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Foundations and Proof Theory of 3-valued Logics

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by

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Many valued logics in general and 3-valued logic in particular is an old subject which had its beginning in the work of Lukasiewicz [Luk]. Recently there is a revived interst in this topic, due to its potential applications in several areas of computer science, like: proving correctness of programs ([Jo]), knowledge bases ([CP]) and Artificial Inteligence ([Tu]). There is, however, a huge number of 3-valued systems which logicans have studied throughout the years. The motivation behind them and their properties are not always clear and their proof theory is frequently not well developed. Our goal in this work is to try to facilitate both the use of and the research on 3-valued logic by providing a unified treatment, within a quite general framework, of the most important ones. These include the 3-valued logics of Lukasiewicz, Kleene and Sobociński, the logic LPF used in the VDM project, the Logic RM_3 from the relevance family and the paraconsistent 3-valued logic. We shall present a point of view from which all these logics appear quite natural and closely related to each other. It will turn out, for example, that Lukasiewicz 3-valued logic and RM_3 (the strongest logic in the family of relevance logics) are in a strong sense dual to each other, and that both are derivable by the same natural general construction from, respectively, Kleene 3-valued logic and the 3-valued paraconsistent logic. We shall present also a unique 4-valued natural logic in which all the various 3-valued systems can naturally be incorporated.

On the more technical side, we shall provide also a proof-theoretical analysis of all the 3-valued systems we discuss. This will include:

- Hilbert type representations with M.P. as the sole rule of inference of almost every system (or fragment thereof) which includes an appropriate implication connective in its language.[1]

- Cut-free Gentzen-type formulations of *all* the systems we discuss. In the

[1] RM_3 and its fragments with either $\wedge$ or $\vee$ are the only exceptions.

cases of Lukasiewicz and RM_3 this will be possible only by employing a calculus of *hypersequents*, which are finite sequences of ordinary sequents.

All the 3-valued systems we consider below are based on the following *basic structure*:

- Three truth-values :T, F and $\perp$. T and F correspond to the classical two truth values.
- An operation $\neg$, which is defined on these truth-values. It behaves like classical negation on $\{T, F\}$, while $\neg \perp = \perp$.

The language of all the systems we consider includes a negation connective, also denoted by $\neg$, which corresponds to the operation above. Most of them include also the connectives $\wedge$ and $\vee$. The corresponding truth tables are defined as follows: $a \wedge b = min(a, b)$, $a \vee b = max(a, b)$, where $F < \perp < T$. We shall see that from a certain point of view the introduction of these connectives as well as the way they are defined are dictated by the interpretation of the operation $\neg$ as *negation.*

Traditionally, the differences between the various systems are with respect to:

- What other connectives are taken as basic. Especially: what is the official "implication" connective of the language.
- What truth-values are taken to be designated.

Examples:

Kleene 3-valued logic: This logic has, essentially, the basic connectives we describe above with the same truth tables. In addition its standard presentation includes also a connective $\Rightarrow$ defined by

$$a \Rightarrow b = \neg a \vee b$$

T is here the only designated value.

LPF: This is an extension of Kleene's logic wich was developed within the VDM project (see [BCJ], [Jo]). On the propositional level it is obtained from Kleene by adding a connective Δ such that:

$$\Delta(a) = \begin{cases} F & a = I \\ T & a = T, F \end{cases}$$

Lukasiewicz: This was the first 3-valued logic ever to be invented. Besides the basic 3 connectives above it has also an implication connective $\rightarrow$ so that:

$$a \rightarrow b = \begin{cases} \neg a \vee b & a > b \\ T & \text{otherwise} \end{cases}$$

Again T is taken as the only designated value.

RM$_3$: This is the strongest logic in the family of relevance logics ([AB],[Du]). It has *both* T and $\perp$ as designated. Besides the basic 3 connectives above it has also an implication connective $\rightarrow$ (first introduced in [Sob]) so that:

$$a \rightarrow b = \begin{cases} \perp & a = b = \perp \\ F & a > b \\ T & \text{otherwise} \end{cases}$$

3-valued paraconsistent logic: This logic also has both T and $\perp$ as designated and has one extra implication connective $\supset$ besides the three basic ones. It is defined as follows:

$$a \supset b = \begin{cases} T & a = F \\ b & \text{otherwise} \end{cases}$$

The truth table for this connective was introduced in [dC]. The corresponding logic was investigated and axiomatized in [Av3], where it is shown to be a maximal paraconsistent logic (i.e. a logic in which contradictions do not imply everything).

For obvious reasons, all these systems take T as designated and none takes F. This leads into two main directions, corresponding to whether or not we take $\perp$ as designated. The decision depends, of course, on the intended intuitive interpretation of $\perp$. If it corresponds to some notion of *incomplete* information, like "undefined" or "unknown" then usually it is not taken as designated. If, on the other hand, it corresponds to *inconsistent* information (so its meaning is something like "known to be both true and false") then it does. Accordingly, the logics below will be devided into two classes, corresponding to these two interpretations. We shall see that each class has one basic logic from which all the rest are derivable by general methods. We shall show also how the two interpretations can be merged into one, coherent four-valued logic.

The above two criterions do not really suffice for characterizing the various logics we discuss. We shall see below, for example, that LPF and Lukasiewicz 3-valued logic have exactly the same expressive power: every primitive or definable connective of one is also a primitive or definable connective of the other. Also both have T as the only designated value. The only difference is therefore with respect to what connectives are taken as primitive. Usually this is not taken as an essential issue [2], unless this choice reflects something deeper. This can only be (especially when we are dealing with "implication" connectives) a difference with respect to the *consequence relation* associated with the logic. We shall begin therefore our discussion with this crucial notion as our starting point.

