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by

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Abstract

By introducing 'convergence predicates' on the set of terms of the language of *CCS*, behavioural preorders based upon the idea of bisimulation but refining the notion of bisimulation equivalence and incorporating explicit treatments of 'divergence' are defined. The associated precongruence relations are characterized and complete axiomatizations of these relations for the sets of closed finite terms and regular expressions are presented.

Introduction

In developing his Calculus of Communicating Systems (*CCS*), Milner introduced [M1] the notion of ***observational equivalence.*** If $\mathbf{P}$ is the set of closed terms of the language of *CCS* and $\{\stackrel{u}{\Longrightarrow} \mid u \in Act\}$ a family of binary 'experiment' relations on $\mathbf{P}$, then defining a functional $\mathcal{H} : \mathcal{P}(\mathbf{P} \times \mathbf{P}) \to \mathcal{P}(\mathbf{P} \times \mathbf{P})$ by setting for $\mathcal{R} \in \mathcal{P}(\mathbf{P} \times \mathbf{P})$ and $P, Q \in \mathbf{P}$,

$P\mathcal{H}(\mathcal{R})Q$ if and only if for all $u \in Act$

(i) if $P \stackrel{u}{\Longrightarrow} P'$ then $\exists Q'.Q \stackrel{u}{\Longrightarrow} Q' \wedge P'\mathcal{R}Q'$,

(ii) if $Q \stackrel{u}{\Longrightarrow} Q'$ then $\exists P'.P \stackrel{u}{\Longrightarrow} P' \wedge P'\mathcal{R}Q'$,

observational equivalence is defined in [M1] to be the relation $\approx_\omega$ where $\approx_0 =_{df} \mathbf{P} \times \mathbf{P}$, for $n < \omega$, $\approx_{n+1} =_{df} \mathcal{H}(\approx_n)$ and $\approx_\omega =_{df} \bigcap_{n<\omega} \approx_n$.

A refinement of this definition was suggested by Park [P]. Observing that the functional $\mathcal{H}$ is monotonic, ***bisimulation equivalence***, $\approx$, may be defined to be its maximal fixed point. In addition to being a fixed point of $\mathcal{H}$, which $\approx_\omega$, in general, is not, $\approx$ also admits of a more manageable proof technique: for defining $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$ to be a ***bisimulation*** if and only if whenever $P\mathcal{R}Q$ then for all $u \in Act$ conditions (i) and (ii) above hold, then for all $P, Q \in \mathbf{P}$, $P \approx Q$ if and only if for some bisimulation $\mathcal{R}$, $P\mathcal{R}Q$.

One form of criticism of bisimulation equivalence, noted by Milner in [M2], concerns the treatment of the phenomenon of ***divergence.*** This term may be interpreted in more than one way. However it might be expected that the *CCS*-term $\mu X(\tau.X)$ would be regarded as divergent, since $\mu X(\tau.X) \stackrel{\tau}{\longrightarrow} \mu X(\tau.X)$. If so then the fact that $\mu X(\tau.X) \approx Nil$ may be interpreted as indicating that for some purposes bisimulation equivalence may not provide an entirely satisfactory formalization of the notion of the 'behavioural equivalence' of processes.

This paper is an investigation of some ways in which bisimulation equivalence may be refined to obtain behavioural preorders and equivalences which incorporate explicit treatments of divergence. These refinemnets are obtained by the introduction of a convergence predicate $\Downarrow$ and a family of parameterized convergence predicates $\{\Downarrow u \mid u \in Act\}$ defined over terms. The former was employed by Milner [M2] and the latter by Hennessy and Plotkin [HP]. Stirling [S] has also utilized such predicates in his work on modal logics for communicating systems. Abramsky [A] obtained a charaterization in terms of 'indistinguishability by tests' of the preorder of [M2] and a preorder closely related to that of [HP].

After a short preliminary section setting out notation used throughout the paper, the preorders and their associated precongruences are introduced. It seems to be the case that to obtain a satisfactory treatment using only the single convergence predicate $\Downarrow$, it is necessary to use the derived transition relations $\stackrel{a}{\Longrightarrow}_0 =_{df} (\stackrel{\tau}{\longrightarrow})^* \stackrel{a}{\longrightarrow}$ of [M2]. Using the family of convergence predicates $\{\Downarrow u \mid u \in Act\}$ however, it is possible to use either the above relations or the 'standard' derived transition relations, $\stackrel{a}{\Longrightarrow} =_{df} (\stackrel{\tau}{\longrightarrow})^* \stackrel{a}{\longrightarrow} (\stackrel{\tau}{\longrightarrow})^*$.

(Some of the results in the paper illuminate some observations of Abramsky [A, section 6] on this subject.) In the third section complete axiomatizations of the precongruence relations for the set of finite closed terms are presented. The final section is devoted to proofs of the existence of complete axiomatizations of the precongruence relations for the set of 'finite-state' (or 'regular') expressions.

1. Preliminaries

This preliminary section sets out some notations used throughout the paper. For explanations of their intended interpretations see [M1].

Let A be a set of (atomic action) symbols not containing τ or ε and let $Act = A \cup \{\tau\}$. It is assumed that there is a bijection $\bar{\cdot} : A \to A$ which is its own inverse. The letters a,b range over A, u over $Act \cup \{\varepsilon\}$ and f over the set RF of *renaming functions*, i.e. of those $f : Act \to Act$ such that $f(\bar{a}) = \overline{f(a)}$, $f(\tau) = \tau$ and $f(a) \neq \tau$. The language of CCS consists of an infinite set Var of variables ranged over by X, Y, constant symbols $\mathbf{0}$ and $\perp$, unary function symbols $u.$ $(u \in Act)$, $\backslash a$ $(a \in A)$, $[f]$ $(f \in RF)$ and μX $(X \in Var)$, and binary function symbols $+$ and $|$. The set $\mathbf{E}$ of CCS-expressions ranged over by E, F is the set given by the following BNF-style description:

$$E \ ::= \ X \mid \mathbf{0} \mid \perp \mid u.E \mid E + F \mid E|F \mid E\backslash a \mid E[f] \mid \mu XE.$$

The syntax here differs from that in [M4] only in the use of $\mathbf{0}$ in place of Nil, the use of μ in place of fix, and the presence of the constant symbol $\perp$ which is intended to represent a totally undefined process. If $\overline{X} = \langle X_0, \ldots, X_m \rangle$ then $\mu\overline{X}E$ abbreviates $\mu X_0 \ldots \mu X_m E$. The usual notions of free and bound occurrences of variables in expressions are assumed and $fv(E)$ denotes the set of variables which occur free in E. $\mathbf{P}$ is the set of all closed expressions, or agents, ranged over by P, Q and R.

An operational semantics is given for the language by means of families of transition relations $\{\xrightarrow{u} \mid u \in Act\}$ and of convergence predicates $\{\Downarrow\} \cup \{\Downarrow a \mid a \in A\}$.

$\{\xrightarrow{u} \mid u \in Act\}$ is the smallest family of binary relations on $\mathbf{E}$ such that:
(a) $u.E \xrightarrow{u} E$,
(b) if $E \xrightarrow{u} E'$ then $E + F \xrightarrow{u} E'$, $F + E \xrightarrow{u} E'$, $E|F \xrightarrow{u} E'|F$, $F|E \xrightarrow{u} F|E'$, and $E[f] \xrightarrow{f(u)} E'[f]$,
(c) if $E \xrightarrow{u} E'$ and $u \neq a$ then $E\backslash a \xrightarrow{u} E'\backslash a$,
(d) if $E \xrightarrow{a} E'$ and $F \xrightarrow{\bar{a}} F'$ then $E|F \xrightarrow{\tau} E'|F'$,
(e) if $E\{\mu XE/X\} \xrightarrow{u} F$ then $\mu XE \xrightarrow{u} F$, where if $\overline{H} = \langle H_i \rangle$ and $\overline{Y} = \langle Y_i \rangle$ are lists of expressions and variables respectively then $G\{\overline{H}/\overline{Y}\}$ is the expression obtained from G by simultaneously substituting H_i for Y_i for each i, with change of bound variables if

necessary.

The derived transition relations $\stackrel{u}{\Longrightarrow}$ ($u \in Act$) and $\stackrel{\varepsilon}{\Longrightarrow}$ are defined as follows: $\stackrel{u}{\Longrightarrow} =_{df} (\stackrel{\tau}{\longrightarrow})^* \stackrel{u}{\longrightarrow} (\stackrel{\tau}{\longrightarrow})^*$ and $\stackrel{\varepsilon}{\Longrightarrow} =_{df} (\stackrel{\tau}{\longrightarrow})^*$, where $(\stackrel{\tau}{\longrightarrow})^*$ is the reflexive and transitive closure of $\stackrel{\tau}{\longrightarrow}$ and juxtaposition of relations denotes forward relational composition. Note that $\stackrel{\tau}{\Longrightarrow}$ is the transitive closure of $\stackrel{\tau}{\longrightarrow}$. The following derived transition relations will also be of importance: $\stackrel{\tau}{\Longrightarrow}_0 =_{df} \stackrel{\tau}{\Longrightarrow}$, $\stackrel{\varepsilon}{\Longrightarrow}_0 =_{df} \stackrel{\varepsilon}{\Longrightarrow}$ and $\stackrel{a}{\Longrightarrow}_0 =_{df} (\stackrel{\tau}{\longrightarrow})^* \stackrel{a}{\longrightarrow}$ ($a \in A$).

The convergence predicates, and their corresponding divergence predicates, are defined as follows.

$\downarrow$ is the smallest relation on $\mathbf{E}$ such that:
(a) $X\downarrow$, $\mathbf{0}\downarrow$ and $u.E\downarrow$,
(b) if $E\downarrow$ and $F\downarrow$ then $E+F\downarrow$, $E|F\downarrow$, $E\backslash a\downarrow$ and $E[f]\downarrow$,
(c) if $E\{\mu XE/X\}\downarrow$ then $\mu XE\downarrow$.

$\uparrow$ is the relation complementary to $\downarrow$. $\Downarrow$ is the least relation on $\mathbf{E}$ such that if $E\downarrow$ and whenever $E \stackrel{\tau}{\longrightarrow} F$, $F\Downarrow$, then $E\Downarrow$. $\Uparrow$ is the relation complementary to $\Downarrow$. Also $\Downarrow\tau =_{df} \Downarrow$, $\Downarrow\varepsilon =_{df} \Downarrow$ and $E\Downarrow a$ iff $E\Downarrow$ and whenever $E \stackrel{a}{\Longrightarrow} F$, $F\Downarrow$. $\Uparrow u$ is the relation complementary to $\Downarrow u$. It is convenient to introduce $\hat{}: Act \to A \cup \{\varepsilon\}$ by $\hat{a} = a$ and $\hat{\tau} = \varepsilon$. Finally, if $\mathcal{R}$ is a binary relation on $\mathbf{P}$, then the *pointwise extension* to $\mathbf{E}$ of $\mathcal{R}$ is the relation (also denoted) $\mathcal{R}$ defined on $\mathbf{E}$ by setting $E\mathcal{R}F$ iff whenever $\overline{X}$ is a list of variables containing all those occurring free in either E or F, then for all $\overline{P} \in \mathbf{P}$, $E\{\overline{P}/\overline{X}\}\mathcal{R}F\{\overline{P}/\overline{X}\}$.

2. Preorders and precongruences

In seeking to refine the notion of bisimulation equivalence to take account of the presence of $\perp$ and the existence of the family of convergence predicates $\{\Downarrow u \mid u \in Act\}$, there are several possibilities. Rather than consider them all at once it seems preferable to concentrate initially on one. The others will be considered later.

Let Rel be the set of all binary relations on $\mathbf{P}$ and define $\mathcal{F} : Rel \to Rel$ by setting for $\mathcal{R} \in Rel$:

$P\,\mathcal{F}(\mathcal{R})\,Q$ iff
(i) $\forall u.$ if $P \stackrel{u}{\longrightarrow} P'$ then $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge P'\mathcal{R}Q'$,
(ii) $\forall u.$ if $P\Downarrow u$ then
(a) $Q\Downarrow u$,
(b) if $Q \stackrel{u}{\longrightarrow} Q'$ then $\exists P'.P \stackrel{\hat{u}}{\Longrightarrow} P' \wedge P'\mathcal{R}Q'$.

It is straight forward to verify that $\mathcal{F}$ is a monotonic functional. Hence $\mathcal{F}$ has a maximal fixed point $\sqsubseteq = \bigcup\{\mathcal{R} \mid \mathcal{R} \subseteq \mathcal{F}(\mathcal{R})\}$. Moreover $\sqsubseteq$ is a preorder, i.e. is reflexive and

transitive. A relation $\mathcal{R}$ such that $\mathcal{R} \subseteq \mathcal{F}(\mathcal{R})$ is called a *prebisimulation* and $\sqsubseteq$ a *prebisimulation ordering.* As described in the following lemma $\sqsubseteq$ is preserved by many of the operators of the calculus.

<u>Lemma 1</u> If $P \sqsubseteq Q$ then
(a) $u.P \sqsubseteq u.Q$,
(b) $P|R \sqsubseteq Q|R$,
(c) $P\backslash a \sqsubseteq Q\backslash a$,
(d) $P[f] \sqsubseteq Q[f]$.

Proof: (a) Let $\mathcal{R}$ be a prebisimulation such that $P\mathcal{R}Q$. Then $\mathcal{S} = \{(u.P, u.Q)\} \cup \mathcal{R}$ is a prebisimulation.

(b) Let $\mathcal{R} = \{(P|R, Q|R) \mid P \sqsubseteq Q\}$. It suffices to prove that $\mathcal{R}$ is a prebisimulation.

(i) Suppose $P|R \xrightarrow{u} P'|R'$.
(1) $P \xrightarrow{u} P' \wedge R' \equiv R$.
Then $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$, so $Q|R \stackrel{\hat{u}}{\Longrightarrow} Q'|R$.
(2) $R \xrightarrow{u} R' \wedge P' \equiv P$.
Then $Q|R \xrightarrow{u} Q|R'$.
(3) $u = \tau \wedge \exists a.P \xrightarrow{a} P' \wedge R \xrightarrow{\overline{a}} R'$.
Then $\exists Q'.Q \stackrel{a}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$, so $Q|R \stackrel{u}{\Longrightarrow} Q'|R'$.

(ii) Suppose $P|R\Downarrow u$. Then $P\Downarrow u$ and $R\Downarrow u$.
(b) Suppose $Q|R \xrightarrow{u} Q'|R'$.
(1) $Q \xrightarrow{u} Q' \wedge R' \equiv R$.
Then since $P\Downarrow u$, $\exists P'.P \stackrel{\hat{u}}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$, so $P|R \stackrel{\hat{u}}{\Longrightarrow} P'|R$.
(2) $R \xrightarrow{u} R' \wedge P' \equiv P$.
Then $P|R \xrightarrow{u} P\mathcal{R}'$.
(3) $u = \tau \wedge \exists a.Q \xrightarrow{a} Q' \wedge R \xrightarrow{\overline{a}} R'$.
Then $P \Downarrow a$, for if not then $\exists P.P \stackrel{a}{\Longrightarrow} P' \wedge P'\Uparrow$, when $P|R \stackrel{\tau}{\Longrightarrow} P'|R'$ and $P'|R'\Uparrow$, a contradiction. Hence $\exists P'.P \stackrel{a}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$, so $P|R \stackrel{\tau}{\Longrightarrow} P'|R'$.
(a) Suppose $Q|R\Uparrow u$.
(1) Suppose $u = \tau$. Then
either (A) $\exists Q', R'.Q|R \stackrel{\varepsilon}{\Longrightarrow} Q'|R' \wedge Q'|R'\Uparrow$
or (B) (A) fails and $Q|R(\xrightarrow{\tau})^{\omega}$.
Now since $P|R\downarrow$, $Q|R\downarrow$, so in (A), 'ε' may be replaced by 'τ'. Suppose $Q|R \stackrel{\tau}{\Longrightarrow} Q'|R' \wedge Q'|R'\uparrow$. Then $\exists k$, $0 < k < \omega$ and $\exists\langle Q_i\rangle_{i\leq k}$, $\langle R_i\rangle_{i\leq k}$ and $\langle a_i\rangle_{i<k}$ such that $Q_0 \equiv Q$, $R_0 \equiv R$ and for $i < k$, $Q_i \stackrel{a_i}{\Longrightarrow} Q_{i+1}$, $R_i \stackrel{\overline{a_i}}{\Longrightarrow} R_{i+1}$, $Q_k \equiv Q'$ and $R_k \equiv R'$.. Note that $k > 0$ since $Q\Downarrow$ and $R\Downarrow$. Then

(α) There is $\langle P_i\rangle_{i\leq k}$ such that $P_0 \equiv P$ and for $i < k$, $P_i \stackrel{a_i}{\Longrightarrow} P_{i+1}$ and for $i \leq k$, $P_i \sqsubseteq Q_i$.

