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Simple Consequence Relations

by

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1 Introduction

This paper has several purposes. From a very general perspective it aims to help clarifying questions like: What is a logic? What is a formal inference system? What are the differences between the usual kinds of formal systems and what is common to them? Are they the only possible useful ones? What is so special about the usual connectives and how does one characterize them? etc. At a first glance these questions seem to have only (?) philosophical interest. Their practical importance is immediately realized, though, when one is trying to design a general framework for implementing systems of logic on a computer. This is exactly the aim of the current Edinburgh LF-system (see [HHP], [AHM]). The present paper resulted from an attempt to solve basic problems which were encountered while developing this system. Our point of view is therefore a completely practical one, and we do not claim to solve the deep problems that exist concerning the foundations of logic and the meanings of the logical constants.

Another purpose, intimately connected with the first, is to suggest new methods for representing logics, that might make search for proofs, proof checking and implementation easier. The new notions and representation methods introduced in it, as well as various characterizations of more familiar ones have already contributed to the further development of the LF system and I hope it will be of help for any future effort at the same direction.

The structure of the paper is as follows. In section 2. we explain our notion of a simple consequence relation and why we take it to be the fundamental concept of Logic. Various examples of different consequence relations that are frequently associated with classical, modal and many-valued logics are then provided in section 3. In section 4. we investigate two groups of connectives that seem to be natural within the general setting we offer. We then proceed to characterize several well-known propositional calculii (including classical, intuitionistic, linear and relevant) in terms of these general connectives. In section 5. we discuss the standard methods for uniformly representing consequence relations: Hilbert type

systems, Natural Deduction and Gentzen type systems. We define all three in precise terms. According to our definitions Hilbert systems are a special case of Natural Deduction systems. The latter are, in turn, a special kind of Gentzen-type systems. The advantages and disadvantages of using each system and what should be taken as *good* representations in each case are explained. The notion of a *pure* system is found to be crucial in this context, especially from the implementation point of view. Various ways in which systems may fail to be pure are described, with examples, at the end of this section. Finally, in the last section we briefly outline (with examples) some methods for developing non-uniform, but still efficient, representations of consequence relations in the case where good uniform representations are not available.

Because of the general nature of the ideas discussed below it is very difficult to trace the origin of each of them or to give credit. Nevertheless, I like to mention at least the papers [Sc1], [Sc2] of Scott, in which the notion of a consequence relation (of the type dealt with here) was first introduced (as well as many other important ideas), [Ha], [Be] and above all — ([Gen]). Exactly like Hacking in [Ha], I see my paper just as a collection of footnotes to this brilliant work of Gentzen.

2 The Notion of a Consequence Relation

2.1 Axiomatic Systems

Traditionally a "formal system" is understood to include the following components:

1. A formal language L with several syntactic categories, one of which is the category of "well formed formulae " (wff).

2. An effective set of wffs called "axioms".

3. An effective set of rules (called "inference rules") for deriving theorems from the axioms.

The set of "theorems" is usually taken to be the minimal set of wffs which includes all the axioms and is closed under the rules of inference.

Systems of this sort have many names in the literature. Here we shall call them *Axiomatic systems* (or, sometimes: Hilbert-systems for theoremhood). Undoubtly they constitute the most basic kind of formal system and so their importance cannot be denied. One can argue that in fact all other, more complicated deduction formalisms reduce to systems of this sort. This is true, though, for *every*

recursively-defined system. Take for example the wffs in the propositional calculus. One can regard them as the "theorems" of the axiomatic system in which the "wffs" are strings of symbols, the "axioms" are the propositional variables and the "inference rules"- the usual formation rules.[1] The concept of theoremhood in systems of the above sort is not sufficient, therefore, to characterize the notion of a *Logic*. It is too broad a concept. On the other hand the notion of theoremhood of wffs is, at the same time, also too narrow to characterize what a logical system concerning these wffs is all about.

Let us make our last point clearer by a few very simple examples. Consider Kleene's 3-valued logic. It has 3 "truth-values": 1,0 and -1, of which 1 is taken as the only designated one. The operations corresponding to the usual connectives are: $\neg a = -a, a \vee b = max(a,b), a \wedge b = min(a,b)$. Suppose that L is the language of propositional calculus where the wffs are defined as usual. It is immediate then that *no* wff is a theorem of this logic (i.e. there is no wff that gets a designated value under all assignments). The notion of theoremhood seems to be vacuous for this logic. One might ask therefore in what sense it is a "logic".

On the other hand, consider the case in which we take both 1 and 0 to be designated. It is easy then to see that a wff is a theorem of the new logic iff it is a classical tautology. From the point of view of theoremhood there is no difference between this logic and the classical, two-valued one. But are they really the same? Obviously not, a major difference is, e.g. , that the "new" 3-valued logic is *paraconsistent*: It is possible for inconsistent theory to be non-trivial in this logic. (it is possible, e.g. , for p and $\neg p$ to be both "true" while q is "false").

2.2 Consequence Relations

Both examples above show that sets of "logical truths" are not enough for characterizing logics. The second example indicates that what is really important is what wffs follow from what theories. Indeed, in modern treatments of logic another concept, that of a *consequence relation* (C.R.) [2] is taken as fundamental. Logic might be defined, in fact, as the science of consequence relations. Unfortunately, the notion of a C.R. has in the literature several (similar, but not identical) meanings. We shall define first the one which we are going to use here (which is rather general) and then discuss some possible reasonable variations.

Definition A *consequence relation* (C.R.) on a set Σ of formulas is a binary relation $\vdash$ between finite multisets of formulas s.t:

[1] In some recent systems of typed constructive mathematics (see, e.g., [ML]) this resemblance is taken rather seriously and both "proposition" and "theorem" are taken as (different) "judgements" so that there is no significant difference between possible proofs of these "judgements"!

[2] See, e.g. [Sc1],[Ur],[Ga],[Ha].

(I) **Reflexivity:** $A \vdash A$ for every formula A.

(II) **Transitivity, or "Cut":** if $\Gamma_1 \vdash \Delta_1, A$ and $A, \Gamma_2 \vdash \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

2.3 Remarks and variations

1. We use above the notion of a "multiset". By this we mean "sets" in which the number of times each element occurs is significant, but not the order of the elements. Thus, for example, $[A, A, B] = [A, B, A] \neq [A, B]$. In this example we use $[\cdot]$ to denote a multiset. We shall also use "," for denoting the operation of multiset-union (so $[A, A, B], [A, B] = [A, A, B, A, B]$), and omit the "[ ]" whenever there is no danger of confusion.

   It is more customary to take a C.R. to be a relation between *sets*, rather than multisets. This is undoubtly more intuitive and so preferable wherever possible. We define, accordingly, a C.R. to be *regular* if it can be taken to be between sets. There are, however, logics the full understanding of which requires us to make finer distinctions that only the use of multisets enable us to make. Examples are: Relevance logics, Girard's Linear logic and the finite valued logics of Lukasiewicz (see examples below). It is possible, of course, to go one step further and to take C.R.'s to be between *sequences* of formulas (as Gentzen himself did). This, however, will considerably complicate the transitivity condition (II), and the need for it seems to be very rare indeed. We choose, therefore, not to be *so* general (at least in the present paper).

2. We define a C.R. to be a relation between *finite* (multi)sets. This means that we are assuming *compactness* for all the logics we consider. This rules out many "model-theoretic logics". Our excuse for this elimination is that we are primarily interested in formal systems that can (at least in principle) be *computerized*. Now effective rules with infinite number of premises *are* possible, but the amount of information needed for applying such rules in any particular case should always be finite. Accordingly, we believe that any effective presentation of such rules should be within the scope of our framework. This issue might (and should) be opened to further investigations. What *is* clear is that only a logic which admits a decent proof theory is interesting from the present point of view. In contrast, proof theory is not seen (according to [BF]) as being particularly central to the subject of model-theoretic logics since many of these logics do not even have an r.e. set of valid sentences.

3. In most definitions of a C.R. that one can find in the literature there is a third condition besides the two formulated above. This is the *weakening*

condition, according to which if $\Gamma \vdash \Delta$ then also (i) $\Theta, \Gamma \vdash \Delta$ and (ii) $\Gamma \vdash \Delta, \Theta$. (Some times only condition (i) is demanded, especially when one is interested only in a single-conclusioned C.R., i.e. if $\Gamma \vdash \Delta$ then Δ consists of a single formula). Again, this is a very natural restriction but it fails for many systems. In fact it fails for every system for non-monotonic reasoning, as well as some C.R. based on Relevance logics and Linear Logic.

In the sequel, a C.R. which is regular and is closed under weakening will be called *ordinary*.

4. The above definition of a C.R should more accurately be taken as a definition of a *simple* C.R.. In [Ga] there is another requirement: *uniformity*. This means that $\vdash$ should be "closed under substitutions". This condition involves the inner structure of wffs and so is a little vague. Although its meaning is quite obvious in particular cases, it is less obvious how to define it in general. One might ask: substitutions of what for what? In order to provide a precise definition (and for other reasons as well) one should define a C.R. to be a *ternary* relation $\Gamma \vdash_{\vec{x}} \Delta$ where Γ and Δ are as before and $\vec{x}$ is a finite set of *variables of the language*. (It is assumed, therefore, that some of the syntactic categories of the language include special subcategories of variables for these categories). For example: $\forall x A \vdash_{A,y} A(y/x)$ intuitively means that for every formula ϕ and any individual term t which is free for x in ϕ, $\phi(t/x)$ follows from $\forall x \phi$. The use of this general notion of a C.R. requires extending the cut condition to some version of resolution (i.e.: unification should be incorporated), and it involves delicate problems concerning substitutions. We prefer therefore to postpone treating these problems to the second part of this paper, and here treat only simple C.R's.