1 General Considerations

The notion of a consequence relation was first introduced in [Sc1] and [Sc2]. It is extensively used in [Ur] for characterizing many-valued logics. In what follows we shall need, however, a generalization from [Av1] of the original definition (see there for explanations and motivations):

Definition: A *consequence relation* (C.R.) on a set Σ of formulas is a binary relation $\vdash$ between finite multisets of formulas s.t:

(I) Reflexivity: $A \vdash A$ for every formula A.

(II) Transitivity, or "Cut": if $\Gamma_1 \vdash \Delta_1, A$ and $A, \Gamma_2 \vdash \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

It is more customary to take a C.R. to be a relation between *sets*, rather than multisets (which are "sets" in which an element may occure more than once). We define, accordingly, a C.R. to be *regular* if it can be viewed in this way (equivalently, if it is closed under contraction and its converse). There are, however, logics the full understanding of which requires us to make finer distinctions that only the use of multisets enable us to make. Examples are provided in [Av1] and below. Another standard condition that we find necessary to omit is closure under *weakening*. In what follows we shall call *ordinary* any regular C.R. which satisfies this condition[3]

Other concepts from [Av1] that will be of great importance below are those of *internal* and *combining* connectives. The internal connectives are connectives that make it possible to transform a given sequent to an equivalent one that has a special required form. The combining connectives, on the other hand, make it possible to combine certain pairs of sequents into a single one, which is valid iff the original two are valid. In [Av1] we characterized several logics (including classical, intuitionistic, relevant and linear logic) in terms of the internal and combining connectives available in them and the structural rules under which they are closed. We repeat here the definitions of the internal negation and

[2]In the literature one can find a lot of different formulations of classical logic with different choices of the primitive connectives— and they all are generally taken to be equivalent!

[3]The concept of ordinary C.R. coincides with the original concept of a C.R. due to Scott.

implication and of the combining conjunction and disjunction:

Definition: Let $\vdash$ be a C.R.

Internal Negation: We call a unary connective $\neg$ a right internal negation if for all Γ, Δ, A:

$$\Gamma, A \vdash \Delta \quad \text{iff} \quad \Gamma \vdash \Delta, \neg A \ .$$

We call a unary connective $\neg$ a left internal negation if for all Γ, Δ, A:

$$\Gamma \vdash \Delta, A \quad \text{iff} \quad \Gamma, \neg A \vdash \Delta \ .$$

It can easily be shown that $\neg$ is a right internal negation iff it is a left one. We shall use therfore just the term *internal negation* to denote both. We shall call a C.R. which has an internal negation **symmetrical**.

Internal Implication: [4] We call a binary connective $\rightarrow$ an internal implication if for all Γ, Δ, A, B:

$$\Gamma, A \vdash \Delta, B \quad \text{iff} \quad \Gamma \vdash \Delta, A \rightarrow B \ .$$

Combining Conjunction: We call a connective $\wedge$ a combining conjunction iff for all Γ, Δ, A, B:

$$\Gamma \vdash \Delta, A \wedge B \quad \text{iff} \quad \Gamma \vdash \Delta, A \ \text{ and } \ \Gamma \vdash \Delta, B \ .$$

Combining Disjunction: We call a connective $\vee$ a combining disjunction iff for all Γ, Δ, A, B:

$$A \vee B, \Gamma \vdash \Delta \quad \text{iff} \quad A, \Gamma \vdash \Delta \ \text{ and } \ B, \Gamma \vdash \Delta \ .$$

The following facts were shown in [Av1]:

1. $\neg$ is an internal negation iff $\vdash$ is closed under the rules:

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \quad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \ .$$

2. $\wedge$ is a combining conjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \quad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \ .$$

[4] This was called strong intensional implication in [Av1]. We believe that the present terminology is better.

3. $\vee$ is a combining disjunction iff $\vdash$ is closed under the rules:

$$\frac{\Gamma\vdash\Delta,A}{\Gamma\vdash\Delta,A\vee B}\quad\frac{\Gamma\vdash\Delta,B}{\Gamma\vdash\Delta,A\vee B}\qquad\frac{\Gamma,A\vdash\Delta\quad\Gamma,B\vdash\Delta}{\Gamma,A\vee B\vdash\Delta}.$$

4. $\rightarrow$ is an internal implication iff $\vdash$ is closed under the rules:

$$\frac{\Gamma,A\vdash B,\Delta}{\Gamma\vdash A\rightarrow B,\Delta}\quad\frac{\Gamma_1\vdash\Delta_1,A\quad B,\Gamma_2\vdash\Delta_2}{\Gamma_1,\Gamma_2,A\rightarrow B\vdash\Delta_1,\Delta_2}.$$

The most important of these connectives (for our present purposes) is the internal negation. In its presence the existence of internal implication suffices for having all the other internal connectives we have considered in [Av1] (like internal conjunction and disjunction), while the existence of either combining conjunction or combining disjunction suffices for having all the other combining connectives. All the C.R.s we discuss in this work are either ordinary or symmetrical (i.e., have an internal negation), but not both. The only exception is, of course, classical logic (which can, in fact, be ***characterized*** by these two properties). We note also that for *ordinary* C.R. a connective is a combining conjunction iff it is an internal conjunction, and similar relations hold for the other connectives. This is not true in general, though.

