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ABSTRACT

This paper is about Set Theory as it was originally intended: i.e. as a theory of the extensions of properties. We investigate, and prove relatively consistent to ZF, several restrictions to Frege's inconsistent Comprehension Principle in Set Theory different from Zermelo's "Limitation of Size" Principle.

More precisely we discuss models for highly self referential and self descriptive Set Theories. In these theories many interesting classes such as the membership relation or the class of all sets are themselves sets but sets are nonetheless closed under interesting operations (e.g. intersection, union and power set). We deal also with models of non purely set theoretical theories for the foundations of Mathematics.

Our models carry naturally a peculiar topological structure. In fact any of them can be viewed as a $\varkappa$-compact $\varkappa$-metric space which coincides with the space of its closed subsets equipped with Hausdorff's $\varkappa$-metric. The comprehension properties of these models are a consequence of this topological structure.

Ideas and techniques in the theory of non-well-founded sets play a crucial role in this paper. Techniques similar to these have been widely used also in the theory of transition systems. In fact, the analogy of "sets as processes" establishes a correspondence between many concepts in these two areas, e.g. f-admissible relations and strong bisimulations, greatest f-admissible relation and strong observational congruence. We hope therefore that this paper might be of inspiration to the theory of process algebras, and many of the constructions developed in this paper might be fruitfully carried to that domain.

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MODELS OF SELF-DESCRIPTIVE SET THEORIES

Marco Forti (Cagliari) and Furio Honsell (Edinburgh)

dedicated to Ennio De Giorgi on his 60th birthday

Introduction

It is well known that Zermelo-Fraenkel set theory has a limited self-descriptive power. In fact most of the basic set-theoretic relations, operations and properties (e.g. membership, union, sethood) cannot be represented as sets since the classes which correspond to them are too large.

Many attempts have been made to define set theories consistent relative to ZF, which allow as sets many interesting classes having the size of the universe. Apart from W.V.O.QUINE's NF [16], whose consistency strength is still unknown, we can mention the theories (all equiconsistent with ZF) considered by A.CHURCH [1], H.FRIEDMAN [11], E.MITCHELL [14], and A.OBERSCHELP [15]. These, however, are in some sense unsatisfactory, since each of them is not closed under some basic construction.

A very interesting class of set-theoretical models, closed under many basic operations but still possessing a lot of large sets, was introduced by R.J.MALITZ in his thesis [13]. Unfortunately, he considered only wellfounded universes, thus utterly weakening the actual power of his construction. In fact the most interesting properties of the models he defined depended on a conjecture which is now almost completely disproved (see section 2).

However, by simply performing Malitz's construction inside a non-wellfounded universe verifying a suitable "Free Construction Principle", the first author [7] succeeded in proving the consistency, relative to ZF, of the axiom schema GPK. This is a general "Positive Comprehension Schema", which postulates the existence of the set $\{x \mid \Phi(x)\}$ for any non-negative formula Φ (for a precise definition of the generalized positive comprehension GPK, see [7] and section 3 below).

On a different ground, wider self-referential power can also be achieved by considering non purely set-theoretic foundational theories. In these theories basic objects such as *properties, relations* and *operations* are considered as primitive notions and are not identified with their usual set-theoretic reductions. We refer in particular to the work inspired by E.DE GIORGI and developed since the late seventies by him and several researchers attending his Seminar on Logic and Foundations at the Scuola Normale Superiore, Pisa (see [2], [3], [4]).

In this paper we discuss in depth, from topological and set-theoretical viewpoints, the constructions of [7]. Using techniques from the theory of infinitary trees we provide a number of counterexamples to Malitz's conjecture. We also generalize the construction of [7] to universes with (universe-many) urelements. The models thus obtained provide a suitable environment for modelling theories for the Foundations of Mathematics like [2], [3], [4]. In section 3 we explore the possibility of modelling significant sublists of the strong axioms of [2,§VI]. Theorems 3.3 and 3.4 are first results in this direction; a more detailed account of this will be given in [10].

It is well known that comprehension principles entailing the existence of universe-sized sets are often inconsistent with principles of choice (see [6]). In the last section of this paper we discuss various classical choice principles in connection to our models. We obtain *inter alia* the relative consistency of the axiom of choice and of the well-ordering principle with respect to the generalized positive comprehension schema GPK plus an axiom of infinity.

Finally the authors would like to express how deeply they are indebted to Ennio De Giorgi for his constant help and encouragement throughout their set-theoretic and foundational research.

1. The Basic Construction

We work in a non-wellfounded Zermelo-Fraenkel like set theory with urelements. We assume the axiom of choice and, instead of the axiom of foundation, a suitable *free construction principle*.

The axioms of our set theory are the following:[1]

ZF_0^- - *Pairing* Pair, *Union* Un, *Power-set* PS, *Replacement* Rpl, and *Infinity* Inf as in *Zermelo-Fraenkel's theory* ZF.

AC - *Zermelo's axiom of choice.*

WE - *Weak Extensionality with respect to a* (possibly empty) *set U of atoms,* i.e.

$(x \in U \longrightarrow t \notin x)$ & $(\exists x \notin U\, \forall t\; t \notin x)$ &

$(x \notin U$ & $y \notin U$ & $\forall t (t \in x \longleftrightarrow t \in y) \,.\!\longrightarrow.\, x = y)$.

FC - *Unique Free Construction with respect to a set U of atoms,* i.e.

(1) For definitions and standard results on set theory we usually refer to [12]; when we adhere instead to the notations of [2] or [8], we shall mention it explicitly.

Given a function $f\colon X \longrightarrow \mathcal{P}(X) \cup U$ *such that* $f(a) = a$ *for any* $a \in X \cap U$, *there is a unique function* $g\colon X \longrightarrow T$ *verifying* [2]

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \in U \\ \hat{g}(f(x)) & \text{otherwise.} \end{cases}$$

The axiom FC generalizes the free construction axiom X_1 of [8] to set theories with atoms. A straightforward modification of the argument in the proof of Theorem 3 of [9] yields:

THEOREM 1.0

Given any model $\mathfrak{M}$ *of* $ZF_0^- + WE$ *there is, up to isomorphism, exactly one (inner) model* $\mathfrak{N}$ *of* $ZF_0^- + WE + FC$ *with the same atoms and the same well-founded sets of* $\mathfrak{M}$.

Therefore, as far as relative consistency and mutual interpretability are concerned, our theory $ZF_0^- + WE + FC$ is equivalent to ZF. The same holds for any extension of both theories obtained by adding any large cardinal axiom or any choice principle, in particular AC (cfr [9]).

An easy consequence of the axiom FC is *the absence of nontrivial atom-preserving* $\in$*-homomorphisms*. This property of *atomic rigidity*, analogous to the rigidity property implied by the axiom of Foundation, will be of some importance in the sequel, so we formulate it explicitly:

AR - *If* T *is transitive and* $h\colon T \longrightarrow S$ *verifies* $h(x) = x$ *for* $x \in T \cap U$ *and* $h(x) = \hat{h}(x)$ *for* $x \in T \setminus U$, *then* h *is the identity on* T.

In particular, AR implies the following axiom of *strong*

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[2] We denote by $\hat{g}(x)$ the image of x under the function g; more generally, we put throughout the paper $\hat{x}(y) = \{v \mid \exists u \in y\ (u,v) \in x\}$.

extensionality up to atoms:

SextA - *If two transitive sets are $\in$-isomorphic under an isomorphism which leaves any atom fixed, then they are equal.*

In defining our models, we shall use topological notions.[3] In fact we need *a uniform topology with a nested uniformity basis made up by equivalences*. To this aim, we fix a regular cardinal $\varkappa$ and we assume that the set U of the atoms carries a *$\varkappa$-hypermetric*, i.e. a distance $d: U^2 \longrightarrow {}^*R$, where ${}^*R$ is any nonstandard model of the real numbers with cofinality $\varkappa$, satisfying the following properties:

(i) $d(a,b) = 0$ *iff* $a = b$;

(ii) $d(a,b) = d(b,a) \geq 0$ *for all* $a,b \in U$;

(iii) $d(a,b) \leq \max\{d(a,c),d(b,c)\}$ *for all* $a,b,c \in U$.

We are interested only in the uniform structure induced by d. We assume ${}^*R$ to be a model of the reals only for sake of suggestivity. Actually, all that is needed is simply an ordered set of type $1+\eta$, with $\text{cof}\ \eta^* = \varkappa$.

Therefore we fix a *strictly decreasing $\varkappa$-sequence* $\langle \varepsilon_\alpha \mid \alpha < \varkappa \rangle$ with infimum 0, and we define for any ordinal $\alpha \leq \varkappa$ the *α-equivalence* $\approx_\alpha$ on U by

$$a \approx_\alpha b \quad \text{iff} \quad d(a,b) < \varepsilon_\beta \quad \textit{for any} \quad \beta < \alpha. \tag{1.1}$$

Thus $\approx_0$ and $\approx_\varkappa$ are respectively the trivial equivalence U^2 and the equality. Moreover, the chain $\langle \approx_\alpha \mid \alpha \leq \varkappa \rangle$ is *weakly decreasing and continuous* (i.e. $\approx_\alpha \subseteq \approx_\beta$ whenever $\alpha > \beta$, and $\approx_\lambda = \bigcap \{\approx_\alpha \mid \alpha<\lambda\}$ for limit λ), and *generates the uniformity* $\mathcal{U}$

[3] We shall only sketch some of the topological arguments in this paper. All properties we shall state and use are straightforward modifications of standard results and methods of the theory of metric and of compact spaces. We refer to [5], where also a detailed treatment of general uniform spaces can be found.

associated to d, which is therefore either *discrete* or of *weight* $\varkappa$.

Note also that the above defined sequence is made up of equivalences by virtue of the hypermetric inequality (iii), which implies that the set of all balls of any fixed radius is a partition of U. However, this condition is restrictive only for $\varkappa = \omega$, since in the uncountable case any $\varkappa$-distance d verifying the usual triangular in- equality can be replaced by a uniformly equivalent one satisfying the hypermetric inequality (iii). Actually, it is easy to see that it is possible to define such a $\varkappa$-hypermetric for any uniform space having a *nested uniformity basis of uncountable cofinality* $\varkappa$. Only when $\varkappa = \omega$, i.e. when $\mathcal{U}$ is *metrizable*, one has to check the sup- plementary condition that *no pair of different points can be connected by a finite set of arbitrarily small non-disjoint balls* (see [18] for more details about $\varkappa$-metric spaces).

Following [7], we extend inductively on α the equivalences $\approx_\alpha$ to the whole universe V by

$$x \approx_\alpha y \;\; iff \quad \forall\beta<\alpha \; \forall s\in x \; \forall u\in y \; \exists t\in y \; \exists v\in x \quad s \approx_\beta t \;\; \& \;\; u \approx_\beta v. \tag{1.2}$$

Note that the sequence $\langle \approx_\alpha \rangle_{\alpha<\varkappa}$ is now *strictly* decreasing and continuous. Moreover, $\approx_0$ is again the trivial relation V^2, but $\approx_\varkappa$ is no more the equality (e.g. all ordinals greater then $\varkappa$ are $\varkappa$-equivalent, see [13]). However, we can extend the distance d to V^2 by putting

$$d(x,y) = \begin{cases} \varepsilon_\alpha & \text{if} \;\; \exists\alpha<\varkappa \;\; x \approx_\alpha y \;\; \& \;\; x \not\approx_{\alpha+1} y \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

thus obtaining a *pseudo-$\varkappa$-hypermetric*, which verifies only conditions (ii) and (iii) above.

In order to obtain the corresponding $\varkappa$-metric space we need to single out just one point from each ball of radius zero, but we can neither invoke the axiom of choice nor simply take the quotient, since many 0-balls are proper classes. We cannot even apply Scott's trick as in [13], since we are working in a non-well-founded universe; we use here instead the method of [7], the axiom FC playing the role of X_1 in the presence of urelements.

First of all we intoduce the *ϰ-membership* $\in_\varkappa$ on V by

$$x \in_\varkappa y \qquad iff \qquad \begin{cases} \forall\alpha < \varkappa \quad \exists z \in y \quad x \approx_\alpha z, \\ \qquad \text{or equivalently} \\ \exists x' \approx_\varkappa x \quad \exists y' \approx_\varkappa y \quad x' \in y'. \end{cases} \tag{1.4}$$

Both ϰ-membership and ϰ-equivalence on V have nice topological characterizations, namely

LEMMA 1.1

Two points $x, y \in V\backslash U$ *are ϰ-equivalent iff they have the same closure (considered as subsets of* V*), and the ϰ-members of any point* $x \in V\backslash U$ *are precisely the ordinary members of its closure.*

PROOF

By the first definition of ϰ-membership, any ϰ-member of x is the limit of a ϰ-sequence of members of x, hence belongs to the closure of x; conversely, any point of the closure of x is such a limit, and the second assertion of the lemma follows.

Moreover, by the second definition of ϰ-membership, ϰ-equivalent sets have the same ϰ-members, hence the same closure. On the other hand, although the closure $y = \overline{x}$ of a given set x is possibly a proper class, nevertheless it satisfies *in toto* the condition (1.2) for being ϰ-equivalent to x; the lemma is thus completely proved.

Q.E.D.

By the above lemma, if the closure of any set were itself a set, then the closed sets together with the atoms would be a complete set of representatives for the $\varkappa$-equivalence classes. Apparently, this is not the case, but our goal can be achieved if we restrict ourselves to a suitable subspace.

LEMMA 1.2

Suppose that the set X *meets all* $\varkappa$*-equivalence classes, i.e. that for any* $y \in V$ *there is* $x \in X$ *such that* $x \approx_{\varkappa} y$.

Then there are a unique transitive set N *and a unique function* $g\colon X \longrightarrow N$ *verfying the following conditions:*

(i) $x \approx_{\varkappa} g(x)$ *for any* $x \in X$;

(ii) $x \approx_{\varkappa} y$ *iff* $g(x) = g(y)$ *for any* $x,y \in X$;

(iii) $x \in_{\varkappa} y$ *iff* $g(x) \in g(y)$ *for any* $x,y \in X$.