3 Some Examples of Abstract Consequence Relations

3.1 Classical Propositional Logic

Truth: [3] $A_1, \ldots, A_n \vdash_t B_1, \ldots, B_m$ iff every valuation which makes all the A_i's true makes one of the B_j's true as well.

Validity: [4] $A_1, \ldots, A_n \vdash_v B_1, \ldots, B_m$ iff any substitution of sentences for atomic propositional sentences which makes all the A_i's tautologies does the same to one of the B_i's.

[3] the name "Truth-functional" might, perhaps, be better.
[4] the name "Tautological" is also possible.

Reduction: $A_1, \ldots, A_n \vdash_r B_1, \ldots, B_m$ iff $A_1 \wedge A_2 \wedge \cdots \wedge A_n \rightarrow B_1 \vee B_2 \vee \cdots \vee B_m$ is a tautology.

If we limit ourselves above to the case $m = 1$ we get the single-conclusioned counterparts of these three C.R.'s. It is not difficult to see that in this case all three are in fact identical. In the multiple-conclusioned case, however, the "validity" C.R. differs from the other two, and is the *minimal* C.R. which extends the single-conclusioned one.

3.2 First-order logic

Let $A_1, \ldots, A_n$ and B be formulas of some first-order language L (i.e. they may contain free variables).

Truth: $A_1, \ldots, A_n \vdash_t B$ iff B is true in every model relative to any assignment which makes all the A_i's true.

Validity: $A_1, \ldots, A_n \vdash_v B$ iff B is valid in any model (for L) in which all the A_i's are valid (by "valid" we mean: true relative to all assignments).

The above are examples of two important *single-conclusioned* C.R.'s which are frequently associated with first order logic. Unlike the propositional case, they are *not* identical. $\forall x p(x)$ follows, for example, from $p(x)$ according to the second, but not according to the first. On the other hand, the classical deduction theorem holds for the first but not for the second. The two consequence relations *are* identical, though, from the point of view of *theoremhood*: $\vdash_v A$ iff $\vdash_t A$ (and in fact if all formulas of Γ are closed then $\Gamma \vdash_t A$ iff $\Gamma \vdash_v A$). Both C.R.'s defined above can be extended to multiple-conclusioned C.R.'s in more than one interesting way, but we do not go into the details.

3.3 Propositional Modal Logic

Truth: $A_1, \ldots, A_n \vdash_t B$ iff for any frame and for any valuation in this frame, B is *true* in every *world* in this frame in which all the A_i's are true.

Validity: $A_1, \ldots, A_n \vdash_v B$ iff B is valid (i.e. true in all worlds) in every frame relative to any valuation which makes all the A_i's valid.

Again we consider here two important single-conclusioned C.R.'s. The situation concerning them is similar to that in the previous case: $A \vdash_v \Box A$ but $A \not\vdash_t \Box A$. The deduction theorem obtains for $\vdash_t$ but not for $\vdash_v$. Again the two C.R.'s are identical as far as *theorems* are concerned.

3.4 Three-Valued Logic

Assume again a propositional language with the connectives $\neg, \vee, \wedge$. Let corresponding operations on the truth-values {-1,0,1} be defined as in section 2.1. We define now 5 different C.R.'s based on the resulting structure. In these definitions v denotes an assignment of truth values to formulas which respects the operations, $\Gamma = A_1, \ldots, A_m$ and $\Delta = B_1, \ldots, B_n$.

Kl: $\Gamma \vdash_{Kl} \Delta$ iff $v(B_i) = 1$ for some i or $v(A_j) \in \{-1, 0\}$ for some j.

Pac: $\Gamma \vdash_{Pac} \Delta$ iff either $v(B_i) \in \{1, 0\}$ for some i or $v(A_j) = -1$ for some j.

Lt: $\Gamma \vdash_{Lt} \Delta$ iff for some i,j $v(B_i) = 1$ or $v(A_j) = -1$ or $v(B_i) = v(A_j) = 0$.

Sob: $\Gamma \vdash_{Sob} \Delta$ iff either $v(B_i) = 1$ for some i or $v(A_j) = -1$ for some j or $v(A_i) = v(B_j) = 0$ for *all* i, j.

Luk: $\Gamma \vdash_{Luk} \Delta$ iff either $v(B_i) = 1$ for some i or $v(A_j) = -1$ for some j or at least *two* formulas in Γ, Δ get 0 (under v).

Notes:

1. $\vdash_{Kl}$ corresponds to taking 1 as the only designated value, and so it is the obvious C.R. defined dy Kleene's 3-valued logic. As remarked above, it has no theorems.

2. $\vdash_{Pac}$ corresponds to taking both 1 and 0 as designated. As noted above, it has the same set of theorems as classical propositional calculus, but it is *paraconsistent* ($P, \neg P \not\vdash_{Pac} Q$).

3. $A_1, \ldots, A_m \vdash_{Lt} B_1, \ldots, B_n$ iff for every v, $v(A_1 \wedge A_2 \wedge \ldots \wedge A_m) \leq v(B_1 \vee B_2 \vee \ldots \vee B_n)$. This C.R. also has no theorems. In fact if $\Gamma \vdash_{Lt} \Delta$ then both Γ and Δ are non-empty.

4. $A_1, \ldots, A_n \vdash_{Sob} B$ iff $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$ is valid when $\rightarrow$ is defined as in Sobociński 3-valued logic, (See [Sob] or [AB], pp. 148-9). Moreover, $A_1, \ldots A_m \vdash_{Sob} B_1, \ldots B_n$ iff $\vdash_{RM_3} A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_m \rightarrow (B_1 + \ldots + B_n)) \ldots)$ [5], where RM_3 (the 3-valued extension of RM- see [AB] or [Du]) is the strongest in the family of logics created by the Relevantists school. *Weakening fails for this C.R. on both sides* (this is our first example of this sort!).

[5] $A + B \stackrel{\text{def}}{=} \neg A \rightarrow B$. $B_1 + \cdots + B_n \stackrel{\text{def}}{=} \perp$ in case n=0.

5. $A_1 \ldots, A_n \vdash_{Luk} B$ iff $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$ is valid in Lukasiewicz three-valued logic. (using negation, it is easy to give a corresponding interpretation for *every* sequent). Its main property is that *contraction fails* (on both sides). It is therefore completely necessary to work with *multisets* within this C.R. .

6. The classical C.R. (our first example) can also be characterized in the present framework by : $\Gamma \vdash \Delta$ iff for every v either $v(A_i) = -1$ or $v(B_j) = 1$ or at least one formula in $\Gamma \cup \Delta$ gets 0 (the proof of this claim uses known proof-theoretical reductions). Note that this C.R. is *not* the union of $\vdash_{Kl}$ and $\vdash_{Pac}$. For example, if we take $\Gamma = \{C \vee \neg P, P \vee R\}, \Delta = \{C, S, R \wedge \neg S\}$ then classically $\Gamma \vdash \Delta$, but this is not the case if we interpret $\vdash$ as either $\vdash_{Pac}$ or $\vdash_{Kl}$!

All the consequence relations described above were defined in abstract terms, using semantics. The rest of this paper is devoted to various methods for *syntactically* characterizing consequence relations. It is possible, of course, for a C.R. to be defined directly in syntactical terms, and often this is (or was) the case. Thus, the two kinds of modal C.R.'s given above were in use (at least for important special cases) long before the semantic description was known!

4 Classification of C.R. according to their Basic Connectives

Our main object in this section is to provide a syntactic characterization of some propositional C.R.'s in terms of the connectives which are definable in them. We examine for this purpose two classes of connectives that are important in the present framework and see what rules they should obey. All these connectives and rules will be quite familiar. In fact, the set of primitive connectives of practically any seriously investigated logic forms (more or less) a subset of the set of the general connectives introduced below. We then use these connectives for characterizing several well-known logics.

4.1 Intensional connectives

The main class of connectives which we discuss is that of intensional connectives relative to a given C.R.. They can be characterized as connectives that make it possible to transform a given sequent to an equivalent one that has a special

required form. By "equivalent" here we mean that one sequent obtains iff the other does (but in most important cases it can be interpreted in a much stronger sense).

In what follows assume $\vdash$ to be a fix C.R., all notions defined are taken to be relative to $\vdash$.

Intensional Disjunction: We call a binary connective $+$ an intensional disjunction if for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A, B \quad \text{iff} \quad \Gamma \vdash \Delta, A + B \ .$$

Intensional Conjunction: We call a binary connective $\circ$ an intensional conjunction if for all Γ, Δ, A, B:

$$\Gamma, A, B \vdash \Delta \quad \text{iff} \quad \Gamma, A \circ B \vdash \Delta \ .$$

Intensional Negation: We call a unary connective $\neg$ a right intensional negation if for all Γ, Δ, A:

$$\Gamma, A \vdash \Delta \quad \text{iff} \quad \Gamma \vdash \Delta, \neg A \ .$$

We call a unary connective $\neg$ a left intensional negation if for all Γ, Δ, A:

$$\Gamma \vdash \Delta, A \quad \text{iff} \quad \Gamma, \neg A \vdash \Delta \ .$$

Since it can be shown that $\neg$ is a right intensional negation iff it is a left one, we use the term *intensional negation* to mean either.