Suppose that $\vdash$ is a C.R., and that $\neg$ is a unary connective in its language. How can we reasonably change $\vdash$ to make $\neg$ an internal negation? There are two possible directions in which a solution to this problem may be sought. One involves weakening $\vdash$, the other involves strengthening it. Specifically, call a sequent $\Gamma'\vdash\Delta'$ a *version* of $\Gamma\vdash\Delta$ if it can be obtained from the later by finite number of steps, in each of which a formula is transfered from one side of a sequent to the other while removing a $\neg$ symbol from its beginning or adding one there. If we define a sequent to be w-valid iff some version of it is valid in $\vdash$ then the minimal C.R. for which all w-valid sequents obtain is also the minimal C.R. which extends $\vdash$ and relative to which $\neg$ is an internal negation. Classical Logic is obtained from Intuitionistic Logic in this way. Alternatively we might try to restrict $\vdash$ by demanding a sequent to be strongly valid iff *every* version of it is valid. Unfortunately, this is *too* strong: Unless $\neg$ is already an internal negation even the reflexivity condition fails for this new relation. Nevertheless, if we demand the new relation to be a strengthening only of the *single-conclusioned* fragment of the old one then under certain natural conditions we can do better:

Definition: Let $\vdash$ be a C.R. so that both $A\vdash\neg\neg A$ and $\neg\neg A\vdash A$ (these conditions will be called below *the symmetry conditions for negation*). Define $\vdash^S$, the *derived symmetrical version of* $\vdash$, as follows: $\Gamma\vdash^S\Delta$ iff every single-conclusioned version of $\Gamma\vdash\Delta$ obtains.

Proposition:

1. $\vdash^S$ is a C.R..
2. If $\Gamma \vdash^S A$ then $\Gamma \vdash A$.
3. $\neg$ is an internal negation with respect to $\vdash^S$.
4. $\vdash^S$ is the maximal C.R. having the above properties.
5. $\vdash$ and $\vdash^S$ have the same *logical theorems*, i.e. for any A, $\vdash A$ iff $\vdash^S A$.
6.

$$A_1, \ldots, A_m \vdash^S B_1, \ldots, B_n$$

iff for every $1 \leq i \leq m$ and $1 \leq j \leq n$ we have:

$$A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_m, \neg B_1, \ldots, \neg B_n \vdash \neg A_i$$

$$A_1, \ldots, A_m, \neg B_1, \ldots, \neg B_{j-1}, \neg B_{j+1}, \ldots, \neg B_n \vdash B_i$$

We leave the easy proof of this proposition to the reader. We note that the last claim in it provides an *effective* alternative definition of the derived symmetrical C.R.. It is also easy to see that the symmetry conditions for negation are in fact necessary for getting a C.R. from this construction. They are obviously satisfied by any C.R. based on the above 3-valued semantics (with respect, of course, to the connective $\neg$ defined there).

Our next goal is to find conditions on $\vdash$ which insure that $\vdash^S$ has the other connectives we have defined.

Proposition: Let $\wedge$ be a combining conjunction for $\vdash$. Suppose also that $\vdash$ is closed under the rules:

$$\frac{\Gamma, \neg A \vdash \Delta \quad \Gamma, \neg B \vdash \Delta}{\Gamma, \neg(A \wedge B) \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \neg A}{\Gamma \vdash \Delta, \neg(A \wedge B)} \quad \frac{\Gamma \vdash \Delta, \neg B}{\Gamma \vdash \Delta, \neg(A \wedge B)}$$

(we shall call these conditions *the symmetry conditions for conjunction*). Then $\wedge$ is a combining conjunction for $\vdash^S$.

Proof: Suppose $\Gamma \vdash^S \Delta, A$ and $\Gamma \vdash^S \Delta, B$. We want to show that $\Gamma \vdash^S \Delta, A \wedge B$. Let, accordingly, $\Gamma' \vdash C$ be a single-conclusioned version of $\Gamma \vdash \Delta, A \wedge B$. We want to prove that this sequent is true. There are two possible cases to consider:

1. C is $A \wedge B$.
   By our assumptions, $\Gamma' \vdash A$ and $\Gamma' \vdash B$ are both true. Hence also $\Gamma' \vdash A \wedge B$ is true, since $\wedge$ is a combining conjunction for $\vdash$.

2. $\neg(A \wedge B)$ is in Γ'.
   In this case our assumptions and the first symmetry condition for $\wedge$ easily entail that $\Gamma' \vdash C$.

For the converse, we should show that if $\Gamma \vdash^S \Delta, A \wedge B$ then $\Gamma \vdash^S \Delta, A$ and $\Gamma \vdash^S \Delta, B$. The proofs are again splitted into two cases. The second symmetry condition for $\wedge$ is used for one of them, the other part of the definition of a combining conjunction— for the other. Details are left to the reader.

Analogous symmetry conditions for the existence of a combining disjunction can easily be formulated, but in the presence of an internal negation and a combining conjunction such a connective is available anyway.

We next turn our attention to the problem of having an internal implication for $\vdash^S$. If $\rightarrow$ is such a connective then $\vdash^S A \rightarrow B$ iff $A \vdash^S B$ iff $A \vdash B$ and $\neg B \vdash \neg A$. Suppose now that $\vdash$ has an internal implication $\supset$ and a combining conjunction $\wedge$. Then the last two conditions are together equivalent to $\vdash (A \supset B) \wedge (\neg B \supset \neg A)$. This, in turn, is equivalent to $\vdash^S (A \supset B) \wedge (\neg B \supset \neg A)$ (by 5. of the last proposition). Hence the last formula provides an obvious candidate for defining $\rightarrow$. Our next proposition contains natural conditions for this candidate to succeed.

Proposition: Suppose $\wedge$ is a combining conjunction for $\vdash$ which satisfies (in $\vdash$) the corresponding symmetry conditions. Suppose also that $\supset$ is an internal implication for $\vdash$ and that $\vdash$ is closed under the following rules:

$$\frac{\Gamma, A, \neg B \vdash \Delta}{\Gamma, \neg(A \supset B) \vdash \Delta} \quad \frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash \Delta_2, \neg B}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \neg(A \supset B)}$$

(These two rules will be called below *the symmetry conditions for implication*). Define:

$$A \rightarrow B =_{Df} (A \supset B) \wedge (\neg B \supset \neg A)$$

Then $\rightarrow$ is an internal implication for $\vdash^S$.

The proof of this proposition is left to the reader. We only note that the naturalness of the above symmetry conditions for implication can most clearly be seen by working out the details of this proof.

