. in [2] can be replaced by a completely standard cut-elimination proof.

We assume in what follows an acquaintance with at least the introductory section of [1].

1 The system GSH

(Gentzen type formulation of S.H.'s system)

1.1 The language

As customary while trying to get rid of discharge functions, we start by introducing a new *formal* symbol $\vdash$ into the language (in [1] this symbol is used only in the meta-language):

Formulas:

$$A$$

Rules:

$$R ::= \; A \mid (R_1, \ldots, R_n \Rightarrow A)$$

Sequents:

$$S ::= \; \vdash A \mid R_1, \ldots, R_n \vdash A$$

We shall use A, B as syntactic variables for formulas, R for rules, Γ, Δ for finite sets of rules. $\Gamma \Rightarrow A$ just means A in case Γ is empty.

1.2 The pure system:

Logical axioms:

$$A \vdash A$$

Weakening:

$$\frac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}$$

($\Rightarrow\vdash$):

$$\frac{\Delta_1, \Gamma_1 \vdash A_1 \ \ldots \ \Delta_n, \Gamma_n \vdash A_n}{\Delta_1, \Delta_2, \ldots, \Delta_n, ((\Gamma_1 \Rightarrow A_1), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow A) \vdash A}$$

1.3 General GSH systems

S.H. permits in [1] the addition of various basic "rules" to the basic system. In this way we really get a *family* of systems. Now, for each system in this family we can construct a corresponding *general GSH-system* by adding to pure GSH local inference rules as follows: Whenever $(\Gamma_1 \Rightarrow A_1), (\Gamma_2 \Rightarrow A_2), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow A$ is a "basic rule" add to pure GSH the non-logical rule:

$$\frac{\Delta_1, \Gamma_1 \vdash A_1 \ \ldots \ \Delta_n, \Gamma_n \vdash A_n}{\Delta_1, \ldots, \Delta_n \vdash A}$$

Note that since we are using sequents rather than formulas in our calculi, we do not need the concept of "discharge functions" which was used by S.H.!

2 The relations between GSH and S.H. system

A natural translation of a proof-tree in S.H. formalism into a tree of sequents can be obtained, as usual, by replacing each formula A in the original tree by the sequent $\Gamma \vdash A$,where Γ is the set of assumptions (in the original proof) on which A depends. It is immediately seen that any *direct* applications of an assumption rule is transformed in this way into an application of $(\Rightarrow\vdash)$, while every direct application of a basic rule is transformed into an application of the corresponding non-logical inference rule. In general, however, the resulting tree of sequents is *not* a proof tree in GSH, since S.H. permits in his formalism also *indirect* applications of a rule, through a direct application of one of its subrules (see definition below). Nevertheless, we shall show that the resulting tree of sequents can be converted into a GSH- proof-tree of the root sequent.

We reformulate first S.H. notion of a "subrule":

definitions:

1. (a) R is a subrule of itself.

   (b) If every element of Γ'_i is a subrule of some element of Γ_i $(1 \leq i \leq n)$ then

   $$(\Gamma'_1 \Rightarrow B_1), \ldots, (\Gamma'_n \Rightarrow B_n) \Rightarrow A$$

   is a subrule of

   $$(\Gamma_1 \Rightarrow B_1), \ldots, (\Gamma_n \Rightarrow B_n) \Rightarrow A$$

2. Let R be the rule $(R_1, \ldots, R_n \Rightarrow A)$.
   $\Delta \vdash R$ is an abbreviation for $\Delta, R_1, \ldots R_n \vdash A$.

3. $\Delta \vdash_G R$ means that $\Delta \vdash R$ is provable.

Note: The notation $\Delta \vdash R$ was used already by S.H.(in [1]).

Lemma 1: $R \vdash_G R$.

Proof: By induction on the complexity of R.

Theorem 1:("cut elimination") If $\Gamma \vdash_G R$ and $\Delta, R \vdash_G R'$ then $\Gamma, \Delta \vdash_G R'$.
Proof: By double induction on the complexity of R and on the sum of the lengths of the proofs of the two given sequents. Details are standard, and we omit them.

Note: In the case of the pure calculus Theorem 1 presents a true cut-elimination theorem, implying, e.g. ,the sub-formula property. [1] The fact that this cut elimination is true also for the general GSH systems with non-logical rules is due to the fact that such rules can never introduce in an antecedent a rule that has

[1] It would have been better to call it the "sub-rule" property here. Unfortunately, this name has already another meaning!

not been already present at one of the antecedents of the premises. However, decidability and the subformula property are not gauranteed in the general case by this cut elimination, and Theorem 1 may have therefore little significance in case there is an infinite number of non-logical rules (as is the case when the set of basic rules is defined using schemes—recall that S.H.'s notion of a "rule" is a local one and is identical to what is usually taken as *an instance* of a rule of inference!). In what follows we shall have the opportunity to charaterize an important class of cases in which a ***significant*** version of cut-elimination does obtain.