Intensional Implications: We call a binary connective $\rightarrow$ a weak intensional implication if for all A, B:

$$A \vdash B \quad \text{iff} \quad \vdash A \rightarrow B \ .$$

We call a binary connective $\rightarrow$ an intensional implication if for all Γ, A, B:

$$\Gamma, A \vdash B \quad \text{iff} \quad \Gamma \vdash A \rightarrow B \ .$$

We call a binary connective $\rightarrow$ a *strong* intensional implication if for all Γ, Δ, A, B:

$$\Gamma, A \vdash \Delta, B \quad \text{iff} \quad \Gamma \vdash \Delta, A \rightarrow B \ .$$

Intensional Truth: We call a 0-ary connective $\top$ an intensional truth if for all Γ, Δ:

$$\Gamma \vdash \Delta \quad \text{iff} \quad \top, \Gamma \vdash \Delta \ .$$

Intensional Falsehood: We call a 0-ary connective $\perp$ an intensional falsehood if for all Γ, Δ:

$$\Gamma \vdash \Delta \quad \text{iff} \quad \Gamma \vdash \Delta, \perp .$$

Proposition: The following are immediate consequences of the above definitions:

1. If $\vdash$ has a (primitive or definable) intensional disjunction then any sequent $\Gamma \vdash \Delta$ such that Δ is not empty is equivalent to a sequent of the form: $\Gamma \vdash A$. If $\vdash$ has also an intensional falsehood then this is true for *any* sequent.

2. If $\vdash$ has a (primitive or definable) intensional conjunction then any sequent $\Gamma \vdash \Delta$ such that Γ is not empty is equivalent to a sequent of the form: $A \vdash \Delta$. If $\vdash$ has also an intensional truth then this is true for *any* sequent.

3. If $\vdash$ has intensional disjunction, conjunction, falsehood and weak implication or intensional disjunction, implication and falsehood then for all Γ, Δ there is A such that $\Gamma \vdash \Delta$ is equivalent to $\vdash A$. Problems concerning such consequence relations can therefore be reduced to problems about *theoremhood* of formulas.

4. If $\vdash$ has a right (left) intensional negation then any sequent is equivalent to one in which the left (right) hand side is empty. If in addition it has also an intensional disjunction (conjunction) then every non-empty sequent is equivalent to one of the form $\vdash A$ ($A \vdash$).

5. If $\vdash$ has a strong implication then every sequent in which the right-hand side is not empty is equivalent to one with the left-hand side empty.

4.2 Characterizing the Intensional Connectives by Gentzen-Type Rules

In the previous subsection we have introduced several intensional connectives. Their definition can be split into two rules that are the converse of each other. In each case one of the two rules does not have what Gentzen has called "the subformula property". In this subsection we describe a uniform method for deriving rules *with* the subformula property which characterize the same connectives. All the rules we shall find are quite standard in Gentzen-type systems. As an example, we treat the case of strong implication in detail. We then list the rules which are obtained for the other connectives by using the same method.

The definition above directly entails that $\rightarrow$ is a strong implication iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \quad \frac{\Gamma \vdash A \rightarrow B, \Delta}{\Gamma, A \vdash B, \Delta} .$$

The first of these rules already has the subformula property. The second does not. In order to find an appropriate substitute for it we use the reflexivity and transitivity of $\vdash$. The reflexivity condition, applied to a formula with $\rightarrow$ as the principal connective yields:

$$A \rightarrow B \vdash A \rightarrow B .$$

Hence the second condition above implies that:

$$A, A \rightarrow B \vdash B .$$

Taking $A \rightarrow B$ to be the principal formula in the last sequent, we proceed next to eliminate the others using cuts. Suppose, accordingly, that $\Gamma_1 \vdash \Delta_1, A$ and $B, \Gamma_2 \vdash \Delta_2$. Then two cuts of these two sequents with the last sequent above result with $\Gamma_1, \Gamma_2, A \rightarrow B \vdash \Delta_1, \Delta_2$. We obtain, therefore, that in order for $\rightarrow$ to be a strong implication relative to $\vdash$ this relation should be closed under the rule:

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \vdash \Delta_1, \Delta_2} .$$

Conversely, the closure of $\vdash$ under this rule implies the provability of $A, A \rightarrow B \vdash B$, and so if $\Gamma \vdash A \rightarrow B, \Delta$ then also $\Gamma, A \vdash B, \Delta$ (using a cut).

By using the same analysis we get the following

Proposition:

1. $\rightarrow$ is a strong intensional implication iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \quad \frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \vdash \Delta_1, \Delta_2} .$$

Similarly we can show:

2. $+$ is an intensional disjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A + B, \Delta} \quad \frac{A, \Gamma_1 \vdash \Delta_1 \quad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A + B \vdash \Delta_1, \Delta_2} .$$

3. $\circ$ is an intensional conjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A \circ B} \quad \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \circ B \vdash \Delta} .$$

4. $\rightarrow$ is an intensional implication iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad \frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \vdash \Delta_1, \Delta_2}.$$

5. $\perp$ is an intensional falsehood iff $\vdash$ is closed under the rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \perp} \qquad \perp \vdash .$$

6. $\top$ is an intensional truth iff $\vdash$ is closed under the rules:

$$\vdash \top \qquad \frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta}.$$

7. The following conditions are equivalent:

   (a) $\neg$ is an intensional negation.

   (b) $\neg$ is a right intensional negation.

   (c) $\neg$ is a left intensional negation.

   (d) $\vdash$ is closed under the rules:

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \qquad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}.$$

The following observations are immediate from these characterizations. They are familiar facts about the connectives of classical logic. The foregoing discussion and the examples at the end of this section show, however, that they are not peculiar to classical logic or to truth-functional connectives.

- If $\vdash$ has an intensional negation $\neg$ and an intensional disjunction $+$ then (as in classical logic) it also has intensional conjunction and strong implication, defined by $\neg(\neg A + \neg B)$ and $\neg A + B$ respectively. Similar results obtain if $\vdash$ has an intensional negation and either an intensional conjunction or a strong intensional implication.

- If $\vdash$ has a *strong* intensional implication $\rightarrow$ and an intensional falsehood $\perp$ then it also has an intensional negation, defined by $A \rightarrow \perp$.

- If $\vdash$ is closed under weakenings and has *theorems* (i.e. formulas A such that $\vdash A$), then each of these theorems is an intensional truth. (For having theorems it suffices, e.g. that $\vdash$ has a weak intensional implication, since then $A \rightarrow A$ is a theorem for every A.) If such $\vdash$ has also an intensional negation then the negation of each theorem is an intensional falsehood.

4.3 The Combining Connectives

The function of the intensional connectives is to provide certain operations on a single sequent in order to transform it to some desirable form. We now study connectives that have a different function: To enable the combination of two sequents into a single one. One which contains exactly the same information as the original two. [6] In the list of the rules for the intensional connectives there are, of course, rules that take two sequents and return a single one. The resulting combination is not reversible, though: the premises cannot always be recovered from the conclusion. Indeed, one cannot expect the *exact* combination of any two sequents always to be possible. It *is* possible, though, in one important case: When the two sequents to be combined are identical in all formulas except perhaps the two that are actually connected by the combining connective. Now there are three possible positions that these exceptional formulas might occupy and according to them we get the following three combining connectives:

Combining Conjunction: We call a connective $\wedge$ a combining conjunction iff for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A \wedge B \quad \text{iff} \quad \Gamma \vdash \Delta, A \text{ and } \Gamma \vdash \Delta, B\,.$$

(In this case both exceptional formulas are succedents).

Combining Disjunction: We call a connective $\vee$ a combining disjunction iff for all Γ, Δ, A, B:

$$A \vee B, \Gamma \vdash \Delta \quad \text{iff} \quad A, \Gamma \vdash \Delta \text{ and } B, \Gamma \vdash \Delta\,.$$

(In this case both exceptional formulas are antecedents).

Combining Implication: We call a connective $\supset$ a combining implication iff for all Γ, Δ, A, B:

$$A \supset B, \Gamma \vdash \Delta \quad \text{iff} \quad \Gamma \vdash \Delta, A \text{ and } B, \Gamma \vdash \Delta\,.$$

(In this case the exceptional formulas are on different sides of $\vdash$).

The choice of these three connectives was guided by tradition. Obviously, there are three other possibilities. Thus we could characterize the "Sheffer stroke" by:

$$\Gamma, A|B \vdash \Delta \quad \text{iff} \quad \Gamma \vdash \Delta, A \text{ and } \Gamma \vdash \Delta, B\,.$$

[6] The connectives we discuss are usually called "extensional" by the relevantists, while Girard calls them "additive" (see [Gi]). We find the term "combining" more suggestive.

It is not difficult, indeed, to carry the analysis below to either this connective or to the others. We shall be satisfied, though, with the above three (Note, by the way, that if an intensional negation is available then the existence of one combining connective entails the existence of the rest).

By using exactly the same method we have above applied for investigating the intensional connectives we can now show:

Proposition:

1. $\vee$ is a combining disjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}.$$

2. $\wedge$ is a combining conjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}.$$

3. $\supset$ is a combining implication iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A \supset B} \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B} \qquad \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \supset B \vdash \Delta}.$$

It is easy to see that relative to an *ordinary* C.R. (i.e. regular and closed under weakenings) a connective is a combining disjunction (conjunction, implication) iff it is an intensional disjunction (conjunction, strong implication). This is not the case in general, though, as the examples of Linear and Relevance Logics below show.

4.4 Characterization of some Known Logics

As the title suggests, we shall try to use the various notions introduced so far for characterizing several well-known logics. We shall examine these logics according to three criteria:

- Regular or not.
- Closure under weakening.
- The intensional and combining connectives which are available in them.