1.1 The Basic System

By collecting the various conditions at which we arrive in this section we get a Gentzen-type system for the minimal *ordinary* C.R. for which all these conditions obtain. This system, with or without its implicational rules, will provide the basis for all the formal representations of the *ordinary* C.R.s we present below. It has a 4-valued semantics which will be discussed later. Cut-elimination can be shown for it rather easily.

Axioms:

$$A \Rightarrow A$$

Rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \; \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \; \frac{\Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \neg A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \; \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)} \; \frac{\Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \quad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B} .$$

$$\frac{\Gamma, A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg B}{\Gamma \Rightarrow \Delta, \neg(A \supset B)}$$

And the usual structural rules of Exchange, Weakening and Contraction.

2 Consequence Relations based on 3-valued Semantics

2.1 The "Undefined" Interpretation

In this section we investigate several C.R.s in which $\perp$ is taken as corresponding to a truth gap, and so T is the only designated value. We start with the basic relation which naturally corresponds to this interpretation. As we shall see, all the others are essentially based on it.

Definition: $\vdash_{Kl}$ is the C.R. defined by:

> $\Gamma \vdash_{Kl} \Delta$ iff any valuation v (in the basic 3-valued structure) which assigns (the designated value) T to all the sentences in Γ assigns it also to at least one of the sentences in Δ.

Two obvious facts about this C.R. are:

- $\vdash_{KI}$ is an ordinary C.R..
- $\neg$ satisfies the symmetry conditions for negation, but it is not an internal negation.

We next check how can we define operations on the basic structure so that we get combining conjunction and internal implication, both satisfying the corresponding symmetry conditions. The main conclusion is that these requirements completely determine the truth-tables for such connectives.

Proposition:

1. The connective $\wedge$ which was described in the introduction ($a \wedge b = min(a, b)$ where $F < \perp < T$) is a combining conjunction for $\vdash_{KI}$ which satisfies the symmetry conditions. Moreover, it is the only possible connective on this structure which has these properties. Similar results hold for $\vee$ from the introduction with respect to disjunction.

2. Define a connective $\supset$ on the basic 3-valued structure as follows:

$$a \supset b = \begin{cases} b & a = T \\ T & \text{otherwise} \end{cases}$$

   Then $\supset$ defines an internal implication for $\vdash_{KI}$ which satisfies the symmetry conditions. Moreover, $\supset$ is the only possible connective on this structure which has these properties.

Proof:

1. For any many-valued ordinary C.R. $\vdash$ the conditions: $A \wedge B \vdash A$, $A \wedge B \vdash B$ and $A, B \vdash A \wedge B$ (which obtain in any *ordinary* C.R. for which $\wedge$ is a combining conjunction) entail that $A \wedge B$ gets a designated value iff both A and B do. Similarly, the conditions: $\neg A \vdash \neg(A \wedge B)$, $\neg B \vdash \neg(A \wedge B)$, $\neg(A \wedge B) \vdash \neg A, \neg B$ (which follow from the symmetry conditions) entail that $\neg(A \wedge B)$ gets a designated value iff either $\neg A$ or $\neg B$ does. In the present case these observations determine a unique truth-table for $\wedge$, and it is easy to see that the corresponding connective is really a combining conjunction which satisfies the symmetry conditions. Similar argument works for $\vee$.

2. If $\supset$ is an internal implication for an ordinary C.R. $\vdash$ then $A \vdash B \supset A$, $\vdash A, A \supset B$ and $A, A \supset B \vdash B$ all obtain. These conditions entail, in a

many-valued C.R., that $A \supset B$ gets a designated value iff B does or else if A does not. In the present case this observation alone determines 7 out of the 9 entries in a possible truth-table for $\supset$, and reduce to 2 the number of possibilities in each of the two remaining cases. The requirements that $\neg(A \supset B) \vdash \neg B$ and $A, \neg B \vdash \neg(A \supset B)$ determine these two cases as well. It is easy to see that the connective which corresponds to the resulting truth-table really meets the requirements.

We now investigate some known logics that are obtained using the connectives of the last proposition and the general constructions of the previous section.

2.1.1 Kleene's 3-valued logic

This logic can now be characterized as $\vdash_{Kl}$ in a language which has, besides $\neg$, also the above unique combining conjunction (or disjunction) that satisfies the symmetry conditions.

An important property of this logic is that it has no logical theorems: $\vdash_{Kl} A$ for no A in its language. This means, first of all, that no corresponding internal implication exists in its language (since at least $A \to A$ should be a theorem for any possible candidate $\to$) [5]. Since an internal disjunction *is* available it follows also that no possible internal negation is definable (and so not only the official $\neg$ fails to be one).

The official $\Rightarrow$ usually associated with this logic is not an implication connective in any sense, and it is just one out of many connectives that are definable from $\neg$ and $\wedge$.

2.1.2 LPF

This logic is $\vdash_{Kl}$ in a language which has, in addition to Kleene's connectives, also the internal implication defined above. It is, of course, an ordinary conservative extension of the original logic of Kleene, and the basic connectives of Kleene retain in it their properties.

At the introduction we follow [BCJ] and define LPF in terms of another connective, Δ. We have, however, the following relations between this connective and our $\supset$:

$$\Delta A = \neg(A \equiv \neg A) \quad \text{where} \quad A \equiv B =_{Df} (A \supset B) \wedge (B \supset A)$$

$$A \supset B = \Delta A \wedge A \Rightarrow B = \neg \Delta A \vee \neg A \vee B.$$

[5]The same consideration will apply to any possible C.R. which is based on Kleene's connectives.

These relations mean that the expressive power of the two languages are the same. Since the C.R. associated with both is $\vdash_{Kl}$, [6] the two versions are equivalent. The present version seems to us more natural, though, and it opens the door to intersting observations, like the one given in our next proposition.