Lemma 2: If R' is a subrule of R then $R \vdash_G R'$.
Proof: By induction on the complexity of R:

case a) : $R' = R$. Apply Lemma 1.

case b) :

$$R = (\Gamma_1 \Rightarrow B_1), \ldots, (\Gamma_n \Rightarrow B_n) \Rightarrow A$$
$$R' = (\Gamma'_1 \Rightarrow B_1), \ldots, (\Gamma'_n \Rightarrow B_n) \Rightarrow A$$

where every rule of Γ'_i is a subrule of a rule in Γ_i.

Suppose $\Gamma'_i = R'_{i,1}, R'_{i,2}, \ldots, R'_{i,m_i}$ (i=1,...,n). By the induction hypothesis and weakenings we have:

$$\Gamma_i \vdash_G R'_{i,j} \qquad (1 \leq i \leq n, 1 \leq j \leq m_i)$$

By repeatedly applying theorem 1 to these sequents and to $(\Gamma'_i \Rightarrow B_i), \Gamma'_i \vdash_G B_i$ (provable by lemma 1) we get, for each i, that:

$$(\Gamma'_i \Rightarrow B_i), \Gamma_i \vdash_G B_i \quad (1 \leq i \leq n)$$

Applying $(\Rightarrow\vdash)$ to these sequents we get $R \vdash_G R'$.□

We are now ready to prove:

Theorem 2: Γ ⊢ A in a S.H.-system iff $\Gamma \vdash_G$ A in the corresponding GSH.
(Note that "Γ ⊢ A" is a metaproposition for S.H. systems, while it is a formal assertion in the corresponding GSH. "$\Gamma \vdash_G$ A" is again a metaproposition for GSH).
Proof: One direction is immediate by the discussion which follows the definition of a GSH.

For the converse, suppose $\Gamma \vdash A$ in S.H.. Take a proof tree of A from Γ in S.H. and convert it into a tree of sequents as described at the same discussion.

We show now by induction that every sequent of the new tree is provable in the GSH. The case of the tip nodes is obvious. For the induction step we need two subcases:

1) The formula in the corresponding node of the original tree was obtained from its immediate premises by the *assumption* rule:

$$R = ((\Gamma_1 \Rightarrow A_1), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow A)$$

Then the sequent at that node of the new tree is of the form: $R, \Delta_1, \ldots, \Delta_n \vdash A$, while the sequents in the immediate ancestors are $\Delta_i, \Gamma_i' \vdash A_i$, where every rule in Γ_i' is a subrule of some rule in Γ_i. The last fact, lemma 2 and theorem 1 entail that $\Gamma_i, \Delta_i \vdash_G A_i$ $(1 \leq i \leq n)$,and from these sequents follows,by $(\Rightarrow\vdash)$,the provability of $R, \Delta_1, \ldots, \Delta_n \vdash A$.

2) A basic rule was used in the corresponding place in the original tree—similar.□

The limitation of the language to sequents of the form $\Gamma \vdash A$ is, to our opinion, artificial. Notationally, in fact, we have already abandoned it (and so did S.H. himself!). The next theorem "officially" removes this limitation:

Theorem 3: $\Gamma \vdash_G R$ iff $\Gamma \vdash R$ is provable in the system obtained by generalizing the concept of a sequent to allow *rules* on the right side of the $\vdash$ and by adding to GSH the inference-rule:

$$\frac{\Gamma, R_1, \ldots, R_n \vdash A}{\Gamma \vdash (R_1, \ldots, R_n \Rightarrow A)}$$

moreover: Theorem 1 is true for the extended system.
we leave the proof to the reader.□

3 GSH and intuitionistic implicational calculus

Theorem 3 above indicates that S.H.'s system is just a new (somewhat strange) formulation of the pure implicational intuitionistic calculus. We claim that there is no real difference between S.H.'s $\Rightarrow$ and the intuitionistic $\supset$. The following definition and proposition make this claim precise:

definition: Let RU be the set of S.H. rules, IMP— the set of sentences of the pure implicational calculus, $Int_{\rightarrow}$—the pure intuitionistic implicational calculus.
Define v:RU$\rightarrow$IMP, u:IMP$\rightarrow$RU as follows:

1. $v(\mathrm{A}) = \mathrm{A}$

2. $v(R_1, \ldots, R_n \Rightarrow \mathrm{A}) = v(R_1) \supset (v(R_2) \supset (\cdots \supset (v(R_n) \supset \mathrm{A}))\ldots)$

3. $u(p) = p$ (p atomic)

4. $u(A_1 \supset (A_2 \supset (\cdots (A_n \supset p))\ldots)) = u(A_1), \ldots, u(A_n) \Rightarrow p$ (p atomic)

Note that since$\{R_1, \ldots, R_n\}$ is a set, v is multiple-valued!