"Multiplicative" Linear Logic: [7] This is the logic which corresponds to the *minimal* C.R. which includes all the intensional connectives (half of them

[7] see [Gi].

will do, of course). It is possible to delete the intensional truth and falsehood to get the minimal C.R. which includes the others. (The system is conservative with respect to this fragment). In both versions the system is neither regular nor closed under weakenings.

Propositional Linear Logic (without the "exponentials" and the propositional constants T and F): This is the minimal consequence relation which contains all the connectives introduced above. Again— it is neither regular nor closed under weakening. It is important to note that its intensional connectives behave quite differently from its combining ones!

$R_{\underset{\rightarrow}{\sim}}$ **-the Intensional Fragment of the Relevant Logic** R: [8] This is the minimal C.R. which contains all the intensional connectives (the intensional truth and falsehood are again optional) and is *closed under contraction.* It is still not regular since the converse of contraction (and so also weakening) fails for it. (A word is in order here of what C.R. associated with $R_{\underset{\rightarrow}{\sim}}$ we have in mind, since there is more than one candidate. The answer is: that which the standard Gentzen type formulation of this system defines. An equivalent definition is: $A_1, \ldots, A_n \vdash_{R_{\underset{\rightarrow}{\sim}}} B_1, \ldots, B_m$ iff $A_1 \rightarrow (A_2 \rightarrow \cdots (A_n \rightarrow B_1 + \cdots + B_m) \ldots)$ is a theorem of this system, where $A + B$ is $\sim A \rightarrow B$ and $(A_n \rightarrow B_1 + \cdots + B_m)$ is $\sim A_n$ in case $m = 0$. The last interpretation is the one we have in mind also with respect to the other systems in the Relevance/Linear family).

R^t **without Distribution:** This corresponds to the minimal C.R. which contains all the connectives which were described above and is closed under contraction. $\perp$ and $\top$ are again optional.

$RMI_{\underset{\rightarrow}{\sim}}$: [9] This corresponds to the minimal *regular* C.R. which contains intensional negation, disjunction, conjunction and strong implication.

$RM^t_{\underset{\rightarrow}{\sim}}$: [10] This corresponds to the minimal regular C.R. which includes *all* intensional connectives described above.

In contrast to Linear Logic and $R_{\underset{\rightarrow}{\sim}}$, $RM_{\underset{\rightarrow}{\sim}}$ is *not* a conservative extension of $RMI_{\underset{\rightarrow}{\sim}}$. This means that the addition of the intensional truth (or falsehood) to $RMI_{\underset{\rightarrow}{\sim}}$ forces new sequents in the language of this system to obtain. The reason is that the intensional constants bring with them part of the power

[8] See [AB] or [Du].

[9] see [AB] pp. 148-9 and [Av1].

[10] This is the intensional fragment of the system RM^t (see [AB] or [Du]). Without the propositional intensional constants this is Sobocinski 3-valued logic (see **2.4** above), also called $RMI^1_{\underset{\rightarrow}{\sim}}$ in [Av1].

of weakening. This happens to be harmless in the context of Linear logic and R, but together with the converse of contraction, (which is available, of course in every regular C.R. but can also be taken as a *very* special case of weakening) it causes sequents like $A, B \Rightarrow A, B$ to be provable. This is true in fact for every regular C.R. which has an intensional implication: starting with $\Rightarrow \top, \top$, two applications of $(\rightarrow\Rightarrow)$ give $\top \rightarrow A, \top \rightarrow B \Rightarrow A, B$. But since $\top, A \Rightarrow A$ is provable, so are $A \Rightarrow \top \rightarrow A$ and $B \Rightarrow \top \rightarrow B$. Hence two cuts give $A, B \Rightarrow A, B$ (In order to get a cut-free representation of this system it is necessary to add the following "mingle" rule: from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$)[11].

RM^t without Distribution: [12] This corresponds to the minimal regular C.R. which has all the connectives described above. It is not closed under weakening. Again, the intensional constants here are optional.

Classical Propositional Logic: This of course corresponds to the minimal ordinary C.R. which has all the above connectives. Needless to say, there is no difference in it between the combining connectives and the corresponding intensional ones.

Intuitionistic Logic: It is a common belief that the main difference between classical and intuitionistic logic is that the later is single-conclusioned while the former is essentially multiple-conclusioned. The origin of this belief is the way in which Gentzen has formulated his sequential version of intuitionistic logic. That version differs from his formalism for classical logic only by limiting sequents to have at most one formula in the succedent. Assuming for a moment this belief is true, it is straightforward to define for the single-conclusioned case the notions of a regular and an ordinary C.R., the combining connectives and the intensional conjunction, implication, truth and falsehood (but not strong implication!). It is easy then to see that:

> The usual single-conclusioned C.R. associated with intuitionistic logic is the minimal ordinary single-conclusioned C.R. which has intensional implication and falsehood and combining disjunction and conjunction.

We proceed now to offer what we believe to be a deeper analysis, in which we need not treat intuitionistic logic differently from the other logics we have considered so far. What we seek therefore is a multiple-conclusioned conservative extension of the above single-conclusioned C.R., which has the

[11] For a proof and for more information about this system see [Av2].
[12] see [AB] or [Du].

same types of basic connectives. The unique solution to this problem is easy to find once we recall that for *ordinary* C.R. a combining disjunction is also an intensional one. Accordingly, we define:

> $A_1, \ldots, A_n \vdash_{Int} B_1, \ldots, B_m$ iff there is a proof of $B_1 \vee \cdots \vee B_m$ from $A_1, \ldots, A_n$ in one of the usual formalisms for intuitionistic logic.

It is easy now to see that:

> $\vdash_{Int}$ is the minimal ordinary C.R. which has intensional disjunction, conjunction, falsehood and implication. [13]

It is illuminating to compare this to the following possible characterization of classical logic:

> The standard C.R. associated with classical logic is the minimal ordinary C.R. which has intensional disjunction, conjunction, falsehood and *strong* implication.

According to these two characterizations classical and intuitionistic logic differ mainly with respect to their *implication* connective, while their disjunctions are the same! Indeed, it is easy to show that exactly the same sequents which involve only $\vee, \wedge$ and $\perp$ are valid in both (note that there are no theorems, i.e. sequents of the form $\vdash A$, among these sequents!). There is however another crucial difference between the two logics that is somewhat hidden in these characterizations: *Intuitionistic logic does not contain any intensional negation:* There is no sentence $N(p)$ in the $\{\vee, \wedge, \rightarrow, \perp\}$-language (in which only the atomic formula p occurs) such that $p, N(p) \vdash$ and $\vdash p, N(p)$ are both valid (This is an immediate consequence of the disjunction property of $\vdash_{Int}$). The usual definition of $\neg A$ as $A \rightarrow \perp$ works well when $\rightarrow$ is a strong implication (as in the classical case) but not otherwise. This fact might explain why the rules for negation look so nasty compared to the other rules in the context of intuitionistic natural deduction, and why authors like Prawitz and Schroeder-Heister prefer to take $\perp$ rather than negation as a primitive connective of intuitionistic logic. [14]

4.5 On the Meanings of the Propositional Connectives

There is a long tradition, originated already in Gentzen ([Gen]), about the introduction and elimination rules of Natural Deduction as providing the meanings of

[13] This characterization corresponds to Maehera's multiple-conclusioned, cut-free Gentzen-type formulation of Int. logic which appears in ch. 1 of [Tak].

[14] See [Pra2] and [SH].

the propositional connectives. The famous [Pri] has forced the followers of this tradition to be more careful about this issue, and so today the emphasis is usually put on the *introduction* rules as those which define the meaning of a connective. Concerning the elimination rules the general principle is that one should not be able to get more out of a formula than the introduction rules can put into it. This principle was used, e.g., by Schroeder-Heister in [SH] for developing an explicit method for *deriving* the (unique) elimination rule for a connective from the corresponding set of introduction rules. [15]

This all is very nice. Unfortunately, it does not seem to work beyond the realm of intuitionistic logic. A particularly important connective that seems to escape this type of characterization is the negation. One illuminating fact about it, in this respect, is that intuitionists usually raise the above principle to attack the excluded middle, whereas relevantists (like Dunn in [Du] p. 152) use it for justifying *their* rejection of the disjunctive syllogism. Neither the intuitionists nor the relevantists reject both laws, though.

The intuitionists usually try to avoid the problem of characterizing negation by introducing instead as primitive a (quite artificial) constant for intensional falsehood, and then defining (even more artificially) negation as $A \rightarrow \perp$[16]. As we have shown above, this might work well in the context of *classical* logic, but in no way defines an *appropriate* negation in the intuitionistic case (nor is there any other way of defining a decent negation in the context of intuitionistic logic).

The present paper suggests another method of taking rules as defining the meaning of connectives. It had the following two main properties:

- The meaning of a connective is always something which is *relative* to some C.R..
- What defines a connective is not a set of "introduction rules" but a single rule which is *reversible*.

The reversible rule which defines a connective might introduce it in either of the succedent or the antecedent of a sequent. In the first case it usually corresponds to an "introduction" rule of N.D., in the second—to an elimination rule. There is no priority therefore, at least in this context, to introduction rules over elimination rules. Indeed, the combining disjunction, for example, is characterized by what is usually taken as the *elimination* rule for disjunction (indeed, actual uses of disjunctions are usually made when one wants to infer something from them without being in a position to assert either of the disjuncts!).

[15]For more explanations about this tradition—see [Sud2] and the extensive literature cited there.

[16]I should admit that I do not believe that in an ordinary discourse, when someone denies something, he means that what he denies implies "der false"...