This is established as follows. Since $P|R\Downarrow$, $P\downarrow a_1$. Hence since $Q \stackrel{a_1}{\Longrightarrow} Q_1$, $\exists P_1.P \stackrel{a_1}{\Longrightarrow} P_1 \wedge P_1 \sqsubseteq Q_1$. The sequences are constructed by iterating this argument.

But $P_k|R_k\Downarrow$ so $Q_k|R_k\downarrow$, a contradiction.
Suppose (A) fails. Then there are sequences $\langle Q_i\rangle_{i<\omega}$, $\langle R_i\rangle_{i<\omega}$ and $\langle a_i\rangle_{i<\omega}$ such that $Q_0 \equiv Q$, $R_0 \equiv R$ and for $i<\omega$, $Q_i \stackrel{a_i}{\Longrightarrow} Q_{i+1}$, $R_i \stackrel{\overline{a_i}}{\Longrightarrow} R_{i+1}$, $Q_i\Downarrow$ and $R_i\Downarrow$. Then

(β) There is $\langle P_i\rangle_{i<\omega}$ such that $P_0 \equiv P$ and for $i<\omega$, $P_i \stackrel{a_i}{\Longrightarrow} P_{i+1}$ and $P_i \sqsubseteq Q_i$.
This is established by constructing the sequence as in the proof of (α) above. Having constructed $\langle P_i\rangle_{i<j}$ to define P_j note that $P|R \stackrel{\tau}{\Longrightarrow} P_{j-1}|R_{j-1}$ and so since $R_{j-1} \stackrel{\overline{a_{j-1}}}{\Longrightarrow} R_j$, $P_{j-1}\Downarrow a_{j-1}$. Hence since $Q_{j-1} \stackrel{a_{j-1}}{\Longrightarrow} Q_j$, $\exists P_j.P_{j-1} \stackrel{a_j}{\Longrightarrow} P_j \wedge P_j \sqsubseteq Q_j$.

Hence $P|R(\stackrel{\tau}{\longrightarrow})^\omega$, a contradiction.

(2) Now suppose $u = a$. By the above argument $Q|R\Downarrow$. Thus $\exists Q', R'.Q|R \stackrel{a}{\Longrightarrow} Q'|R' \wedge Q'|R'\Uparrow$. Then there are $k<\omega$ and $\langle P_i\rangle_{i\le k}$, $\langle R_i\rangle_{i\le k}$ and $\langle a_i\rangle_{i<k}$ such that $Q_0 \equiv Q$, $R_0 \equiv R$, for $i<k$, $Q_i \stackrel{a_i}{\Longrightarrow} Q_{i+1}$ and $R_i \stackrel{\overline{a_i}}{\Longrightarrow} R_{i+1}$, and $Q_k \stackrel{a}{\Longrightarrow} Q'$ and $R_k \stackrel{\epsilon}{\Longrightarrow} R'$, or $R_k \stackrel{a}{\Longrightarrow} R'$ and $Q_k \stackrel{\epsilon}{\Longrightarrow} Q'$. Then

(γ) There is $\langle P_i\rangle_{i\le k}$ such that $P_0 \equiv P$ and for $i<k$, $P_i \stackrel{a_i}{\Longrightarrow} P_{i+1}$ and for $i\le k$, $P_i \sqsubseteq Q_i$.
This sequence is constructed by arguments similar to those above.

Since $P|R \stackrel{\tau}{\Longrightarrow} P_k|R_k$ and $P|R\Downarrow a$, $P_k\Downarrow a$ and $\neg\exists R''.R_k \stackrel{a}{\Longrightarrow} R''$. Hence $Q_k \stackrel{a}{\Longrightarrow} Q'$ and so since $P_k \sqsubseteq Q_k$, $\exists P'.P_k \stackrel{a}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$. Then $P|R \stackrel{a}{\Longrightarrow} P'|R'$ with $P' \sqsubseteq Q'$, a contradiction.

This completes the proof of (2), hence of (a) and hence of (ii). □(b)

(c) Let $\mathcal{R} = \{(P\backslash a, Q\backslash a) \mid P \sqsubseteq Q\}$. Then $\mathcal{R}$ is a prebisimulation.
(i) If $P\backslash a \stackrel{u}{\longrightarrow} R$ then $\exists P'.P \stackrel{u}{\longrightarrow} P' \wedge R \equiv P'\backslash a$ and $u \ne a$. So $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$, so $Q\backslash a \stackrel{\hat{u}}{\Longrightarrow} Q'\backslash a$.
(ii) Suppose $P\backslash a\Downarrow u$. If $u = a$ then certainly $Q\backslash a\Downarrow u$ since $Q\Downarrow$ and $\neg\exists Q''.Q \stackrel{a}{\Longrightarrow} Q''$. If $u \ne a$ then since $P\Downarrow u$, $Q\Downarrow u$ and so $Q\backslash a\Downarrow u$. Suppose $Q\backslash a \stackrel{u}{\longrightarrow} R$. Then $u \ne a$ and $\exists Q'.Q \stackrel{u}{\longrightarrow} Q' \wedge R \equiv Q'\backslash a$. So $\exists P'.P \stackrel{\hat{u}}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$, so $P\backslash a \sqsubseteq P'\backslash a$.

(d) Let $\mathcal{R} = \{(P[f], Q[f]) \mid P \sqsubseteq Q\}$. Then by an argument similar to that in the proof of (c) above it may be established that $\mathcal{R}$ is a prebisimulation. For (ii), if $P[f]\Downarrow a$ then for all b such that $f(b) = a$, $P\Downarrow b$, so $Q\Downarrow b$ for all such b and hence $Q[f]\Downarrow a$. □

$\sqsubseteq$ is not however preserved by $+$, as e.g. $\tau.\mathbf{0} \sqsubseteq \mathbf{0}$ but $\tau.\mathbf{0} + a.\mathbf{0} \not\sqsubseteq \mathbf{0} + a.\mathbf{0}$. Let $\sqsubseteq^c$ be the largest relation included in $\sqsubseteq$ which is a precongruence relation with respect to all the operators of the calculus, including the fixed point operators in the sense that if $\sqsubseteq^c$

is extended pointwise to $\mathbf{E}$, then if $E \sqsubseteq^c F$ then $\mu XE \sqsubseteq^c \mu XF$.

By analogy with one of the characterizations of bisimulation congruence, the largest congruence relation included in bisimulation equivalence, it might be expected that $\sqsubseteq^c = \sqsubseteq^+$ where $\sqsubseteq^+$ is defined by setting $P \sqsubseteq^+ Q$ iff $\forall R. P + R \sqsubseteq Q + R$. To prove that this is indeed the case, and to obtain another characterization useful in demonstrating the existence of sound and complete axiomatizations of $\sqsubseteq^c$ on certain subclasses of $\mathbf{E}$, it is convenient to introduce the relation $\sqsubseteq^*$ defined as follows:

$P \sqsubseteq^* Q$ iff
(i) $\forall a.$ if $P \xrightarrow{a} P'$ then $\exists Q'. Q \stackrel{a}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$,
(ii) if $P \stackrel{\tau}{\Longrightarrow} P'$ then
  (a) if $P' \Downarrow$ then $\exists Q'. Q \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$,
  (b) if $P' \Uparrow$ then $\exists Q'. Q \stackrel{\epsilon}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$,
(iii) $\forall u.$ if $P \Downarrow u$ then
  (a) $Q \Downarrow u$,
  (b) if $Q \xrightarrow{u} Q'$ then $\exists P'. P \stackrel{u}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$.

Let $\simeq$, $\simeq^c$, $\simeq^+$, $\simeq^*$ be the equivalence relations generated by $\sqsubseteq$, $\sqsubseteq^c$, $\sqsubseteq^+$, $\sqsubseteq^*$ respectively. The following observations are useful.

<u>Lemma 2</u> (a) If $P \Uparrow$ then $\forall Q. P \sqsubseteq P + Q$.
(b) If $P \Uparrow$, $P \sqsubseteq Q$ and $\neg \exists P'. P \stackrel{\tau}{\Longrightarrow} P' \wedge P' \Downarrow$ then $\forall R. P \sqsubseteq Q + R$.
(c) $P + \mathbf{0} \simeq^* P$.
(d) $\tau.(P + \bot) \simeq^+ P + \bot$.
(e) If $\mu \overline{X} E$ is closed then $\mu \overline{X} E \simeq^+ E\{\mu \overline{X} E / \overline{X}\}$.

Proof: (a) Let $\mathcal{R} = \{(P, P + Q) \mid P \Uparrow\} \cup Id_{\mathbf{P}}$. Then $\mathcal{R}$ is a prebismulation.

(b) Let $\mathcal{R}$ be a prebisimulation such that $P \mathcal{R} Q$. Then $\mathcal{S} = \mathcal{R} \cup \{(P', Q + R) \mid P \stackrel{\epsilon}{\Longrightarrow} P' \wedge P' \mathcal{R} Q\}$ is a prebisimulation.

(c) Trivial.

(d) (1) $\tau.(P + \bot) + R \sqsubseteq P + \bot + R$.
If $\tau.(P + \bot) + R \xrightarrow{u} R'$ then either $R \xrightarrow{u} R'$ when $P + \bot + R \xrightarrow{u} R'$, or $u = \tau$ and $R' \equiv P + \bot$. Then $P + \bot + R \stackrel{\epsilon}{\Longrightarrow} P + \bot + R$ and $P + \bot \sqsubseteq P + \bot + R$ by (a) above. Since $\tau.(P + \bot) + R \Uparrow$, (1) follows.
(2) $P + \bot + R \sqsubseteq \tau.(P + \bot + R)$.
If $P + \bot + R \xrightarrow{u} R'$ then $\tau.(P + \bot) + R \stackrel{u}{\Longrightarrow} R'$. Since $P + \bot + R \Uparrow$, (2) follows.

(e) Immediate from the definitions. □

In [M4] it is established that if $=$ denotes the relation of bisimulation congruence then $P = Q$ iff (a) $\forall u$. if $P \xrightarrow{u} P'$ then $\exists Q'.Q \stackrel{u}{\Longrightarrow} Q' \wedge P' \approx Q'$, and (b) $\forall u$. if $Q \xrightarrow{u} Q'$ then $\exists P'.P \stackrel{u}{\Longrightarrow} P' \wedge P' \approx Q'$. The following result suggests that $\sqsubseteq^*$ may be the appropriate analogue of this characterisation for $\sqsubseteq$.

<u>Lemma 3</u> $P \sqsubseteq^* Q$ iff $P \sqsubseteq^+ Q$.

Proof: In this proof it is assumed that A is infinite.

($\Rightarrow$) Suppose that $\neg P \sqsubseteq^* Q$. Then there is R such that $\neg P + R \sqsubseteq Q + R$. To see this consider the various clauses in the definition of $\sqsubseteq^*$.
(i) $\exists a.\exists P'.P \xrightarrow{a} P' \wedge \forall Q'$. if $Q \stackrel{a}{\Longrightarrow} Q'$ then $\neg P' \sqsubseteq Q'$.
Take $R \equiv \mathbf{0}$.
(ii) $P \stackrel{\tau}{\Longrightarrow} P'$.
(a) $P' \Downarrow \wedge \forall Q'$ if $Q' \stackrel{\tau}{\Longrightarrow} Q'$ then $\neg P' \sqsubseteq Q'$.
Take $R \equiv a.\mathbf{0}$ where $a \notin sort(P) \cup sort(Q)$. Then if $Q + R \stackrel{\varepsilon}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ then $Q' \equiv Q + R$. But $P' \Downarrow a$ since $P' \Downarrow$ and $a \notin sort(P)$, so $Q + R \Downarrow a$, and since $\exists R'.Q + R \xrightarrow{a} R'$, $\exists P''.P' \stackrel{a}{\Longrightarrow} P''$, a contradiction.
(b) $P' \Uparrow \wedge \forall Q'$. if $Q \stackrel{\varepsilon}{\Longrightarrow} Q'$ then $\neg P' \sqsubseteq Q'$.
Take $R \equiv \mathbf{0}$.
(iii) $P \Downarrow u$.
(a) $Q \Uparrow u$.
Take $R \equiv \mathbf{0}$.
(b) $Q \Downarrow u \wedge \exists Q'.Q \xrightarrow{u} Q' \wedge \forall P'$. if $P \stackrel{u}{\Longrightarrow} P'$ then $\neg P' \sqsubseteq Q'$.
Take $R \equiv a.\mathbf{0}$ where $a \notin sort(P) \cup sort(Q)$. Then $P + R \Downarrow u$ so since $Q + R \xrightarrow{u} Q'$, if $P + R \sqsubseteq Q + R$ then $\exists P'.P + R \stackrel{\hat{u}}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$. But then $u = \tau$ and $P' \equiv P + R$. But $P + R \xrightarrow{a} \mathbf{0}$ while $\neg\exists Q''.Q' \stackrel{a}{\Longrightarrow} Q''$, a contradiction.

($\Leftarrow$) Suppose that $P \sqsubseteq^* Q$ and that R is arbitrary.
(i) Suppose $P + R \xrightarrow{a} P'$. If $R \xrightarrow{a} P'$ then $Q + R \xrightarrow{a} P'$. If $P \xrightarrow{a} P'$ then $\exists Q'.Q \stackrel{a}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ so $Q + R' \stackrel{a}{\Longrightarrow} Q'$. Suppose $P + R \xrightarrow{\tau} P'$. Again if $R \xrightarrow{\tau} P'$ then $Q + R \xrightarrow{\tau} P'$. Suppose $P \xrightarrow{\tau} P'$. If $P' \Downarrow$ then $\exists Q'.Q \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ so $Q + R \stackrel{\tau}{\Longrightarrow} Q'$. Suppose $P' \Uparrow$.
(1) $\neg\exists P''.P' \stackrel{\tau}{\Longrightarrow} P'' \wedge P'' \Downarrow$.
If $\exists Q'.Q \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ then $Q + R \stackrel{\tau}{\Longrightarrow} Q'$. Otherwise $P' \sqsubseteq Q$ and by Lemma 2(b), $P' \sqsubseteq Q + R$.
(2) $\exists P''.P' \stackrel{\tau}{\Longrightarrow} P'' \wedge P'' \Downarrow$.
If $\exists Q'.Q \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ then $Q + R \stackrel{\tau}{\Longrightarrow} Q'$. If not then $P' \sqsubseteq Q$. Then since $P \stackrel{\tau}{\Longrightarrow} P''$, $\exists Q'.Q \stackrel{\tau}{\Longrightarrow} Q' \wedge P'' \sqsubseteq Q'$. Then $Q + R \stackrel{\tau}{\Longrightarrow} Q'$.
(ii) If $P + R \Downarrow u$ then $Q + R \Downarrow u$, and if $Q + R \xrightarrow{u} Q'$ then either $R \xrightarrow{u} Q'$ when $P + R \xrightarrow{u} Q'$, or $Q \xrightarrow{u} Q'$ when $\exists P'.P \stackrel{u}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$, and so $P + R \stackrel{u}{\Longrightarrow} P'$. $\square$

The next two lemmas establish that (the pointwise extension to $\mathbf{E}$ of) $\sqsubseteq^*$ is a pre-

congruence relation with respect to all the operators of the calculus.

<u>Lemma 4</u> If $P \sqsubseteq^* Q$ then
(a) $u.P \sqsubseteq^* u.Q$,
(b) $P + R \sqsubseteq^* Q + R$,
(c) $P|R \sqsubseteq^* Q|R$,
(d) $P\backslash a \sqsubseteq^* Q\backslash a$,
(e) $P[f] \sqsubseteq^* Q[f]$.

Proof: For (a) and (b) it is convenient to use Lemma 3.

(a) It is straight forward to show that for any R, $u.P + R \sqsubseteq u.Q + R$.

(b) It is easy to show that $\sqsubseteq$ is associative. Then for any R and R', $(P + R) + R' \sqsubseteq P + (R + R') \sqsubseteq Q + (R + R') \sqsubseteq (Q + R) + R'$.

(c) (i) If $P|R \xrightarrow{a} P'|R$ then $\exists Q.Q \stackrel{a}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ when $Q|R \stackrel{a}{\Longrightarrow} Q'|R'$ and $P'|R \sqsubseteq Q'|R$, while if $P|R \xrightarrow{a} P|R'$ then $Q|R \xrightarrow{a} Q|R'$ and $P|R' \sqsubseteq Q|R'$.
(ii) Suppose $P|R \stackrel{\tau}{\Longrightarrow} P'|R'$. Then there are k with $0 < k < \omega$ and $\langle P_i\rangle_{i\leq k}$, $\langle R_i\rangle_{i\leq k}$ such that $P_0 \equiv P$, $R_0 \equiv R$, $P_k \equiv P'$, $R_k \equiv R'$, and for $i < k$, $P_i|R_i \xrightarrow{\tau} P_{i+1}|R_{i+1}$. Then it is routine to establish, by considering the possible cases, that there is Q' such that either $P'|R'\Downarrow$ and $Q|R \stackrel{\tau}{\Longrightarrow} Q'|R'$, or $P'|R'\Uparrow$ and $Q|R \stackrel{\epsilon}{\Longrightarrow} Q'|R'$.
(iii) Suppose $P|R\Downarrow u$. Then by arguments very similar to those in the proof of Lemma 1(b) it is possible to establish that $Q|R\Downarrow u$ and if $Q|R \xrightarrow{u} Q'|R'$ then $\exists P'.P|R \stackrel{\tau}{\Longrightarrow} P'|R'$. The most significant point is that if $u = \tau$ then $P|R \stackrel{\tau}{\Longrightarrow} P'|R'$, and not simply $P|R \stackrel{\epsilon}{\Longrightarrow} P'|R'$.