Theorem 4:

1. $u(v(R)) = R$

2. $\vdash_{Int_{\to}} v(u(A)) \equiv A$

3. If $R_1, \ldots, R_n \vdash_{S.H.} R$ then $v(R_1), \ldots, v(R_n) \vdash_{Int_{\to}} v(R)$.

4. If $A_1, \ldots, A_n \vdash_{Int_{\to}} B$ then $u(A_1), \ldots, u(A_n) \vdash_{S.H.} u(B)$.

proof: Easy.□

Theorem 4 and the formulation of GSH suggest a new Gentzen-type formulation of intuitionistic logic,in which the usual ($\supset\vdash$) rule is replaced by:

$$\frac{\Gamma_1 \vdash A_1 \cdots \Gamma_n \vdash A_n}{\Gamma_1, \ldots, \Gamma_n, (A_1 \supset (A_2 \supset \ldots \supset B)\ldots) \vdash B}$$

It is not difficult to show that this formulation is correct and that it admits cut-elimination. It might even seem more intuitive than the ordinary one. The trouble with it is that the new rule is not exactly a rule in the ordinary sense, not even a rule schema: It includes an infinite number of rule-schemes (for each n there is a corresponding rule with exactly n premises). A natural question to ask is therefore: What should be done in order to replace this infinite number by a finite number of rules with a fix number of premises? Well, the answer should allow us to derive every n-instance of the new "rule". This sould naturally be done by induction. For the base case we need the rule:

$$\frac{\Gamma \vdash A}{\Gamma, A \supset B \vdash B}$$

For the induction step we need a rule which will permit us to pass from:

$$\Gamma, A_n \supset (A_{n-1} \supset \cdots \supset (A_1 \supset B)\ldots) \vdash B$$

and from:

$$\Gamma \vdash A_{n+1}$$

to

$$\Gamma, A_{n+1} \supset (A_n \supset \cdots \supset (A_1 \supset B)\ldots) \vdash B$$

If we denote A_{n+1} by A, $A_n \supset (\cdots \supset (A_1 \supset B)\ldots)$ by C we get that the rule needed is:

$$\frac{\Gamma \vdash A \quad \Gamma, C \vdash B}{\Gamma, A \supset C \vdash B}$$

This is, of course, the ordinary ($\supset\vdash$) rule in Gentzen systems. (Since $B \vdash B$ is an axiom, the rule needed for the base case is also covered by this rule!)

4 On introduction and elimination rules

S.H. presents in his paper the following method for adding new n-ary operators to a language L:

Let $\Phi_1(A_1,\ldots,A_n),\ldots,\Phi_m(A_1,\ldots,A_n)$ $(m \geq 0)$ be a list of lists of rule-*schemes*. A new n-operator S, expressing the "common content" of $\Phi_1,\ldots,\Phi_m$, can then be introduced by the following rule schemes:

introduction rules:

$$\Phi_i(A_1,\ldots,A_n) \Rightarrow S(A_1,\ldots,A_n) \quad (1 \leq i \leq m)$$

elimination rule:

$$(\Phi_1 \Rightarrow A),(\Phi_2 \Rightarrow A),\ldots,(\Phi_m \Rightarrow A),(S(A_1,\ldots,A_n)) \Rightarrow A$$

(where $\Phi_i = \Phi_i(A_1,\ldots,A_n)$ $i = 1,\ldots,m$)

S.H. shows then that the validity of these rules is a necessary and sufficient condition for the following to be true:

(*)For all $A_1,\ldots,A_n$ and for every R:

$S(A_1,\ldots,A_n) \vdash R$ iff for all $1 \leq i \leq m$ $\Phi_i(A_1,\ldots,A_n) \vdash R$.