Therefore N *is a transitive set of representatives for the* $\varkappa$*-equivalence classes, and* $\varkappa$*-membership agrees on* N *with ordinary membership; thus* N *provides a sort of "transitive collapse" of the "quotient structure"* $(V/_{\approx_{\varkappa}}, \in_{\varkappa}/_{\approx_{\varkappa}})$.

PROOF

Note that $X \supseteq U$, since any atom is the only member of its $\varkappa$-equivalence class; define the function $f\colon X \longrightarrow \mathcal{P}(X)\cup U$ by $f(u) = u$ for $u \in U$ and $f(x) = X\cap\overline{x}$ for $x \in X\setminus U$.

Let $g\colon X \longrightarrow N$ be the unique function, given by the axiom FC, which is the identity on U and equal to $\hat{g}\circ f$ on $X\setminus U$. By definition g has transitive range and is the identity on U: therefore, in proving (i)-(iii), we can restrict ourselves to consider only sets.

We shall prove by induction on α that $x \approx_{\alpha} g(x)$ for any $\alpha < \varkappa$: note that the assertion is trivial for $\alpha = 0$ and that the limit steps are true by definition.

Assume now $t \approx_{\alpha} g(t)$ for any $t \in X$ and pick $x \in X\setminus U$: then

$g(x) = \hat{g}(X \cap \bar{x}) = \{g(t) \mid t \in X \ \& \ t \in_{\varkappa} x\}$. First pick $s \in x$: by hypothesis there is some $t \in X$ with $t \approx_{\varkappa} s$, and surely $t \in_{\varkappa} x$, hence $g(t) \in g(x)$. Conversely, pick $g(t) \in g(x)$: since $t \in_{\varkappa} x$, t is the limit of some $\varkappa$-sequence in x, hence there is some $s \in x$ such that $s \approx_{\alpha} t$. In both cases we have $s \approx_{\alpha} g(t)$, whence $x \approx_{\alpha} g(x)$.

The implication $g(x) = g(y) \Longrightarrow x \approx_{\varkappa} y$ is an immediate consequence of (i), and the remaining part of (ii) follows from the fact that $\varkappa$-equivalent sets, having the same closure, have the same image under f.

Finally, what we have shown above, namely that $g(y) = N \cap \{g(x) \mid x \in_{\varkappa} y\}$, is a mere rephrasing of (iii).

Up to now, we have only used the existential part of the axiom FC. The uniqueness of g and N follows from the rigidity property AR, which is a consequence of the uniqueness part of FC.

Q.E.D.

We obtained the set N starting from a set X where all $\varkappa$-equivalence classes were represented. But the role of X in FC is merely that of a *parameter set for defining the real membership* on the set one is looking for.

Therefore all that we need in order to get N is a set which *parametrizes all $\varkappa$-equivalence classes*, i.e. a set Y together with a mapping $\tau: V \longrightarrow Y$ inducing the identity on U and verifying $x \approx_{\varkappa} y$ whenever $\tau(x) = \tau(y)$.

Then we can put $f(y) = \{z \in Y \mid \exists u,v \ u \in_{\varkappa} v \ \& \ \tau(u)=z \ \& \ \tau(v)=y\}$ for $y \in Y \setminus U$ and $f(y) = y$ for $y \in U$; taking the function g given by the axiom FC and putting $\sigma = g \circ \tau$ we obtain a function which satisfies conditions (i), (ii), and (iii) for all $x,y \in V$ and has therefore the same range N.

There are several ways of defining such a mapping τ, and we choose one that gives supplementary information about all "quotients" $V/\approx_\alpha$ for $\alpha \le \varkappa$.

We fix functions $\tau_\alpha : U \longrightarrow U$ for $\alpha \le \varkappa$, in such a way that $x \approx_\alpha \tau_\alpha(x)$ for any $x \in U$ and any $\alpha \le \varkappa$; we extend them inductively to V by putting, for $x \in V\setminus U$

$\tau_0(x) = \tau_0(u)$ for some $u \in U$,

$\tau_{\alpha+1}(x) = \hat{\tau}_\alpha(x) = \{ \tau_\alpha(y) \mid y \in x \}$ for any $\alpha < \varkappa$, and

$\tau_\lambda(x) = \langle \tau_\alpha(x) \rangle_{\alpha<\lambda} = \{ (\alpha, \tau_\alpha(x)) \mid \alpha<\lambda \}$ for limit λ.

Note that we can impose to the original τ_α's the supplementary condition that $\tau_\alpha(x) = \tau_\alpha(y)$ whenever $x \approx_\alpha y$, thus getting a sequence of *choice functions for the α-equivalence classes* of U; but we can as well take all τ_α's to be the identity on U, in order to make our construction independent of the axiom of choice.

In any case we obtain

LEMMA 1.3

(i) *If* $\tau_\alpha(x) = \tau_\alpha(y)$, *then* $x \approx_\alpha y$;

(ii) *If* $x \approx_\alpha y$ *implies* $\tau_\alpha(x) = \tau_\alpha(y)$ *for* $x,y \in U$, *then the same holds for any* $x,y \in V$.

(iii) $\hat{\tau}_\alpha(V)$ *is a set and* $|\hat{\tau}_\alpha(V)| \le \exp_{\alpha+1}(|\hat{\tau}_\alpha(U)|)$ *for any* $\alpha \le \varkappa$.[4]

PROOF

(i) We proceed by induction on α, assuming $x,y \notin U$ since (i) is true by definition for $x,y \in U$, and the hypothesis is never true when $x \in U$ and $y \in V\setminus U$.

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[4] We define inductively the *iterated exponential* $\exp_\alpha(\varkappa)$ in the usual way: $\exp_0(\varkappa) = \varkappa$, $\exp_{\alpha+1}(\varkappa) = 2^{\exp_\alpha(\varkappa)}$ and, for limit λ, $\exp_\lambda(\varkappa) = \sup\{\exp_\alpha(\varkappa) \mid \alpha<\lambda\}$. We also put $\beth_\alpha = \exp_\alpha(\aleph_0) = \exp_{\omega+\alpha}(0)$

The case $\alpha = 0$ is trivial. For limit α it suffices to recall that the equivalences $\approx_\beta$ are a decreasing and continuous sequence, hence $x \approx_\alpha y$ iff $x \approx_\beta y$ for all $\beta<\alpha$, whereas, by definition, $\tau_\alpha(x) = \tau_\alpha(y)$ iff $\tau_\beta(x) = \tau_\beta(y)$ for all $\beta<\alpha$.

Finally, assume (i) true for α and $\tau_{\alpha+1}(x) = \tau_{\alpha+1}(y)$, i.e. $\hat{\tau}_\alpha(x) = \hat{\tau}_\alpha(y)$; then we have that for any $s \in x$ there is some $t \in y$ such that $\tau_\alpha(s) = \tau_\alpha(t)$ and symmetrically, hence $x \approx_{\alpha+1} y$.

(ii) We proceed inductively on α, and once more both the initial and the limit steps are straightforward.

Assuming (ii) true for α and $x \approx_{\alpha+1} y$ with $x, y \notin U$, we have for any $s \in x$ some $t \in y$ such that $s \approx_\alpha t$, hence $\tau_\alpha(s) = \tau_\alpha(t)$, and symmetrically starting from $t \in y$.

Hence $\tau_{\alpha+1}(x) = \hat{\tau}_\alpha(x) = \hat{\tau}_\alpha(y) = \tau_{\alpha+1}(y)$.

(iii) Again we proceed by induction on α, the assertion being trivial for $\alpha = 0$, and we put $\varkappa_\alpha = |\hat{\tau}_\alpha(U)|$ and $\nu_\alpha = |\hat{\tau}_\alpha(V)|$.

Since $\tau_{\alpha+1} = \hat{\tau}_\alpha$ on $V \setminus U$, we have $\hat{\tau}_{\alpha+1}(V\setminus U) \subseteq \mathcal{P}(\hat{\tau}_\alpha(V))$, hence $\nu_{\alpha+1} \leq \varkappa_{\alpha+1} + 2^{\nu_\alpha} \leq \exp_{\alpha+2}(\varkappa_{\alpha+1})$ by induction hypothesis.

For limit α, we have $\hat{\tau}_\alpha(V\setminus U) \subseteq \prod_{\beta<\alpha} \hat{\tau}_\beta(V)$, hence

$$\nu_\alpha \leq \varkappa_\alpha + \prod_{\beta<\alpha} \nu_\beta \leq \varkappa_\alpha + (\exp_\alpha(\varkappa_\alpha))^{|\alpha|} \leq \exp_{\alpha+1}(\varkappa_\alpha),$$

since, by induction hypothesis, $\nu_\beta \leq \exp_\alpha(\varkappa_\alpha)$ for any $\beta<\alpha$.

Q.E.D.

We can summarize the preceding results as follows:

THEOREM 1.4

Suppose that the set U *of all urelements carries a* $\varkappa$*-hypermetric structure, let* $\langle \approx_\alpha \rangle_{\alpha \leq \varkappa}$ *be the chain of equivalences on the universe* V *associated to it according to* (1.1)-(1.2), *and let* $\mathfrak{U}$ *be the* *generated uniformity.*

Then there are a unique transitive set $N = N_{\varkappa}(\mathcal{U})$ *and a unique projection* $\sigma\colon V \longrightarrow N$ *verifying the following conditions:*

(i) $x \approx_{\varkappa} \sigma(x)$ *for any* $x \in V$;

(ii) $\sigma(x) = \sigma(y)$ *iff* $x \approx_{\varkappa} y$ *for any* $x,y \in V$;

(iii) $\sigma(x) \in \sigma(y)$ *iff* $x \in_{\varkappa} y$ *for any* $x,y \in V$.

In particular $\varkappa$*-membership agrees on* N *with ordinary membership.*

The equivalences $\langle \approx_{\alpha} \cap N^2 \rangle_{\alpha<\varkappa}$ *are a nested basis for a uniformity on* N, *compatible with the* $\varkappa$*-hypermetric obtained by restricting (1.3) to* N. *In the corresponding uniform topology,* $\sigma(X)$ *is the closure of* X *for any subset* X *of* N. *Therefore* N *is the disjoint union of its clopen subsets* U *and* $N\backslash U$, *the latter being exactly the set of all closed subsets of* N.

PROOF

The assertions about σ are merely a restatement of the above lemmata. Moreover, since $\approx_{\varkappa}$ is the equality on N, the distance defined by (1.3) verifies also condition (i), hence is a $\varkappa$-hypermetric on N, whose induced uniformity admits the equivalences $\approx_{\alpha}$ as a basis.

It remains to prove that, whenever $X \subseteq N$, $\sigma(X)$ is the closure of X in N, for the remaining assertions are easy consequences of this fact.

Given any set $x \in V\backslash U$, denote by $\bar{\bar{x}}$ its closure in V, and if $x \subseteq N$ put $\bar{x} = \bar{\bar{x}} \cap N$, thus $\bar{x}$ is the closure of x in N.

By Lemma 1.1, $\bar{\bar{x}}$ is saturated w.r.t. the equivalence $\approx_{\varkappa}$, hence in particular $\bar{\bar{x}} \supseteq \hat{\sigma}(\bar{\bar{x}})$. Then, if $x \subseteq N$, we have

$$\sigma(x) = \{\sigma(y) \mid y \in_{\varkappa} x\} = \{\sigma(y) \mid y \in \bar{\bar{x}}\} = \hat{\sigma}(\bar{\bar{x}}) \subseteq \bar{\bar{x}} \cap N = \bar{x} = \hat{\sigma}(\bar{x}) \subseteq \hat{\sigma}(\bar{\bar{x}});$$

therefore both inclusions are equalities, whence $\sigma(x) = \bar{x}$.

Q.E.D.

Since $N\backslash U$ is precisely the set of its closed subsets, it is natural to consider on it the *Hausdorff* $\varkappa$*-metric*

$$h(x,y) = \max \left\{ \sup_{s\in x} \inf_{t\in y} d(s,t), \quad \sup_{t\in y} \inf_{s\in x} d(t,s) \right\}$$

By means of (1.1)-(1.3) it is easy to compare the distances d and h, and obtain $h(x,y)^{+} \le d(x,y) \le h(x,y)$, where $\varepsilon_{\alpha}^{+} = \varepsilon_{\alpha+1}$: hence d and h are *uniformly equivalent*.

The same conclusion is reached by comparing the product distance of pairs $d_2((x,y),(u,v)) = \max \{d(x,u), d(y,v)\}$ with the distance d between the same pairs (intended à la Kuratowski) considered as subsets of N; hence $N \times N$ is a *closed uniform subspace* of N, and the same is true for any power N^n.

If we consider general function spaces, the situation is not so nice. However, from Theorem 1.4 and the above remarks, we can conclude that a function (or relation) graph belongs to N iff it is a closed subset of the product space $N\times N$; in particular all *continuous functions with closed domains* belong to N.

Moreover the uniformity induced by N on any function space is given by the Hausdorff distance of the graphs; in particular, on spaces of functions with the same domain, it agrees with the *uniformity of uniform convergence*.

Since the situation becomes neater when the space N is $\varkappa$*-compact*, we shall give a topological characterization of the function spaces which are members of N (Lemma 3.1) only after dealing with $\varkappa$-compactness in section 2.

We conclude this section with some useful remarks.

REMARK 1.5

For $A \subseteq U$, define the *cumulative hierarchy* $\Pi(A) = \bigcup_{\alpha \in \mathrm{Ord}} \Pi_\alpha(A)$ of the *sets wellfounded over A* by putting

$$\Pi_0(A) = A, \qquad \Pi_{\alpha+1}(A) = \Pi_\alpha(A) \cup \mathcal{P}(\Pi_\alpha(A)), \qquad \text{and} \qquad \Pi_\lambda(A) = \bigcup_{\alpha<\lambda} \Pi_\alpha(A)$$

for limit λ.

In particular $\Pi_\alpha = \Pi_\alpha(\emptyset)$ is the set of all well founded sets of rank less than α.

Call x α-*isolated* if it is the only element of its $\approx_\alpha$-class, and let U_α be the set of all α-isolated points of U.

It is easily seen by induction on β that any $x \in \Pi_\beta(U_\alpha)$ is $(\alpha+\beta)$-isolated: in particular any wellfounded set of rank less than α is α-isolated (cfr. [7], [13]). It follows that $N \supseteq \bigcup_{\alpha<\varkappa} \Pi_\varkappa(U_\alpha)$.