(d) (i) If $P\backslash a \xrightarrow{b} P'\backslash a$ where $P \xrightarrow{b} P'$ then $Q\backslash a \stackrel{b}{\Longrightarrow} Q'\backslash a$ where $Q \stackrel{b}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ so $P'\backslash a \sqsubseteq Q'\backslash a$.
(ii) If $P\backslash a \stackrel{\tau}{\Longrightarrow} P'\backslash a$ where $P \stackrel{\tau}{\Longrightarrow} P'$ then either $P'\Downarrow$ and $\exists Q'.Q \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ when $Q\backslash a \stackrel{\tau}{\Longrightarrow} Q'\backslash a$ and $P'\backslash a \sqsubseteq Q'\backslash a$, or $P'\Uparrow$ and $\exists Q'.Q \stackrel{\epsilon}{\Longrightarrow} Q' \wedge P' \sqsubseteq Q'$ when $Q\backslash a \stackrel{\epsilon}{\Longrightarrow} Q'\backslash a$ and $P'\backslash a \sqsubseteq Q'\backslash a$.
(iii) Suppose $P\backslash a\Downarrow u$. If $u = a$ then certainly $P\backslash a\Downarrow u$ since $Q\Downarrow$ and $\neg\exists Q'.Q\backslash a \stackrel{a}{\Longrightarrow} Q'$. If $u \neq a$ then since $P\Downarrow u$, $Q\Downarrow u$ and so $Q\backslash a\Downarrow u$. If $Q\backslash a \xrightarrow{u} Q'\backslash a$ where $Q \xrightarrow{u} Q'$, then since $P'\Downarrow u$, $\exists P'.P \stackrel{u}{\Longrightarrow} P' \wedge P' \sqsubseteq Q'$, so $P\backslash a \stackrel{u}{\Longrightarrow} P'\backslash a$ and $P'\backslash a \sqsubseteq Q'\backslash a$.

(e) The proof is similar to that of (d) above. □

<u>Lemma 5</u> If $E \sqsubseteq^* F$ and $\mu\overline{X}E$ and $\mu\overline{X}F$ are closed then $\mu\overline{X}E \sqsubseteq^* \mu\overline{X}F$.

Proof: The proof is given only for the case when $\overline{X}$ consists of a single variable. The proof is entirely similar in the general case. It is convenient to introduce the following definition.

$\mathcal{R}$ is a *prebisimulation up to* $\sqsubseteq$ iff $\sqsubseteq \mathcal{R} \sqsubseteq$ is a prebisimulation.

It is straightforward to verify that if $\mathcal{R}$ is a prebisimulation up to $\sqsubseteq$ and $P\mathcal{R}Q$ then $P \sqsubseteq Q$. Futhermore the following assertion, whose proof is omitted, is easily established.

<u>Proposition</u> Suppose that $\mathcal{R}$ is such that whenever $P\mathcal{R}Q$ then
(i) $\forall u.$ if $P \xrightarrow{u} P'$ then $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$,
(ii) $\forall u.$ if $P \Downarrow u$ then
  (a) $Q \Downarrow u$
  (b) if $Q \xrightarrow{u} Q'$ then $\exists P' P \stackrel{\hat{u}}{\Longrightarrow} P' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$.
Then $\mathcal{R}$ is a prebisimulation up to $\sqsubseteq$.

Set $A \equiv \mu XE$ and $B \equiv \mu XF$. Then setting $\mathcal{R} = \{(G\{A/X\}, G\{B/X\}) \mid fv(G) \subseteq \{X\}\}$, it suffices to show that $\mathcal{R}$ is a prebisimulation up to $\sqsubseteq$, for then for any agent R, $(X+R)\{A/X\} \sqsubseteq (X+R)\{B/X\}$, i.e. $A+R \sqsubseteq B+R$, so $A \sqsubseteq^+ B$ and hence $A \sqsubseteq^* B$ by Lemma 3. It does not seem to be possible to establish that $\mathcal{R}$ is a prebisimulation up to $\sqsubseteq$ in a 'direct' way. However the following sequence of assertions suffices to establish the result. In what follows $G(A)$ and $G(B)$ abbreviate $G\{A/X\}$ and $G\{B/X\}$ respectively.

<u>Lemma 5.1</u> If $G(A) \xrightarrow{a} P$ then $\exists Q.G(B) \stackrel{a}{\Longrightarrow} Q \wedge P \sqsubseteq \mathcal{R} \sqsubseteq Q$.

<u>Lemma 52.</u> If $G(A) \xrightarrow{\tau} P \wedge P \Downarrow$ then $\exists Q.G(B) \stackrel{\tau}{\Longrightarrow} Q \wedge P \sqsubseteq \mathcal{R} \sqsubseteq Q$.

<u>Lemma 5.3</u> If $G(A) \xrightarrow{\tau} P \wedge P \Uparrow \wedge \neg\exists P'.P \stackrel{\tau}{\Longrightarrow} P' \wedge P' \Downarrow$ then $\exists Q.G(B) \stackrel{\epsilon}{\Longrightarrow} Q \wedge P \sqsubseteq \mathcal{R} \sqsubseteq Q$.

<u>Lemma 5.4</u> If $G(A) \xrightarrow{\tau} P \wedge P \Uparrow \wedge \exists P'.P \stackrel{\tau}{\Longrightarrow} P' \wedge P' \Uparrow$ then $\exists Q.G(B) \stackrel{\epsilon}{\Longrightarrow} Q \wedge P \sqsubseteq \mathcal{R} \sqsubseteq Q \wedge$ (if $\neg\exists R.G(B) \stackrel{\tau}{\Longrightarrow} R \wedge P \sqsubseteq \mathcal{R} \sqsubseteq R$ then if $P \stackrel{\tau}{\Longrightarrow} P' \wedge P' \Downarrow$ then $\exists Q'.G(B) \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$).

Each of Lemmas 5.1-5.4 is proved by induction on the depth of the inference by which $G(A) \xrightarrow{a} P$ or $G(A) \xrightarrow{\tau} P$ is established. In each case there are seven cases (c1)-(c7) as follows:
(c1) $G \equiv X$, (c2) $G \equiv u.H$, (c3) $G \equiv H + J$, (c4) $G \equiv H|J$, (c5) $G \equiv H\backslash a$, (c6) $G \equiv \mu XH$, (c7) $G \equiv H[f]$.

The inductions for Lemmas 5.1 and 5.2 are straight forward. For Lemma 5.3 the induction is straight forward except for (c3) when if, say, $H(A) \xrightarrow{\tau} P$ and $\neg\exists Q.H(B) \stackrel{\tau}{\Longrightarrow} Q \wedge P \sqsubseteq \mathcal{R} \sqsubseteq Q$, so that $P \sqsubseteq \mathcal{R} \sqsubseteq H(B)$, then $P \sqsubseteq \mathcal{R} \sqsubseteq G(B)$ by Lemma 2(b). For Lemma 5.4, again the induction is straight forward except for (c3). Suppose $H(A) \xrightarrow{\tau} P$. Then if $\neg\exists Q.H(B) \stackrel{\tau}{\Longrightarrow} Q \wedge P \sqsubseteq \mathcal{R} \sqsubseteq Q$ then $P \sqsubseteq \mathcal{R} \sqsubseteq H(B)$ and if $P \stackrel{\tau}{\Longrightarrow} P' \wedge P' \Downarrow$ then $\exists Q'.H(B) \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$. Then setting $Q \equiv G(B)$ it follows that $P \sqsubseteq \mathcal{R} \sqsubseteq Q$.

<u>Lemma 5.5</u> Let $u \in Act$. If $G(A) \Downarrow u$ then
(a) $G(B) \Downarrow u$
(b) if $G(B) \stackrel{u}{\Longrightarrow} Q$ then $\exists P.G(A) \stackrel{u}{\Longrightarrow} P \wedge P\mathcal{R} \sqsubseteq Q$.

Proof of Lemma 5.5: By induction on the depth of inference by which $G(A) \Downarrow u$ is established. (It does not seem to be possible directly to establish (b) with $\stackrel{u}{\longrightarrow}$ in place of $\stackrel{u}{\Longrightarrow}$.)
(c1) If $A \Downarrow u$ then $E(A) \Downarrow u$ by a shorter inference so $E(B) \Downarrow u$ and hence since $E \sqsubseteq^* F$, $F(B) \Downarrow u$ and so $B \Downarrow u$. If $B \stackrel{u}{\Longrightarrow} Q$ then $F(B) \stackrel{u}{\Longrightarrow} Q$ so since $E \sqsubseteq^* F$ and $E(B) \Downarrow u$, $\exists R.E(B) \stackrel{u}{\Longrightarrow} R \wedge R \sqsubseteq Q$. Then by induction hypothesis $\exists P.E(A) \stackrel{u}{\Longrightarrow} P \wedge P\mathcal{R} \sqsubseteq R$. So $A \stackrel{u}{\Longrightarrow} P \wedge P\mathcal{R} \sqsubseteq Q$.

(c2) Suppose that $G \equiv \tau.H$. If $G(A) \Downarrow u$ then $H(A) \Downarrow u$ by a shorter inference so $H(B) \Downarrow u$ and so $G(B) \Downarrow u$. If $G(B) \stackrel{u}{\Longrightarrow} Q$ then either $u = \tau \wedge Q \equiv H(B)$ or $H(B) \stackrel{u}{\Longrightarrow} Q$ whence either $G(A) \stackrel{u}{\Longrightarrow} H(A)\mathcal{R}Q$ or $H(A) \stackrel{u}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$, so $G(A) \stackrel{u}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$. Now suppose that $G \equiv a.H$. Only the case $u = a$ is nontrivial. If $G(A) \Downarrow a$ then $H(A) \Downarrow \varepsilon$ by a shorter inference so $H(B) \Downarrow \varepsilon$ and so $H(B) \Downarrow a$. Also if $G(B) \stackrel{a}{\Longrightarrow} Q$ then $H(B) \stackrel{\varepsilon}{\Longrightarrow} Q$ so $\exists P.H(A) \stackrel{\varepsilon}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$, so $G(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$.

(c3) If $G(A) \Downarrow u$ then $H(A) \Downarrow u$ and $J(A) \Downarrow u$ by shorter inferences and so $H(B) \Downarrow u$ and $J(B) \Downarrow u$, so $G(B) \Downarrow u$. If $G(B) \stackrel{u}{\Longrightarrow} Q$ then $H(B) \stackrel{u}{\Longrightarrow} Q$ or $J(B) \stackrel{u}{\Longrightarrow} Q$ so $\exists P.H(A) \stackrel{u}{\Longrightarrow} P \vee J(A) \stackrel{u}{\Longrightarrow} P) \wedge P\mathcal{R} \sqsubseteq Q$, $G(A) \stackrel{u}{\Longrightarrow} P \wedge P\mathcal{R} \sqsubseteq Q$.

(c4) First consider (b). Suppose that $G(B) \stackrel{u}{\Longrightarrow} Q$.
<u>Case 1</u> $u = a$.

There are two subcases:

<u>Case 1.1</u> $G(B) \stackrel{a}{\longrightarrow} Q' \stackrel{\varepsilon}{\Longrightarrow} Q$.
There are two sub-subcases:
<u>Case 1.1.1</u> $H(B) \stackrel{a}{\longrightarrow} Q_1' \wedge Q' \equiv Q_1'|J(B)$, and
<u>Case 1.1.2</u> $J(B) \stackrel{a}{\longrightarrow} Q_2' \wedge Q' \equiv H(B)|Q_2'$.
Consider Case 1.1.1 (Case 1.1.2 is similar). Since $G(A) \Downarrow a$, $H(A) \Downarrow a$ by a shorter inference and so $\exists P_1'.H(A) \stackrel{a}{\Longrightarrow} P_1' \wedge P_1'\mathcal{R} \sqsubseteq Q_1'$. Then $G(A) \stackrel{a}{\Longrightarrow} P'\mathcal{R} \sqsubseteq Q'$ where $P' \equiv P_1'|J(A)$. Say $P' \equiv K(A)\mathcal{R}K(B) \sqsubseteq Q'$. Since $K(A) \Downarrow \varepsilon$ by a shorter inference than that of $G(A) \Downarrow a$, and since, as $K(B) \sqsubseteq Q'$, $\exists R.K(B) \stackrel{\varepsilon}{\Longrightarrow} R \sqsubseteq Q$, by induction hypothesis $\exists P.K(A) \stackrel{\varepsilon}{\Longrightarrow} P\mathcal{R} \sqsubseteq R \sqsubseteq Q$. So $G(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$. $\square$(Case 1.1)

<u>Case 1.2</u> $G(B) \stackrel{\tau}{\longrightarrow} Q' \stackrel{a}{\Longrightarrow} Q$.
There are four sub-subcases:
<u>Case 1.2.1</u> $H(B) \stackrel{\tau}{\longrightarrow} Q_1' \wedge Q' \equiv Q_1'|J(B)$.
<u>Case 1.2.2</u> $J(B) \stackrel{\tau}{\longrightarrow} Q_2' \wedge Q' \equiv H(B)|Q_2'$.

Case 1.2.3 $b \neq a \wedge H(B) \xrightarrow{b} Q_1' \wedge J(B) \xrightarrow{\bar{b}} Q_2' \wedge Q' \equiv Q_1'|Q_2'$.
Case 1.2.4 $H(B) \xrightarrow{a} Q_1' \wedge J(B) \xrightarrow{\bar{a}} Q_2' \wedge Q' \equiv Q_1'|Q_2'$.
For Case 1.2.1: Since $H(A) \Downarrow \varepsilon$, by induction hypothesis, $\exists P_1'.H(A) \stackrel{\tau}{\Longrightarrow} P_1'\mathcal{R} \sqsubseteq Q_1'$ so $\exists P'.G(A) \stackrel{\tau}{\Longrightarrow} P'\mathcal{R} \sqsubseteq Q'$, say $P' \equiv K(A)\mathcal{R}K(B) \sqsubseteq Q'$. Then $K(A) \Downarrow a$ by a shorter inference than that of $G(A) \Downarrow a$, so since, as $K(A) \Downarrow a$, $\exists R.K(B) \stackrel{a}{\Longrightarrow} R \sqsubseteq Q$, $\exists P.K(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq R \sqsubseteq Q$. So $G(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$.
Case 1.2.2 is similar to Case 1.2.1.
For Case 1.2.3: In this case $H(A) \Downarrow b$ and $J(A) \Downarrow \bar{b}$; for suppose $H(A) \Uparrow b$ and $J(A) \Downarrow \bar{b}$. Then since $H(A) \Downarrow \varepsilon$, $\exists P.H(A) \stackrel{b}{\Longrightarrow} P \wedge P \Uparrow \varepsilon$. Since if $J(A) \stackrel{\bar{b}}{\Longrightarrow} R$ then $G(A) \stackrel{\tau}{\Longrightarrow} P|R$ so $P|R \Downarrow a$ by a shorter inference and hence $R \Downarrow$ by a shorter inference, $J(A) \Downarrow \bar{b}$ by a shorter inference than that of $G(A) \Downarrow a$ and so by the induction hypothesis, $\exists R.J(A) \stackrel{\bar{b}}{\Longrightarrow} R\mathcal{R} \sqsubseteq Q_2'$. So $G(A) \stackrel{\tau}{\Longrightarrow} P|R$ and $P|R \Uparrow \varepsilon$, a contradiction. Similarly the assumption that $H(A) \Downarrow b$ and $J(A) \Uparrow \bar{b}$ leads to a contradiction. Finally the assumption that $H(A) \Uparrow b$ and $J(A) \Uparrow \bar{b}$ leads to a contradiction, since $\exists P, R.H(A) \stackrel{b}{\Longrightarrow} P \wedge P \Uparrow \varepsilon \wedge J(A) \stackrel{\bar{b}}{\Longrightarrow} R \wedge R \Uparrow \varepsilon$, so $G(A) \stackrel{\varepsilon}{\Longrightarrow} P|R \wedge P|R \Uparrow \varepsilon$, a contradiction. Hence by induction hypothesis $\exists K, L.H(A) \stackrel{b}{\Longrightarrow} K(A)\mathcal{R}K(A) \sqsubseteq Q_1' \wedge J(A) \stackrel{\bar{b}}{\Longrightarrow} L(A)\mathcal{R}L(B) \sqsubseteq Q_2'$. Then $G(A) \stackrel{\tau}{\Longrightarrow} (K|L)(A)\mathcal{R}(K|L)(B) \sqsubseteq Q'$. So since $(K|L)(A) \Downarrow a$ by a shorter inference, $\exists P.(K|L)(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$. So $G(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$.
For Case 1.2.4: In this case $J(A) \Downarrow \bar{a}$, for if not then $\exists R.J(A) \stackrel{\bar{a}}{\Longrightarrow} R \wedge R \Uparrow \varepsilon$ and since $H(A) \Downarrow a$, by induction hypothesis $\exists P.H(A) \stackrel{a}{\Longrightarrow} P$ so $G(A) \stackrel{\tau}{\Longrightarrow} P|R \wedge P|R \Uparrow \varepsilon$, a contradiction. Thus by induction hypothesis $\exists K, L.H(A) \stackrel{a}{\Longrightarrow} K(A)\mathcal{R}K(B) \sqsubseteq Q_1' \wedge J(A) \stackrel{\bar{a}}{\Longrightarrow} L(A)\mathcal{R}L(B) \sqsubseteq Q_2'$. Then $G(A) \stackrel{\tau}{\Longrightarrow} (K|L)(A)\mathcal{R}(K|L)(B) \sqsubseteq Q'$ and by the induction hypothesis applied to $K|L \Downarrow a$, $\exists P.(K|L)(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$. Hence $G(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$.
$\square$ (Case 1)