5 Uniform Representations of C.R's

The notion of a C.R., as defined in the first section and exemplified in the second is an *abstract* one. We have seen above several ways, semantical as well as syntactical, of defining or characterizing C.R.'s. However, in order to use a certain abstract C.R. in practice one needs a *concrete* way of *representing* it. This is usually done by using a formal system [17]. There are two basic demands that such representations of a C.R. $\vdash$ should meet. These are:

Faithfulness: If the representation can be used to show that $\Gamma \vdash \Delta$ then this is actually the case.

Effectiveness: If someone uses the representation to show that $\Gamma \vdash \Delta$ then it can mechanically be checked that he really does so.

There is also a third property that we would like an adequate representation of $\vdash$ to have, but is not always achievable:

Completeness: Whenever $\Gamma \vdash \Delta$ the representation can be used to show this.

Note: If we accept Church's thesis, then the effectivity demand means that the set of of sequents that can be shown to hold by a formal representation is an r.e. set. Completeness can in principle be achieved, therefore, only if the represented C.R. is r.e.. Otherwise we can only expect a *partial* representation.

As was emphasized several times before, we understand a C.R. $\vdash$ to be an abstract relation, and so whenever we write a sequent $\Gamma \vdash \Delta$ this is a metaclaim about $\vdash$. Now it is important to realize that while dealing with a C.R., $\vdash$, the premises of a sequent are no less important than the conclusions. A full representation of $\vdash$ should reflect this. Hence it should include in its language a formal counterpart "$\Gamma \vdash \Delta$" for each $\Gamma \vdash \Delta$, so that officially the system can show that $\Gamma \vdash \Delta$ iff "$\Gamma \vdash \Delta$" is derivable in it. Obviously, the most direct way of achieving such a formal counterpart for each abstract sequent is to have in the formal language a formal symbol which directly corresponds to $\vdash$. We shall use henceforth "$\Rightarrow$" to denote such a symbol, reserving "$\vdash$" to denote (in the metalanguage of our discussion) the represented C.R. [18]. Accordingly we shall call creatures of the form $\Gamma \Rightarrow \Delta$ "formal sequents". In order to unify our treatment of the usual various formal systems, we shall assume below that such a formal symbol alway exists. If officially it does not, then this assumption means at worst that we are considering an *extended* language in which it does, and an *extended* formal system in which the connections between the old one

[17] In fact, Hodges ([Ho],p.26) defines a formal system (or a "formal proof calculus") to be "a device for proving sequents in a language L."

[18] It is also customary to use $\models$ for the C.R. and $\vdash$ for its formal counterpart.

and $\vdash$ are made a part of the formal machinery (while in the original system they should be explained in the meta-language). For example, an explanation in the metalanguage of the form: "$A_1, \ldots, A_n \vdash B$ iff there is a proof of B from $A_1, \ldots, A_n$" will be translated (in the extended system) to: "From a formal proof of B from $A_1, \ldots, A_n$ infer $A_1, \ldots, A_n \Rightarrow B$". Formulas, *formal* sequents and proofs (in the original system) of formulas from other formulas will all be formal objects of such an extended formal system, possibly with different types. This might look a little bit complex, but it is absolutely necessary (in some form or another) for a real full representation (that can, e.g. be computerized). Anyway, in the usual cases a much simpler translation will be provided below. It will allow us to regard Hilbert-type systems for provability, natural deduction systems and Gentzen type systems all as *axiomatic systems* (see 1.1) in which the "theorems" are formal sequents.

5.1 Using Axiomatic Systems for Representations

The oldest way of representing a C.R. $\vdash$ is by using an axiomatic system (see **1.1**) which has the same language L (i.e. the same well-formed formulas) as $\vdash$. Now axiomatic systems are designed to prove *theorems*, and so they can be used only indirectly for representing C.R.'s . There are two main methods for doing this:

The Interpretation Method: One defines a correspondence between sequents of L and sets of wffs of L so that $\Gamma \vdash \Delta$ iff at least one of the sentences of the corresponding set (or perhaps all of them) is a theorem of the corresponding axiomatic system. Usually, the corresponding set is just a singleton, and so every sequent is translated into some formula of the language. Such a translation is straightforward if $\vdash$ has the required intensional connectives. This was really the case for all the logics that were investigated in previous stages of the development of mathematical logic (like classical and intuitionistic logics) and so it suffices at these stages to concentrate on the notion of theoremhood while doing logic. An interpretation using intensional connectives has the property that the interpretation of the formal sequent $\Rightarrow A$ is just A. This should not always be the case, though. A famous example for this is Gödel's interpretation of classical logic within intuitionistic logic. An example in which the interpretation does not use just singletons may be provided by the intuitionistic pure implicational C.R.. Here $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ iff *for some i*, $A_1 \to (A_2 \to \ldots \to (A_n \to B_i) \ldots)$ is a theorem (of intuitionistic logic).

The Extension Method: Here we say that $A_1, \ldots, A_n \vdash B$ iff B is a theorem of the axiomatic system which is obtained by adding $A_1, \ldots, A_n$ to the axioms

of the given one. This second method is the origin and the prototype of what is known as Hilbert-type systems. In the next subsection we shall treat this type of systems in detail. For the time being let us just mention two properties of every C.R. that can be represented using this method: first, such a C.R. is single-valued. Second, it is what we call above ordinary, i.e: closed under weakenings (on the left) and relates sets (rather than multisets) of formulas.

5.2 Hilbert Type Representations

If one uses an axiomatic system AS together with the extension method in order to show that $A_1, \ldots A_n \vdash B$, then one should provide a proof of B in the new axiomatic system $AS + \{A_1, \ldots A_n\}$. Of course, we do not seriously mean that we are using an infinite number of new axiomatic systems. We just use one which is, each time, temporarily augmented by a finite number of axioms in order to derive some sequent. If we want both to make this explicit, and to unify our handling of this "infinite number" of axiomatic systems, then we should work directly with formal sequents. What we get is the following *axiomatic system for sequents:*

Axioms:

1. $A \Rightarrow A$ for every A.
2. $\Rightarrow A$ whenever A is an axiom of AS.

Rules:

1.

$$\frac{\Gamma \Rightarrow A}{\Delta, \Gamma \Rightarrow A} \quad \text{(weakening)}$$

2.

$$\frac{\Gamma_1 \Rightarrow A_1 \ldots, \Gamma_n \Rightarrow A_n}{\Gamma_1, \ldots \Gamma_n \Rightarrow B}$$

where

$$\frac{A_1, \ldots, A_n}{B}$$

is (an instance of) a rule of AS.

The main fact to note concerning this sequential system is that all the "activity" is made on the right-hand side of the $\Rightarrow$. Besides the structural rule of weakening and the axioms which reflect the reflexivity condition— the set of wffs on the left-hand side of a conclusion of a rule is always the union of the left-hand sides of the premises. This is, in fact, the main thing that is common to all "Hilbert-type formalisms" that can be found in the literature. Accordingly we can define this kind of formalism quite generally as follows:

Definition: *A Hilbert-type system for consequence* in the language L is an axiomatic system the "formulas" of which are formal sequents of L and:

1. The axioms includes $A \Rightarrow A$ for all A. All the other axioms are of the form $\Rightarrow A$.

2. With the possible exception of cut and weakening, the *set* of formulas which appear on the left-hand side of a conclusion of a rule is the union of the *sets* of formulas which appear on the left-hand side of the premises. (We shall call this property the *left-hand side property*).

If we take axioms as rules with 0 premises then Hilbert representations can be characterized as those systems which *have besides the basic reflexivity and transitivity rules only structural rules and/or rules with the left-hand side property.*

It is important to realize in what ways does this definition of a Hilbert-type representation really generalize from the above Hilbert-type system which was derived by the extension method. There are, in fact three new issues involved:

- The above system was ordinary. In the general case we deal, however, with multisets. We have not demanded, though, that the multiset of formulas on the left-hand side of a conclusion should be the *multiset union* of the l.h.s. of the premises. We require these multisets only to be identical as *sets*. This allows for rules like
$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$
(which characterizes the combining conjunction and is important in the context of, e.g., linear and relevance logics).

- The system above has the property that whenever its rules allow (for some $\Delta, \Gamma_1, \ldots, \Gamma_n$) the inference of $\Delta \Rightarrow B$ from $\Gamma_1 \Rightarrow A_1, \ldots \Gamma_n \Rightarrow A_n$ then for *all* $\Gamma'_1 \ldots, \Gamma'_n$ we may infer $\Gamma'_1, \ldots, \Gamma'_n \Rightarrow B$ from $\Gamma'_1 \Rightarrow A_1, \ldots, \Gamma'_n \Rightarrow A_n$.

We shall call Hilbert type systems with this property *pure* . In such systems we can just take all these inferences to be instances of the schema:

$$\frac{A_1 \ldots A_n}{B} .$$

Obviously pure systems are the most frequent type of systems that one finds in practice. As we shall see, this is true also for natural deduction and Gentzen- type systems, so we delay the discussion of this crucial property until we have it defined also for the other types of systems. For the moment let us give only one quite famous example of an impure system. Take some normal modal logic with the associated *truth* C.R. (see 2.3). A standard Hilbert-type representation will have some axioms and the following two general rules of inference:

$$\frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow A \rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}$$

$$\frac{\Rightarrow A}{\Rightarrow \Box A} .$$

The first one (M.P) is pure [19], the other is not. Hence this Hilbert-type system is impure (The material-implication $\rightarrow$ of S4, say, is an intensional strong implication relative to *this* C.R. and so it is quite common to represent this C.R. by using the interpretation method, applied to the standard axiomatic system for S4-theorems).