On the other hand all ordinals greater than α are α-equivalent to each other, hence in particular $\Pi \cap N = \Pi_\varkappa$.

More generally one has that the elements of $\Pi_\alpha(A)$ are pairwise $(\beta+\alpha)$-inequivalent whenever the elements of A are pairwise β-inequivalent. Hence, putting $\varkappa_\alpha = |U/\approx_\alpha|$ and $\nu_\alpha = |N_\varkappa(\mathcal{U})/\approx_\alpha|$, one obtains the inequality $\exp_\alpha(\varkappa_\beta) \le \nu_{\beta+\alpha}$ for any $\alpha, \beta < \varkappa$.

Better estimates can be obtained by observing that if $\langle S_\alpha \mid \alpha<\lambda \rangle$ (λ limit) is an increasing sequence of sets of representatives for the α-equivalence on N, then all elements of $\mathcal{P}(\bigcup_{\alpha<\lambda} S_\alpha)$ are pairwise λ-inequivalent, and $\mathcal{P}(S_\alpha)$ is a set of representatives for the $(\alpha+1)$-equivalence on $N \setminus U$.

Therefore

$$\nu_{\alpha+1} = 2^{\nu_\alpha} + \varkappa_{\alpha+1} \qquad \text{and} \qquad \nu_\lambda = 2^{\sup\{\nu_\alpha | \alpha<\lambda\}} \qquad (\lambda \text{ limit})$$

In particular we obtain that, if $\varkappa_{\alpha+1} \le 2^{\varkappa_\alpha}$ for $\varkappa > \alpha > \beta$, then

$$\nu_\alpha = \beth_{\alpha+1} \qquad \text{for any} \quad \alpha > \beta.$$

REMARK 1.6

If there are no urelements, i.e. $U = \emptyset$, the model $N_\varkappa = N_\varkappa(\emptyset)$ is exactly the same as the one introduced in [7]. This is more comprehensive than the corresponding model $M_\varkappa$ of [13], which contains only the representatives of the $\varkappa$-equivalence classes of wellfounded sets.

Restricting the construction of Theorem 1.4 to all wellfounded sets, we can obtain a *transitive collapse* $N_\varkappa^{wf}$ of the model $M_\varkappa$ with the $\varkappa$-membership $\in_\varkappa$. Similarly, we can perform our construction only for the class $\Pi(U)$ of all wellfounded sets over the atoms. However, in view of the $\varkappa$-compactness results of the next section, the full model $N_\varkappa$ seems more interesting.

REMARK 1.7

If we consider models $N = N_\alpha(U)$ for α any limit ordinal, as in [7] and [13], the uniformity of N has then weight $\nu = \text{cof } \alpha$ and will therefore never be ν-compact or even ν-bounded, when α is singular (see section 2).

Moreover, stopping the construction of $N_\varkappa(\mathcal{U})$ at $\alpha < \varkappa$ amounts to starting it with a set of atoms isometric to the quotient space $U/\approx_\alpha$ (which is either discrete or has weight $\nu = \text{cof } \alpha$). On the other hand, proceeding up to $\alpha > \varkappa$ is equivalent to starting with a discrete set of atoms, which can even be assumed pairwise 1-inequivalent.

The only interesting possibility is therefore to take, instead of the basic $\varkappa$-sequence $\langle \varepsilon_\alpha \rangle_{\alpha < \varkappa}$, a new λ-sequence $\langle \varepsilon'_\alpha \rangle_{\alpha < \lambda}$, where λ is any limit ordinal of cofinality $\varkappa$.

In this way the uniform structure of U is preserved, and all wellfounded sets of rank less than λ are now present in $N_\lambda(\mathcal{U})$. However, in view of the results of the next section (Theorem 2.7), Cauchy completeness is preserved only for $\varkappa = \text{cof}\ \lambda = \omega$. Moreover, $\varkappa$-compactness is lost for singular λ, and with it many interesting comprehension properties of the model (see section 3).

2. Cauchy completeness and $\varkappa$-compactness

As pointed out before, most of the interesting features of our models depend on additional topological properties, which, for $\varkappa$-hypermetric spaces, can be characterized as follows:

DEFINITION 2.0

Let N be a $\varkappa$-hypermetric space:

N is *Cauchy complete* iff any Cauchy $\varkappa$-sequence of N converges in N;

N is *$\varkappa$-bounded* iff there are less than $\varkappa$ balls of any fixed radius;

N is *$\varkappa$-compact* iff any $\varkappa$-sequence in N has a convergent $\varkappa$-subsequence.

It is easily seen that the above definition of $\varkappa$-compactness is equivalent to each of the following classical properties:

(i) any open cover of N has a subcover of cardinality less than $\varkappa$;

(ii) any strictly descending $\varkappa$-chain of closed sets has non-empty intersection.

Moreover, any *$\varkappa$-compact* $\varkappa$-metric space is both $\varkappa$-bounded and Cauchy complete, but the converse implication can fail for uncountable $\varkappa$, e.g. for the tree T' defined in the proof of Lemma 2.3 (see also [13]).

We shall see below that $\varkappa$-boundedness, Cauchy completeness and $\varkappa$-compactness of the space $N = N_\varkappa(\mathcal{U})$ can be obtained by combining the same properties of the subspace U of all atoms with suitable combinatorial properties of the cardinal $\varkappa$.

We begin by considering $\varkappa$-boundedness:

LEMMA 2.1

$N = N_\varkappa(\mathcal{U})$ is $\varkappa$-bounded iff U is $\varkappa$-bounded and $\varkappa$ is strongly inaccessible.

PROOF

First of all, N is the disjoint union of its clopen subsets U and $N\setminus U$, and $N\setminus U$ includes the set $\Pi_\varkappa$ of all hereditarily well-founded sets of rank less than $\varkappa$, by Remark 1.5.

Since any point $x \in \Pi_\alpha$ is α-isolated, the $\varkappa$-boundedness of N yields both that U is $\varkappa$-bounded and that $\beth_\alpha < \varkappa$ for any $\alpha < \varkappa$; therefore the given condition is necessary.

On the other hand, by Lemma 1.3, the number of distinct α-equivalence classes in the whole universe V does not exceed $\exp_{\alpha+1}(\varkappa_\alpha)$, where $\varkappa_\alpha$ is the number of α-equivalence classes in U, and $\varkappa_\alpha < \varkappa$ when U is $\varkappa$-bounded.

Therefore, if $\varkappa$ is inaccessible, the number of ε_α-balls in the whole universe is strictly less than $\varkappa$, for $\exp_\eta(\xi) < \varkappa$ whenever both ξ and η are less than $\varkappa$.

Q.E.D.

We shall now investigate the notion of Cauchy completeness.

LEMMA 2.2

Let $\varkappa = \lambda^+$ be a successor cardinal. Then $N = N_\varkappa(\mathcal{U})$ is not Cauchy complete.

PROOF

We define a "universal" 2^λ-ary tree T of subsets of $\varkappa+1$ in the following way:[5] for $\alpha < \varkappa$, put

$$I_\alpha = \{ \lambda\cdot\alpha + \gamma \mid 0 \le \gamma < \lambda \},$$

$$T_\alpha = \{ x \cup \{\varkappa\} \mid x \in \mathcal{P}(\lambda\cdot\alpha) \;\&\; x \cap I_\beta \ne \emptyset \quad \forall\beta<\alpha \},$$

and, for $x, y \in T = \bigcup_{\alpha<\varkappa} T_\alpha$, put

$$x <_T y \quad \text{iff} \quad \exists\alpha<\varkappa \quad x \setminus \{\varkappa\} = y \cap \lambda\cdot\alpha.$$

Clearly $(T,<_T)$ is a tree of height $\varkappa$, whose α^{th} level is T_α; it is a universal 2^λ-ary tree, since any of its nodes has exactly 2^λ immediate successors and any of its branches of limit length has exactly one immediate successor, hence any 2^λ-ary tree is (isomorphically) embeddable into T, and the embedding can be taken level-preserving.

Moreover, recalling that all ordinals $\ge \alpha$ are α-equivalent, whereas those $< \alpha$ are pairwise α-inequivalent, we get that, for any $x \in T_\alpha$ and any $y \in T$, $x <_T y$ holds iff $x \approx_{\lambda\alpha+1} y$. Hence there is a natural correspondence between $\varkappa$-branches of T and Cauchy $\varkappa$-sequences of elements of T.

Let S be the set of all *bounded strictly increasing* α*-sequences* (with $\alpha < \varkappa$) of elements of the *lexicografically ordered set* $Q = \{s\in\lambda^\omega \mid \exists m\ \forall n>m \quad s_n = 0 \}$ *of all eventually* 0 *sequences* of ordinals less than λ, and arrange S in a tree by inclusion.

S is a classical λ*-ary tree of height* $\varkappa$ *without any* $\varkappa$*-branch*, and it is *homogeneous* in the sense that it is isomorphic to each of its full subtrees obtained by taking all successors of any node.

Let T' be a subtree of T isomorphic to S, and let x_α be the α^{th} level of T', i.e. the set of all nodes of T' corresponding to

---

[5] Recall that a partially ordered set $(T,<_T)$ is a *tree* iff the predecessors of any element (*node*) $x \in T$ are wellordered by $<_T$, their *order type* (*length*) being the *level* of x.

T is $\varkappa$*-ary* iff any *node* of T has *at most* $\varkappa$ *immediate successors* and any *branch of limit length at most one*.

α-sequences of S. The $\varkappa$-sequence $\langle x_\alpha \rangle_{\alpha < \varkappa}$ is Cauchy, and in fact $x_\alpha \approx_{\lambda\alpha+2} x_\beta$ for $\alpha < \beta$, as can be seen by considering, through any $s \in x_\alpha$, a branch of T' of length greater than β.

But the sequence $\langle x_\alpha \rangle$ cannot have a limit, since otherwise, putting $x = \lim x_\alpha$ and picking some $s \in x$, we would find for any $\alpha < \varkappa$ elements $s_\alpha \in x_\alpha$ verifying $s_\alpha \approx_{\lambda\alpha+1} s$, and these would constitute a $\varkappa$-branch of the tree T'.

Q.E.D.

LEMMA 2.3

Let $\varkappa$ *be inaccessible and assume that* $\varkappa \not\longrightarrow (\varkappa)^2_2$.(6) *Then the space* $N = N_\varkappa(\mathcal{U})$ *is not Cauchy complete.*

PROOF

In order to reach our conclusion, we follow closely the argument used in the case of a successor cardinal.

We define a tree T of subsets of $\varkappa+1$ by putting, for any infinite cardinal $\lambda < \varkappa$,

$$I_\lambda = \{ \gamma \mid \lambda \le \gamma < \lambda^+ \},$$

$$T_\lambda = \{ x \cup \{\varkappa\} \mid x \in \mathcal{P}(\lambda) \ \& \ \forall \xi < \lambda \ \ x \cap I_\xi \ne \emptyset \},$$

and, for $x, y \in T = \bigcup_{\lambda < \varkappa} T_\lambda$,

(6) Recall that the *partition property* $\varkappa \longrightarrow (\varkappa)^2_2$ holds iff given any partition of all doubletons from $\varkappa$ into two parts, there is a $\varkappa$-sized subset of $\varkappa$ all of whose doubletons belong to the same part.
It is well known (cfr.[12]) that $\varkappa \longrightarrow (\varkappa)^2_2$ is equivalent to the *binary tree property,* saying that any binary tree of cardinal $\varkappa$ has a $\varkappa$-branch, and that it implies that $\varkappa$ is strongly inaccessible.
For a strongly inaccessible cardinal $\varkappa$, the property $\varkappa \longrightarrow (\varkappa)^2_2$ is also equivalent to the *tree property* (which says that any tree of size $\varkappa$ all whose levels have sizes less than $\varkappa$ has a $\varkappa$-branch), as well as to *weakly compactness* (which says that any $\varkappa$-complete filter over a $\varkappa$-complete field $\mathfrak{F}$ of subsets of $\varkappa$ is included in a $\varkappa$-complete ultrafilter on $\mathfrak{F}$).
We *include* ω among the strongly inaccessible cardinals.

$x <_T y$ iff $\exists\lambda<\varkappa \quad x \setminus \{\varkappa\} = y \cap \lambda$.

Clearly $(T,<_T)$ is a tree of height $\varkappa$, whose α^{th} level is $T_{\aleph_\alpha}$; any node of T_λ has exactly 2^{λ^+} immediate successors and any branch of limit length in T has exactly one immediate successor.

Since, as above, for any $x \in T_\lambda$ and any $y \in T$, $x <_T y$ holds iff $x \approx_{\lambda+1} y$, there is again a natural correspondence between $\varkappa$-branches of T and Cauchy $\varkappa$-sequences of elements of T.

Assume now that $\varkappa$ is weakly, but not strongly inaccessible. Then, for some $\lambda < \varkappa$, $\varkappa$ is less than 2^λ, and one can embed isomorphically into T any $\varkappa$-ary tree of height $\varkappa$.

Define a tree S in the following way:
put $Q = \{(\lambda,\alpha) \mid \alpha<\lambda<\varkappa \ \&\ \lambda \text{ is an infinite cardinal}\}$, and let $S = Q^{<\omega}$ be the set of all finite sequences of elements of Q.

Given $s = \langle(\lambda_0,\alpha_0),\ldots,(\lambda_m,\alpha_m)\rangle$ and $t = \langle(\mu_0,\beta_0),\ldots,(\mu_n,\beta_n)\rangle$ put $s < t$ iff $m \le n$, $\lambda_i = \mu_i$ for $i \le m$, $\alpha_i = \beta_i$ for $i<m$ and $\alpha_m \le \beta_m$.

Clearly S becomes a $\varkappa$-ary tree of height $\varkappa$ without any $\varkappa$-branch. In fact any node of S has exactly $\varkappa$ immediate successors and the whole tree S is isomorphically embeddable into the subtree of all successors of any of its nodes.

Let T' be a subtree of T isomorphic to S, and let x_α be the α^{th} level of T': the very same argument of the previous lemma now works and proves that the $\varkappa$-sequence $\langle x_\alpha \rangle_{\alpha<\varkappa}$ is Cauchy, but cannot have any limit.