Proposition: The positive fragment of LPF (i.e. the $\{\vee, \wedge, \supset\}$-fragment) is identical to the corresponding classical one. In particular every classical positive tautology is valid in it.

The proof of this fact is by showing that every axiom and rule of the standard Gentzen-type representation of positive classical logic is valid in the 3-valued semantics (the converse is obvious). All these rules are included in the basic system of the previous section, the rules of which are all valid here.

2.1.3 The 3-valued C.R. of Lukasiewicz

As observed above, $\vdash_{Kl}$ is not symmetrical. Nevertheless, the various symmetry conditions concerning $\neg, \vee, \wedge$ obtain for it, and those concerning implication hold for $\supset$ in the extended version. We can apply therefore our general constructions to get the symmetrical versions of both. We shall denote the symmetrical version of Kleene basic logic by $\vdash_{WLuk}$ and that of its extension with $\supset$ by $\vdash_{Eluk}$. When we mean either we shall just use $\vdash_{Luk}$. We give first a semantical characterization of this C.R.:

Proposition: $\Gamma \vdash_{Luk} \Delta$ iff for every assignment, either one of the sentences in Δ gets T, or one of the sentences in Γ gets F, or at least two (occurances of) sentences in Γ, Δ get $\perp$.

Proof: Suppose first that the condition holds. Let $\Gamma' \vdash A$ be a single conclusioned version of $\Gamma \vdash \Delta$ and v an assignment for which all the sentences in Γ' get T. This means that the third possibility mentioned in the proposition does not obtain, since at most the ancestor of A can gets $\perp$. On the other hand, each of the other two possibilities obviously gaurantees that A gets T in case all the sentences in Γ' get T.

For the converse, suppose that v is an assignment for which the condition above fails for the sequent $\Gamma \vdash \Delta$. If there is no sentence in Γ or Δ which gets $\perp$ then *no* single-conclusioned version of $\Gamma \vdash \Delta$ belongs to $\vdash_{Kl}$. Otherwise let $\Gamma' \vdash A$ be the single conclusioned version of $\Gamma \vdash \Delta$ in which A is the unique sentence in $\Gamma \vdash \Delta$ which gets $\perp$ (if it occures in Δ) or its negation (if it occures

[6] In the case of the original LPF this is obvious from the natural deduction system presented in [BCJ].

in Γ). The failure of the condition entails that all the sentences in Γ' get T, and so the resulting single-conclusioned version does not belong to $\vdash_{Kl}$, and $\Gamma \vdash \Delta$ does not belong to $\vdash_{Luk}$.

Our next propsition just summerizes the properties which $\vdash_{Luk}$ has according to the general discussion of the previous section:

Proposition:

1. If $\Gamma \vdash_{Luk} A$ then $\Gamma \vdash_{Kl} A$.
2. $\neg$ is an internal negation for $\vdash_{Luk}$.
3. $\wedge$ and $\vee$ are, respectively, combining conjunction and disjunction for $\vdash_{Luk}$.
4. $\vdash_{Eluk}$ is a conservative extension of $\vdash_{WLuk}$.
5. Define:
$$A \rightarrow B =_{DF} (A \supset B) \wedge (\neg B \supset \neg A)$$
Then $\rightarrow$ is an internal implication for $\vdash_{Eluk}$.

The relation between the derived symmetrical version of $\vdash_{Kl}$ and Lukasievicz 3-valued logic (which justifies the name $\vdash_{Luk}$) is given in the next proposition and its corollary:

proposition: $\rightarrow$ of the previous proposition is exactly Lukasiewicz' implication.

Corollary: $A_1, \ldots, A_n \vdash_{Luk} B$ iff $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B)\ldots)$ is valid in Lukasiewicz 3-valued logic.

We show now that the difference between Lukasiewicz 3-valued logic and LPF is only with respect to the associated C.R.:

Proposition: Lukasiewicz 3-valued logic and LPF have the same expressive power.

Proof: We have seen already that Lukasiewicz implication is definable using $\neg, \wedge$ and $\supset$. for the converse something even stronger holds: $\supset$ is definable from $\rightarrow$ alone. In fact, we have:

$$a \supset b = a \rightarrow (a \rightarrow b)$$

It is worth to recall at this point that $\vee$ is also definable from $\rightarrow$ alone, since $a \vee b = (a \rightarrow b) \rightarrow b$. Hence the languages of $\{\neg, \rightarrow\}$ and that of LPF are equivalent.

We note, finally a quite remarkable property of $\vdash_{Luk}$:

Proposition: $\vdash_{Luk}$ is not closed under contraction. Hence it is not regular (note, however, that it is still closed under weakening).

Proof: We have, e.g., that $\neg A \wedge A, \neg A \wedge A \vdash_{WLuk} B$ is valid while $\neg A \wedge A \vdash_{WLuk} B$ is not.

The last proposition is reflected in the fact that $(A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B$ is not a theorem of Lukasiewicz logic. Note, however, that the example we gave is not connected with $\rightarrow$ at all, and applies also to $\vdash_{WLuk}$.

2.2 The "Inconsistent" Interpretation

In this section we investigate several C.R.s in which the meaning of $\perp$ is "both true and false", and so $\perp$ will be designated. The discussion will parallel that of the "unknown" case, and there will be a lot of similarities. We start it, as before, by introducing the basic associated C.R..

Definition: $\vdash_{Pac}$ is the C.R. defined by:

> $\Gamma \vdash_{Pac} \Delta$ iff every valuation v (in the basic three-valued structure) which assigned either T or $\perp$ to all the sentences of Γ does the same to at least one sentence of Δ.

Again it is obvious that $\vdash_{Pac}$ is an ordinary C.R. in which $\neg$ satisfies the symmetry conditions (but is not an internal negation). Another aspect in which $\vdash_{Pac}$ resembles $\vdash_{Kl}$ is that for $\vdash_{Pac}$ too there is exactly one possible way to define internal implication and combining conjunction (or disjunction) which satisfy the symmetry conditions. For the combining connectives exactly the same truth-tables do the job as before, with a very similar proof. We shall see, however, that for the implication a new truth-table will be needed.