- In the general definition of a Hilbert-type system we allow cut to be one of the primitive rules. This is unnecessary for pure systems, since they are always closed under this rule (this can easily be proved by induction on the length of the proof of that premise of the cut in which the cut formula occurs on the left). It is true also for many impure systems, like that for the modal logics described above. This might explain why the cut is never discussed in the context of Hilbert-type systems, and I believe that this state of affairs should be taken as normative: a reasonable Hilbert-type formalism should always allow cut-elimination.

Hilbert-type systems, as defined above, always represent a single conclusioned C.R.. There is nothing in the discussion above that depends on this feature, though. A natural generalization will be, therefore, to remove this limitation and to allow multiple-conclusioned Hilbert-type systems. (In the last section we shall

[19] It should be clear from the definition above what the definition of a pure rule is, and so we omit it here and elsewhere.

present a practical use of such a system.)

Remark: It is a common belief that Hilbert-type systems have only to do with *provability*, not with deducibility. [20] Sometimes this belief is expressed just before the author proceeds to prove a deduction theorem about deducibility in some Hilbert-type system... The reason for this belief are partially historical: in the past logicians happened to be interested mainly in proving logical theorems and used for this Hilbert-systems for theoremhood, i.e. axiomatic systems. Thus most textbooks about modal logics present them as axiomatic systems and are quite confused about what amounts to a deduction from assumptions in them. This confusion is due to the fact that there might exist *more than one reasonable way* to define a C.R. which is compatible with a given axiomatic system (i.e.: has the same set of *theorems*). The pure one, obtained by the extension method, is always a candidate, but there might be more. We have seen already the example of the impure Hilbert-type representations of the truth C.R. in normal modal logics (the pure extension, by the way, corresponds to the *validity* C.R.), and we shall see more below. What makes this phenomenon possible is that in a Hilbert-type system a proof of a sequent $\Rightarrow \Delta$ always consists of sequents of the same form. This allows for a certain degree of freedom while deciding what *other* sequents should be taken as valid!

5.3 Natural-Deduction Representations

The next type of formal systems that we examine is natural-deduction. We start with a very general definition of this type of system which closely resembles that given above for Hilbert-type formalisms:

Definition: *A* Natural-deduction system in the language L is an axiomatic system such that:

1. The formulas are formal sequents of L.
2. $A \Rightarrow A$ is an axiom for each formula A.
3. All the other rules (including 0-premises rules!) have the following property: the set of formulas that appear on the left hand side of their conclusion is a *subset* of the union of the left-hand side of the premises.

If we compare this definition to that of Hilbert-type systems we see that the only difference is that in N.D. (Natural-Deduction) systems we allow certain formulas

[20] see, e.g., [sud1] pp. 134-5.

of the left-hand side of a premise of a rule to be omitted (or to "be *discharged*") from the left-hand side of the conclusion.

The notion of a "pure" system can now be generalized to N.D. systems as follows:

Definition: We call a N.D. system "pure" if whenever $\Gamma \Rightarrow \Delta$ can be derivable from $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ then there are sub-multisets $\Gamma'_1, \ldots, \Gamma'_n, \Delta'_1, \ldots, \Delta'_n$ of $\Gamma_1, \ldots, \Gamma_n, \Delta_1, \ldots, \Delta_n$ (respectively) and a submultiset Δ' of Δ such that: for every $\Gamma''_1, \ldots, \Gamma''_n, \Delta''_1, \ldots, \Delta''_n$ we can infer

$$\Gamma''_1, \ldots, \Gamma''_n \Rightarrow \Delta', \Delta''_1, \ldots, \Delta''_n$$

from

$$(\Gamma''_1, \Gamma'_1 \Rightarrow \Delta'_1, \Delta''_1), \ldots, (\Gamma''_n, \Gamma'_n \Rightarrow \Delta'_n, \Delta''_n) \ .$$

Pure N.D. systems are again the most usual kind of N.D. systems. In such systems we can take all the inferences to be applications of rules of the form:

$$\frac{\begin{matrix}[\Gamma'_1] \\ \Delta'_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Gamma'_n] \\ \Delta'_n\end{matrix}}{\Delta'}$$

(where the Γ'_1 etc. are like in the above definition). A well known example of a N.D. which is not pure is Prawitz N.D. system for S4 (see [Pra1]). In this system one can infer $\Gamma \Rightarrow \Box A$ from $\Gamma \Rightarrow A$ only iff *all* the formulas in Γ begin with $\Box$, and so this rule is impure.

There are important differences that should be noted between pure Hilbert systems and pure N.D. systems. While using pure Hilbert systems one can (and does) write down in a proof-tree of a sequent only the r.h.s of the sequents which participate in that proof. One can then quite easily discover at the end which *sequent* was actually proved, by just looking at the root and the leaves of the tree. One need not know for this what intermediate sequents were proved before the final one was derived. Moreover: one can check each part of the proof separately without needing to examine what happened prior to that part. All these are not true for N.D. proofs—not even pure ones. Here one is *forced to keep track at each stage of a proof of what sequent was actually derived in it.* This is the case regardless of whether formal sequents are employed while implementing the system or other devices are used for this goal. Whatever the method is, the fact remains that N.D. systems are *essentially sequents calculii.*

Since natural-deduction systems enable us to manipulate sequents and not only formulas, they provide us, within the formal machinery, an access to methods of proofs that necessarily belong to the *meta*-theory in the case of Hilbert-type

systems. An obvious example of this ability is the deduction theorem. For many Hilbert-type systems this is an important meta-theorem which is extensively used for indirectly showing that something is provable, without actually proving it. In natural deduction systems this method of proof is usually incorporated into the system as one of its rules. The ability to do such things is the main source of power for N.D. systems and their crucial advantage over Hilbert-type systems.

At this point a natural objection may be raised against our definition of natural deduction systems: according to our presentation, every Hilbert-type system is a N.D. system, but not vice versa. This seems to render as pointless the frequent problem of finding a natural-deduction presentation for a C.R. to which a Hilbert-type representation is known. Something essential seems to be missing: the notion of introduction and elimination rules which [Sud1], for example, takes to be the second major component (besides the possibility of discharging assumptions) of natural deduction systems. I was unable, though, to find a sufficiently general definition of this notion which will not rule out from the class of N.D. formalisms systems that are taken to be such in the literature. In fact the very first N.D. representation of classical logic in [Gen] included as axiom the excluded middle, which cannot be characterized in terms of introduction and elimination rules. It really seems that almost only intuitionistic logic, together with a careful choice of the basic connectives, admits single-conclusioned N.D. representation consisting only of matching introduction and elimination rules (Actually, some fragments of Linear and relevance logics admit such as well, but the corresponding C.R. is not ordinary). As for classical logic, the only way to do so is by using *multiple-conclusioned* N.D. systems, but this certainly is not the *standard* procedure!

In my opinion, therefore, the demand for using matching introduction and elimination rules belongs to the methodology of constructing *good* N.D. representations, not to their definition. This raise the question what makes one formal representation of a C.R. better then another. These can be judged according to the following two criterions:

- ease of finding (and also checking) proofs of sequents.
- usefulness for (constructively) showing significant properties of the represented C.R. (An example of such a property in many logics is Craig's interpolation theorem).

Past experience indicates that in the context of N.D. systems the above two goals are best achieved by first formulating such a system using introduction and (matching) elimination rules. Then using these rules for defining a notion of a "normal proof" with nice properties, and finally proving a "normalization theorem" to the effect that every proof can be converted into a normal proof of the same end sequent. This method seems to be successful only when the

connectives involved are intensional [21] and when the represented C.R. can be characterized in terms of them— as in the examples of the last section. The method is even more limited if we confine ourselves to single-conclusioned C.R.'s. The *exact* characterization of the C.R.'s to which this method is applicable is an interesting topic which we are not going to pursue here. Let us just mention that some authors, especially those with intuitionistic tendencies, believe the method of introduction and elimination rules, as well as the concept of a normal proof, to be of a crucial philosophical importance. They believe, accordingly, that there are deeper reasons for the success of this method than the above description suggests. The interested reader is referred to the enormous literature on the subject like: [Pra1], [Pra2], [Sud2], [SH] and (of course!) the original paper of Gentzen ([Gen]).

A final point which we like to discuss in this subsection is the status of the cut rule in the context of N.D. systems. It can easily be checked that the definition we gave above does not exclude cut from being one of the rules of a N.D. system. For pure N.D. system its eliminability is as easy to show as for pure Hilbert-type systems (with the same method of proof). For impure systems it might be less easy and should not be taken for granted. Let us give an example of such a system for which cut-elimination is true but not *completely* trivial: Consider the $\{\neg, \Box\}$ fragment of the N.D. system for S4. It is immediate that in this system we have that $\neg\neg\Box A \vdash \Box A$ and that $\Box A \vdash \Box\Box A$. It is not, at first glance, obvious that $\neg\neg\Box A \vdash \Box\Box A$, since because of the side conditions on the $(\Rightarrow \Box)$ rule, the proofs of these two sequents cannot directly be combined to produce a proof of the third (This is strongly related to the fact that normalization fails for this version of the system- see [Pra1]). The system does admit cut-elimination, though, but this requires at least *some* efforts. It is not inconceivable that more complicated impure N.D. systems might offer more serious difficulties while proving cut- elimination, or even that it just may fail for them!