Case 2 $u = \tau$.
Suppose $G(B) \xrightarrow{\tau} Q' \stackrel{\varepsilon}{\Longrightarrow} Q$. There are three subcases:
Case 2.1 $H(B) \xrightarrow{\tau} Q_1' \wedge Q' \equiv Q_1'|J(B)$.
Case 2.2 $J(B) \xrightarrow{\tau} Q_2' \wedge Q' \equiv H(B)|Q_2'$.
Case 2.3 $H(B) \xrightarrow{b} Q_1' \wedge J(B) \xrightarrow{\bar{b}} Q_2' \wedge Q' \equiv Q_1'|Q_2'$.
For Case 2.1: Since $H(A) \Downarrow$, by induction hypothesis $\exists K.H(A) \stackrel{\tau}{\Longrightarrow} K(A)\mathcal{R}K(B) \sqsubseteq Q_1'$, so $G(A) \stackrel{\tau}{\Longrightarrow} (K|J)(A)\mathcal{R}(K|J)(B) \sqsubseteq Q'$, so since $(K|J)(A) \Downarrow$, by induction hypothesis $\exists P.(K|J)(A) \stackrel{\varepsilon}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$. So $G(A) \stackrel{\tau}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$.
Case 2.2 is similar to Case 2.1.
For Case 2.3: In this case $H(A) \Downarrow b$ and $J(A) \Downarrow \bar{b}$. For suppose $H(A) \Uparrow b$ and $J(A) \Uparrow \bar{b}$. Since $H(A) \Downarrow$ and $J(A) \Downarrow$, $\exists P, R.H(A) \stackrel{b}{\Longrightarrow} P \wedge P \Uparrow \wedge J(A) \stackrel{\bar{b}}{\Longrightarrow} R \wedge R \Uparrow$, when $G(A) \stackrel{\tau}{\Longrightarrow} P|R \wedge P|R \Uparrow$, a contradiction. Suppose $H(A) \Uparrow b$ and $J(A) \Downarrow \bar{b}$. Then $\exists P.H(A) \stackrel{b}{\Longrightarrow} P \wedge P \Uparrow$, so by induction hypothesis applied to $J(A) \Downarrow \bar{b}$, $\exists R.J(A) \stackrel{\bar{b}}{\Longrightarrow} R$, so $G(A) \stackrel{\tau}{\Longrightarrow} P|R \wedge P|R \Uparrow$, a contradiction. Similarly the assumption that $H(A) \Downarrow b$ and $J(A) \Uparrow \bar{b}$ leads to a contradiction. So by induction hypothesis, $\exists P, R.H(A) \stackrel{b}{\Longrightarrow} P\mathcal{R} \sqsubseteq$

$Q_1' \wedge J(A) \stackrel{\bar{b}}{\Longrightarrow} R\mathcal{R} \sqsubseteq Q_2'$, so $G(A) \stackrel{\tau}{\Longrightarrow} P|R\mathcal{R} \sqsubseteq Q'$. Then by induction hypothesis applied to $P|R\Downarrow\varepsilon$, $\exists P'.P|R \stackrel{\varepsilon}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$. So $G(A) \stackrel{\tau}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$. $\square$(Case 2) $\square$(b)

Now consider (a). If $u = \tau$ then certainly $G(B)\Uparrow$, since $H(A)\Downarrow$ and $J(A)\Downarrow$ so $H(B)\Downarrow$ and $J(B)\Downarrow$. Also if $G(B) \stackrel{\tau}{\Longrightarrow} Q$ then $\exists P.G(B) \stackrel{\tau}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$ and $P\Downarrow\varepsilon$ by a shorter inference so $Q\Downarrow$. If $u = a$ then $G(B)\Downarrow\varepsilon$, for if $G(B) \stackrel{\tau}{\Longrightarrow} Q$ then $\exists P.G(A) \stackrel{\tau}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$ with $P\Downarrow a$ by a shorter inference, while if $G(B) \stackrel{a}{\Longrightarrow} Q$ then $\exists P.G(A) \stackrel{a}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$ with $P\Downarrow\varepsilon$ by a shorter inference, so $Q\Downarrow\varepsilon$. $\square$(a)

This completes the proof of (c4).

(c5) If $u = a$ then (a) and (b) are easily seen to hold. Suppose that $u \neq a$. If $G(A)\Downarrow u$ then $H(A)\Downarrow u$ by a shorter inference so $H(B)\Downarrow u$ and hence $G(B)\Downarrow u$. If $G(B) \stackrel{u}{\Longrightarrow} Q$ then $\exists Q'.H(B) \stackrel{u}{\Longrightarrow} Q' \wedge Q \equiv Q'\backslash a$. By induction hypothesis $\exists P'.H(A) \stackrel{u}{\Longrightarrow} P' \wedge P\mathcal{R} \sqsubseteq Q'$. Then $G(A) \stackrel{u}{\Longrightarrow} P'\backslash a \wedge P'\backslash a\mathcal{R} \sqsubseteq Q$.

(c6) If $G(A)\Downarrow u$, i.e. $\mu Y(H(A))\Downarrow u$, then $H(A)\{\mu Y(H(A))/Y\}\Downarrow u$ by a shorter inference so $(H\{\mu Y(H/Y\})(A)\Downarrow u$ by a shorter inference and so $(H\{\mu YH/Y\})(B)\Downarrow u$, so $G(B)\Downarrow u$. If $G(B) \stackrel{u}{\Longrightarrow} Q$ then $(H\{\mu YH/Y\})(B) \stackrel{u}{\Longrightarrow} Q$ so by induction hypothesis $\exists P.(H\{\mu YH/Y\})(A) \stackrel{u}{\Longrightarrow} P\mathcal{R} \sqsubseteq Q$ when $G(A) \stackrel{u}{\Longrightarrow} P$.

(c7) If $G(A)\Downarrow u$ then $H(A)\Downarrow v$ for all v such that $f(v) = u$, so $H(B)\Downarrow v$ for all such v and hence $G(B)\Downarrow u$. If $G(B) \stackrel{u}{\Longrightarrow} Q$ then $\exists v.\exists Q'.f(v) = u \wedge H(B) \stackrel{v}{\Longrightarrow} Q' \wedge Q \equiv Q'[f]$, so $\exists P'.H(A) \stackrel{v}{\Longrightarrow} P' \wedge P'\mathcal{R} \sqsubseteq Q'$ so $G(A) \stackrel{u}{\Longrightarrow} P'[f] \wedge P'[f]\mathcal{R} \sqsubseteq Q$.

This completes the proof of Lemma 5.5 and hence of Lemma 5. $\square$

<u>Corollary 6</u> $\sqsubseteq^c = \sqsubseteq^+ = \sqsubseteq^*$.

Proof: Since $\sqsubseteq^*$ is a precongruence, $\sqsubseteq^* \subseteq \sqsubseteq^c$. But $\sqsubseteq^c \subseteq \sqsubseteq^+ = \sqsubseteq^*$. $\square$

In considering alternatives to the definition of $\sqsubseteq$ the following, using only the single convergence predicate $\Downarrow$, is one possibility. Define $\mathcal{F}^g : Rel \rightarrow Rel$ (g for the single 'global' convergence predicate) by setting for $\mathcal{R} \in Rel$:

$P\mathcal{F}^g(\mathcal{R})Q$ iff

(i) $\forall u$. if $P \stackrel{u}{\longrightarrow} P'$ then $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge P'\mathcal{R}Q'$,

(ii) if $P\Downarrow$ then

(a) $Q\Downarrow$,

(b) $\forall u$. if $Q \stackrel{u}{\longrightarrow} Q'$ then $\exists P'.P \stackrel{\hat{u}}{\Longrightarrow} P' \wedge P'\mathcal{R}Q'$.

$\mathcal{F}^g$ is a monotonic functional and hence has a maximal fixed point $\sqsubseteq^g = \bigcup\{\mathcal{R} \mid \mathcal{R} \subseteq \mathcal{F}^g(\mathcal{R})\}$ which is a preorder. Moreover:

<u>Lemma 7</u> $\sqsubseteq^g$ is a proper subset of $\sqsubseteq$.

Proof: That $\sqsubseteq^g \subseteq \sqsubseteq$ is an immediate consequence of the following observation.
(1) If $\mathcal{R} \subseteq \mathcal{F}^g(\mathcal{R})$ then $\mathcal{R} \subseteq \mathcal{F}(\mathcal{R})$.
To establish this suppose that $\mathcal{R} \subseteq \mathcal{F}^g(\mathcal{R})$ and $P\mathcal{R}Q$. If $P \xrightarrow{u} P'$ then since $P\mathcal{F}^g(\mathcal{R})Q$, $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge P'\mathcal{R}Q'$. Suppose $P\Downarrow$. Then since $P\mathcal{F}^g(\mathcal{R})Q$, $Q\Downarrow$ and if $Q \xrightarrow{\tau} Q'$ then $\exists P'.P \stackrel{\epsilon}{\Longrightarrow} P' \wedge P'\mathcal{R}Q'$. Suppose $P\Downarrow a$. If $Q\Uparrow a$ then since $Q\Downarrow$, $\exists Q'.Q \stackrel{a}{\Longrightarrow} Q' \wedge Q'\Uparrow$. Then since $P\mathcal{F}^g(\mathcal{R})Q$, $\mathcal{R} \subseteq \mathcal{F}^g(\mathcal{R})$ and $P\Downarrow$, $\exists P'.P \stackrel{a}{\Longrightarrow} P' \wedge P'\mathcal{R}Q'$, a contradiction since $P'\Downarrow$, $Q'\Uparrow$ and $P'\mathcal{F}^g(\mathcal{R})Q'$. Hence $Q\Downarrow a$. If $Q \xrightarrow{a} Q'$ then since $P\mathcal{F}^g(\mathcal{R})Q$ and $P\Downarrow$, $\exists P'.P \stackrel{a}{\Longrightarrow} P' \wedge P'\mathcal{R}Q'$.

To see that the inclusion is proper it suffices to note that $a.(b.\mathbf{0}+\bot) \sqsubseteq a.(b.\mathbf{0}+\bot)+a.\bot$ whereas it is not the case that $a.(b.\mathbf{0}+\bot) \sqsubseteq^g a.(b.\mathbf{0}+\bot)+a.\bot$. □

Define $\sqsubseteq^{g,+}$ by setting $P \sqsubseteq^{g,+} Q$ iff $\forall R.P+R \sqsubseteq^g Q+R$, and $\sqsubseteq^{g,*}$ by setting $P \sqsubseteq^{g,*} Q$ iff
(i) $\forall a$. if $P \xrightarrow{a} P'$ then $\exists Q'.Q \stackrel{a}{\Longrightarrow} Q' \wedge P' \sqsubseteq^g Q'$,
(ii) if $P \stackrel{\tau}{\Longrightarrow} P'$ then
  (a) if $P'\Downarrow$ then $\exists Q'.Q \stackrel{\tau}{\Longrightarrow} Q' \wedge P' \sqsubseteq^g Q'$,
  (b) if $P'\Uparrow$ then $\exists Q'.Q \stackrel{\epsilon}{\Longrightarrow} Q' \wedge P' \sqsubseteq^g Q'$,
(iii) if $P\Downarrow$ then
  (a) $Q\Downarrow$,
  (b) $\forall u$. if $Q \xrightarrow{u} Q'$ then $\exists P'.P \stackrel{u}{\Longrightarrow} P' \wedge P' \sqsubseteq^g Q'$.
Then by an argument similar to that in the proof of Lemma 3 it may be established that $\sqsubseteq^{g,*}=\sqsubseteq^{g,+}$. However if $\sqsubseteq^{g,c}$ is the largest precongruence relation included in $\sqsubseteq^g$ then it is not the case that $\sqsubseteq^{g,*}=\sqsubseteq^{g,c}$, for $b.\mathbf{0}+\bot \sqsubseteq^{g,*} b.\mathbf{0}+\tau.\bot$ but $a.(b.\mathbf{0}+\bot) \not\sqsubseteq^{g,*} a.(b.\mathbf{0}+\tau.\bot)$. No characterization of $\sqsubseteq^{g,c}$ is, as far as I am aware, known.

To obtain a more satisfactory treatment using only the single convergence predicate $\Downarrow$, recall the transition relations $\stackrel{u}{\Longrightarrow}_0$ and define $\mathcal{F}_0^g : Rel \to Rel$ by setting for $\mathcal{R} \in Rel$:
$P\mathcal{F}_0^g(\mathcal{R})Q$ iff
(i) $\forall u$. if $P \xrightarrow{u} P'$ then $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow}_0 Q' \wedge P'\mathcal{R}Q'$,
(ii) if $P\Downarrow$ then
  (a) $Q\Downarrow$,
  (b) $\forall u$. if $Q \xrightarrow{u} Q'$ then $\exists P'.P \stackrel{\hat{u}}{\Longrightarrow}_0 P' \wedge P'\mathcal{R}Q'$.
$\mathcal{F}_0^g$ is a monotonic functional and its maximal fixed point $\sqsubseteq_0^g = \bigcup\{\mathcal{R} \mid \mathcal{R} \subseteq \mathcal{F}_0^g(\mathcal{R})\}$ is a preorder.

<u>Lemma 8</u> $\sqsubseteq_0^g$ is a proper subset of $\sqsubseteq^g$.

Proof: That $\sqsubseteq_0^g \subseteq \sqsubseteq^g$ follows from the observation that if $\mathcal{R} \subseteq \mathcal{F}_0^g(\mathcal{R})$ then $\mathcal{R} \subseteq \mathcal{F}^g(\mathcal{R})$, which in turn follows from the fact that if $P \stackrel{\hat{u}}{\Longrightarrow}_0 Q$ then $P \stackrel{\hat{u}}{\Longrightarrow} Q$. To see that the inclusion is proper it suffices to note that $a.(b.\mathbf{0}+\tau.c.\mathbf{0}) \sqsubseteq^g a.(b.\mathbf{0}+\tau.c.\mathbf{0})+a.c.\mathbf{0}$ whereas it is not the case that $a.(b.\mathbf{0}+\tau.c.\mathbf{0}) \sqsubseteq_0^g a.(b.\mathbf{0}+\tau.c.\mathbf{0})+a.c.\mathbf{0}$. □

By arguments similar to those in the proof of Lemma 1 the following may be established.

Lemma 9 If $P \sqsubseteq_0^g Q$ then
(a) $u.p \sqsubseteq_0^g u.Q$,
(b) $P|R \sqsubseteq_0^g Q|R$,
(c) $P\backslash a \sqsubseteq_0^g Q\backslash a$,
(d) $P[f] \sqsubseteq_0^g Q[f]$.

As with $\sqsubseteq$, $\sqsubseteq_0^g$ is not preserved by $+$. However if $\sqsubseteq_0^{g,c}$ is the largest precongruence relation included in $\sqsubseteq_0^g$, $\sqsubseteq_0^{g,+}$ is defined by setting $P \sqsubseteq_0^{g,+} Q$ iff $\forall R.P + R \sqsubseteq_0^g Q + R$, and $\sqsubseteq_0^{g,*}$ is defined from $\sqsubseteq_0^g$ as $\sqsubseteq^{g,*}$ is defined from $\sqsubseteq^g$, with $\stackrel{u}{\Longrightarrow}_0$ replacing $\stackrel{u}{\Longrightarrow}$ throughout the definition, then by proofs similar to those of Lemmas 3, 4 and 5 and Corollary 6 the following may be established.

Lemma 10 $P \sqsubseteq_0^{g,*} Q$ iff $P \sqsubseteq_0^{g,+} Q$.