We shall examine now the associated and derived logics.

2.2.1 The basic 3-valued paraconsistent logic

This logic is $\vdash_{Pac}$ in the language of the usual $\neg$ and $\wedge$. $\neg, \wedge$ and $\vee$ have in it exactly the same properties they have in Kleene's logic. On the other hand, unlike $\vdash_{Kl}$ (which has no logical theorems at all) $\vdash_{Pac}$ has a very distinguished set of logical theorems:

Proposition: $\vdash_{Pac} A$ iff A is a classical tautology.

Proof: One direction is trivial. For the converse, suppose that v is a 3-valued valuation. Let w be the two-valued valuation which assigns T to an atomic variable p iff $v(p)$ is designated. It is easy to prove by induction on the complexity of A that if $w(A) = T$ then $v(A) \in \{T, \perp\}$, and if $w(A) = F$ then $v(A) \in \{F, \perp\}$. It follows that if $w(A) = T$ for every two-valued valuation w then $v(A)$ is designated for every 3-valued v.

An alternative proof is to note that the classical equivalences which are used for reducing a sentence to its conjunctive normal form are valid in $\vdash_{Pac}$ in the strong sense that both sides of each equivalence always have the same truth-value. It is also easy to see that a sentence in such normal form is classicallly valid if it is valid in the present 3-valued semantics.

It is important to note that despite the last proposition classical logic and the basic $\vdash_{Pac}$ are *not* identical. In classical logic, e.g., contradictions entail everything. This is not the case for $\vdash_{Pac}$: in general $\neg A, A \not\vdash_{Pac} B$. This means that $\vdash_{Pac}$ is *paraconsistent* in the sense of [dC]. Moreover, the basic $\vdash_{Pac}$ have no logical contradictions : $A \vdash_{Pac}$ for no A . This entails immediately (since we have an internal conjunction in the language) that no definable internal negation is available. It is also possible to show that no internal implication is definable.

2.2.2 3-valued Paraconsistent Logic with Internal Implication

Like in the $\vdash_{Kl}$ case, our next goal is to enrich the language of $\vdash_{Pac}$ with an internal implication. Again, demanding also the symmetry conditions for this connective determines it completely:

- The condition $A, A \supset B \vdash_{Pac} B$ implies that $a \supset F = F$ if $a \in \{T, \perp\}$ (i.e., if a is designated).

- The conditions $B \vdash_{Pac} A \supset B$ and $\vdash_{Pac} A \supset A$ imply that $a \supset b$ is designated in all other cases.

- The conditions $\neg(A \supset B) \vdash_{Pac} A$ and $\neg(A \supset B) \vdash_{Pac} \neg B$ imply, respectively, that $F \supset a = T$ and $a \supset T = T$

- The condition $A, \neg B \vdash_{Pac} \neg(A \supset B)$ implies that if a is designated and $b =\perp$ then $a \supset b$ cannot be T. Since by the second fact it cannot be F either, it should be $\perp$.

The above facts leads us to a single candidate: the $\supset$ of the 3-valued paraconsistent logic of page 2. It is not difficult to show that this $\supset$ does really meet the requirements. The situation is therefore completely analogous to the one in the case of $\vdash_{Kl}$. This is clearly reflected also in the next proposition, which summerizes the main properties of $\vdash_{Pac}$ in the full language of the 3-valued paraconsistent logic:

Proposition: In the extended language for $\vdash_{Pac}$ we have:

1. $\neg$ satisfies the symmetry conditions (but again $A, \neg A \not\vdash_{Pac} B$).
2. $\wedge$ and $\vee$ are combining conjunction and disjunction, respectively. Both satisfy the symmetry conditions.
3. $\supset$ is an internal implication which satisfies the symmetry conditions.
4. The positive fragment of $\vdash_{Pac}$ is identical to the corresponding fragment of the classical, two-valued C.R..

It follows from the last proposition that $\vdash_{Pac}$ and $\vdash_{Kl}$ have quite similar properties concerning $\wedge, \vee, \supset$, and the differences are all connected with their negation connective!

2.2.3 RM_3 and Sobociński C.R..

Exactly like $\vdash_{Kl}$, $\vdash_{Pac}$ is not symmetrical, but all the needed symmetry conditions hold for it. Hence we can apply our general construction again to get the symmetrical versions of it in both the basic language and its extension with $\supset$. We shall denote these versions, respectively, by $\vdash_{WSob}$ and $\vdash_{ESob}$, and use $\vdash_{Sob}$ to denote either. The semantical characterzation this time (the proof of which we leave to the reader) is the following:

Proposition: $\Gamma \vdash_{Sob} \Delta$ iff for every assignment, either one of the sentences in Γ gets F, or one of the sentences in Δ gets T, or the sequent is not empty and *all* its sentences get $\perp$.

$\vdash_{Sob}$ has the same basic properties of $\vdash_{Luk}$ which were described in the second proposition of **2.1.3**, and its internal implication was again known and used before:

Proposition: The internal implication of $\vdash_{ESob}$, defined as usual by

$$A \to B =_{Df} (A \supset B) \wedge (\neg B \supset \neg A)$$

is identical to the $\to$ of RM_3 (i.e., it is Sobociński 3-valued implication).

Corollary: $A_1, \ldots, A_n \vdash_{Sob} B$ iff $A_1 \to (A_2 \to \ldots \to (A_n \to B)\ldots)$ is a theorem of RM_3.

Proposition: The languages of RM_3 and $\vdash_{Pac}$ have the same expressive power.

Proof: It is enough to note that $\supset$ is definable in RM_3 by:

$$a \supset b = b \vee (a \to b)$$

The most remarkable property of $\vdash_{Sob}$, and the main aspect in which it differs from $\vdash_{Luk}$ is given in the following

Proposition: $\vdash_{Sob}$ is a regular C.R. but it is not ordinary: Weakening fails for it.