Finally, if $\varkappa$ is strongly inaccessible, then by hypothesis there is a binary tree S of height $\varkappa$ without any $\varkappa$-branch. Let T' be a subtree of T isomorphic to S, and put

$x_\alpha = \{ t \in T' \cap T_{\aleph_\alpha} \mid t \text{ has } \varkappa \text{ successors}\}$.

Since $\varkappa$ is strongly inaccessible, all levels of T have size less than $\varkappa$, hence any element of x_α has successors in each x_β with $\beta>\alpha$. Therefore $\langle x_\alpha\rangle_{\alpha<\varkappa}$ is again a Cauchy $\varkappa$-sequence without limit.

Q.E.D.

A criterion for Cauchy completeness can now be given, namely:

LEMMA 2.4

Suppose that the atom space U *is* $\varkappa$*-bounded. Then* $N = N_\varkappa(\mathcal{U})$ *is Cauchy complete iff both* $\varkappa \longrightarrow (\varkappa)^2_2$ *(i.e.* $\varkappa$ *is strongly inaccessible and weakly compact) and* U *is Cauchy complete.*

PROOF

By the lemmata above, the given conditions are necessary for Cauchy completeness. They are also trivially sufficient when $\varkappa$-sequences of urelements are considered.

Thus assume $\varkappa \longrightarrow (\varkappa)^2_2$ and let $\langle x_\alpha\rangle_{\alpha<\varkappa}$ be any Cauchy $\varkappa$-sequence in $N\setminus U$: we can suppose w.l.o.g. that $x_\alpha \approx_\alpha x_\beta$ whenever $\alpha<\beta<\varkappa$.

Put $S = \{x \in N^\varkappa \mid x_\alpha \approx_\alpha x_\beta \ \forall\alpha<\beta<\varkappa\}$.

Define the function $f\colon S \longrightarrow \mathcal{P}(S)\cup U$ by setting

$f(x) = \{y \in S \mid y_\alpha \in x_{\alpha+1} \ \forall\alpha<\varkappa\}$ if x is eventually outside U,

$f(x) = \lim x_\alpha$ otherwise (the limit exists in U by hypothesis).

Let g be the unique function given by the axiom FC. We claim that $g(y) \approx_\alpha y_\alpha$ for all $\alpha<\varkappa$ and all $y \in S$. Then, in particular, $g(x) = \lim x_\alpha$, and we are done.

Our claim is trivial for $\alpha = 0$, and easily verified for any limit $\lambda<\varkappa$, provided it holds for all $\alpha<\lambda$. Moreover, it holds by definition if x is eventually atomic: so we only need to prove the induction step from α to $\alpha+1$ when all y_α's are non-empty sets.

By definition, we have, for any such $y \in S$,

$$g(y) = \hat{g}(f(y)) = \{g(z) \mid z \in S \ \& \ \forall\gamma<\varkappa \ \ z_\gamma \in y_{\gamma+1} \},$$

hence, by induction hypothesis,

$$g(z) \approx_\alpha z_\alpha \in y_{\alpha+1} \quad \text{for any} \quad g(z) \in g(y).$$

Conversely, given $t \in y_{\alpha+1}$, we need to find $z \in S$ such that $t \approx_\alpha z_\alpha$ and $z_\gamma \in y_{\gamma+1}$ for all $\gamma<\varkappa$. Then $t \approx_\alpha z_\alpha \approx_\alpha g(z)$ by induction hypothesis, hence $y_{\alpha+1} \approx_{\alpha+1} g(y)$, and our goal is achieved.

In order to find the $\varkappa$-sequence z, we define a tree $T = \bigcup_{\alpha<\varkappa} T_\alpha$ as follows:

$$T_\alpha = \{\alpha\} \times \{B_\alpha(x) \mid x \in y_{\alpha+1}\}, \quad \text{where} \quad B_\alpha(x) = \{z \in N \mid z \approx_\alpha x\},$$

and $(\alpha,A) <_T (\beta,B)$ iff $\alpha < \beta$ & $A \supseteq B$.

Any node of T lies on a $\varkappa$-branch, since its successors constitute a tree of height $\varkappa$ with levels of size less than $\varkappa$, by our hypotheses.

Pick a $\varkappa$-branch $\langle(\gamma,C_\gamma) \mid \gamma<\varkappa\rangle$ of T through $(\alpha,B_\alpha(t))$. Any $\varkappa$-sequence z such that $z_\gamma \in y_{\gamma+1} \cap C_\gamma$ for all $\gamma<\varkappa$ is now suitable for our purposes.

Q.E.D.

We are now able to state the main result of this section, which generalizes to the present context Theorem 4.4 of [7]:

THEOREM 2.5

The space $N = N_\varkappa(\mathcal{U})$ *is* $\varkappa$*-compact iff both* $\varkappa \longrightarrow (\varkappa)^2_2$ *(i.e.* $\varkappa$ *is strongly inaccessible and weakly compact) and* U *is* $\varkappa$*-compact.*

PROOF

As in Lemma 2.4, the conditions are obviously necessary.

Thus assume $\varkappa \longrightarrow (\varkappa)^2_2$ and let $\langle x_\alpha \rangle_{\alpha<\varkappa}$ be any $\varkappa$-sequence in N. Arrange the pairs (α,x_α) in a tree T in the following way: suppose the levels T_δ of T are already defined for $\delta < \gamma < \varkappa$, consider the

γ-equivalence classes of all elements x_α such that (α, x_α) is not yet arranged at any level $\delta < \gamma$ and, for each class, put in T_γ the pair (β, x_β) having the least index β.

Given (α, x_α) at level δ and (β, x_β) at level γ, put

$(\alpha, x_\alpha) <_T (\beta, x_\beta)$ iff both $\delta < \gamma$ and $x_\alpha \approx_\delta x_\beta$.

It is immediate to verify that T becomes a tree whose γ^{th} level is indeed T_γ. Since the elements of the α^{th} level of T are pairwise α-inequivalent and N is $\varkappa$-bounded by Lemma 2.1, T is a tree of size $\varkappa$ with all levels of size less than $\varkappa$, hence of height $\varkappa$. It has therefore a $\varkappa$-branch, since by hypothesis $\varkappa$ has the tree property.

By definition, the second components of any $\varkappa$-branch of T constitute a Cauchy $\varkappa$-subsequence of the original sequence, which is convergent since N is Cauchy complete by Lemma 2.4: the proof is thus complete.

Q.E.D.

Clearly, when the atom space U is discrete, it is $\varkappa$-compact iff its size is less than $\varkappa$. Hence if $|U| < \varkappa$, then $N_\varkappa(\mathcal{U})$ is $\varkappa$-compact iff $\varkappa$ has the partition property.

On the other hand, a $\varkappa$-metric space is $\varkappa$-compact iff it is uniformly isomorphic to a closed subspace of the "universal" space $2^\varkappa$ of all $\varkappa$-sequences of 0's and 1's, equipped with the *first difference $\varkappa$-hypermetric* $d(x,y) = \varepsilon_\alpha$ iff $\alpha = \min\{\beta \mid x_\beta \neq y_\beta\}$.

It follows that if $N_\varkappa(\mathcal{U})$ is $\varkappa$-compact, then $|U| \leq 2^\varkappa$.

We conclude this section with some remarks about Cauchy completeness.

REMARK 2.6

The space $N_\omega(\mathcal{U})$ is Cauchy complete whenever U is a complete metric space, since the argument of the proof of Lemma 2.4 works without the $\varkappa$-boundedness hypothesis for $\varkappa = \omega$. In fact, using the notation of that proof, given $y \in S$ and $t \in y_{n+1}$ one can always pick a sequence $z \in S$ with $z_n = t$ verifying $z_m \in y_{m+1}$ for all $m \in \omega$.

On the contrary, $\varkappa$-boundedness is a necessary condition for Cauchy completeness for any uncountable $\varkappa$, as we shall show below.

Assume U $\varkappa$-unbounded and let A be a set of $\varkappa$ pairwise α-inequivalent atoms for some $\alpha < \varkappa$. We shall assume, for sake of simplicity, that the elements of A are already 1-inequivalent.

For $a \in A$, consider the "generalized ordinals" a_α defined by

$$a_0 = a, \quad a_{\alpha+1} = a_\alpha \cup \{a_\alpha\} \quad \text{and} \quad a_\lambda = \bigcup_{\alpha<\lambda} a_\alpha \quad \text{for limit } \lambda.$$

It is easy to verify that generalized ordinals built up over different atoms from A are pairwise 2-inequivalent. Moreover

$$a_\alpha \approx_\alpha a_\beta \quad \text{and} \quad a_\alpha \not\approx_{\alpha+1} a_\beta \quad \text{for} \quad \alpha < \beta.$$

Put $A_\alpha = \bigcup_{a\in A} a_\alpha$, $I_\alpha = A_{\alpha+1} \setminus A_\alpha$ and define a tree $T = \bigcup_{\alpha<\varkappa} T_\alpha$

by $$T_\alpha = \{\ x\cup I_\varkappa \mid x\subseteq A_\alpha \ \& \ x\cap I_\beta \neq \emptyset \quad \forall\beta<\alpha\ \} \quad \text{and}$$

$$s <_T t \quad \text{iff} \quad \exists\alpha<\varkappa \quad s \in T_\alpha \ \& \ s \cap A_\alpha = t \cap A_\alpha.$$

Clearly T is a tree whose α^{th} level is T_α and any node of T has exactly $2^\varkappa$ immediate successors. Therefore one can embed in T the tree S defined in the proof of Lemma 2.3. Since for $s,t \in T$ one has

$$s \approx_{\alpha+1} t \quad \text{iff} \quad s \cap A_\alpha = t \cap A_\alpha\ ,$$

the argument of the proof applies and gives a non-convergent Cauchy $\varkappa$-sequence.

REMARK 2.7

Let $\varkappa$ be a strongly inaccessible weakly compact cardinal, and let λ be any limit ordinal of cofinality $\varkappa$.

According to the last part of Remark 1.7, we can build up the model $N_\lambda(\mathcal{U})$ so as to include all wellfounded sets of rank less than λ. Then we can fix an increasing $\varkappa$-sequence $\langle\gamma_\alpha\rangle_{\alpha<\varkappa}$ of ordinals cofinal in λ and use the γ_α-equivalence instead of $\approx_\alpha$.

If $\varkappa = \mathrm{cof}\ \lambda = \omega$, we can easily modify the initial argument of Remark 2.6 so as to obtain that $N_\lambda(\mathcal{U})$ is Cauchy complete iff its atom space U is.

On the other hand, if $\lambda > \varkappa = \mathrm{cof}\ \lambda > \omega$, we can argue as in the proofs of Lemma 2.2 and 2.3. Namely, the $\varkappa$-sequence $\langle\ \gamma_\alpha\ \rangle_{\gamma<\varkappa}$ can replace the ordinals less than $\varkappa$ in defining suitable trees of parts of $\lambda+1$, so as to provide inside $N_\lambda(\mathcal{U})$ counterexemples to Cauchy completeness.

Summing up all results on Cauchy completeness we obtain the following general criterion:

THEOREM 2.7

$N_\lambda(\mathcal{U})$ is a Cauchy complete metric space iff λ has countable cofinality and U is Cauchy complete.

If λ has uncountable cofinality $\varkappa$, then the space $N_\lambda(\mathcal{U})$ is Cauchy complete iff $\lambda \longrightarrow (\lambda)^2_2$ (hence $\lambda = \varkappa$) and U is both Cauchy complete and $\varkappa$-bounded.

In particular the models N_α of [7] are complete iff either $\mathrm{cof}\ \alpha = \omega$ or $\alpha \longrightarrow (\alpha)^2_2$.

REMARK 2.8

In his thesis [13], R.J.Malitz calls *crowded* a $\varkappa$-metric space where any $\varkappa$-sequence has a Cauchy $\varkappa$-subsequence. Clearly, crowdedness implies $\varkappa$-boundedness, whereas $\varkappa$-compactness is equivalent to the conjunction of crowdedness and Cauchy completeness.

Many of the most relevant properties of the models M_α of [13] depend on the existence of some ordinal α such that M_α is both crowded and Cauchy complete (such ordinals are called *Malitz ordinals* in [7]), and Malitz conjectured that *all regular uncountable cardinals* have this property. However, since the counterexamples employed for the negative parts of the above theorems make use only of wellfounded sets, they apply also to Malitz' models. Therefore, if $\varkappa$ is *Malitz,* then $\varkappa \longrightarrow (\varkappa)^2_2$.

On the contrary, a free construction principle (although not necessarily FC) plays the essential role in proving the positive parts of Theorems 2.5 and 2.7. Thus all that one obtains from the argument of Theorem 2.5 is that M_α is *crowded* if and only if $\alpha \longrightarrow (\alpha)^2_2$.

Malitz himself proved that M_ω, unlike our N_ω, is not Cauchy complete. His argument can easily be carried out for any ordinal of countable cofinality, following the pattern of the proof of Theorem 2.7. The question as to whether Malitz cardinals exist at all is still open. As a matter of fact, the opinions of the authors are split in conjecturing an answer to this question. A positive solution would yield that the corresponding $M_\varkappa$ has many of the comprehension properties of our models $N_\varkappa(\mathcal{U})$.

3. Comprehension properties of $\varkappa$-compact models

As we noticed in the first section, it is easier to study functions and function spaces in the model $N_\varkappa(\mathcal{U})$ when this is a $\varkappa$-compact space.

In fact most properties of compact metric spaces have perfect analogues for any uncountable $\varkappa$. E.g. the graph of a function f is closed in the product topology if and only if f is continuous and dom f is closed, and in this case f is a closed uniformly continuous map; the $\varkappa$-compact-open topology on the space of all continuous functions is induced by the uniformity of uniform convergence; a set of continuous functions with the same domain is closed in the $\varkappa$-compact-open topology iff it is equicontinuous, etc..

We summarize the results which are relevant in determining the comprehension properties of our models in the following lemma, and we refer to Chapter 8 of [5] for detailed proofs and more information on this topic (see in particular [5,8.2.4-10]).