5.4 Gentzen-Type Representations

As we argue above, already pure N.D. systems essentially carry us from proofs of formulas to proofs of sequents. The next obvious step is therefore to take full advantage of the use of sequents by allowing the rules of a system to make significant changes also in the antecedent of a sequent:

Definition: A Gentzen-type representation of a given C.R. $\vdash$ in a language L is an axiomatic system such that:

[21] To a lesser degree—also to combining connectives. The resulting N.D. system is usually impure, though, when the combining connectives are not also intensional—as is the case in relevance and linear logics.

1. The formulas of it are formal sequents of L.

2. A formal sequent $\Gamma \Rightarrow \Delta$ is a theorem iff $\Gamma \vdash \Delta$.

The concept of a *pure* representation can be naturally extended to Gentzen -type representations as follows:

Definition: We call a Gentzen-type system *pure* if whenever its rules allow the inference of $\Gamma_0 \Rightarrow \Delta_0$ from $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ then there are subsets Γ'_i, Δ'_i of Γ_i, Δ_i (respectively) such that for every Γ''_i, Δ''_i we can infer $\Gamma''_1, \ldots, \Gamma''_n, \Gamma'_0 \Rightarrow \Delta'_0, \Delta''_1, \ldots, \Delta''_n$ from $\Gamma''_i, \Gamma'_i \Rightarrow \Delta'_i, \Delta''_i$ $(i = 1, \ldots, n)$.

The definition of pure Gentzen-type systems and of pure N.D. systems are very close, the only difference being with respect to the possible existence of Γ'_0 and the possibility (unless we deal with a single-conclusioned C.R.) for Δ'_0 to be empty. Accordingly we can, if we wish, use for pure rules almost the same notation that is frequently used for N.D. systems, but in which symmetry is restored between what is allowed to be a premise and what is allowed to be a conclusion. In fact pure rules in Gentzen systems may be written as follows:

$$\frac{\begin{matrix}[\Gamma'_1] \\ \Delta'_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma'_n] \\ \Delta'_n\end{matrix}}{\begin{matrix}[\Gamma'_0] \\ \Delta'_0\end{matrix}}$$

(where the Γ'_i, Δ'_i $(0 \leq i \leq n)$ are like in the definition above).

Examples:

1. the standard $\vee-$ elimination rule of intuitionistic (single- conclusioned) N.D. system, usually written as

$$\frac{A \vee B \quad \begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{C}$$

will have in Gentzen-type systems the form:

$$\frac{\begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{\begin{matrix}[A \vee B] \\ C\end{matrix}}.$$

And in classical logic just the form: (when ϕ is the empty set):

$$\frac{\begin{matrix}[A] & [B] \\ \phi & \phi\end{matrix}}{\begin{matrix}[A \vee B] \\ \phi\end{matrix}} .$$

The introduction rule in this context will be just:

$$\frac{A, B}{A \vee B} .$$

2. The usual antecedent rule for the conjunction is:

$$\frac{\begin{matrix}[A, B] \\ C\end{matrix}}{\begin{matrix}[A \wedge B] \\ C\end{matrix}}$$

in the intuitionistic case, [22] while in classical logic we can replace C above with the empty set. For both systems the succedent rule for conjunction is just:

$$\frac{A \quad B}{A \wedge B} .$$

In my opinion, it is just a historical accident that people are used to write the full sequents involved while doing proofs in Gentzen-type system, while more economical methods are used for N.D. systems. Repeating passive formulas again and again is really what make the use of Gentzen-systems tedious. It is highly desirable therefore to develop methods for avoiding it in *practical* implementations of such systems. A promising recent contribution in this direction is Girard's concepts of *proof-nets* (see [Gi]). The use of tableaux systems can also be regarded as a method of this kind.

To summerize: in pure Hilbert system the rules [23] are of the form

$$\frac{\Delta_1, \ldots, \Delta_n}{\Delta} .$$

[22]This reminds the form which some rules take in Schroeder-Heister's calculus of higher-order rules. More on the connection between this calculus and Gentzen type calculi can be found in [Av3].

[23]The notion of a "rule" is ambiguous in papers discussing topics of the kind dealt with in the present one, since it has both a global and a local meaning. Thus one can talk about the global rule of adjunction which permits the inference of $A \wedge B$ from A and B for every A and B (here A and B are metavariables), and can also (as in [SH]) regard any inference of $A \wedge B$ from *particular* A and B as application of a local rule which allow inferring $A \wedge B$ from A and B for these particular A and B . We have tried our best here to formulate our definitions and claims to be independent of the interpretation of the term "rule" (but at least for axiomatic systems we usually have in mind the global interpretation).

In pure N.D. systems they have the form:

$$\frac{\begin{matrix}[\Gamma_1] \\ \Delta_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Gamma_n] \\ \Delta_n\end{matrix}}{\Delta}.$$

And in pure Gentzen-type systems:

$$\frac{\begin{matrix}[\Gamma_1] \\ \Delta_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Gamma_n] \\ \Delta_n\end{matrix}}{\begin{matrix}[\Gamma] \\ \Delta\end{matrix}}.$$

A major fact about Gentzen-type systems, even pure ones, is that unlike pure Hilbert-type and N.D. systems, the transitivity (= cut) is never obvious. In the previous types of systems it was a direct result of the asymmetry in rules between antecedents and succedents. In Gentzen-type systems this asymmetry is abolished, and so the cut-rule should either be taken as an explicit rule (as frequently it should) or else be proved admissible. This explains the fact that cut-elimination which is crucial for *every* formal system, was (and is) investigated only in the context of Gentzen-type systems.

Usually, a Gentzen-type representation of a C.R. has the following form: It has as axioms the reflexivity axioms $A \Rightarrow A$ (sometimes it suffices to take only a subset off these and then derive the rest). The rules are then divided into two groups:

- Structural rules, which operate on the multisets, but do not change the formulas in them (though it may add or delete some). Frequent examples are: weakening, contraction, anticontraction, mingle (from $\Gamma_1 \Rightarrow \Delta_1$ and $\Gamma_2 \Rightarrow \Delta_2$ infer $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$) and, of course, cut.
- Rules concerning the constants of the language. Usually they are divided into succedent rules, which specify how a logical constant may be introduced in the succedent of a sequent, and antecedent rules, which do the same for the antecedent.

In order to judge how good a given Gentzen-type presentation of a given C.R. is, we should apply exactly the same criterions as we did for N.D. systems. However, since we allow more in Gentzen-type systems we are usually entitled to demand more. Hence we usually should expect it to be easier to demonstrate the validity of a formal sequent in a Gentzen-type formalism than it is in a corresponding N.D. system. Also we can expect nice proof-theoretic properties to

hold. Such properties are usually proved by induction, the major step of which is showing that the various rules preserve the property under discussion. Usually the cut rule is the major obstacle while providing such a proof. A very good example for this is the important subformula property. This property is preserved by all the rules we found as characteristic for the various intentional and combining connectives. It is preserved also by all the structural rules considered in the last section—except cut. If we delete cut from the set of rules of each of the systems we discuss in the last section we get therefore formal systems with the subformula property. The cut-elimination theorem (which obtains for each of those systems) mean that these new systems are still representations of C.R.'s and that the C.R.'s they represent are identical to the original ones.

Note: Since the antecedent and succedent rules completely specify how a connective can be introduced into sequents, it can be argued that they operationally define that connective. This claim is strongly related to the analogous claim concerning introduction and elimination rules in N.D. systems. I believe that a much better case can be made to this claim in the present context than in the previous one.[24] Nevertheless, I still think that it is tenable only for special types of connectives. After all, the rules of Gentzen-type systems (or even N.D. systems) do not necessarily treat one connective at a time. The following rule, e.g. is frequently a useful one (see next section),but it treats two:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \neg A \quad \Gamma_2 \Rightarrow \Delta_2, \neg B}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \neg(A \wedge B)}.$$

In systems with rules like the last one we might be forced to say that the various connectives are simultaneously defined by the set of the rules (taken as a whole). This does not seem to be very helpful, though. Moreover, the distinction between antecedent and succedent rules is not so sharp. In modal logics, for example, we frequently have rules which introduce new connectives on *both* sides of a sequent, like the following rule of the minimal normal modal logic K:

$$\frac{A_1, \ldots, A_n \Rightarrow B}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}.$$

It might be claimed that this rule completely defines the box in the context of K, but such a claim might be more difficult to swallow than similar claims for more regular rules. Note that rules like the last one are not available in N.D. formalisms!

[24]See, in relation to this claim the first chapter of [Ga] and the paper of Hacking [Ha].

5.5 Three Degrees of Impurity

In the previous subsections a great deal of importance was attached to what we call pure systems, The most general definition of which was given while discussing Gentzen-type representations. Now the reasons for the importance of pure systems is due to the fact that they are relatively easy to implement in an economical way, and it is easy also to check proofs in them. [25] However, there are also known representations which are not pure. Our purpose now is to try to identify the most usual forms that impurity may take. We assume, accordingly, that we are dealing with a formal system of one of the types described above. Moreover, we assume that the inference rules leading from sequents to sequents are given by some global rule-schemes of the form described after the definition of a pure Gentzen-type system (in the last subsection), but that there are side conditions on the applicability of some of these schemes which make them impure. We shall examine three possible ways in which this might happen:

Level 1: At this level the side conditions are related to the *structure* of the multisets of the side-formulas (Γ_i'', Δ_i'' above). Examples of such side conditions are:

- Demanding that there be no side formulas. In Hilbert-type systems rules with this conditions are usually known as *rules of proof*. The best known example is the necessitation rule in traditional formulations of normal modal logics. Other example is the adjunction rule of many relevance logics (from A and B infer $A \wedge B$) which frequently is taken to be only a rule of proof.
- Demanding, in a multiple-conclusioned C.R., the antecedents of the conclusion and the hypotheses of a rule to be singletons. Examples are the rule for the $\Box$ in the Gentzen-type system for the minimal normal modal logic K (see above) and the succedent rule for the intuitionistic implication in the multiple-conclusioned Gentzen-type version which we mention in section 4.4.
- Demanding all the hypotheses of a rule to have *exactly* the same side-formulas. Examples are provided by the rules for the combining connectives in section 4.3.. For logics without contraction or weakening, like relevance logics and linear logic, these rules are impure (and this explains why Girard has found (in [Gi]) the "multiplicative" fragment

[25] Every ordinary, pure single-conclusioned N.D. system can, e.g., quite easily be implemented on the Edinburgh LF, which is a general proof-development environment, based on a general Logical Framework which was developed in the computer science department of Edinburgh university.

of his logic, which is pure, to be better understood than the "additive" one, which is not).