Examples:

Conjunction: Here $m = 1$, $\Phi_1(A,B) = \{A,B\}$. The rules are:

intro. : $A,B \Rightarrow A \wedge B$

elim. : $(A,B \Rightarrow C),(A \wedge B) \Rightarrow C$

disjunction: Here $m = 2$, $\Phi_1(A,B) = \{A\}$, $\Phi_2(A,B) = \{B\}$.

intro. : $A \Rightarrow A \vee B \qquad B \Rightarrow A \vee B$

elim. : $(A \Rightarrow C), (B \Rightarrow C), (A \vee B) \Rightarrow C$

implication: Here $m = 1$, $\Phi_1(A,B) = \{A \Rightarrow B\}$.

intro. : $(A \Rightarrow B) \Rightarrow A \supset B$

elim. : $((A \Rightarrow B) \Rightarrow C), (A \supset B) \Rightarrow C$

Turning now to our Gentzen-type version, suppose $\Phi_i = \{R_{i,1}, \ldots, R_{i,m_i}\}$ $(i = 1, \ldots, m)$. Then for each i, the i-introduction rule of S.H. is naturally translated into the basic rule:

$$\frac{\Delta_1 \vdash R_{i,1}(A_1,\ldots,A_n) \;\; \Delta_2 \vdash R_{i,2}(A_1,\ldots,A_n) \;\cdots\; \Delta_{m_i} \vdash R_{i,m_i}(A_1,\ldots,A_n)}{\Delta_1, \Delta_2, \ldots, \Delta_{m_i} \vdash S(A_1,\ldots,A_n)}$$

The elimination rule, on the other hand, becomes:

$$\frac{\Gamma_1, \Phi_1 \vdash A \;\; \Gamma_2, \Phi_2 \vdash A \;\cdots\; \Gamma_m, \Phi_m \vdash A \quad \Delta \vdash S(A_1,\ldots,A_n)}{\Delta, \Gamma_1, \Gamma_2, \ldots, \Gamma_m \vdash A}$$

In the context of Gentzen-type systems it is, however, much more natural to replace the natural-deduction-style " elimination" rules by *introduction* rules in the antecedent. This can be achieved by substituting $\{S(A_1,\ldots,A_n)\}$ for Δ at the above version of the elimination rules. Using the axioms we then obtain:

$$\frac{\Gamma_1, \Phi_1 \vdash A \;\; \Gamma_2, \Phi_2 \vdash A \;\cdots\; \Gamma_m, \Phi_m \vdash A}{S(A_1,\ldots,A_n), \Gamma_1, \Gamma_2, \ldots, \Gamma_m \vdash A}$$

Using cuts it is not difficult to show that the two formulations above are in fact equivalent.

Examples:

disjunction: The rules we get are:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad\qquad \frac{\Gamma_1, A \vdash C \quad \Gamma_2, B \vdash C}{\Gamma_1, \Gamma_2, A \vee B \vdash C}$$

conjunction: We get:

$$\frac{\Gamma_1 \vdash A \quad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

implication: We get:

$$\frac{\Gamma \vdash A \Rightarrow B}{\Gamma \vdash A \supset B} \quad \left(\equiv \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\right) \qquad \frac{\Gamma, A \Rightarrow B \vdash C}{\Gamma, A \supset B \vdash C}$$

Note that the rules we got for conjunction and disjunction are the usual Gentzen -type rules for them. This is true even for conjunction, the elimination rule for which looks somewhat unusual in S.H.'s formulation. For $\supset$ we just get the identity between $\Rightarrow$ and $\supset$. Again the corresponding rules are more intuitive than in the setting of S.H., but the fact that S.H. treats $\Rightarrow$ as basic forces us in this case to derive the usual rules for $\supset$ by the detour through $\Rightarrow$ which we made in the previous section.

Using standard methods, it is not difficult to show that any system which can be defined by gradually introducing new operators, using the above two kinds of introduction rules, admits cut-elimination. From this we can easily deduce S.H.'s result concerning the conservative character of his introduction and elimination rules (theorem 4.8 of [1]). S.H. himself used for this a normalization theorem which he proved in [2]. Normalization is, of course,the natural-deduction counterpart of cut-elimination. However, cut-elimination is in this case much easier to show and to use, since we are free from the complications caused by the notions of subrules and discharge functions.

Finally, a remark about S.H.'s characterization of the intuitionistic connectives and their definability power: Since, as we show above, there is no real difference between $\Rightarrow$ and the intuitionistic $\supset$, it is obvious that the "common content" of $\Phi_1, \ldots, \Phi_m$ is given by

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{m_i} v(R_{i,j}) \quad (\text{where } \Phi_i = \{R_{i,1}, \ldots, R_{i,m_i}\})$$

This is the real content of S.H.'s theorem about the definability within the intuitionistic propositional calculus of all the connectives which can gradually be defined using the method of intro. and elim. rules ($\perp$ corresponds to the case $m = 0$). I have some reservations, though, concerning this characterization, since it seems to me to force us,e.g., to regard negation as a derived rather than a primitive connective, since I see no way of directly defining it by intro. and elim. rules without introducing $\perp$ first.

References

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[2] P. Schroeder-Heister *Untersuchungen zur regellogischen Deutung von Aussagenverknüpfungen.* Dissertation, Bohn, 1981.