Lemma 11 If $P \sqsubseteq_0^{g,*} Q$ then
(a) $u.P \sqsubseteq_0^{g,*} u.Q$,
(b) $P + R \sqsubseteq_0^{g,*} Q + R$,
(c) $P|R \sqsubseteq_0^{g,*} Q|R$,
(d) $P\backslash a \sqsubseteq_0^{g,*} Q\backslash a$,
(e) $P[f] \sqsubseteq_0^{g,*} Q[f]$.

Lemma 12 Let $\sqsubseteq_0^{g,*}$ be extended pointwise to $\mathbf{E}$. Then if $E \sqsubseteq_0^{g,*} F$ and $\mu\overline{X}E$ and $\mu\overline{X}F$ are closed then $\mu\overline{X}E \sqsubseteq_0^{g,*} \mu\overline{X}F$.

Corollary 13 $\sqsubseteq_0^{g,c} = \sqsubseteq_0^{g,+} = \sqsubseteq_0^{g,*}$.

From Lemmas 7 and 8 it follows that $\sqsubseteq_0^g$ is a proper subset of $\sqsubseteq$. Moreover:

Lemma 14 $\sqsubseteq_0^{g,c}$ is a proper subset of $\sqsubseteq^c$.

Proof: That $\sqsubseteq_0^{g,c} \subseteq \sqsubseteq^c$ follows from the fact that $\sqsubseteq_0^g \subseteq \sqsubseteq$. To see that the inclusion is proper it suffices to note that $a.(b.0 + \tau.c.0) \sqsubseteq^c a.(b.0 + \tau.c.0) + a.c.0$ whereas it is not the case that $a.(b.0 + \tau.c.0) \sqsubseteq_0^{g,c} a.(b.0 + \tau.c.0) + a.c.0$. □

Combining the use of the family of convergence predicates in defining $\sqsubseteq$ with the transition relations $\stackrel{u}{\Longrightarrow}_0$ gives rise to the final refinement of bisimulation equivalence considered in this paper. Define $\mathcal{F}_0 : Rel \rightarrow Rel$ by setting for $\mathcal{R} \in Rel$:

$P\mathcal{F}_0(\mathcal{R})Q$ iff
(i) $\forall u.$ if $P \stackrel{u}{\longrightarrow} P'$ then $\exists Q'.Q \stackrel{\hat{u}}{\Longrightarrow}_0 Q' \wedge P'\mathcal{R}Q'$,

(ii) $\forall u$. if $P\Downarrow$ then

(a) $Q\Downarrow$,

(b) $\forall u$. if $Q \xrightarrow{u} Q'$ then $\exists P'.P \overset{\hat{u}}{\Longrightarrow}_0 P' \wedge P'\mathcal{R}Q'$.

$\mathcal{F}_0$ is a monotonic functional and its maximal fixed point $\sqsubseteq_0 = \{\mathcal{R} \mid \mathcal{R} \subseteq \mathcal{F}_0(\mathcal{R})\}$ is a pre-order.

<u>Lemma 15</u> (a) $\sqsubseteq_0^g$ is a proper subset of $\sqsubseteq_0$.
(b) $\sqsubseteq_0$ is a proper subset of $\sqsubseteq$.

Proof: (a) This may be proved by an argument similar to that in the proof of (1) in the proof of Lemma 7.
(b) That $\sqsubseteq_0 \subseteq \sqsubseteq$ follows from an argument similar to that in the first part of the proof of Lemma 8. To see that the inclusion is proper it suffices to note that $a.(b.0 + \tau.c.0) \sqsubseteq a.(b.0+\tau.c.0)+a.c.0$ whereas it is not the case that $a.(b.0+\tau.c.0) \sqsubseteq_0 a.(b.0+\tau.c.0)+a.c.0$.
□

Define $\sqsubseteq_0^c$, $\sqsubseteq_0^+$ and $\sqsubseteq_0^*$ from $\sqsubseteq_0$ as $\sqsubseteq^c$, $\sqsubseteq^+$ and $\sqsubseteq^*$ are defined from $\sqsubseteq$ with $\overset{\hat{u}}{\Longrightarrow}_0$ replacing $\overset{\hat{u}}{\Longrightarrow}$ in the latter case. Then by arguments almost identical to those in the proofs of Lemmas 3,4 and 5 and Corollary 6, with $\overset{\hat{u}}{\Longrightarrow}_0$ replacing $\overset{\hat{u}}{\Longrightarrow}$ throughout, the following may be established.

<u>Lemma 16</u> $\sqsubseteq_0^c = \sqsubseteq_0^+ = \sqsubseteq_0^*$.

It is not clear whether or not either of $\sqsubseteq_0^g = \sqsubseteq_0$ and $\sqsubseteq_0^{g,c} = \sqsubseteq_0^c$ holds. It will be established however that the latter equality holds for the classes of finite agents and finite-state expressions considered in the next two sections.

3. Axiomatization for finite agents

Let $\mathbf{F}$ be the set of closed expressions which contain no fixed point operators μX. Thus $\mathbf{F}$ is given by the following *BNF*-style description:

$$P ::= \mathbf{0} \mid \bot \mid u.P \mid P + Q \mid P|Q \mid P\backslash a \mid P[f].$$

This section is devoted to demonstrations of the existence of complete axiomatizations of the precongruence relations $\sqsubseteq^c$, $\sqsubseteq_0^{g,c}$ and $\sqsubseteq_0^c$ for the class $\mathbf{F}$ of finite agents. The axiomatizations for $\sqsubseteq_0^{g,c}$ and $\sqsubseteq_0^c$ coincide showing the identity of these relations for this class of agents. Throughout this section P, Q and R range over $\mathbf{F}$. First consider the problem of characterizing $\sqsubseteq^c$.

Let $\leq$ and $=$ be the smallest pair of binary relations on $\mathbf{F}$ such that $\leq$ is reflexive and transitive and preserved by $u.$, $+$, $|$, $\backslash a$ and $[f]$, and $P = Q$ iff $P \leq Q$ and $Q \leq P$,

and such that:

(S1) $P + Q = Q + P$
(S2) $P + (Q + R) = (P + Q) + R$
(S3) $P + P = P$
(S4) $P + \mathbf{0} = P$

(τ1) $u.\tau.P = u.P$
(τ2) $P + \tau.P = \tau.P$
(τ3) $u.(P + \tau.Q) = u.(P + \tau.Q) + u.Q$

(R1) $\mathbf{0}\backslash a = \mathbf{0}$
(R2) $\bot\backslash a \leq \bot$
(R3) $(P + Q)\backslash a = P\backslash a + Q\backslash a$
(R4) $(u.P)\backslash a = \mathbf{0}$ if $u = a$, $u.(P\backslash a)$ if $u \neq a$

(Rl1) $\mathbf{0}[f] = \mathbf{0}$
(Rl2) $\bot[f] \leq \bot$
(Rl3) $(P + Q)[f] = P[f] + Q[f]$
(Rl4) $(u.P)[f] = f(u).(P[f])$

($\bot$1) $\bot \leq P$
($\bot$2) $\tau.(P + \bot) \leq P + \bot$

(Exp) if $P \equiv \sum u_i.P_i \ \{+\bot\}$ and $Q \equiv \sum v_j.Q_j \ \{+\bot\}$ then
$P|Q = \sum_{\overline{u_i}=v_j} \tau.(P_i|Q_j) + \sum u_i.(P_i|Q) + \sum v_j.(P|Q_j) \ \{+\bot\}$
where an empty sum is understood as $\mathbf{0}$, $\{+\bot\}$ indicates that $\bot$ may or may not be a summand of a term and $\bot$ is a summand of the right hand side iff it is a summand either of P or of Q.

<u>Remark</u> By ($\bot$1), (τ2) and ($\bot$2), the following holds:
($\bot$2′) $\tau.(P + \bot) = P + \bot$.

<u>Theorem 17</u> $P \leq Q$ iff $P \sqsubseteq^c Q$.

The proof stretches over several lemmas.

<u>Lemma 18</u> If $P \leq Q$ then $P \sqsubseteq^c Q$.

Proof: The proof is omitted. It is straight forward to check that replacing $\leq$ and $=$ by $\sqsubseteq^+$ and $\simeq^+$ respectively in each of the axioms produces a true assertion. □

The proof of completeness follows a standard pattern involving the isolation of a 'nor-

mal form' for the elements of $\mathbf{F}$. First the class of 'sumforms' is given by the following recursive definition.

P is a *sumform* iff P is of the form $\sum u_i.P_i\{+\bot\}$ where each P_i is a sumform.

<u>Lemma 19</u> For each P there is a sumform Q such that $P = Q$.

Proof: The proof, which is omitted, is by induction on the structure of P and uses the following fact which, setting $d(Q)$ = the depth of the derivation tree of Q, is established by induction on $d(Q) + d(R)$ using the axioms (R1)-(R4), (Rl1)-(Rl4) and (Exp):
(1) If Q and R are sumforms then there are sumforms Q', Q'' and Q''' such that $Q|R = Q'$, $Q\backslash a = Q''$ and $Q[f] = Q'''$. □

The class of 'normal forms' is given by the following recursive definition.

P is a *normal form* iff P is a sumform $\sum u_i.P_i\{+\bot\}$ such that
(a) each P_i is a normal form,
(b) if $u_i = \tau$ then $P_i\Downarrow$,
(c) if $P\Downarrow$ and $P\Uparrow a$ then $a.\bot$ is a summand of P,
(d) if $P \stackrel{u}{\Longrightarrow} Q$ then for some i, $u.Q \equiv u_i.P_i$.

Note that if P is a normal form then
(i) if $P \stackrel{\tau}{\Longrightarrow} Q$ then $Q\Downarrow$, and
(ii) if $P\Uparrow$ then $\bot$ is a summand of P.

<u>Lemma 20</u> For each sumform P there is a normal form Q such that $P = Q$ and $d(P) = d(Q)$.

Proof: By induction on $d(P)$. If $d(P) = 0$ then $P = \mathbf{0}$ or $P = \bot$ and the result is immediate. Suppose $d(P) > 0$ and $P = \sum u_i.P_i\ \{+\bot\}$ where by induction hypothesis each P_i is a normal form.

Suppose $u_i = \tau$ and $P_i\Uparrow$. Then for some normal form P_i', $P_i \equiv P_i' + \bot$ and hence

$$\begin{aligned} P &= \textstyle\sum u_j.P_j\ \{+\bot\} \\ &= \textstyle\sum_{j\neq i} u_j.P_j + \tau.(P_i' + \bot)\ \{+\bot\} \\ &= \textstyle\sum_{j\neq i} u_j.P_j + P_i' + \bot \end{aligned}$$

by $(\bot 2')$. Repeating this process yields a sumform Q' satisfying (a) and (b) of the definition of 'normal form' with $P = Q'$ and $d(P) = d(Q')$.

Suppose $P\Downarrow$ and $P\Uparrow a$. Then there are two possibilities.
(a) $\exists P'.P \xrightarrow{\tau} P' \wedge P'\Uparrow a$.
Then by induction hypothesis there is a normal form $P'' \equiv P''' + a.\bot$ such that $P' = P''$. Then for some P_0,

$$\begin{aligned} P &= P_0 + \tau.P' \\ &= P_0 + \tau.(P''' + a.\bot) \\ &= P_0 + \tau.(P''' + a.\bot) + a.\bot \\ &= P_0 + \tau.P' + a.\bot \\ &= P + a.\bot \end{aligned}$$

using ($\tau 2$).
(b) $\exists P'.P \xrightarrow{a} P' \wedge P'\Uparrow$.
Then by induction hypothesis there is a normal form $P'' = P''' + \bot$ with $P' = P''$. Then for some P_0,

$$\begin{aligned} P &= P_0 + a.P' \\ &= P_0 + a.(P''' + \bot) \\ &= P_0 + a.(P''' + \tau.\bot) \\ &= P_0 + a.P' + a.\bot \\ &= P + a.\bot \end{aligned}$$

using ($\tau 3$) and the fact that $\tau.\bot = \bot$ (by ($\bot 2'$)). Repeating this process yiels a sumform Q'' satisfying (a), (b) and (c) of the definition of 'normal form' with $P = Q''$ and $d(P) = d(Q'')$.

Suppose $Q'' \stackrel{u}{\Longrightarrow} R$ and $u.R$ is not a summand of Q''.
(1) $Q'' = Q'' + u.R$.
Suppose that $\exists P'.Q'' \xrightarrow{u} P' \wedge P' \stackrel{\tau}{\Longrightarrow} R$. Then by induction hypothesis $P' = P' + \tau.R$. Then

$$\begin{aligned} Q'' &= Q'' + u.P' \\ &= Q'' + u.(P' + \tau.R) \\ &= Q'' + u.(P' + \tau.R) + u.R \\ &= Q'' + u.P' + u.R \\ &= Q'' + u.R \end{aligned}$$

using ($\tau 3$) and (S3).

Suppose that $\exists P'.P \xrightarrow{\tau} P' \wedge P' \stackrel{u}{\Longrightarrow} R$. Then by induction hypothesis $P' = P' + u.R$. Then

$$\begin{aligned} Q'' &= Q'' + \tau.P' \\ &= Q'' + \tau.P' + u.R \\ &= Q'' + u.R \end{aligned}$$

using ($\tau 2$).

Repeated use of (1) yields a normal form Q such that $P = Q$ and $d(P) = d(Q)$. □

<u>Lemma 21</u> If P and Q are normal forms and $P \sqsubseteq^* Q$ then $P \leq Q$.

Proof: By induction on $d = d(P) + d(Q)$. If $d = 0$ it is immediate. Suppose $d > 0$ and $P \equiv \sum u_i.P_i \ \{+\bot\}$ and $Q \equiv \sum v_j.Q_j \ \{+\bot\}$. The following observation is useful.

(1) If $R \sqsubseteq R'$ then $R \sqsubseteq^* R'$ or $\tau.R \sqsubseteq^* R'$ or $R \sqsubseteq^* \tau.R'$.
To establish this it suffices to examine the following possibilities.
(a) $\exists R''.R \xrightarrow{\tau} R'' \wedge R'' \sqsubseteq R'$.
Then $R \sqsubseteq^* \tau.R'$.
(b) $\exists R''.R' \xrightarrow{\tau} R'' \wedge R \sqsubseteq^* R''$.
Then $\tau.R \sqsubseteq^* R'$.
(c) Neither (a) nor (b) holds.
Then $R \sqsubseteq^* R'$.

Suppose $P \xrightarrow{u} P'$. Then since Q is a normal form and $P \sqsubseteq^* Q$, for some Q' such that $P' \sqsubseteq Q'$, $u.Q'$ is a summand of Q. Then considering the cases in (1):
(a) $P' \sqsubseteq^* \tau.Q'$.
Let Q'' be a normal form with $Q'' = \tau.Q'$ and $d(Q'') = d(\tau.Q')$. Then by induction hypothesis $P' \leq \tau.Q'$ and so by (τ1), $u.P' \leq u.Q'' = u.\tau.Q' = u.Q'$.
(b) $\tau.P' \sqsubseteq^* Q'$.
Let P'' be a normal form with $P'' = \tau.P'$ and $d(P'') = d(\tau.P')$. Then by induction hypothesis $P'' \leq Q'$ and so by (τ1), $u.P' = u.\tau.P'' = u.P'' \leq u.Q'$.
(c) $P' \sqsubseteq^* Q'$.
Then by induction hypothesis $P' \leq Q'$ and so $u.P' \leq u.Q'$.

Hence for each summand $u.P'$ of P there is a summand $u.Q'$ of Q such that $u.P' \leq u.Q'$. If $P \Uparrow$ then $\perp$ is a summand of P and so by ($\perp$1), $P \leq Q$. Suppose $P \Downarrow$. If $P \Downarrow a$ then since $P \sqsubseteq^* Q$, for each summand $a.Q'$ of Q there is a summand $a.P'$ of P with $P' \sqsubseteq Q'$. By considering the cases (a), (b) and (c) of (1) as above it may be established that $a.P' \leq a.Q'$. If $P \Uparrow a$ then for each summand $a.Q'$ of Q there is a summand $a.P'(\equiv a.\perp)$ of P with $a.P' \leq a.Q'$.

It follows that $P \leq Q$. □

By Lemmas 19, 20 and 21 and Corollary 6:

<u>Lemma 22</u> If $P \sqsubseteq^c Q$ then $P \leq Q$.

This completes the proof of Theorem 17.

Now consider the problem of axiomatizing $\sqsubseteq_0^{g,c}$ and $\sqsubseteq_0^c$. Of the axioms for $\sqsubseteq^c$ only (τ3), $u.(P + \tau.Q) = u.(P + \tau.Q) + u.Q$, is not valid when '=' is replaced by the equivalence relation $\simeq_0'^c$ generated by $\sqsubseteq_0^{g,c}$. Let $\leq_0$ and $=_0$ be the smallest pair of relations satisfying the conditions in the definitions of $\leq$ and $=$ above with (τ3) deleted. Then by arguments similar to those in the proof of Lemma 18 the following may be established.

<u>Lemma 23</u> If $P \leq_0 Q$ then $P \sqsubseteq_0^{g,c} Q$.