LEMMA 3.1

Let $N = N_{\varkappa}(\mathcal{U})$ *be* $\varkappa$*-compact. Then*

(i) *A function* f *belongs to* N *iff it is continuous and its domain is closed, and in this case* f *is a closed uniformly continuous map. More generally, if* $A \subseteq N$, *a function* $g: A \longrightarrow N$ *is* $\varkappa$*-equivalent to a function* $f \in N$ *iff it is uniformly continuous on* A *(the domain of* f *being then the closure of* A *in* N*).*

(ii) *For any* $X \in N \setminus U$ *and any* $Y \subseteq N$, *the space* $Y^X \cap N$ *with the induced uniformity is precisely the set* $\mathcal{U}(X,Y)$ *of all uniformly continuous functions from* X *into* Y *with the uniformity of uniform convergence (which induces the* $\varkappa$*-compact-open topology).*

(iii) *A set* $F \subseteq N^X$ *belongs to* N *iff* F *is equicontinuous and* X *is closed. In particular, if* $|Y| > 1$, *then* $\mathcal{U}(X,Y) = Y^X \cap N$ *belongs to* N *iff* X *is closed and discrete, i.e. iff* $|X| < \varkappa$, *and then* $\mathcal{U}(X,Y) = Y^X$.

PROOF

(i) Since N is $\varkappa$-compact, the graph of f is closed in $N\times N$ iff f is continuous and dom f is closed; moreover any subset of N is closed iff it is $\varkappa$-compact, hence any continuous function maps closed sets onto closed sets, and is uniformly continuous on any closed set.

On the other hand, if g is uniformly continuous on A, then it has a unique uniformly continuous extension to $\overline{A}$, whose graph is clearly the representative of g in N.

(ii) As in the ordinary compact case, it is easy to see that if the space X is $\varkappa$-compact, then the Hausdorff distance of the graphs induces on the space $\mathcal{U}(X,Y)$ of all (necessarily uniformly) continuous functions from X into Y both the uniformity of uniform convergence and the $\varkappa$-compact-open topology. Since $\mathcal{U}(X,Y) = Y^X \cap N$ by (i) above, (ii) follows.

(iii) By Ascoli's theorem extended to $\varkappa$-compact $\varkappa$-metric spaces, if X is $\varkappa$-compact, then a closed set $F \subseteq Y^X$ is $\varkappa$-compact iff F is equicontinuous and $\{f(x) \mid f\in F\}$ has $\varkappa$-compact closure for any $x\in X$. Since in this context closed and $\varkappa$-compact are synonyms, we conclude the first assertion.

As to the second one, the condition is obviously sufficient, for then the points of X are α-isolated for some α, hence the set of all functions on X is equicontinuous.

To prove the converse, let x be a cluster point of X, pick two different points z_0, z_1 in Y, and, for any $y \in X$ put $f_\alpha(y) = z_0$ if $y \approx_\alpha x$, $f_\alpha(y) = z_1$ otherwise.

Clearly, $\langle f_\alpha \rangle_{\alpha < \varkappa}$ is a $\varkappa$-sequence of uniformly continuous functions on X whose pointwise limit is not continuous, hence it cannot have any uniformly convergent subsequence. Therefore $\mathcal{U}(X,Y)$ is not $\varkappa$-compact.

Q.E.D.

We shall now illustrate the selfdescriptive power of a $\varkappa$-compact model $N = N_{\varkappa}(\mathcal{U})$. As pointed out before, these models are closed under many basic operations. Simultaneously many interesting large classes are closed subsets of N, hence they belong to N.

We begin by stating a theorem which transposes to the present situation, where urelements are allowed, all the assertions of Theorem 4 of [7 §4.2].

Following [7], we define the class GPF of the *generalized positive formulae* as the least class which includes the *atomic formulae* and is closed under *conjunction, disjunction, existential* and *universal quantification* as well as under the following rules of bounded quantification (which, strictly speaking, are non-positive):

if ϕ *is* GPF, *then both* $\forall x(x \in y \longrightarrow \phi)$ *and* $\forall x(\theta(x) \longrightarrow \phi)$ *are* GPF, where θ is *any* formula with *exactly one* free variable.

The *Generalized Positive Comprehension Principle* GPK is the axiom schema (denoted by *Comp(GPF)* in [7]) which postulates the existence of the set $\{x \mid \phi\}$ for any generalized positive formula ϕ,

GPK - $\exists x \forall y\ (y \in x \longleftrightarrow \phi)$ where ϕ is GPF and x is not free in ϕ.

Since the formula $z = (x,y)$ is GPF, the generalized positive comprehension principle GPK yields both the existence of many *fundamental graphs* (e.g. membership, inclusion, identity, singleton and power-set maps, projections, permutations and all natural manipulations of n-tuples, etc.) and the stability under many *basic operations* (e.g. union, intersection, cartesian product, domain, range, inversion, composition and fibred product of graphs, etc.).

In the light of the above remark, the strength of the following theorem will now be evident.

THEOREM 3.2

Assume that $N = N_{\varkappa}(\mathcal{U})$ *is* $\varkappa$*-compact. Then*

(i) *any subset of* N *of size less than* $\varkappa$ *belongs to* N *together with its complement w.r.t.* N*. Both the product of less than* $\varkappa$ *elements of* $N\setminus U$ *and the intersection of arbitrarily many of them belong to* N.

(ii) N *satisfies the Generalized Positive Comprehension schema* GPK.

(iii) *The "cumulative cardinals"* $c_{<\lambda} = \{x\in N \mid |x|<\lambda\}$ *belong to* N *for any cardinal* $\lambda < \varkappa$, *whereas no "Frege-Russell cardinal"* $f_{\lambda} = \{x\in N \mid |x|=\lambda\}$ *belongs to* N *for* $\lambda > 1$.

PROOF

First of all, any set $X \subseteq N$ of size less than $\varkappa$ is well-spaced, i.e. its points are all α-isolated for some $\alpha < \varkappa$, hence X is both clopen and discrete.

Now the first part of (i) is immediate. The second one follows directly from Lemma 3.1, any function on X being uniformly continuous and any set of functions on X equicontinuous. The last assertion of (i) is obvious, since any intersection of closed sets is closed.

In order to prove (ii), we make use of an analogue of the classical Bernays' theorem for Gödel-Bernays class theory, proved in [7,§4.1]. Namely, GPK holds in N provided that the following sets belong to N:

(1) $I = \{(x,y)\in N^2 \mid x=y\}$ and $E = \{(x,y)\in N^2 \mid x\in y\}$;

(2) $X^{-1} = \{(x,y)\in N^2 \mid (y,x)\in X\}$, $Q(X) = \{(x,y)\in N^2 \mid \forall z\in x\ ((x,y),z)\in X\}$, $\{X,Y\}$, $\hat{X}(Y)$, and $X*Y = \{((x,y),z)\in N^2 \mid (x,z)\in X\ \&\ (y,z)\in Y\}$, for all X,Y in N;

(3) $\check{X}(Y) = \{z\in N \mid \forall y\in Y\ (y,z)\in X\}$ for any $X \in N$ and any $Y \subseteq N$.

Since a $\varkappa$-sequence of pairs (x_{α},y_{α}) converges to (x,y) iff both

$\lim x_\alpha = x$ and $\lim y_\alpha = y$, it is easy to check that the identity I is closed, as well as X^{-1} and $X*Y$ whenever X and Y are closed.

Moreover, any fiber $\tilde{X}(y) = \{z \mid (y,z) \in X\}$ is closed provided X is closed; this yields (3), for $\check{X}(Y) = \bigcap \{\tilde{X}(y) \mid y \in Y\}$.

On the other hand, if both X and Y are closed, hence $\varkappa$-compact, then also $\hat{X}(Y) = \hat{P}_2(X \cap Y \times V)$, being the image of a $\varkappa$-compact set under the continuous map P_2 (second projection), is $\varkappa$-compact, hence closed.

In order to obtain (ii), it remains to prove that E and $Q(X)$ are closed.

Let (x_α, y_α) be any $\varkappa$-sequence in E, and suppose that $(x,y) = \lim (x_\alpha, y_\alpha)$. We may assume w.l.o.g. that $x_\alpha \approx_{\alpha+1} x$ and $y_\alpha \approx_{\alpha+1} y$; since $x_\alpha \in y_\alpha$, there is for any $\alpha < \varkappa$ some $s_\alpha \in y$ such that $x_\alpha \approx_\alpha s_\alpha$. Then $\lim s_\alpha = \lim x_\alpha = x$ belongs to y and so E is closed.

Finally, let (x,y) be the limit of a $\varkappa$-sequence (x_α, y_α) in $Q(X)$, and assume again that $x_\alpha \approx_{\alpha+1} x$ and $y_\alpha \approx_{\alpha+1} y$; pick $z \in x$ and, for any α, $z_\alpha \in x_\alpha$ such that $z_\alpha \approx_\alpha z$: by definition $((x_\alpha, y_\alpha), z_\alpha) \in X$ for any α, hence also $((x,y),z) \in X$, and we are done.

It remains to prove (iii): to this aim, assume that we are given a $\varkappa$-sequence x_α such that $|x_\alpha| < \lambda$ and $x_\alpha \approx_{\alpha+1} x$ for any $\alpha < \varkappa$.

By considering the quotients modulo α-equivalence, we get $|x/{\approx_\alpha}| = |x_\alpha/{\approx_\alpha}| < \lambda$ for any $\alpha < \varkappa$. If $|x| \geq \lambda$, consider any subset y of x of size λ: since $\lambda < \varkappa$, the elements of y are pairwise β-inequivalent for some $\beta < \varkappa$, hence $|y/{\approx_\beta}| \geq \lambda$, contradiction.

Finally, for fixed $\lambda < \varkappa$, let $x_\alpha = \{\alpha + \gamma \mid \gamma < \lambda\}$.

Clearly, $\lim x_\alpha = \lim \{\alpha\} = \{\bar{\varkappa}\}$ (where $\bar{\varkappa} = \lim \alpha = \varkappa \cup \{\bar{\varkappa}\}$). Hence no Frege-Russell cardinal greater than 1 is closed.

Q.E.D.

Although compact $N_{\varkappa}(\mathcal{U})$'s are highly self-referential, nevertheless interesting open relations, like *non-identity*, and discontinuous operations, like *binary intersection*, fail to be elements of the model. The most serious self-descriptive deficiencies of these models are ultimately due to the fact that full function spaces in general are not elements. In fact the set $X^Y \cap N = \mathcal{U}(Y,X)$ *of all uniformly continuous maps* from Y into X is not closed whenever $|Y| \geq \varkappa$ and $|X|>1$.

However these deficiencies appear only if we continue to focus on the usual set theoretic reductions of the fundamental notions of operation and relation. We shall show now that, assuming a non purely set theoretic foundational framework like the *Ample theory* (theory A) of [2], many of these deficiencies can be partially amended.

It was with this in mind that we assumed urelements in our models. These have played no role up to now. We intend here to *activate* them as *relations, operations* and *qualities*, not withstanding the fact that many of these notions *do not have a corresponding set (graph, extension) in the model.* This method is similar to that of Oberschelp [15].

We assume for the rest of this section that $\varkappa$ is an uncountable strongly inaccessible weakly compact cardinal and that the space U of the urelements is isometric to the universal $\varkappa$-compact space $2^{\varkappa}$ of all $\varkappa$-sequences of 0's and 1's, endowed with the *first-difference* hypermetric (see section 2).

Since the mapping $x \mapsto x \cup \{\bar{\varkappa}\}\times\{0,1\}$ provides an isometry of $2^{\varkappa}$ onto a closed subspace L of N, we can fix an isometric inner labelling of the atoms by elements of L, say $\ell\colon U \longrightarrow L$. We fix also a uniformly isomorphic embedding of $N = N_{\varkappa}(\mathcal{U})$ into U, which will be denoted by j. Note that both ℓ and j, being continuous, belong to N. Further conditions on the embedding j will be specified later.

From now onwards, we assume the reader acquainted with the definitions and the notation of [2], which we shall adopt without further explanation for lack of space. In particular we shall deal freely with the over 200 distinguished objects of the Ample theory, which will be denoted by the names of the corresponding constants. Similarly, we shall refer to any axiom of the seventeen groups constituting the theory A by simply quoting the reference number it received in [2].[7]

Our goal is to extend the set-theoretic structure $(N,\in)$ to an *Ample structure* with domain *N*. An ample structure is a first order relational structure capable of accomodating the interpretation of the constants and predicates of the ample theory. The fundamental predicates of the theory A are:

x is a quality, *x enjoys the quality y;*

x is a relation, *x is in the relation z with y;*

x is an operation, *y is the result of the operation z on x;*

z is the pair with first component x and second component y.

If the axioms 1.A-J hold, the ample structure is uniquely

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[7] We modify slightly the formulation of the axioms 1.K and 17.F of [2], in order to make it closer to the common use and meaning of the objects involved. Namely, we do not reduce the quality of being a *q-r-structure* to the simple extensional condition given in [2];consequently we take in the corresponding axiom only a one-sided implication (as done in [2] for the quality *quniv*):

1.K - *qqrs* $\mathfrak{S}$ *implies that* $\mathfrak{S}=(q,r)$, *qqual q, qrel r and, if for some y yrx, then qx.*

On the other hand, we replace the functional relation *rbid* by an operation *bid* which associates to any pair (x,y) and to any relation *r* the proposition "*xry*". This seems to represent better the act of *bidding* some opinion.

17.F - *qop bid, if y rval bid then qprop y, and x rdom bid iff* $x=((u,v),r)$ *and qrel r.*

(due to a misprint, the axioms on propositions received in [2] the number 16 instead of 17).

determined by the interpretations of the distinguished constants and of the ternary predicate *x is in the relation z with y* (see [2, §I Appendix]).

We call *natural* an ample structure with domain N if its structure of collections is *standard*, i.e. if the extension of the quality *qcoll* is $N \cap \mathcal{P}(N)$ and the relation *rcoll* is the *true membership* restricted to N. We proceed now to sketch a first extension of $(N,\in)$ to a natural ample structure with universe N.