The last proposition entails that $A \to (B \to A)$ is not a theorem of RM_3. This is a characteristic feature of a *Relevance* logic. RM_3 is indeed the strongest logic in the family of logics which were created by the relevantists school (see [AB] and [Du]).

3 Merging The Two Interpretations

In this section we investigate C.R.s which are based on a *four*-valued structure, in which both the "undefined" and "inconsistent" interpretations of $\bot$ have a counterpart.

Definition: The lattice $KB4$ consists of the four elements $T, F, \bot_N, \bot_B$, together with the order relation $\leq$ defined by the following diagram:

$$T$$

$$\bot_N \qquad \bot_B$$

$$F$$

(i.e: $F \leq \bot_N, \bot_B \leq T$).

We define the operations $\neg, \vee, \wedge$ on $KB4$ as follows: $\vee$ and $\wedge$ are the usual lattice operations. $\neg T = F$, $\neg F = T$, $\neg \bot_N = \bot_N$, $\neg \bot_B = \bot_B$.

Historically a structure which closely resembles $KB4$ [7] was first introduced in order to characterize the valid relevant first-degree entailments (f.d.e.). These are the theorems of the usual relevant logics (R and E - see [AB] , [Du]) which have the form $A \rightarrow B$ where $\rightarrow$ (the "relevant implication") occures in neither A nor B (i.e the only connectives occuring in A or in B are $\neg, \vee, \wedge$). The characterization is given in the following:

Fact: A f.d.e. $A \rightarrow B$ is provable in the relevance systems R and E iff $v(A) \leq v(B)$ for every valuation v in $KB4$.

In [Be1] and [Be2] Belnap suggests the use of this 4-valued structure for knowledge bases. He describes there the intuitive meaning of the four truth values relative to some knowledge base as follows:

1. A proposition A has the truth value T if A can be deduced from the knowledge base, but $\neg A$ cannot (i.e. A is "true only" according to the knowledge base).

2. A proposition A has the truth value F if $\neg A$ can be deduced from the knowledge base but A cannot (i.e. A is "false only").

3. A proposition A has the truth value $\bot_N$ if neither A nor $\neg A$ can be deduced from the knowledge base.

4. A proposition A has the valus $\bot_B$ if both A and $\neg A$ can be deduced from the knowledge base.

Obviously, $\bot_N$ corresponds to kleene's $\bot$ while $\bot_B$ corresponds to that of $\vdash_{Pac}$. We take, accordingly, T and $\bot_B$ as designated, and define the corresponding C.R. in the obvious way:

Definition: $\Gamma \vdash_{Be} \Delta$ iff every valuation which makes all the sentences in Γ true (i.e. assigns to them either T or $\bot_B$) makes at least one of the sentences in Δ true.

$\vdash_{Be}$ has the familiar properties of $\vdash_{Kl}$ and $\vdash_{Pac}$: it is ordinary. $\neg$ is not internal negation for it but it satisfies the symmetry conditions. $\wedge$ and $\vee$ are combining conjunction and disjunction for it which satisfies the symmetry conditions, and they are the only possible connectives with these properties (proof — as usual). Like $\vdash_{Kl}$, $\vdash_{Be}$ has no logical theorems, and like $\vdash_{Pac}$ it is paraconsistent. As for the existence of a well-behaved internal implication and the strength of the positive fragment the situation is exactly like in the 3-valued

[7] But in which only T is taken as designated and $\rightarrow$ is differently defined.

fragments, with similar proofs:

Proposition: There is exactly one possible way to define an operation $\supset$ on $KB4$ so that the symmetry conditions for it obtain. It is characterized by the following two principles:

- If a is not designated (i.e. $a = \perp_N, F$) then $a \supset b = T$.
- If a is designated (i.e. $a = \perp_B, T$) then $a \supset b = b$.

Proposition: The positive fragment of $\vdash_{Be}$ in the language with $\supset$ is identical to the classical one.

The proofs of both propositions are very similar to those given in the previous cases, and we leave them to the reader.

Our next step is to introduce, by the usual method, $\vdash^S_{Be}$— the symmetrical version of $\vdash_{Be}$. For this C.R. both weakening and contraction fail. This, and the fact that it has all the standard internal and combining connectives, makes it a very close relative of the *Linear* C.R. [8]. Accordingly, the internal implication of $\vdash^S_{Be}$, defined as usual, has a lot in common with the relevant implication of the Relevance Logic R, and even more— with the linear implication of Girard.[9].

As in the previous case, the two implications, $\supset$ and $\rightarrow$, are equivalent as far as expressive power goes. $A \rightarrow B$ is equivalent, as usual, to $(A \supset B) \wedge (\neg B \supset \neg A)$. $A \supset B$, on the other hand is equivalent this time to $B \vee (A \rightarrow (A \rightarrow B))$.

Another connection between $\rightarrow$ and $\supset$ is provided by the following:

Proposition: If A and B are in the basic language (of $\neg, \wedge$ and $\vee$) then $A \supset B$ is valid iff $A \rightarrow B$ is.

Proof: One direction is trivial. For the other we can easily prove by induction on complexity of sequents that if $A_1, \ldots, A_n, B_1, \ldots, B_m$ are all in the basic language then if $A_1, \ldots, A_n \vdash_{Be} B_1, \ldots, B_m$ then also $\neg B_1, \ldots, \neg B_m \vdash_{Be} \neg A_1, \ldots, \neg A_n$. It follows immediately that if $A \supset B$ is valid then so is $A \rightarrow B$.

Remark: It is quite easy to see that the sentences dealt with in the last proposition are exactly the valid f.d.e. of the usual relevance logics.