Then defining the notions of 'sumform' and 'normal form' as before, except that in clause (d) in the definition of the latter $\overset{u}{\Longrightarrow}$ is replaced by $\overset{u}{\Longrightarrow}_0$, by arguments similar

to those in the proofs of Lemmas 19, 20 and 21, and using Lemma 16, the following may be established.

<u>Lemma 24</u> If $P \sqsubseteq_0^c Q$ then $P \leq_0 Q$.

Combining Lemmas 23 and 24 with the consequence of Lemma 15(a) that $\sqsubseteq_0^{g,c} \subseteq \sqsubseteq_0^c$ yields:

<u>Corollary 25</u> $P \sqsubseteq_0^{g,c} Q$ iff $P \sqsubseteq_0^c Q$ iff $P \leq_0 Q$.

<u>Remark</u> Although the proofs presented here were obtained independently, adapting arguments from [M4], Theorem 17 was known to Hennessy and Plotkin (cf. [HP]).

4. Axiomatization for finite-state expressions

Let **FS** be the set of expressions given by the following *BNF*-style description:

$$E ::= X \mid \mathbf{0} \mid \bot \mid u.E \mid E + F \mid \mu X E.$$

To obtain a complete axiomatization of $\sqsubseteq^c$ on **FS** it is helpful to obtain a characterization of $\sqsubseteq^c$ on **FS** more amenable to manipulation than the definition of pointwise extension. To do so it is necessary to obtain also such a characterization of $\sqsubseteq$.

The strategy and many of the details of the completeness proof are modifications of arguments used by Milner [M3] to obtain a complete axiomatization of bisimulation congruence on **FS**.

Throughout this section E, F and G range over **FS**. A free occurrence of a variable W in an expression E is said to be *guarded* iff it lies within some subexpression of E of the form $a.F$. Otherwise it is said to be *unguarded*. $E \rhd W$ denotes that E contains a free unguarded occurrence of W. Let Rel^o be the set of all binary relations on **FS** and define $\mathcal{G} : Rel^o \to Rel^o$ by setting for $\mathcal{R} \in Rel^o$:

$E\mathcal{G}(\mathcal{R})F$ iff

(i) $\forall u.$ if $E \xrightarrow{u} E'$ then $\exists F'.F \overset{\hat{u}}{\Longrightarrow} F' \wedge E'\mathcal{R}F'$,

(ii) $\forall u.$ if $E \Downarrow u$ then

(a) $F \Downarrow u$

(b) if $F \xrightarrow{u} F'$ then $\exists E'.E \overset{\hat{u}}{\Longrightarrow} E' \wedge E'\mathcal{R}F'$,

(iii) if $E \rhd W$ then $F \rhd W$,

(iv) if $E \Downarrow$ then if $F \rhd W$ then $E \rhd W$.

$\mathcal{G}$ is a monotonic functional and its maximal fixed point $\sqsubseteq^o = \bigcup\{\mathcal{R} \mid \mathcal{R} \subseteq \mathcal{G}(\mathcal{R})\}$ is a

preorder.

<u>Lemma 26</u> $E \sqsubseteq^o F$ iff $E \sqsubseteq F$.

Proof: The proof will not be given in complete detail. Throughout the incomplete proof given here, $G(\overline{P})$ abbreviates $G\{\overline{P}/\overline{X}\}$ and it is assumed that A is infinite, although the result probably holds without this assumption.
($\Rightarrow$) Let $\mathcal{R} = \{(E(\overline{P}), F(\overline{P}) \mid E \sqsubseteq^o F \ \wedge \ \overline{P}, E(\overline{P}), F(\overline{P})$ *closed* $\}$. Then it suffices to show that $\mathcal{R}$ is a prebisimulation. Suppose $E(\overline{P}) \xrightarrow{u} Q$. Then either $\exists E'.E \xrightarrow{u} E' \ \wedge \ Q \equiv E'(\overline{P})$ or $\exists X.E \rhd X \ \wedge \ P \xrightarrow{u} Q$, where P is substituted for X. So either $\exists F'.F \overset{\hat{u}}{\Longrightarrow} F' \ \wedge \ E' \sqsubseteq^o F'$ or $F \rhd X$ when $F(\overline{P}) \overset{u}{\Longrightarrow} Q$. In either case $\exists R.F(\overline{P}) \overset{\hat{u}}{\Longrightarrow} R \ \wedge \ Q\mathcal{R}R$. Suppose $E(\overline{P}) \Downarrow u$. Then $E \Downarrow u$ and so $F \Downarrow u$. Thus if $F \rhd X$ then $E \rhd X$ and it follows that $F(\overline{P}) \Downarrow u$. Suppose that $F(\overline{P}) \xrightarrow{u} R$. Then either $\exists F'.F \xrightarrow{u} F' \wedge R \equiv F'(\overline{P})$, or $\exists X.F \rhd X \wedge P \xrightarrow{u} R$ where P is substituted for X. So since $E \Downarrow u$, either $\exists E'.E \overset{\hat{u}}{\Longrightarrow} E' \ \wedge \ E' \sqsubseteq^o F'$ or $E \rhd X \ \wedge \ E(\overline{P}) \overset{u}{\Longrightarrow} R$. In either case $\exists Q.E(\overline{P}) \overset{\hat{u}}{\Longrightarrow} Q \ \wedge \ Q\mathcal{R}R$.
($\Leftarrow$) Let $R = \{(E,F) \mid$ if $fv(E) \cup fv(F) \subseteq \overline{X}$ then $\forall \overline{P}.E(\overline{P}) \sqsubseteq F(\overline{P})\}$. Then it suffices to show that $\mathcal{R} \subseteq \mathcal{G}(\mathcal{R})$. Suppose $E \xrightarrow{u} E'$. Then $E(\overline{0}) \xrightarrow{u} E'(\overline{0})$ so since $E(\overline{0}) \sqsubseteq F(\overline{0})$, $\exists F'.F \overset{\hat{u}}{\Longrightarrow} F' \ \wedge \ E'(\overline{0}) \sqsubseteq F'(\overline{0})$, when $E'\mathcal{R}F'$. Suppose $E \Downarrow u$. Then $E(\overline{0}) \Downarrow u$, so since $E(\overline{0}) \sqsubseteq F(\overline{0})$, $F(\overline{0}) \Downarrow u$ and hence $F \Downarrow u$. Also if $F \xrightarrow{u} F'$ then $F(\overline{0}) \xrightarrow{u} F'(\overline{0})$, so since $E(\overline{0}) \sqsubseteq F(\overline{0})$ and $E(\overline{0}) \Downarrow u$, $\exists E'.E \overset{\hat{u}}{\Longrightarrow} E' \ \wedge \ E'(\overline{0}) \ \sqsubseteq F'(\overline{0})$, when $E'\mathcal{R}F'$. Suppose $E \rhd X$. Then $F \rhd X$, since if not and $a \notin sort(F)$ then setting $P = E\{a.0/X, \overline{0}/\overline{X}\}$ and $Q = F\{a.0/X, \overline{0}/\overline{X}\}$, then $P \sqsubseteq Q$, $\exists P'.P \overset{a}{\Longrightarrow} P'$ but $\neg\exists Q'.Q \overset{a}{\Longrightarrow} Q'$, a contradiction. Suppose $E \Downarrow$ and $F \rhd X$. Then $E \rhd X$, since if not and $a \notin sort(E)$, then defining P and Q as above, $P \sqsubseteq Q$, $P \Downarrow a$, $\exists Q'.Q \overset{a}{\Longrightarrow} Q'$ but $\neg\exists P'.P \overset{a}{\Longrightarrow} P'$, a contradiction. $\square$

Define $\sqsubseteq^{o,*}$ by setting $E \sqsubseteq^{o,*} F$ iff
(i) if $E \xrightarrow{a} E'$ then $\exists F'.F \overset{a}{\Longrightarrow} F' \ \wedge \ E' \sqsubseteq^o F'$,
(ii) if $E \overset{\tau}{\Longrightarrow} E'$ then
  (a) if $E' \Downarrow$ then $\exists F'.F \overset{\tau}{\Longrightarrow} F' \ \wedge \ E' \sqsubseteq^o F'$
  (b) if $E' \Uparrow$ then $\exists F'.F \overset{\epsilon}{\Longrightarrow} F' \ \wedge \ E' \sqsubseteq^o F'$,
(iii) $\forall u$. if $E \Downarrow u$ then
  (a) $F \Downarrow u$
  (b) if $F \xrightarrow{u} F'$ then $\exists E'.E \overset{u}{\Longrightarrow} E' \ \wedge \ E' \sqsubseteq^o F'$,
(iv) if $E \rhd W$ then $F \rhd W$,
(v) if $E \Downarrow$ then if $F \rhd W$ then $E \rhd W$.

<u>Lemma 27</u> $E \sqsubseteq^{o,*} F$ iff $E \sqsubseteq^* F$.

Proof: It is quite straight forward to establish this result directly. Alternatively, defining $\sqsubseteq^{o,+}$ by setting $E \sqsubseteq^{o,+} F$ iff $\forall G.E + F \sqsubseteq^o F + G$, it follows from Lemma 26 that $E \sqsubseteq^{o,+} F$ iff $E \sqsubseteq^{+} F$. Furthermore, by an argument similar to that in the proof of

Lemma 3, with an extension to handle free, unguarded occurrences of variables, it may be established that $E \sqsubseteq^{o,+} F$ iff $E \sqsubseteq^{o,*} F$. The result then follows by Lemma 3. □

Overriding the use of the symbols in the preceding section let $\leq$ and $=$ be the smallest pair of relations on **FS** such that $\leq$ is reflexive and transitive and preserved by $u.$, $+$ and μX, and $E = F$ iff $E \leq F$ and $F \leq E$, and such that:

(S1) $E + F = F + E$
(S2) $E + (F + G) = (E + F) + G$
(S3) $E + E = E$
(S4) $E + \mathbf{0} = E$

(τ1) $u.\tau.E = u.E$
(τ2) $E + \tau.E = \tau.E$
(τ3) $u.(E + \tau.F) = u.(E + \tau.F) + u.F$

($\perp$1) $\perp \leq E$
($\perp$2) $\tau.(E + \perp) \leq E + \perp$

(R1) if $Y \notin fv(E)$ then $\mu XE = \mu Y(E\{Y/X\})$
(R2) $\mu XE = E\{\mu XE/X\}$
(R3) if $\neg E \rhd X$ then if $F = E\{F/X\}$ then $F = \mu XE$
(R4) $\mu X(E + X) \leq \mu X(E + \perp)$
(R5) $\mu X(\tau.(E + X) + F) \leq \mu X(E + F + \perp)$.

<u>Remark</u> By ($\perp$1), ($\perp$2), (τ2), (R4) and (R5) the following hold:
($\perp$2′) $\tau.(E + \perp) = E + \perp$
(R4′) $\mu X(E + X) = \mu X(E + \perp)$
(R5′) $\mu X(\tau.(E + X) + F) = \mu(E + F + \perp)$.

<u>Theorem 28</u> $E \leq F$ iff $E \sqsubseteq^{o,*} F$.

The proof stretches over several lemmas.

<u>Lemma 29</u> If $E \leq F$ then $E \sqsubseteq^{o,*} F$.

Proof: The result is straight forward except in the case of (R3), (R4) and (R5). For (R3) it suffices to establish the following.

(1) Suppose that $fv(E) \subseteq \overline{X}$, that each X in $\overline{X}$ is guarded in E, that $\overline{P} \simeq^c E\{\overline{P}/\overline{X}\}$ and that $Q \simeq^c E\{\overline{Q}/\overline{X}\}$. Then $\overline{P} \simeq^c \overline{Q}$.

A slightly more general result is obtainable. A free occurrence of a variable X in

an expression $E \in \mathbf{E}$ is said to be *sequential* iff X does not lie within the scope of any operator $|$, $\backslash a$ or $[f]$.

(2) Suppose that $fv(E) \subseteq \overline{X}$, that each X in $\overline{X}$ is both guarded and sequential in E, that $\overline{P} \simeq^c E\{\overline{P}/\overline{X}\}$ and that $\overline{Q} \simeq^c E\{\overline{Q}/\overline{X}\}$. Then $\overline{P} \simeq^c \overline{Q}$.

(2) will be proved in the case when $\overline{X}$ consists of a single variable X. The general case is entirely similar. If G and R are expressions and X a variable then $G(R)$ abbreviates $G\{R/X\}$. First:

(3) Suppose that $fv(E) \subseteq \{X\}$, that X is guarded and sequential in E and that $E(P) \xrightarrow{u} P'$. Then there is E' such that $E \xrightarrow{u} E' \wedge P' \equiv E'(P)$. Moreover X is sequential in E', if $u = \tau$ then X is guarded in E', and for any Q, $E(Q) \xrightarrow{u} E'(Q)$.
The proof, which is omitted, is a straight forward induction on the depth of the inference of $E(P) \xrightarrow{u} P'$.

(4) If $fv(E) \subseteq \{X\}$, X is guarded and sequential in E and $E(P) \Downarrow u$, then for any Q,
(a) $E(Q) \Downarrow u$ and (b) if $E(Q) \xrightarrow{u} Q'$ then there is E' such that $E \xrightarrow{u} E'$ and $Q' \equiv E'(Q)$, and such that $E(P) \xrightarrow{u} E'(P)$, X is sequential in E, and if $u = \tau$ then X is guarded in E'.
The proof, which is omitted, is by induction on the depth of the inference of $E(P) \Downarrow u$.

Now set $\mathcal{R} = \{(G(P), G(Q)) \mid G \text{ sequential}, fv(E) \subseteq \{X\}\}$. To show that $P \sqsubseteq^c Q$ it suffices (cf. the proof of Lemma 5) to establish that $\mathcal{R}$ is a prebisimulation up to $\sqsubseteq$. (A symmetric argument may be used to show that $Q \sqsubseteq^c P$.) For this it suffices to establish:

(5) If $G(P) \xrightarrow{u} P'$ then $\exists Q'.G(Q) \stackrel{u}{\Longrightarrow} Q' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$.
(6) If $G(P) \Downarrow u$ then $G(P) \Downarrow u$ and if $G(Q) \xrightarrow{u} Q'$ then $\exists P'.G(P) \stackrel{u}{\Longrightarrow} P' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$.

To establish (5) and (6) it is convenient first to prove:

(7) If $G(P) \stackrel{\tau}{\Longrightarrow} P'$ then $\exists Q'.Q \stackrel{\epsilon}{\Longrightarrow} Q' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$.
(8) If $G(P) \Downarrow$ then $G(Q) \Downarrow$ and if $G(Q) \stackrel{\tau}{\Longrightarrow} Q'$ then $\exists P'.P \stackrel{\epsilon}{\Longrightarrow} P' \wedge P' \sqsubseteq \mathcal{R} \sqsubseteq Q'$.
For (7): If $G(P) \stackrel{\tau}{\Longrightarrow} P'$ then since $G(P) \sqsubseteq^c G(E(P))$, $\exists P''.G(E(P)) \stackrel{\epsilon}{\Longrightarrow} P'' \wedge P' \sqsubseteq P''$. Since X is guarded and sequential in $G(E)$, by (3) applied repeatedly there is F such that X is guarded and sequential in F, $G(E) \stackrel{\epsilon}{\Longrightarrow} F$ and $P'' \equiv F(P)$. Then $G(E(Q)) \stackrel{\epsilon}{\Longrightarrow} F(Q)$ and so since $G(E(Q)) \sqsubseteq^c E(Q)$, $\exists Q'.E(Q) \stackrel{\epsilon}{\Longrightarrow} Q' \wedge F(Q) \sqsubseteq Q'$. Then $E(Q) \stackrel{\epsilon}{\Longrightarrow} Q' \wedge P' \sqsubseteq F(P)\mathcal{R}F(Q) \sqsubseteq Q'$.
For (8): If $G(P) \Downarrow$ then since $G(P) \sqsubseteq^c G(E(P))$, $G(E(P)) \Downarrow$. Then since X is guarded and sequential in $G(E)$, by (4), $G(E(Q)) \Downarrow$ and hence since $G(E(Q)) \sqsubseteq G(Q)$, $G(Q) \Downarrow$. Also if $G(Q) \stackrel{\tau}{\Longrightarrow} Q'$ then since $G(E(Q)) \sqsubseteq^c G(Q)$ and $G(E(Q)) \Downarrow$, $\exists Q''.G(E(Q)) \stackrel{\tau}{\Longrightarrow} Q'' \wedge Q'' \sqsubseteq Q'$. Then since $G(E(P)) \Downarrow$, by (4) applied repeatedly, $\exists P'.G(E(P)) \stackrel{\epsilon}{\Longrightarrow} P'' \wedge P''\mathcal{R}Q''$. Then since $G(P) \sqsubseteq^c G(E(P))$ and $G(P) \Downarrow$, $\exists P'.G(P) \stackrel{\epsilon}{\Longrightarrow} P' \wedge P' \sqsubseteq P''$. Then