In the rest of this section, we shall use *square brackets* to denote the usual set-theoretic codification à la Kuratowski of *pairs* and *n-tuples*, namely

$$[x,y] = \{\{x\},\{x,y\}\} \quad \text{and} \quad [x_1,\ldots,x_n,x_{n+1}] = [[x_1,\ldots,x_n],x_{n+1}].$$

The standard notation (x,y) and $(x_1,\ldots,x_n)$ will be reserved for denoting *primitive pairs* and *n-tuples*.

Similarly, we shall distinguish between (ordinary) graphs and products, which are collections of *primitive pairs or n-tuples*, and κ*-graphs* or κ*-products*, which are built up à la Kuratowski and marked by a subscript κ. For instance, the κ-product of A and B is $A \times_\kappa B = \{[a,b] \mid a\in A \;\&\; b\in B\}$, and is therefore distinct from the ordinary cartesian product $A \times B = \{(a,b) \mid a\in A \;\&\; b\in B\}$.

First of all, we put

$Coll = N\setminus U$,

$Sys = \{j(x) \mid x \subseteq N\times_\kappa N \;\&\; |x|<\varkappa\}$, and

$Card = \{j(\lambda) \mid \lambda$ a Von Neumann cardinal $<\varkappa\} \cup \{j(\overline{\varkappa})\}$.

We interpret the elements of *Coll* as *collections* with the *ordinary membership* $\in$. We interpret those of *Sys* as *systems*, with the natural rule stating that, for the system $S = j(x)$, uSv holds iff $[u,v] \in j(x)$. Finally we interpret the elements of *Card* as *cardinals*, with $\daleth = j(\bar{\varkappa})$ intended as the size of any *large* collection (the quality *qsmall* meaning having size less than $\varkappa$). The standard cardinal operations and ordering (on *small* cardinals, i.e. below $\varkappa$) are transferred by means of j.

In particular the collection of the *natural numbers* is $\mathbb{N} = \hat{j}(\omega)$, with $0 = j(\emptyset)$, $1 = j(\{\emptyset\})$, $n+1 = j(j^{-1}(n) \cup \{j^{-1}(n)\})$, and $\aleph_0 = j(\omega)$

Moreover, in accord with the axiom 9.B of [2], we define the *natural pair* of x and y by $(x,y) = j(\{[1,x],[2,y]\})$, and similarly the *n-tuple* $(x_1,\ldots,x_n) = j(\{[1,x_1],\ldots,[n,x_n]\})$.

Before proceeding we remark that *Coll* and *Card* are closed, hence are collections themselves, whereas *Sys* is not. We could have chosen as systems the images of *all closed subsets of* $N \times_K N$, thereby obtaining a closed collection, but we prefer to deal only with *small systems*. So doing we allow for the *greatest manageability of systems*, which is a typical feature of the theory A.

In order to encode qualities, relations and operations as suitable definable subsets of U we proceed as follows.

Given a set $A \subseteq \mathcal{P}(N)$, define the *Gödel closure* $\mathcal{C}\ell(A)$ of A as the least superset of A which is closed under all *Gödel operations* (we can take only the operations X^{-1}, $X{*}Y$, $\hat{X}(Y)$ considered in the proof of Theorem 3.2, together with the complement $N \setminus X$).

Let $Q = \mathcal{C}\ell(N)$ be the Gödel closure of N, let $\varphi\colon Q \longrightarrow_K U$ be an injective κ-mapping, and put $\Phi = \{[\varphi(x),y] \mid y \in x \in Q\}$.

Let $R = \mathcal{P}(N \times N) \cap \mathcal{C}\ell(N \cup \{\Phi\})$ be the set of all binary graphs belonging to the Gödel closure of $N \cup \{\Phi\}$, let $\psi\colon R \longrightarrow_K U$ be an injective κ-mapping, and put $\Psi = \{[\psi(x),y] \mid y \in x \in R\}$.

Let $F = N^{\subseteq N} \cap \mathcal{C}\ell(N \cup \{\Phi,\Psi\})$ be the set of all functional graphs belonging to the Gödel closure of $N \cup \{\Phi,\Psi\}$, and let $\chi\colon F \longrightarrow_K U$ be an injective κ-mapping.

The domains N, Q, R and F being overlapping, one can have *more than one* atom associated to a given set by the encoding functions φ, ψ, χ, and j (e.g. each of them is defined at any closed functional graph). We assume that the ranges of φ, ψ, χ, j are *pairwise disjoint* leaving uncovered a large part of U. On the other hand, we assume that there is some *uniform definable combinatorial rule connecting the labels of atoms which correspond to the same set via different encodings.* Since we set no topology on Q, R and F, we have no topological constraints on φ, ψ, χ; further conditions on them will be specified later on.

Now we put

$$Qual_0 = \hat{\varphi}(Q), \qquad Op_0 = \hat{\chi}(F), \qquad Rel_0 = \hat{\psi}(R),$$

and we interpret $q = \varphi(X)$ as a *quality whose extension is X* (hence qx holds iff $x \in X$), $f = \chi(Y)$ as an *operation* and $r = \psi(Z)$ as a *relation whose graphs are Y and Z*, respectively (hence $y = fx$ holds iff $(x,y) \in Y$ and xry iff $(x,y) \in Z$).

We have thus determined the interpretation all basic predicates of the ample theory. More specifically, we have an ample structure

$$\mathfrak{N}_0 = \langle N;\ Qual_0,\ P_1;\ Rel_0,\ P_2;\ Op_0,\ P_3;\ P_4 \rangle$$

where $P_1 = \bigcup_{X \in Q} X \times_K \{\varphi(X)\}$, $P_2 = \bigcup_{X \in R} X \times_K \{\psi(X)\}$, $P_3 = \bigcup_{X \in F} X \times_K \{\chi(X)\}$

and $P_4 = \{[x, y, j(\{[1,x],[2,y]\})] \mid x, y \in N\}$.

In order to complete the definition of the ample structure $\mathfrak{N}_0$ we have to give the interpretation of each constant of the ample theory. Since there are more than two hundred constants to interpret, it would be cumbersome to list explicitly all the corresponding assignements in the model $\mathfrak{N}_0$. We prefer to explain instead the general idea underlying our definitions, namely that of *interpreting any constant of the theory as the "qualified urelement" associated to the subset of N which naturally codes the corresponding mathematical concept.*

In doing this, we make use of the fact that a quality exists in $\mathfrak{N}_0$ iff its extension belongs to Q (i.e. iff it is $\in$-definable). This can be obtained, in many cases, by a suitable choice of the labels of the objects which have to enjoy that quality. E.g. one can settle in this way the interpretation of all "descriptive" qualities like *qqual*, *qrel*, *qop*, *qrefl*, etc., by choosing *a priori* suitable *definable* subsets of U onto which φ, ψ, χ have to map the graphs and extensions enjoying the corresponding properties.

Similarly, a relation (an operation) exists iff its graph belongs to R (resp. F); this can again be obtained by imposing suitable connections between the functions j, ℓ, ψ, χ (in this way are easily settled, among others, the operations *invop*, *invrel*).

In particular any closed subset of N is the extension of a quality, and any closed set of pairs is the graph of a relation and also of an operation when it is functional. It follows that all *small* qualities, operations and relations are present in N.

According to our initial stipulations, we interpret the fundamental structures of collections, systems and cardinals in the natural way, namely:

$$qcoll = \varphi(Coll) \qquad rcoll = \psi(E);$$

$$qsys = \varphi(Sys) \qquad rsys = \psi(\{((x,y),s)\in N^2\times Sys \mid [x,y]\in j^{-1}(s)\});$$

$$qcard = \varphi(Card) \qquad rcard = \psi(\{(x,y)\in Card^2 \mid j^{-1}(x)\subseteq j^{-1}(y)\}).$$

The intended interpretation of most constants is determined by the corresponding axioms of the ample theory, once the *fundamental structures of qualities, relations, operations, collections, systems,* and *cardinals* are given. E.g.:

$$rnid = \psi(N\setminus I), \qquad id = \chi(I), \qquad \mathcal{P} = \chi(\{(A,\mathcal{P}(A)) \mid A \in N\setminus U\}),$$

$$psys = \chi(\{((x,j(X)),y) \mid [x,y]\in X \ \&\ X \text{ is a functional } \kappa\text{-graph }\}).$$

Other constants which are not uniquely determined by the fundamental structures, like *qqrs, quniv, qprop*, will be interpreted below in a natural way.

In fact, all qualities, relations and operations involving only collections, systems and cardinals, like those introduced in Chapters II and IV of [2], exist in $\mathfrak{M}_0$ since all the corresponding subsets belong to *Q*. Moreover, the comprehension properties stated in Theorem 3.2 yield all axioms of the groups 4-8, 13 and 14 of [2]. Similarly there are all relations connecting qualities, for their graphs belong to *R*, and all operations not involving other operations, for their graphs are elements of *F*.

Therefore the following axioms of [2] are directly satisfied by the natural interpretation sketched above:

1.ABC, F, HIJ 2.ABCD, FG, IJ

4.ABCDE 5.ABCDE 6.ABCDEF 7.ABCD 8.ABCDE 9.AB

10.A 11.EFG 13.ABCDEFGHI 14.ABCDEF

Being careful in mapping graphs which are reflexive, symmetric, etc. onto previously determined definable subsets of *U*, we can satisfy also the axioms 3.ABCDE. Paying similar attention in mapping qualities relations and operations of any cardinality $\lambda<\varkappa$, we get 12.A. A suitable choice of a set of less than $\varkappa$ q-r-structures, including those which are explicitly postulated by the ample theory, yields 1.K.

We can satisfy also the axioms 16.ABCD by the following interpretation of the qualities *qinac* and *quniv*:

(i) a cardinal $j(\lambda) < \hbar$ is *inaccessible* iff λ is a strongly inaccessible Von Neumann ordinal;

(ii) a collection V is a *universe* iff $\aleph_0 < |V| = \lambda < \varkappa$ is inaccessible and, for any $X \subseteq V$, if $|X| < \lambda$, then X belongs to V as well as $j(X)$, $\varphi(X)$, $\psi(X)$, and $\chi(X)$ (whenever they are defined).

Finally, for sake of simplicity, we trivialize the structure of *propositions* by allowing only two of them, the *true* proposition t and the *false* proposition f (t and f being two *new* atoms). In this way we easily obtain the validity of the axioms **17.ABCDEFG** (recall that we have replaced the relation *rbid* of [2,§V.2] by the corresponding operation *bid*).

Thus we are left with the problem of assigning a graph in R to each of the relations

rrel, *rop*, *rdom*, *rval*, *rginc*, *rexteq*, (3.1)

and one in F to each of the operations

eval, *oprest*, *hat*, *graph*, *dom*, *img*. (3.2)

The natural assignement is possible for the operation *graph*, since *only continuous operations with closed domains can have a graph in* N, and we can arrange χ so that the set $\{(x,\chi(x)) \mid x \in N \cap F\}$ belongs to Q, hence to F. We can also choose χ in such a way that *the closures of the domain and of the image of each operation are encoded* by suitable $\varkappa$-subsequences into the *label* of the atom corresponding to the operation itself. Moreover, we can make distinguishable those operations whose *domains and/or images are closed*. In this way one finds in Op_0 also the natural interpretation of the operations *dom* and *img*; one can even discover when *a given collection includes the domain of an operation*, thus getting *all trivial restrictions*.

Unfortunately, one cannot find within $Rel_0 \cup Op_0$ the natural full interpretation of the remaining constants (3.1-2). One can find instead homologous operations and relations acting on *qualities of pairs*. Therefore, using the correspondence between the images of the same graph under ϕ, ψ and χ, we decide to interpret the constants above as acting in the natural way only on *relations and operations whose graphs belong to Q.*

Having thus completely defined the *natural ample structure* $\mathfrak{N}_0$, we see that all axioms of the theory A hold in it, but

1.DE 2.EH 10.C 15.CE (B)

Moreover, introducing the qualities *qrequa* and *qopqua* (of being a *relation* and an *operation corresponding to a quality of pairs*), the given interpretation of the constants (3.1-2) satisfies the axioms

$1.D_0E_0$, $2.E_0H_0$, $10.C_0$, $15.C_0E_0$ (B_0)

obtained by restricting the corresponding axioms of A to relations and operations enjoying *qrequa* and *qopqua.*[8]

Let A_0 be the axiomatic theory resulting from A by replacing the axioms (B) by their weakenings (B_0). It is then straightforward to complete the proof of the following theorem (see also [10], where constructions similar to the one sketched above are developed in full details).

THEOREM 3.3

The ample structure $\mathfrak{N}_0$ *is a model of the theory* A_0.

Moreover the following supplementary axioms of extensionality, comprehension and stability hold in $\mathfrak{N}_0$:

I *The fundamental structures of qualities, relations and operations are extensional.*

II *Any system has a graph, any collection is the extension of a*

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[8] Therefore, e.g., the axioms $1.D_0$ and $2.H_0$ are

$1.D_0$ - *(x,y) rrel r iff both qrelqua r and xry.*

$2.H_0$ - *eval* $\downarrow$ *z iff z = (f,x), qopqua f and f* $\downarrow$ *x. In this case eval(f,x) = fx.*

quality, any quality of pairs is associated to a relation, and any functional relation to an operation. Moreover all qualities and relations have characteristic operations.

III *The collections are closed under union, intersection, cartesian product, power-set operator and power, the qualities and the relations are closed under negation and disjunction, the relations and the operations under composition, fibred and tensor product and restriction to any collection.*(9)

The self-referential power of the theory A is seriously weakened by the above restrictions of the axioms 1.DE and 2.EH. In fact, the operations cannot have a complete internal description, since the absence of objects such as *rop* and *eval* is provable, as well as that of many other relations and operations which could replace them in describing the actions of all operations. However one can deal freely with relations and relational pairs inside $\mathfrak{N}_0$, since the *characteristic operations* of the full relations *rrel, rginc* and *resteq* are elements of Op_0.

This lack of self-description is partially balanced by the strong axioms of extensionality, comprehension and stability I, II and III. Actually the wide stability of the model $\mathfrak{N}_0$ goes even beyond the properties III above, which are for themselves already inconsistent with the full theory A, the Antinomy II of [2,§VI.6] being derivable from

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(9) The axioms I-III are particular cases of the "strong axioms" of [2,§VI]. We list here those which hold in $\mathfrak{N}_0$: SA.1 CA.13.2 DC AS.1-3 AO.3 AQ.1 AR.1,2,4,5 NF.1,4-6 R.1,2,4-6 C.1-3,3* SI.