Level 2: Here an applicability of a rule might depend also on the structure of the side-formulas. Examples are:

- The introduction rule for the $\Box$ in Prawitz N.D. system for $S4$. This rule permits the inference of $\Gamma \Rightarrow \Box A$ from $\Gamma \Rightarrow A$ only if all the formulas in Γ begin with $\Box$.
- The introduction rule for $\forall$ in the N.D. systems for classical and intuitionistic logics. Here one can infer $\Gamma \Rightarrow \forall x A$ from $\Gamma \Rightarrow A$ only if x does not occur free in Γ (This is not the whole story, though, since this rule might become pure in the context of non-simple C.R.'s. There are in fact no problems to deal with *this* kind of impurity in the Edinburgh LF!).

Level 3: Here an applicability of a rule might depend not only on its potential premises, but also *on their proofs*. A possible example may be provided by an attempt to base a N.D. system on the following version of the deduction theorem in classical first-order logic: [26] $\Gamma \vdash A \rightarrow B$ iff there is a proof of B from Γ, A in which no inference of $\forall x C$ from C is made in which x is free in A and C depends on A (in that proof).

The third level of impurity carries us, in fact, beyond the class of *uniform* systems [27], with which we were dealing so far. The main properties of uniform systems are that they treat exactly one C.R., and that once a sequent was derived in them one can completely forget about *how* it was derived while using it for deriving other sequents. For systems of the third level of impurity this is no longer the case. Such systems are very inefficient in the time and space required for proof checking. It is advisable, therefore to avoid the use of such systems. In the next section we shall suggest some possible methods for doing this when uniform representations are not available or the available ones are inefficient.

[26] See, e.g., in [Men]. This theorem is true for the pure Hilbert type system for *validity* which is presented there.

[27] A formal definition of a uniform system will be identical to that of a Gentzen-type system and was already given in 5.4.

6 Non-Uniform Representations

As we have seen in the previous section, all the standard proof systems are examples of uniform representations of consequence relations. However, if we accept that a formal system is essentially a device for deriving correct sequents (of a C.R. in which we happen to be interested) then there is no reason to limit ourselves to uniform formal systems. I believe that this observation might open the door to a new area of investigations with a wealth of promising possibilities. To demonstrate the potential of non-uniform representations we shall describe now two methods of developing such representations together with examples of their applicability. We hope that other efficient methods will be developed in the future.

6.1 Treating Several Consequence Relations Simultaneously

A major feature of uniform representations is that they treat only a single C.R.. In mathematics it is often much more efficient to simultaneously solve *several* related problems. The problems of representing related consequence relations should not be an exception in this respect. A good example for this is provided by the two C.R.s which we have associated with modal logics in 3.3 . Obviously, there are strong connections between them. On the other hand it is difficult to provide a nice pure representation of either, since each lacks some important properties (which the other has). It is reasonable, therefore, to try to represent them together in one formal system. This really can be done (at least for natural modal systems like $S4, T, K4$ and so on). Examples of rules of the resulting system in the case of $S4$ are:

$$\frac{\Gamma \vdash_v A}{\Gamma \vdash_v \Box A} \quad \frac{A, \Gamma \vdash_t B, \Delta}{\Gamma \vdash_t A \to B, \Delta} \quad \frac{\Gamma, A \vdash_v B}{\Gamma \vdash_v \Box A \to B} \quad \frac{\Gamma \vdash_t A}{\Gamma \vdash_v A} \quad \frac{\vdash_v A}{\vdash_t A}.$$

In this system $\vdash_t$ is taken as multiple conclusioned while $\vdash_v$ as single conclusioned. All the impurities of the usual representations are eliminated in the combined one (or—depending on the meaning of "pure" in this context— are reduced to impurities of the first degree - see 5.5.)

It is worth noting that a suitable variation of this system was actually used in order to solve the problem of efficiently internalizing $S4$ in the computerized Edinburgh LF system (see [AHM]).

6.2 Higher-Order Sequents

We start with the following observation: Uniform representations are axiomatic systems in which the wffs are formal sequents. Past experience shows that it has frequently been useful to extend such a system to a multiple-conclusioned one even if the ultimate interest remained in proving theoremhood of wffs. It is natural therefore to try applying the same process in the present case. This naturally leads us to consider "hypersequents" which are sequents of sequents. It might be enough to start by considering only one-side hypersequents . Can they be useful? The answer is "yes", and in what follows we provide two examples which demonstrate this claim.

In order to understand the examples, let us return to the five 3-valued consequence relations of 3.4. It is not difficult to provide uniform, cut free Gentzen-type presentations of the first three. It is enough, e.g., to take as logical rules for all of them the usual rules for the (combining) $\vee$ and $\wedge$, as well as the obvious rules for $\neg\vee$, $\neg\wedge$ and $\neg\neg$ (on both sides). The systems will differ then only with respect to the axioms ($P \Rightarrow P$ (P atomic) will be axioms for all. In addition we have $P, \neg P \Rightarrow_{Kl}$, $\Rightarrow_{PAC} P, \neg P$ and $\neg P, P \Rightarrow_{Lt} \neg Q, Q$). For the other two we should work with hypersequents, though! The needed changes are:

1. We should work with multisets of sequents, both sides of which are multisets of formulas.

2. We should add the *external* structural rules:

   External Weakening

   $$\frac{(\Gamma_1 \Rightarrow \Delta_1), \ldots, (\Gamma_n \Rightarrow \Delta_n)}{(\Gamma \Rightarrow \Delta), (\Gamma_1 \Rightarrow \Delta_1), \ldots, (\Gamma_n \Rightarrow \Delta_n)}.$$

   External Contraction

   $$\frac{(\Gamma \Rightarrow \Delta), (\Gamma \Rightarrow \Delta), (\Gamma_1 \Rightarrow \Delta_1), \ldots, (\Gamma_n \Rightarrow \Delta_n)}{(\Gamma \Rightarrow \Delta), (\Gamma_1 \Rightarrow \Delta_1), \ldots, (\Gamma_n \Rightarrow \Delta_n)}.$$

3. We replace the six $\neg$ rules by the usual two rules for internal negation.

4. Internal weakening is not a rule in the case of $\vdash_{Sob}$ while internal contraction is not a rule in the case of $\vdash_{Luk}$.

5. The logical rules and the permitted internal structural rules should be reformulated so that "side-sequents" are allowed. For example $\vee \Rightarrow$ takes the form:

   $$\frac{(A, \Gamma \Rightarrow \Delta), (\Gamma_1 \Rightarrow \Delta_1), \ldots \quad (B, \Gamma \Rightarrow \Delta), (\Gamma'_1 \Rightarrow \Delta'_1), \ldots}{(A \vee B, \Gamma \Rightarrow \Delta), (\Gamma_1 \Rightarrow \Delta_1), \ldots, (\Gamma'_1 \Rightarrow \Delta'_1), \ldots}.$$

6. Instead of the deleted internal structural rules we should add:

   For $\vdash_{Sob}$: The following weaker versions of weakening *(with possible side-sequents)*:

$$\frac{(\Gamma_1 \Rightarrow \Delta_1) \quad (\Gamma_2 \Rightarrow \Delta_2)}{(\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2)}$$

$$\frac{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{(\Gamma_1 \Rightarrow \Delta_1), (\Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta')}$$

   For $\vdash_{Luk}$: We add the following *splitting* rule (again with possible side-sequents):

$$\frac{(\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3) \quad (\Gamma'_1, \Gamma'_2, \Gamma'_3 \Rightarrow \Delta'_1, \Delta'_2, \Delta'_3)}{(\Gamma_1, \Gamma'_1 \Rightarrow \Delta_1, \Delta'_1), (\Gamma_2, \Gamma'_2 \Rightarrow \Delta_2, \Delta'_2), (\Gamma_3, \Gamma'_3 \Rightarrow \Delta_3, \Delta'_3)}.$$

Notes:

1. The proofs that by this method we really get sound and complete representation in which cut-elimination is admissible are not too difficult, since easy reduction steps are available. It is much harder, yet possible, to prove the same properties if we add internal implication to both (getting the full systems RM_3 and Lukasiewicz L_3, respectively). We shall present all these proofs in a forthcoming paper.

2. Cut-free Hypersequential calculi were first introduced in [Pot] and, independently, in [Av2]. In the forthcoming paper mentioned above we shall extend the method to other Logics (e.g. the system LC of Dummet) and supply all the proofs.

3. It is possible, of course, to consider further iterations of the above procedure.[28] The examples we have given should suffice for clarifying the main idea, though.

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[28] In the literature one can already find uses of sequents of arbitrary degree of nesting in [SH] and [Do].

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