$G(P) \stackrel{\epsilon}{\Longrightarrow} P'$ and $P' \sqsubseteq P'' \mathcal{R} Q'' \sqsubseteq Q'$.
For (5): Suppose $G(P) \xrightarrow{u} P'$. Then since $G(P) \sqsubseteq^c G(E(P))$ either $P' \sqsubseteq G(E(P)) \mathcal{R} G(E(Q)) \sqsubseteq G(Q)$, or $\exists P'', P'''.G(E(P)) \stackrel{\epsilon}{\Longrightarrow}\xrightarrow{u} P'' \stackrel{\epsilon}{\Longrightarrow} P''' \wedge P' \sqsubseteq P'''$. Since X is guarded and sequential in $G(E)$, by (2) applied repeatedly there is a sequential H such that $G(E) \stackrel{\epsilon}{\Longrightarrow}\xrightarrow{u} H$, $P'' \equiv H(P)$ and $G(E(Q)) \stackrel{\epsilon}{\Longrightarrow}\xrightarrow{u} H(Q) \equiv Q''$. Then by (7), $\exists Q'''.Q'' \stackrel{\epsilon}{\Longrightarrow} Q''' \wedge P''' \sqsubseteq \mathcal{R} \sqsubseteq Q'''$. Since $G(E(Q)) \sqsubseteq^c G(Q)$, $\exists Q'.G(Q) \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge Q''' \sqsubseteq Q'$. Thus $G(Q) \stackrel{\hat{u}}{\Longrightarrow} Q' \wedge P' \sqsubseteq P''' \sqsubseteq \mathcal{R} \sqsubseteq Q''' \sqsubseteq Q'$.
For (6): Suppose $G(P) \Downarrow u$. Then since $G(P) \sqsubseteq^c G(E(P))$, $G(E(P)) \Downarrow u$ and hence since X is guarded and sequential in $G(E)$, by (4), $G(E(Q)) \Downarrow u$ and hence $G(Q) \Downarrow u$, since $G(E(Q)) \sqsubseteq^c G(Q)$. Suppose $G(Q) \xrightarrow{u} Q'$. Then since $G(E(Q)) \sqsubseteq^c G(Q)$, $\exists Q'', Q'''.G(E(Q)) \stackrel{\epsilon}{\Longrightarrow}\xrightarrow{u} Q'' \stackrel{\epsilon}{\Longrightarrow} Q''' \wedge Q'' \sqsubseteq Q'$. By (4) applied repeatedly there is a sequential H such that $G(E) \stackrel{\epsilon}{\Longrightarrow}\xrightarrow{u} H$, $Q'' \equiv H(Q)$ and $G(E(P)) \stackrel{\epsilon}{\Longrightarrow}\xrightarrow{u} H(P)$. Thus by (8), since $H(P) \Downarrow$, $\exists P'.H(P) \stackrel{\epsilon}{\Longrightarrow} P''' \wedge P''' \sqsubseteq \mathcal{R} \sqsubseteq Q'''$. Then since $G(P) \sqsubseteq^c G(E(P))$ and $G(P) \Downarrow u$, $\exists P'.G(P) \stackrel{u}{\Longrightarrow} P' \wedge P' \sqsubseteq P'''$. Thus $G(P) \stackrel{u}{\Longrightarrow} P' \wedge P' \sqsubseteq P''' \sqsubseteq \mathcal{R} \sqsubseteq Q''' \sqsubseteq Q'$.

This completes the proof that $\mathcal{R}$ is a prebisimulation up to $\sqsubseteq$, hence the proof of (2) and hence the proof of the soundness of (R3).

For (R4) set $A \equiv \mu X(E+X)$ and $B \equiv \mu X(E+\bot)$. Then setting $\mathcal{R} = \{(F(A), F(B)) \mid F \in \mathbf{E}\}$, it suffices to establish the following assertions.
(1) If $F(A) \xrightarrow{u} G$ then $\exists H.F(B) \stackrel{u}{\Longrightarrow} H \wedge G\mathcal{R}H$.
This is established by induction on the inference of $F(A) \xrightarrow{u} G$. The proof is straight forward. For example, if $F \equiv X$ then if $A \xrightarrow{u} G$ then $E(A) \xrightarrow{u} G$ by a shorter inference so $\exists H.E(B) \stackrel{u}{\Longrightarrow} H \wedge G\mathcal{R}H$, when also $B \stackrel{u}{\Longrightarrow} H$.
(2) If $F(A) \Downarrow u$ then $F(B) \Downarrow u$ and if $F(B) \xrightarrow{u} H$ then $\exists G.F(A) \stackrel{u}{\Longrightarrow} G \wedge G\mathcal{R}H$.
This is established by induction on the depth of the inference of $F(A) \Downarrow u$. The case $F \equiv X$ holds vacuously. This completes the sketch of the proof of the soundness of (R4).

For (R5), setting $A \equiv \mu X(\tau.(E+X)+F)$, $B \equiv \mu X(E+F+\bot)$ and $\mathcal{R} = \{(G(A)+H, G(B)+H) \mid G, H \in \mathbf{FS}\}$ it suffices to establish that $\mathcal{R} \subseteq \mathcal{G}(\mathcal{R})$, since then for any $H \in \mathbf{FS}$, $A+H \sqsubseteq^o B+H$ and hence by the observation in the proof of Lemma 27, $A \sqsubseteq^{o,*} B$. For this it suffices to show the following.

(1) If $G(A)+H \xrightarrow{u} J$ then either $\exists K.G(B)+H \stackrel{u}{\Longrightarrow} K \wedge J\mathcal{R} \sqsubseteq^o K$ or $u = \tau \wedge J \Uparrow \wedge J\mathcal{R} \sqsubseteq^o G(B)+H$.
This is proved by induction on the depth of the inference of $G(A)+H \xrightarrow{u} J$.
(2) If $G(A)+H \Downarrow u$ then $G(B)+H \Downarrow u$ and if $G(B)+H \xrightarrow{u} K$ then $\exists J.G(A)+H \stackrel{u}{\Longrightarrow} K \wedge J\mathcal{R} \sqsubseteq^o K$.
This is proved by induction on the depth of the inference of $G(A)+H \Downarrow u$.
The only case considered here is $G \equiv X$. Suppose $A+H \xrightarrow{u} J$. If $H \xrightarrow{u} J$ then $B+H \xrightarrow{u} J$. So suppose $A \xrightarrow{u} J$. Then $\tau.(A+E(A))+F(A) \xrightarrow{u} J$ by a shorter inference. If $F(A) \xrightarrow{u} J$ then either $\exists K.F(B) \stackrel{u}{\Longrightarrow} K \wedge J\mathcal{R} \sqsubseteq^o K$, when $B \stackrel{u}{\Longrightarrow} K$, or

$u = \tau \wedge J\Uparrow \wedge J\mathcal{R} \sqsubseteq^o F(B)$, when $J\mathcal{R} \sqsubseteq E(B) + F(B) + \perp + H = B + H$, since $B\Uparrow$. Suppose $u = \tau \wedge J \equiv A + E(A)$. Then $J\mathcal{R} \sqsubseteq^o B + H$. In this case (2) holds vacuously since $A\Uparrow$. The remaining cases are straight forward.

This completes the proof of Lemma 29. □

An expression of the form μXE (a *recursion*) is said to be *guarded* iff X is guarded in E. An expression F is said to be *guarded* iff every subexpression of F which is a recursion is guarded.

The first step in the completeness proof is to use (R4′) and (R5′) to reduce the problem to that of the completeness of the remaining axioms for guarded expressions.

<u>Lemma 30</u> For each E there is a guarded expression E' such that $E = E'$.

Proof: By induction on the structure of E. The only nontrivial case is when E is a recursion, $E \equiv \mu XF$. For this case first note:

(1) If Y occurs free and unguarded in G then $G = G + Y$.
This is established by a simple induction on the structure of G. If, e.g., $G \equiv \mu ZH$ where $Z \neq Y$ then by induction hypothesis $H = H + Y$ and hence $G = H\{G/Z\} = H\{G/Z\} + Y = G + Y$.

(2) There is a guarded expression F' such that
(a) X is guarded in F',
(b) no free occurrence of a variable Y if F' lies within a recursion,
(c) $\mu XF = \mu XF'$.
This is established by induction on the depth of the nesting of recursions in F. Assume (2) for all expressions in which the depth of nesting of recursions is less than that of F. For each recursion μYG in F which is not nested within any recursion in F, by induction hypothesis there is an expression G' such that Y is guarded in G', no free, unguarded occurrence of a variable Z in G' lies within a recursion in G' and $\mu YG = \mu YG'$. Let F_0 be the result of simultaneously replacing each such 'outermost' recursion μYG in F by $G'\{\mu YG'/Y\}$. Then $F = F_0$ and no free, unguarded occurrence of a variable Z lies within a recursion in F_0. Each free, unguarded occurrence of X in F_0 is 'weakly guarded by τ' to a certain 'depth'. For example in the expression $X+\tau.(\mu Y(a.Y)+X+\tau.(X+\perp))$, the ocurrences of X are 'weakly guarded by τ' to 'depths' 0,1 and 2 (reading from left to right). The following procedure provides a means of obtaining from F_0 an expression F' satisfying (a), (b) and (c) above.

If X is guarded in F take $F' = F$. Suppose X is not guarded in F and choose some free, unguarded occurrence of X in F_0. If the 'depth of weak guarding by τ' of the occurrence is 0 then $F_0 \equiv G + X$ for some G. In this case set $F_1 = G + \perp$. Otherwise the 'depth' is greater than 0, so since no free, unguarded occurrence of X lies within a

recursion in F_0, $F_0 = \tau.G + H$ for some G and H with the chosen occurrence lying in G. By (1), $F_0 = \tau.(G + X) + H$. In this case set $F_1 = G + H + \bot$. Iteration of this process will produce a term F' in which X is guarded, for the procedure does not increase the 'depth' of any free occurrence of X in a given expression. It is clear that no free, unguarded occurrence of a variable Y in F' lies within a recursion. That $\mu XF = \mu XF'$ follows from (R4′) and (R5′).

This completes the proof of Lemma 30. □

Some central concepts in the proof are given in the following sequence of definitions. Let $\overline{X} = \langle X_0, \ldots, X_m \rangle$ and $\overline{W} = \langle W_0, \ldots, W_m \rangle$ be disjoint lists of variables. Let $\overline{H} = \langle H_0, \ldots, H_m \rangle$ be a list of expressions with $fv(H_i) \subseteq \overline{X} \cup \overline{W}$ and with each H_i of the form $\sum_j u_{i_j}.Xi_j + \sum_k W_{i_k} \{+\bot\}$. Then $S = \langle \langle X_0, H_0 \rangle, \ldots, \langle X_m, H_m \rangle \rangle$ is a *standard equation list.* The set of free variables of S is $fv(S) =_{df} \{W \mid W \notin \overline{X}$ and W is a summand of some $H_i\}$. Let $X_i \xrightarrow{u}_S X_j$ denote that $u.X_j$ is a summand of H_i, $X_i \rhd_S W$ that W is a summand of H_i and $X_i \rhd_S \bot$ that $\bot$ is a summand of H_i. S is *cycle-free* iff $\neg\exists i.X_i(\xrightarrow{\tau}_S)^+X_i$. S is *saturated* iff
(i) if $X_i(\xrightarrow{\tau}_S)^* \xrightarrow{u}_S (\xrightarrow{\tau}_S)^*X_j$ then $X_i \xrightarrow{u}_S X_j$,
(ii) if $X_i \xrightarrow{\tau}_S^* X_j$ and $X_j \rhd_S W$ then $X_i \rhd_S W$,
(iii) if $X_i \xrightarrow{\tau}_S^* X_j$ and $X_j \rhd_S \bot$ then $X_i \rhd_S \bot$,
and *fully saturated* iff in addition
(iv) $\neg\exists i.X_0 \xrightarrow{\tau}_S X_i \wedge X_i \rhd_S \bot$.

E *provably satisfies* S iff there is $\overline{E} = \langle E_0, \ldots, E_m \rangle$ such that $E_0 \equiv E$ and for each i, $E_i = H_i\{\overline{E}/\overline{X}\}$. Finally, W is *inaccessible* in S iff $\neg\exists i.X_0 \xrightarrow{\tau}_S^* X_i \wedge X_i \rhd_S W$.

The following two lemmas show that via the notion 'E provably satisfies S' there is a close correspondence between guarded expressions and cycle-free, standard equation lists.

<u>Lemma 31</u> For each guarded expression E with $fv(E) \subseteq \overline{W}$ there is a cycle-free, standard equation list S with $fv(S) \subseteq \overline{W}$ such that E provably satisfies S. Moreover if V is guarded in E then V is inaccessible in S.

Proof: By induction on the structure of E. If $E \equiv W$, $E \equiv \mathbf{0}$ or $E \equiv \bot$, take S to be the single equation $X = W$, $X = \mathbf{0}$ or $X = \bot$ respectively. Suppose $E \equiv u.F$ and $S : \overline{X} = \overline{H}$ is a list for F. Take $S' = \langle Y = u.X_0 \rangle \frown S$. Suppose $E \equiv F + G$ and $S = \overline{X} = \overline{H}$ and $T = \overline{Y} = \overline{J}$ are lists for F and G respectively with $\overline{X} \cap \overline{Y} = \emptyset$. Take $S' = \langle Z = X_0 + Y_0 \rangle \frown S \frown T$. Suppose $E \equiv \mu WF$ where W is guarded in F and $S = \overline{X} = \overline{H}$ is a list for F. Take $S' = \langle X_i = H_i\{H_0/W\} \rangle$. □

<u>Lemma 32</u> For each cycle-free, standard equation list S with $fv(S) \subseteq \overline{W}$ there is an expression $D \equiv D(S)$ with $fv(D) \subseteq \overline{W}$ such that
(a) D provably satisfies S,

(b) if E provably satisfies S and $fv(E) \subseteq \overline{W}$ then $E = D$.

Proof: Suppose $S = \langle \overline{X} = \overline{H} \rangle \frown \langle X_m = H_m \rangle$ where $m \geq 0$. It suffices to prove, by induction on m, the following assertion.

(1) There are expressions $\overline{D}$ with $fv(\overline{D}) \subseteq \overline{W}$ such that $\overline{D} = \overline{H}\{\overline{D}/\overline{X}\}$ and such that if $\overline{E} = \overline{H}\{\overline{E}/\overline{X}\}$ and $fv(\overline{E}) \subseteq \overline{W}$ then $\overline{D} = \overline{E}$.
If $m = 0$ take $D = \mu X_0 H_0$. The result then follows by (R2) and (R3). Suppose $m > 0$. Set $\overline{J} = \overline{H}\{\mu X_m H_m / X_m\}$. Then $T = \langle \overline{X} = \overline{J} \rangle$ is a cycle-free, standard equation list so by the induction hypothesis there are $\overline{D}$ such that $\overline{D} = \overline{J}\{\overline{D}/\overline{X}\}$, $fv(\overline{D}) \subseteq \overline{W}$ and such that if $\overline{E} = \overline{J}\{\overline{E}/\overline{X}\}$ and $fv(\overline{E}) \subseteq \overline{W}$ then $\overline{D} = \overline{E}$. Set $D_m = (\mu X_m H_m)\{\overline{D}/\overline{X}\}$. Then $\overline{D} \frown \langle D_m \rangle = \overline{H}\{\overline{D}/\overline{X}, D_m/X_m\} \frown H_m\{\overline{D}/\overline{X}, D_m/X_m\}$. Now suppose that $\overline{E} \frown \langle E_m \rangle = \overline{H}\{\overline{E}/\overline{X}, E_m/X_m\} \frown H_m\{\overline{E}/\overline{X}, E_m/X_m\}$ where $fv(\overline{E} \frown \langle E_m \rangle) \subseteq \overline{W}$. Then in particular $E_m = H_m\{\overline{E}/\overline{X}, E_m/X_m\}$. Since X_m is guarded in $H_m\{\overline{E}/\overline{X}\}$, by (R3), $E_m = \mu X_m(H_m\{\overline{E}/\overline{X}\}) = (\mu X_m H_m)\{\overline{E}/\overline{X}\}$. Then $\overline{E} = \overline{H}\{\overline{E}/\overline{X}, E_m/X_m\} = \overline{H}\{\mu X_m H_m/X_m\}\{\overline{E}/\overline{X}\} = \overline{J}\{\overline{E}/\overline{X}\}$.

Hence by induction hypothesis $\overline{D} = \overline{E}$ and so $D_m = (\mu X_m H_m)\{\overline{D}/\overline{X}\} = (\mu X_m H_m)\{\overline{E}/\overline{X}\} = E_m$. □

The next result expresses a monotonicity property of provable solutions of cycle-free, standard equation lists.

<u>Lemma 33</u> Suppose that $S = \langle \overline{X} = \overline{H} \rangle$ and $T = \langle \overline{X} = \overline{J} \rangle$ are cycle-free, standard equation lists with $\overline{H} \leq \overline{J}$, i.e. $H_i \leq J_i$ for each i. Then $D(S) \leq D(T)$.