Many more axioms could be satisfied by imposing that suitably chosen subsets of *Q*, *R*, *F* have definable images. However the following axioms cannot be made valid in $\mathfrak{N}_0$:

RA.1-2 IA.1-2 CA.4-7,11,12,14,15 AR.3 NF.2,3 C.4,4*.

A + III.[10]

We conclude this section by expanding $\mathfrak{M}_0$ to a model $\mathfrak{M}'$ of a very highly self-descriptive theory A'. Namely, we will give below an extensive interpretation to the relations and operations (3.1-2), whose domains had been restricted in defining the model $\mathfrak{M}_0$. In doing this, we need to qualify only a finite number of new atoms. Any definable operation acting *directly* on relations (like *domrel*, *invrel*, etc.) will then have a definable extension, which treats correctly all new relations and still belongs to Op_0.

The same argument entails that also the operations *graph, dom* and *img* are still available. It cannot work, however, for relations and operations, like *rginc, hat* and both restrictions, which act on *relational* or *functional pairs*.

Going again through the constants of the theory A, we see that we have to reinterpret, together with the relations and operations (3.1-2), only the four operations

$$cext, \quad syext, \quad gcard, \quad birest \tag{3.3}$$

Therefore we pick two finite sets of new atoms

$$Rel_1 = \{r_1, \ldots, r_{11}\} \quad \text{and} \quad Op_1 = \{f_1, \ldots, f_8\},$$

which will be used to interpret the constants (3.1-3), and we put

$$rdom = r_1 \qquad rval = r_2 \qquad rop = r_3 \qquad rrel = r_4$$

$$rginc = r_5 \qquad rexteq = r_6 \qquad \text{and} \qquad r_{12-n} = r_n^{-1} \;;$$

---

[10] In fact, it has recently been shown by G.LENZI (personal communication) that both theories

A + *any two operations have a composition* and

A + *any two relations have a composition + there is a diagonal relation rdiag s.t. x rdiag y iff y = (x,x)*

are inconsistent.

$eval = f_1$ $\quad bid = f_2$ $\quad hat = f_3$ $\quad oprest = f_4$

$birest = f_5$ $\quad cext = f_6$ $\quad syext = f_7$ $\quad gcard = f_8$.

Since self-description in the theory A is mostly obtained by means of relations, it seems appropriate to pick a third set of atoms $Rel_{-1} = \{r_{-1},\ldots,r_{-11}\}$ to interpret the *negations* of $r_1,\ldots,r_{11}$.(11)

We go now to extend $\mathfrak{N}_0$ to a new natural ample structure

$$\mathfrak{N}' = \langle N;\ Qual_0,\ P_1;\ Rel',\ P_2';\ Op',\ P_3';\ P_4\rangle$$

where $\quad Rel' = Rel_0 \cup Rel_1 \cup Rel_{-1}$, $\quad Op' = Op_0 \cup Op_1$, $\quad$ and

$$P_2' = P_2 \cup \bigcup_{n=1}^{11} G_n \times_K \{r_n\} \cup \bigcup_{n=1}^{11} (N^2 \setminus G_n) \times_K \{r_{-n}\}\ , \qquad P_3' = P_3 \cup \bigcup_{m=1}^{8} F_m \times_K \{f_m\}$$

We interpret the constants (3.1-3) as stipulated above and the remaining ones by extending in the natural way the interpretation given in $\mathfrak{N}_0$. Thus we have only to specify the external graphs G_n and F_m of the new operations and relations.

Due to the simultaneous presence of many *large* and many *non-wellfounded small collections*, providing a model of the whole ample theory A would require particular devices, not only of technical nature. Moreover we want to preserve as much as possible of the properties of comprehension and stability of our previous model $\mathfrak{N}_0$. Last but not least, we are looking for a honest compromise between easy definability and wide applicability of the fundamental operations and relations.

Therefore we decide to maintain the full self-descriptive power of the most important objects, which are

rop, rdom, rval, rrel $\quad$ and $\quad$ *eval, bid,*

---

(11) We shall obtain at once the operation *notr* providing the negation of any relation and satisfying the axiom AR.1 of [2,§VI], namely

$$notr = \chi(\{(\psi(X),\psi(N^2\setminus X)) \mid X\in R\} \cup \{(r_n,r_{-n}) \mid -11\le n\le 11\})$$

by defining their graphs in such a way that the axioms 1.DE 2.EH 17.FG are satisfied.

We slightly weaken instead the actions of the relations *rginc*, *rexteq* and of the operations *oprest*, *birest*, *hat* on pairs involving themselves or other objects r_n, f_m.

Let A' be the axiomatic theory resulting from A by replacing the axioms 10.C and 15.CE by

10.C': *qpreo rginc. If* (r,x) *rginc* (r',x'), *then* $trx \Longrightarrow tr'x'$.

The converse implication holds whenever r, r' *belong to* Rel_0.

15.C'*: If* f *is an operation belonging to* Op_0, *then oprest is defined at* (f,C) *for any collection* C.

If r *is a relation belonging to* Rel_0, *then birest is defined at* $(r,(C,D))$ *for any pair of collections* C,D.

15.E': Like 15.E with the addition: *provided* $f \neq$ *hat, bid, eval.*

Then, by suitably choosing the graphs F_m, G_n, one can prove

THEOREM 3.4

The natural ample structure $\mathfrak{N}$' *is a model of the theory* A' *plus the following axioms of extensionality, comprehension and stability:*[12]

I *The fundamental structures of qualities, relations and operations are extensional.*

II' *Any system has a graph, any collection is the extension of a quality, any quality of pairs is associated to a relation and any functional relation to an operation.*

---

[12] Having replaced the stability axiom III of Theorem 3.3 by the weaker axiom III', the strong axioms R.4 and AR.2,4 are no more valid in $\mathfrak{N}$. However $\mathfrak{N}$ verifies still the axioms SA.1 CA.13.2 AS.1-3 DC AO.3 AQ.1 AR.1,5 NF.1,4-6 R.1,2,5,6 C.1-3,3*.

III' *The collections are closed under union, intersection, cartesian product, power-set operator and power, the qualities are closed under negation and disjunction and the relations under negation.*

SKETCH OF PROOF

We only have to define the graphs F_m $(1 \le m \le 8)$ and G_n $(1 \le n \le 5)$, since G_{12-n} is to be taken equal to G_n^{-1} and $G_6 = G_5 \cap G_7$.

(a) The domains and codomains of all relations r_n, as well as the ranges of all operations f_m are easily determined *a priori* (e.g. the range of *eval* is N, that of *cext* is $N \setminus U$, etc.).

Hence the graph G_2 of *rval* is completely determined.

(b) The operations *cext, syext* and *gcard* have to be reconsidered only on relational pairs (r_n, x), since at any other pair the previous definition works. An easy inspection shows that the operation *cext* (hence a fortiori *syext*) can be made *undefined* in all critical cases, while *gcard* takes on at the corresponding arguments only the value .

Therefore the graphs F_m are determined for $6 \le m \le 8$.

(c) In order to complete the graph of *rdom*, we need only to fix the domains of the operations f_m $3 \le m \le 5$, since dom *bid* is known and dom *eval* $= \bigcup_{n<\omega} D_n$ where $D_0 = \bigcup_{f \ne eval} \{f\} \times \text{dom } f$ and $D_{n+1} = \{eval\} \times D_n$.

According to the axioms 15.C'DE' we put

dom $f_5 = Rel_0 \times Coll^2 \cup \{(r_n, (C,D)) \mid C \times D \ge \text{dom } r_n \times \text{cod } r_n\}$

dom $f_4 = Op_0 \times Coll \cup \{(f_m, C) \mid C \ge \text{dom } f_m\}$

dom $f_3 = \{(f,C) \mid f \ne f_1, f_2, f_3 \ \& \ \hat{f}(C) \in N\} \cup \{(f_m, C) \mid C \ge \text{dom } f_m \ m=1,2,3\}$

Since any collection is closed, $C \ge \text{dom } f$ holds iff *the closure of the domain* of f is included in C. Therefore the above definitions are wellposed, since the closure of dom f_m is easy to specify also when *the domain itself* is yet unknown, provided the atoms f_m are suitably chosen (e.g. as *limits of everywhere defined operations*).

(d) Now the graph of *rdom* is completely defined, while those of the operations f_3, f_4 and f_5 have unique natural definitions once the domains are fixed according to (c). Thus G_1 and F_3, F_4, F_5 are settled.

(e) We define G_5 by stating that $(r,x)\ r_5\ (r',x')$ holds iff either $(r,x) = (r',x')$ or the extensions of the relational pairs (r,x), (r',x') are already completely determined by the preceding stipulations and $trx \Longrightarrow tr'x'$.

(f) Finally we define the graphs of the remaining objects by an inductive procedure involving all of them at once. Namely, we put

$$F_m = \bigcup_{i<\omega} F_m^{(i)} \quad (m=1,2) \qquad \text{and} \qquad G_{\pm n} = \bigcup_{i<\omega} G_{\pm n}^{(i)} \quad (n=3,4)$$

where the six sequences of graphs $G_{\pm n}^{(i)}$ $(n=3,4)$ and $F_m^{(i)}$ $(m=1,2)$ are defined by induction on $i\in\omega$ in the following way:

$$G_3^{(0)} = \{((x,y),f)\in N^2\times Op''' \mid fx=y\}$$

$$G_{-3}^{(0)} = N^2\backslash(N^2\times Op') \cup \bigcup_{f\in Op''} (N\backslash \text{graph } f)\times\{f\} \cup \bigcup_{f\in Op'''} (N\backslash(\text{dom } f \times \text{rng } f))\times\{f\}$$

$$G_4^{(0)} = \{((x,y),r)\in N^2\times Rel''' \mid xry\}$$

$$G_{-4}^{(0)} = N^2\backslash(N^2\times Rel') \cup \{((x,y),r)\in N^2\times Rel''' \mid xr_{-}y\}$$

$$F_1^{(i)} = \{((f,x),y) \mid ((x,y),f)\in G_3^{(i)}\}$$

$$F_2^{(i)} = G_4^{(i)}\times\{\mathrm{t}\} \cup G_{-4}^{(i)}\times\{\mathrm{f}\}$$

$$G_3^{(i+1)} = G_3^{(i)} \cup F_1^{(i)}\times\{f_1\} \cup F_2^{(i)}\times\{f_2\}$$

$$G_{-3}^{(i+1)} = G_{-3}^{(i)} \cup ((\text{dom } F_1^{(i)}\times N)\backslash F_1^{(i)})\times\{f_1\} \cup ((\text{dom } F_2^{(i)}\times N)\backslash F_2^{(i)})\times\{f_2\}$$

$$G_4^{(i+1)} = G_4^{(i)} \cup \bigcup_{n=3}^{4} G_{\pm n}^{(i)}\times\{r_{\pm n}\} \cup (G_{\pm n}^{(i)})^{-1}\times\{r_{\pm(12-n)}\}$$

$$G_{-4}^{(i+1)} = G_{-4}^{(i)} \cup \bigcup_{n=3}^{4} G_{\mp n}^{(i)}\times\{r_{\pm n}\} \cup (G_{\mp n}^{(i)})^{-1}\times\{r_{\pm(12-n)}\}$$

(We have set above $Rel'' = \{r_{\pm n} \mid n=3,4,8,9\}$, $Rel''' = Rel'\setminus Rel''$, $Op'' = \{f_1,f_2\}$, $Op''' = Op'\setminus Op''$. We have also denoted by r_- the *negation* of the relation $r \in Rel'''$).

(g) In order that the axioms 1.DE 2.EH 17.FG hold with the assignements (f), it suffices that $G_n \cup G_{-n} = N^2$ for $n=3,4$, or equivalently that dom f_m = dom F_m for $m=1,2$.

To this aim, we assume that j has been choosen in such a way that one can assign to each object x a *weight* $w(x)$ verifying

$$w(x) = 0 \quad \text{for } x \notin N^2 \cup Rel'' \cup Op'',$$

$$w(x) = 1 \quad \text{for } x \in Rel'' \cup Op'',$$

$$w(x,y) = w(x) + w(y) \quad \text{for } (x,y) \in N^2.$$

Let (x,y) be a pair of least weight not belonging to $G_4 \cup G_{-4}$. Then $x = (u,v)$, $y = r_{\pm k}$ with $k \in \{3,4,8,9\}$ and $(u,v) \notin G_k \cup G_{-k}$. Therefore $k \neq 4,8$, since $w(x,y) > w(u,v)$, and we can assume w.l.o.g. that $y = r_3$ and (u,v) is a pair of least weight outside $G_3 \cup G_{-3}$.

Then $u = (s,t)$, $v = f_m$ with $m \in \{1,2\}$ and $s \in$ dom $f_m \setminus$dom F_m.

If $m=1$, then $s = (f_h, z)$ with $h \in \{1,2\}$ and $z \in$ dom $f_h \setminus$dom F_h. It follows that $((z,t),f_h) \notin G_3 \cup G_{-3}$, contradicting the minimality of (u,v).

If $m=2$, then $s = ((a,b),r) \notin G_4 \cup G_{-4}$, contradicting the minimality of (x,y).

The sketch of the proof is thus concluded.

Q.E.D.

We conjecture that the theory A + I,II',III' is consistent, too. Actually, one can give an inductive simultaneous definition of the graphs F_m $(1 \le m \le 5)$ and $G_{\pm n}$ $(3 \le n \le 9)$, thus expanding the ample structure $\mathfrak{N}'$ to one where all fundamental constants (3.1-3) have a natural interpretation. However it is by no means obvious that one can then assume the strong wellfoundedness property of the encoding of pairs which is needed in order to apply to this wider context the concluding argument (g) sketched above.

4. The axiom of choice

It is well known, since a celebrated result of SPECKER's [17], that the axiom of choice can be inconsistent with set theories admitting large sets: we refer to [6] for a short but exciting review of some negative results.