Proof: By induction on the number of equations in S (and in T). If this number is 1 and $S = \langle X = H \rangle$ and $T = \langle X = J \rangle$ with $H \leq J$ then $D(S) \equiv \mu X H \leq \mu X J \equiv D(T)$. Now suppose $S = \langle \overline{X} = \overline{H} \rangle \frown \langle X_m = H_m \rangle$ and $T = \langle \overline{X} = \overline{J} \rangle \frown \langle X_m = J_m \rangle$ where $m \geq 1$. Let $\overline{H'} = \overline{H}\{\mu X_m H_m / X_m\}$ and $\overline{J'} = \overline{J}\{\mu X_m J_m / X_m\}$. Then $\overline{H'} \leq \overline{J'}$ so setting $S' = \langle \overline{X} = \overline{H'} \rangle$ and $T' = \langle \overline{X} = \overline{J'} \rangle$, by induction hypothesis $D(S') \leq D(T')$. By the proof of Lemma 32, $D(S) = D(S') \frown \langle (\mu X_m H_m)\{D(S')/\overline{X}\} \rangle$ and $D(T) = D(T') \frown \langle (\mu X_m J_m)\{D(T')/\overline{X}\} \rangle$, so $D(S) \leq D(T)$. □

The next two results show that cycle-free, standard equation lists may be fully saturated while preserving provable satisfiability.

<u>Lemma 34</u> Suppose that S is a cycle-free, standard equation list and that E provably satisfies S. Then there is a saturated, cycle-free, standard equation list S' such that E provably satisfies S'.

Proof: Rather than giving a rigorous proof, a method for obtaining S' from S, and for showing that E provably satisfies S', is illustrated by an example. Suppose that S is

$$
\begin{aligned}
X_0 &= a.X_1 + \tau.X_2 \\
X_1 &= \tau.X_2 \\
X_2 &= b.X_3 + \tau.X_3 + \bot \\
X_3 &= a.X_0 + W.
\end{aligned}
$$

S' is obtained from S using $(\tau 1)$, $(\tau 2)$ and $(\tau 3)$ to saturate the equations as follows.

$$
\begin{aligned}
X_0 &= [a.X_1] + \tau.X_2 \\
&= [a.X_1 + a.X_2] + (\tau.X_2) \\
&= a.X_1 + [a.X_2] + (\tau.X_2 + \tau.X_3 + b.X_3 + \bot + W) \\
&= a.X_1 + [a.X_2 + a.X_3] + \tau.X_2 + \tau.X_3 + b.X_3 + \bot + W
\end{aligned}
$$

The brackets [ ] and ( ) indicate the subexpression 'expanded' and the result obtained at successive steps. Thus $a.X_1$ is 'expanded' to $a.X_1 + a.X_2$ at Step 1 and $\tau.X_2$ is 'expanded' to $\tau.X_2 + b.X_3 + \bot + W$ at Step 2, and so on.

$$
\begin{aligned}
X_1 &= [\tau.X_2] \\
&= [\tau.X_2 + b.X_3 + (\tau.X_3) + \bot] \\
&= \tau.X_2 + b.X_3 + (\tau.X_3 + [a.X_0] + W) + \bot \\
&= \tau.X_2 + b.X_3 + \tau.X_3 + [a.X_0 + (a.X_2)] + W + \bot \\
&= \tau.X_2 + b.X_3 + \tau.X_3 + a.X_0 + (a.X_2 + a.X_3) + W + \bot
\end{aligned}
$$

$$
\begin{aligned}
X_2 &= b.X_3 + [\tau.X_3] + \bot \\
&= b.X_3 + [\tau.X_3 + (a.X_0) + W] + \bot \\
&= b.X_3 + \tau.X_3 + (a.X_0 + [a.X_2]) + W + \bot \\
&= b.X_3 + \tau.X_3 + a.X_0 + [a.X_2 + a.X_3] + W + \bot
\end{aligned}
$$

$$
\begin{aligned}
X_3 &= [a.X_0] + W \\
&= [a.X_0 + (a.X_2)] + W \\
&= a.X_0 + (a.X_2 + a.X_3) + W.
\end{aligned}
$$

Thus S' is

$$
\begin{aligned}
X_0 &= a.X_1 + a.X_2 + a.X_3 + \tau.X_2 + \tau.X_3 + b.X_3 + W + \bot \\
X_1 &= \tau.X_2 + b.X_3 + \tau.X_3 + a.X_0 + a.X_2 + a.X_3 + W + \bot \\
X_2 &= b.X_3 + \tau.X_3 + a.X_0 + a.X_2 + a.X_3 + W + \bot \\
X_3 &= a.X_0 + a.X_2 + a.X_3 + W.
\end{aligned}
$$

Since E provably satisfies S there are E_0, E_1, E_2, E_3 such that $E_0 \equiv E$ and

$$
\begin{aligned}
E_0 &= a.E_1 + \tau.E_2 \\
E_1 &= \tau.E_2 \\
E_2 &= b.E_3 + \tau.E_3 + \bot \\
E_3 &= a.E_0 + W.
\end{aligned}
$$

Following the derivation of the equations S' and using $(\tau 1)$, $(\tau 2)$ and $(\tau 3)$ it is possible to check that E provably satisfies S'. □

Lemma 35 Suppose that S' is a saturated, cycle-free, standard equation list and that E provably satisfies S'. Then there is a fully saturated, cycle-free, standard equation list S'' such that E provably satisfies S''.

Proof: As with Lemma 34, rather than giving a rigorous proof, a method for constructing S'', and for showing that E provably satisfies S'', is illustrated by an example. Let S' be as constructed in the proof of Lemma 34. Then S' is not fully saturated since $X_0 \xrightarrow{\tau}_S X_2$ and $X_2 \rhd_S \bot$. S'' is obtained from S' using ($\bot 2'$) and (S3) to reduce the first equation as follows.

$$\begin{aligned} X_0 &= a.X_1 + a.X_2 + a, X_3 + [\tau.X_2] + \tau.X_3 + b.X_3 + W + \bot \\ &= a.X_1 + a.X_2 + a.X_3 + [\tau.(b.X_3 + \tau.X_3 + a.X_0 + a.X_2 + a.X_3 + W + \bot)] \\ &\quad + \tau.X_3 + b.X_3 + W + \bot \\ &= a.X_1 + a.X_2 + a.X_3 + [b.X_3 + \tau.X_3 + a.X_0 + a.X_2 + a.X_3 + W + \bot] + \tau.X_3 \\ &\quad + b.X_3 + W + \bot \\ &= a.X_1 + a.X_2 + a.X_3 + b.X_3 + \tau.X_3 + a.X_0 + W + \bot. \end{aligned}$$

Using ($\bot 2'$) it may be established that E provably satisfies S''. □

The next lemma is the principal step in the proof of completeness.

Lemma 36 Suppose that $E \sqsubseteq^{o,*} F$, that S and T are fully saturated, cycle-free, standard equation lists, that E provably satisfies S and that F provably satisfies T. Then there are cycle-free, standard equation lists $U = \langle \overline{Z} = \overline{K} \rangle$ and $V = \langle \overline{Z} = \overline{L} \rangle$ such that
(a) E provably satisfies U,
(b) F provably satisfies V,
(c) $\overline{K} \leq \overline{L}$.

Proof: By (R1) it may be assumed that $S = \langle \overline{X} = \overline{H} \rangle$ and $T = \langle \overline{Y} = \overline{J} \rangle$ where $\overline{X}$, $\overline{Y}$ and $\overline{W}$ are pairwise disjoint, and that $\overline{E} = \overline{H}\{\overline{E}/\overline{X}\}$ with $E_0 \equiv E$ and $\overline{F} = \overline{J}\{\overline{F}/\overline{Y}\}$ with $F_0 \equiv F$ where $fv(\overline{E})$, $fv(\overline{F}) \subseteq \overline{W}$. Further it may be assumed that every variable in $\overline{X}$ is 'reachable in S' from X_0 and that every variable in $\overline{Y}$ is 'reachable in T' from Y_0.

Since $E \sqsubseteq^{o,*} F$ and S and T are fully saturated
(a) if $X_0 \xrightarrow{u}_S X_i$ then $\exists j.Y_0 \xrightarrow{u}_T Y_j \wedge E_i \sqsubseteq^o F_j$,
(b) if $\neg X_0 \rhd_S \bot$ then $\neg Y_0 \rhd_T \bot$ and if $Y_0 \xrightarrow{\tau}_T Y_j$ then $\exists i.X_0 \xrightarrow{\tau}_S X_i \wedge E_i \sqsubseteq^o F_j$,
(c) if $\neg \exists X_i.X_0 \xrightarrow{a}_S .X_i \wedge X_i \rhd_S \bot$ then $\neg \exists j.Y_0 \xrightarrow{a}_T .Y_j \wedge Y_j \rhd_T \bot$ and if $Y_0 \xrightarrow{a}_T Y_j$ then $\exists i.X_0 \xrightarrow{a}_S X_i \wedge E_i \sqsubseteq^o F_j$,
(d) if $X_0 \rhd_S W$ then $Y_0 \rhd_T W$, and if $\neg X_0 \rhd_S \bot$ then if $Y_0 \rhd_T W$ then $X_0 \rhd_S W$.

Hence since $E \sqsubseteq^{o,*} F$ there is a relation $\mathcal{R} \subseteq \overline{X} \times \overline{Y}$ such that
(A) $\mathrm{dom}(\mathcal{R}) = \overline{X}$, $\mathrm{rng}(\mathcal{R}) = \overline{Y}$,
(B) if $X \mathcal{R} Y$ then
  (a) if $X \xrightarrow{u}_S X'$ then
    either (i) $\exists Y'.Y \xrightarrow{u}_T Y' \wedge X' \mathcal{R} Y'$

or (ii) $u = \tau \wedge X'\mathcal{R}Y$,

(b) if $\neg X \rhd_S \bot$ then $\neg Y \rhd_T \bot$ and if $Y \xrightarrow{\tau}_T Y'$ then $\exists X'.X \xrightarrow{\tau}_S X' \wedge X'\mathcal{R}Y'$ or $X\mathcal{R}Y'$,

(c) if $\neg\exists i.X \xrightarrow{a}_S .X_i \wedge X_i \rhd_S \bot$ then $\neg\exists j.Y \xrightarrow{a}_T .Y_j \wedge Y_j \rhd_T \bot$ and if $Y \xrightarrow{a}_T Y'$ then $\exists X'.X \xrightarrow{a}_S X' \wedge X'\mathcal{R}Y'$,

(d) if $X \rhd_S W$ then $Y \rhd_T W$, and if $\neg X \rhd_S \bot$ then if $Y \rhd_T W$ then $X \rhd_S W$,

(C) $X_0\mathcal{R}Y_0$ and

(a) if $X_0 \xrightarrow{\tau}_S X_i$ then $\neg X_i\mathcal{R}Y_0$,

(b) if $\neg X_0 \rhd_S \bot$ and $Y_0 \xrightarrow{\tau}_T Y_j$ then $\neg X_0\mathcal{R}Y_j$.

Choose a set $\{Z_{ij} \mid (X_i, Y_j) \in \mathcal{R}\}$ of new variables. Define expressions K_{ij} and L_{ij} as follows.

$$\begin{aligned}
K_{ij} &= \sum\{u.Z_{kl} \mid X_i \xrightarrow{u}_S X_k \wedge Y_j \xrightarrow{u}_T Y_l \wedge X_k\mathcal{R}Y_l\} \\
&+ \sum\{\tau.Z_{kj} \mid X_i \xrightarrow{\tau}_S X_k \wedge X_k\mathcal{R}Y_j\} \\
&+ \sum\{\tau.Z_{il} \mid \neg X_i \rhd_S \bot \wedge Y_j \xrightarrow{\tau}_T Y_l \wedge X_i\mathcal{R}Y_l\} \\
&+ \sum\{W \mid X_i \rhd_S W \wedge Y_j \rhd_T W\} \\
&+ \sum\{\bot \mid X_i \rhd_S \bot\}
\end{aligned}$$

$$\begin{aligned}
L_{ij} &= \sum\{u.Z_{kl} \mid X_i \xrightarrow{u}_S X_k \wedge Y_j \xrightarrow{u}_T Y_l \wedge X_k\mathcal{R}Y_l\} \\
&+ \sum\{\tau.Z_{kj} \mid X_i \xrightarrow{\tau}_S X_k \wedge X_k\mathcal{R}Y_j\} \\
&+ \sum\{\tau.Z_{il} \mid \neg X_i \rhd_S \bot \wedge Y_j \xrightarrow{\tau}_T Y_l \wedge X_i\mathcal{R}Y_l\} \\
&+ \sum\{u.Z_{il} \mid X_i \rhd_S \bot \wedge Y_j \xrightarrow{u}_T Y_l \wedge \neg\exists k.X_i \xrightarrow{u}_S X_k \wedge X_k\mathcal{R}Y_l\} \\
&+ \sum\{W \mid (X_i \rhd_S W \wedge Y_j \rhd_T W) \vee (X_i \rhd_S \bot \wedge Y_j \rhd_T W)\} \\
&+ \sum\{\bot \mid X_i \rhd_S \bot \wedge Y_j \rhd_T \bot\}.
\end{aligned}$$

Let $U = \langle \overline{Z} = \overline{K} \rangle$ and $V = \langle \overline{Z} = \overline{L} \rangle$. Then U and V are cycle-free, standard equation lists. Moreover $\overline{K} \leq \overline{L}$ since L_{ij} may be obtained form K_{ij} by replacing zero or more occurrences of $\bot$ by some terms, if $K_{ij} \rhd_U W$ then $L_{ij} \rhd_V W$ and if $\neg K_{ij} \rhd_U \bot$ and $L_{ij} \rhd_V W$ then $K_{ij} \rhd_U W$. Set

$$A_{ij} = \begin{cases} \tau.E_i & \text{if } \exists l.\tau.Z_{il} \text{ is a summand of } K_{ij} \\ E_i & \text{otherwise} \end{cases}$$

Then $A_{00} \equiv E$ by C(b) above and moreover $\overline{A} = \overline{K}\{\overline{A}/\overline{Z}\}$ by the construction, using $(\tau 1)$ and $(\tau 2)$. Set

$$B_{ij} = \begin{cases} \tau.F_j & \text{if } \exists k.\tau.Z_{kj} \text{ is a summand of } L_{ij} \\ F_j & \text{otherwise} \end{cases}$$

Then $B_{00} \equiv F$ by C(a) above and moreover $\overline{B} = \overline{L}\{\overline{B}/\overline{Z}\}$ by the construction. □

<u>Corollary 37</u> If $E \sqsubseteq^{o,*} F$ then $E \leq F$.

Proof: Suppose $E \sqsubseteq^{o,*} F$. By Lemma 30 let E' and F' be guarded expressions such that $E = E'$ and $F = F'$. By Lemma 31 let S and T be cycle-free, standard equation

lists such that E' provably satisfies S and F' provably satisfies T. By Lemmas 34 and 35 let S'' and T'' be fully saturated, cycle-free, standard equation lists such that E' provably satisfies S'' and F' provably satisfies T''. By Lemma 36 let $U = \langle \overline{Z} = \overline{K} \rangle$ and $V = \langle \overline{Z} = \overline{L} \rangle$ be cycle-free, standard equation lists such that E' provably satisfies U, F' provably satisfies V and $\overline{K} \leq \overline{L}$. By Lemma 32, $E' = D(U)$ and $F' = D(V)$. By Lemma 33, $D(U) \leq D(V)$. Hence $E \leq F$. □

This completes the proof of Theorem 28.

By Lemmas 27 and 28, the above axioms provide a sound and complete axiomatization of $\sqsubseteq^c$ on **FS**. The problem of determining such axiomatizations of $\sqsubseteq_0^{g,c}$ and $\sqsubseteq_0^c$ on **FS** may be tackled in a similar manner. Indeed by obtaining characterizations of $\sqsubseteq_0^g$ and $\sqsubseteq_0^{g,*}$, and of $\sqsubseteq_0$ and $\sqsubseteq_0^*$, on **FS** analagous to those of $\sqsubseteq$ and $\sqsubseteq^*$ in Lemmas 26 and 27 and, again overriding the use of the symbols in the preceding section, defining $\leq_0$ and $=_0$ to be the smallest pair of relations on **FS** satisfying the conditions in the definitions of $\leq$ and $=$, except for ($\tau 3$) $u.(P + \tau.Q) = u.(P + \tau.Q) + u.Q$ which is not valid for $\sqsubseteq_0^{g,c}$, then by a sequence of arguments similar to those in the proofs of Lemma 29 to Corollary 37 the following results may be established.

<u>Lemma 38</u> If $E \leq_0 F$ then $E \sqsubseteq_0^{g,c} F$.

<u>Lemma 39</u> If $E \sqsubseteq_0^c F$ then $E \leq F$.

Hence since $\sqsubseteq_0^{g,c} \subseteq \sqsubseteq_0^c$ they have a common axiomatization:

<u>Corollary 40</u> $E \sqsubseteq_0^{g,c} F$ iff $E \sqsubseteq_0^c F$ iff $E \leq_0 F$.

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