We shall briefly discuss here how our models behave w.r.t. various kinds of choice principles. It is worth noticing, in view of the above remark, that we obtain *inter alia* the consistency of the well-ordering principle relative to GPK, the generalized positive comprehension principle. As it is done in most classical analyses of universal choice principles, we consider here in particular the axioms studied in [9]. We phrase them below in a form suitable for set theories with a universal set V:

WoV : *The universe V can be well-ordered.*

E : *The universe V has a choice function.*

H : *Any equivalence has a set of representatives.*

F : *Any relation with domain V includes a function with domain V.*

DCC : *Let R be a relation and X a set such that, for any subset Y of X, there is some $x \in X$ with YRx; then there is a X-valued function f defined at all ordinals and verifying $\hat{f}(\beta)Rf(\beta)$ for any ordinal β.*

DCC_{α}: the same as DCC for ordinals less than α.

DC : *Let R be a relation and X a set such that for any $x \in X$ there is $y \in X$ such that xRy: then there is a function f verifying $f(n)Rf(n+1)$ for any integer n.*

We shall also consider the ordering principle

LoV : *The universe V can be linearly ordered.*

It is easily seen that the generalized comprehension principle GPK yields the implications

$$\mathrm{WoV} \longrightarrow \mathrm{E}\ , \quad \mathrm{H} \longrightarrow \mathrm{F} \longrightarrow \mathrm{E} \quad \text{and} \quad \mathrm{LoV} \longrightarrow \mathrm{AC}_{fin}$$

(AC_{fin} is the axiom of choice for any set of finite sets).
E.g. given a relation *R* with domain *V*, consider the equivalence

$$Q = \{((x,y),(x,z)) \mid xRy \ \&\ xRz\}$$

Any set of representatives for *Q* is clearly a function with domain *V* which is included in *R*.

On the other hand, GPK does not yield either WoV ⟶ F or WoV ⟶ DCC (see Theorem 4.3 below); while both implications hold in Gödel-Bernays class theory, even without foundation.[13]

Since in this context urelements are an inessential complication, we shall consider $U = \emptyset$ throughout this section. The models $N_\varkappa(\mathcal{U})$ are thus the same as the models $N_\varkappa$ of [7]. Moreover, since we are interested in the connections between strong principles of comprehension and choice, we restrict our attention to $\varkappa$-compact models. Therefore we assume that $\varkappa \longrightarrow (\varkappa)^2_2$ throughout this section.

---

(13) The exact stregth of the axiom H is still unknown, even in pure set theory without urelements, when the axiom of foundation fails.
Clearly H follows from WoV and implies F, but the converse implications are open. The authors can only prove that H is strictly stronger than both E and DCC_{Ord} (see [9-II]).

Before proceeding we state in the following lemma a topological property of $N_\varkappa$ which will be useful in the sequel.

LEMMA 4.0

Any accumulation point of $N_\varkappa$ *is complete, i.e. it has* $2^\varkappa$ α*-equivalent points for any* $\alpha < \varkappa$.

PROOF

Let x be an accumulation point of $N = N_\varkappa$. There is an accumulation point y belonging to x, otherwise x would be a set of size less than $\varkappa$ of isolated points, which would therefore be all α-isolated for some $\alpha<\varkappa$: hence x would be $(\alpha+1)$-isolated.

For fixed $\alpha<\varkappa$, let B_α be the set of all points of N which are α-equivalent to y. Pick an injective $\varkappa$-sequence $\langle y_\beta \rangle_{\beta<\varkappa}$ of elements of B_α converging to y and, for any subset s of $\varkappa$, put

$$x_s = (x \setminus B_\alpha) \cup \{y\} \cup \{y_\beta \mid \beta \in s\}.$$

The sets x_s are clearly α- (indeed at least $\alpha+1$) equivalent to x.

Q.E.D.

We begin by defining in $N_\varkappa$ a linear ordering of the universe.

LEMMA 4.1

There is a closed subset O *of* $N_\varkappa^2$ *such that*

(i) O *is reflexive, antisymmetric and transitive;*

(ii) if $x \not\equiv_\alpha y$, *then either* $B(x,\varepsilon_\alpha) \times B(y,\varepsilon_\alpha)$ *or* $B(y,\varepsilon_\alpha) \times B(x,\varepsilon_\alpha)$ *is included in* O.

PROOF

Let $<_\alpha$ be a linear ordering of the set B_α of all closed ε_α-balls of N. Since B_α is a partition of $N_\varkappa$, it is possible to choose a

$\varkappa$-sequence $\langle <_\alpha \mid \alpha<\varkappa \rangle$ in such a way that, given $b <_\alpha b'$, if c, c' are any ε_β-balls $(\beta>\alpha)$ included in b, b' respectively, then $c <_\beta c'$.

Define $O = \{(x,y) \mid x=y \text{ or } \exists\beta\ B(x,\varepsilon_\beta) <_\beta B(y,\varepsilon_\beta)\}$: then O verifies (ii) by construction and (i) since all $<_\alpha$'s are linear orderings (note that if $B(x,\varepsilon_\beta) <_\beta B(y,\varepsilon_\beta)$ holds for some β, then it holds for any β for which they are different).

Therefore we only need to prove that O is closed. Let x_α, y_α be $\varkappa$-sequences such that, for any $\alpha < \varkappa$, $x_\alpha \approx_\alpha x$, $y_\alpha \approx_\alpha y$ and $(x_\alpha,y_\alpha) \in O$. If for all α $x_\alpha \approx_\alpha y_\alpha$, then $x = y$; otherwise for some β $B(x,\varepsilon_\beta) <_\beta B(y,\varepsilon_\beta)$, hence in any case $(x,y) \in O$.

Q.E.D.

We shall show now that neither O nor other relations on $N_\varkappa$ can be wellorderings. Note that the statement of Lemma 4.2 below refers to *external true* wellordrings. We shall see later that if $\varkappa = \omega$ the above defined relation O is a wellordering *in the sense of* N_ω.

LEMMA 4.2

There are no (standard) closed well-orderings of $N_\varkappa$.

PROOF[14]

Assume $N = N_\varkappa$ wellordered and fix an (external) indexing of N by ordinals, say $N = \{x_\alpha \mid \alpha<\lambda\}$. The corresponding graph belongs to N iff it is closed, i.e. iff whenever $\alpha_\iota \le \beta_\iota$ for any $\iota<\varkappa$ and $x = \lim_{\iota\to\varkappa} x_{\alpha_\iota}$, $y = \lim_{\iota\to\varkappa} x_{\beta_\iota}$, also $x \le y$.

Assume that the given wellordering is closed. Then $\lim_{\alpha\to\varkappa} x_\alpha = x_\varkappa$, since any two convergent $\varkappa$-subsequences of $\langle x_\alpha\rangle_{\alpha<\varkappa}$ have the same limit.

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[14] This proof is essentially due to Malitz [13].

Now pick another $\varkappa$-sequence converging to $x_\varkappa$, whose indices are greater than $\varkappa$, which exists since $x_\varkappa$ is a complete accumulation point of N. Let $\langle x_{\gamma_\iota} \rangle_{\iota<\varkappa}$ be any $\varkappa$-subsequence of it with increasing indices γ_ι in the fixed indexing: then we would have $x_{\gamma_\iota} \le x_{\gamma_\nu}$ whenever $\iota \le \nu$, whence $x_{\gamma_\iota} \le x_\varkappa = \lim x_{\gamma_\nu}$, and simultaneously $x_\varkappa < x_{\gamma_\iota}$ for any ι, since $\varkappa < \gamma_\iota$ for any $\iota<\varkappa$.

Therefore no closed wellordering of N can exist.

Q.E.D.

Since all α-sequences for $\alpha<\varkappa$ are elements of $N_\varkappa$, any internal wellordering of $N_\varkappa$ would be a *real* wellordering when $\varkappa$ is uncountable. Hence WoV fails in $N_\varkappa$ for any uncountable $\varkappa$.

This is also a consequence of the following theorem, which summarizes the main choice properties of our models:

THEOREM 4.3

(i) *The axioms* LoV *and* $\forall\alpha \mathrm{DCC}_\alpha$ *(hence also* AC_{fin}*) hold in* $N_\varkappa$ *and the axioms* F *and* DCC *(hence also* H*) fail in* $N_\varkappa$ *for any* $\varkappa$*.*

(ii) *The axiom* WoV *(hence also* E*) holds, whereas the axiom* DC *fails in* N_ω.

PROOF

(i) The set O of the above lemma witnesses that $N_\varkappa \models$ LoV.

Taking into account that the ordinals of $N_\varkappa$ are exactly those which are less than $\varkappa$, we get immediately $N_\varkappa \models \forall\alpha \mathrm{DCC}_\alpha$, since all functions of size less than $\varkappa$ belong to $N_\varkappa$.

In order to prove the failure of DCC, let $\bar\varkappa = \varkappa \cup \{\bar\varkappa\}$ be the closure of $\varkappa$ in $N_\varkappa$, put $C = \bar\varkappa \times \{0,1\}$ and consider the closed relation $R \subseteq \mathcal{P}(C)\times C$ defined by

$$(x,(\alpha,i)) \in R \quad \text{iff} \quad \alpha\times\{0,1\} \subseteq x, \quad (\alpha,i)\notin x \quad \text{and} \quad \begin{cases} (\alpha,1-i)\in x \\ \text{or} \\ i = 0 \end{cases}$$

$$(C,(\bar{\varkappa},i)) \in R \quad \text{for } i = 0,1.$$

Any function in $N_\varkappa$ which is defined at all ordinals less than $\varkappa$ must be uniformly continuous and defined at $\bar{\varkappa}$.

Let $f\colon A \longrightarrow C$ verify $\hat{f}(\alpha)Rf(\alpha)$ for any $\alpha<\varkappa$ (hence A includes $\bar{\varkappa}$). Then $f(0) = (0,0)$, and f proceedes by taking alter- nately all values $(\alpha,0)$ and $(\alpha,1)$, with α increasing without any jump.

Therefore, by continuity, both $(\bar{\varkappa},0)$ and $(\bar{\varkappa},1)$ have to be taken as values of f at $\bar{\varkappa}$, so f is not a function.

Note that the same relation R provides a counterexample also for the axiom F, since any continuous function included in R should associate to C both $(\bar{\varkappa},0)$ and $(\bar{\varkappa},1)$.

The fact that dom R is not the whole universe is easily settled, since there is a projection of $N_\varkappa$ onto any closed set.[15] So we can use such a projection onto $\mathcal{P}(C)$ and transform R into a relation with universal domain.

In order to find in N a projection of N onto the closed set A, working from outside we associate to each α-ball B_α meeting A a point $\sigma(B_\alpha) \in B_\alpha \cap A$, in such a way that $\sigma(B_\beta) = \sigma(B_\alpha)$ whenever B_β is a β-ball (with $\beta>\alpha$) to which $\sigma(B_\alpha)$ belongs.

For $x\notin A$, let $\alpha+1$ be the least (necessarily successor) ordinal such that x is $(\alpha+1)$-inequivalent to each element of A, and put

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[15] Note that we can easily obtain from the existence of projections another choice-like property of $N_\varkappa$, namely that *the injective ordering of cardinalities is coarser than the surjective one.*

In fact, if $j\colon A \longrightarrow B$ is an injective continuous mapping and both A and B are closed, then by $\varkappa$-compactness j is a homeomorphism between A and $A'=\hat{j}(A)$. If p is a continuous projection of N onto A', then $j^{-1}\circ p_{|B}$ is a projection of B onto A.

We do not know whether the converse property holds in $N_\varkappa$.

$p(x) = \sigma(B_\alpha(x))$. Since, for $x,y \notin A$ and $z \in A$, $x \approx_\alpha y \approx_\alpha z$ implies $p(x) \approx_\alpha p(y) \approx_\alpha z$, we extend p by the identity on A and obtain a continuous projection of $N_\varkappa$ onto A.

(ii) We get the wellordering principle in N_ω by showing that any closed set has a least element w.r.t. the ordering O of Lemma 4.1.

Let $A \subseteq N_\omega$ be closed and, for any $n<\omega$, let B_n be the least n-ball meeting A (least in the ordering induced by O). By definition of O, the balls B_n are a descending chain under inclusion, which has non-empty intersection by Cauchy completeness.

Again by definition of O, the unique point lying in the intersection of all balls B_n is the least element of A (and belongs to A as limit of a sequence of points of A).

The proof that DC fails in N_ω could be omitted, since DC is equivalent to DCC for $\varkappa=\omega$. However it is easy to verify that the closed relation

$$S = \{((n,0),(n,1)) \mid n\in\omega\} \cup \{((n,1),(n+1,0)) \mid n\in\omega\} \cup \{\overline{\omega}\} \times \{0,1\}$$

does not admit a continuous function f with closed domain verifying $f(n)Sf(n+1)$ for any $n\in\omega$, since $f(\overline{\omega})$ should be simultaneously 0 and 1.

Q.E.D.

Finally $N_\varkappa$ verifies the axiom of *strong extensionality*

Sext - *Transitive $\in$-isomorphic sets are equal.*

This is a consequence of AR and of the fact that $N_\varkappa$ is a transitive set without urelements.

Therefore we can considerably improve the consistency results of [7] by putting together Theorems 3.2 and 4.3, so as to obtain

COROLLARY 4.4

(i) Con(ZF) $\Longrightarrow$ Con(GPK + Sext + Inf + WoV).

(ii) $\mathrm{Con}(\mathrm{ZFC} + \exists\varkappa{>}\omega\ \varkappa \longrightarrow (\varkappa)^2_2) \Longrightarrow \mathrm{Con}(\mathrm{GPK} + \mathrm{Sext} + \mathrm{SInf} + \mathrm{LoV} + \forall\alpha \mathrm{DCC}_\alpha)$

where Inf *is the usual axiom of infinity*

Inf - $\exists w\ (\emptyset \in w\ \&\ (x \in w \longrightarrow x\cup\{x\} \in w))$,

while Sinf *is some strong axiom of infinity, e.g.*

Sinf - $\forall\alpha\ \exists\mu{>}\alpha$ *μ is a strongly (hyper-hyper-...) Mahlo cardinal.*

We conjecture that $\forall\alpha \mathrm{DCC}_\alpha$ in (ii) above can be replaced by AC, but at present we do not even know whether the axiom of choice holds in $N_\varkappa$ for some uncountable $\varkappa$.

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