[8]Linear Logic was introduced in [Gi]. Its C.R. is characterized in [Av1]. Its connections with Relevance Logic are explained in [Av2].

[9]One difference is that for $\vdash^S_{Be}$ the converse of contraction is valid, while for Linear Logic and the standard Relevance logics it is not. *RM* is the most famous exception in this respect. *RMI* of [Av4] is another.

4 Proof Theory of The Ordinary C.R.s

4.1 Gentzen-type Systems

In this section we provide Gentzen-type systems for the various *ordinary* C.R.s we introduce above. They all are based on the basic system of section 1.1. This system itself corresponds to the 4-valued C.R. of the last section.

Theorem:

1. $\Gamma \vdash_{Be} \Delta$ iff $\Gamma \vdash \Delta$ is provable in the basic system.
2. By adding $A, \neg A \vdash$ to the basic system we get a Gentzen-type formulation for $\vdash_{Kl}$.
3. By adding $\vdash A, \neg A$ to the basic system we get a Gentzen-type formulation for $\vdash_{Pac}$.
4. By adding both $\vdash A, \neg A$ and $A, \neg A \vdash$ to the basic system we get a Gentzen-type formulation for classical logic.

Proof: For this proof we replace first, in the usual way, each of the pairs of rules for $(\wedge \vdash), (\neg\vee \vdash), (\vdash \vee)$ and$(\vdash \neg\wedge)$ by a single rule (the possibility of doing so is due to the soundness of weakening and contraction). The rules of the resulting system(s) are all easily seen to be reversible for all the C.R.s under consideration. It follows that for any given sequent we can construct a finite set of sequents, consisting only of atomic formulas or their negations, so that the given sequent is valid iff all the sequents in the corresponding set are, and provable iff all of them are. It remains to check that a sequent of this form is valid in one of the above C.R.s iff it is provable in the corresponding system. This is easy.

The above theorem is true for both the basic and the extended versions of the C.R.s under consideration (i.e., with or without $\supset$) and for any of their fragments. It is worthwhile to note, however, that in $\vdash_{Be}$ and $\vdash_{Kl}$ the Gentzen-type system for the basic langauage has an important property which that for the extended language lack:

Theorem: Any sequent $\Gamma \vdash \Delta$ of the basic language, which is provable in the calculus for $\vdash_{Kl}$ or that for $\vdash_{Be}$ and in which Δ has at most one formula, has a cut-free proof consisting of sequents with the same property.

Proof: This time we replace only the pairs for $(\wedge \vdash)$ and of $(\neg\vee \vdash)$ by a single rule. In what follows we shall call *single-conclusioned* any proof in the resulting systems in which all the occuring sequents have at most one formula on the r.h.s of the $\vdash$. "Provable" will mean provable in either the basic system or in its extension to $\vdash_{Kl}$.

Lemma 1. The r.h.s of any sequent which occures in a proof of a sequent of the form $\Gamma \vdash$ is empty.

Lemma 2. Suppose that Γ consists only of atomic formulas or their negation, $|\Delta| \geq 1$ and $\Gamma \vdash \Delta$. Then there exists a formula $B \in \Delta$ such that $\Gamma \vdash B$ has a single-conclusioned proof.

The proofs of both lemmas is by an easy induction on the length of cut-free proofs of the given sequents (Lemma 1 is needed in the proof of Lemma 2).

Corollary: If Γ is like in Lemma 2, $|\Delta| \leq 1$ and $\Gamma \vdash \Delta$ is provable then this sequent has a single-conclusioned proof.

Lemma 3. For any Γ there exists sets Γ_i $(i = 1 \ldots n)$ such that:

1. For every Δ, $\Gamma \vdash \Delta$ is provable iff for every i $\Gamma_i \vdash \Delta$ is.
2. For every i, Γ_i consists of atomic formulas or negations of such.
3. There is a cut-free proof of $\Gamma \vdash \Delta$ from $\Gamma_i \vdash \Delta$ in which Δ is the r.h.s of all the sequents involved.

The proof of Lemma 3 is by induction on the complexity of Γ, using the fact that all the r.h.s. rules of the system above are reversible and the active formulas involved in them belong to the r.h.s of the premises. The theorem itself is an easy consequenc of lemma 3 and the corollary above.

The last theorem shows that the single-conclusioned fragment of these formal systems is completely independent of its multiple-conclusioned version. This is in sharp contrast to the known Gentzen-type formulations of classical logic, in which the restriction to at most one formula on the r.h.s of a sequent leads to intuitionistic logic. Thus although all three logics have internal disjunction ($\vee$ in the case of $\vdash_{Be}$ and $\vdash_{Kl}$) and so can be reduced to their single-conclusioned counterparts, only in $\vdash_{Be}$ and $\vdash_{Kl}$ this can be done *within the formal system.* Hence, unlike classical logic, we can take these C.R.s to be *essentially* single-conclusioned. It should be emphsized again, that the above theorem is false once $\supset$ is introduced. Any proof of $\vdash A \vee (A \supset B)$, for example, necessarily involves sequents in which the l.h.s. contains more than one formula.

4.2 Hilbert-type formulations

The system HBe

Defined connective: $A \equiv B =_{Df} (A \supset B) \wedge (B \supset A)$

Axioms: I1 $\quad A \supset B \supset A$

I2 $(A \supset B \supset C) \supset (A \supset B) \supset (A \supset C)$

I3 $((A \supset B) \supset A) \supset A$

C1 $A \wedge B \supset A$

C2 $A \wedge B \supset B$

C3 $A \supset B \supset A \wedge B$

D1 $A \supset A \vee B$

D2 $B \supset A \vee B$

D3 $(A \supset C) \supset (B \supset C) \supset A \vee B \supset C$

N1 $\neg(A \vee B) \equiv \neg A \wedge \neg B$

N2 $\neg(A \wedge B) \equiv \neg A \vee \neg B$

N3 $\neg\neg A \equiv A$

N4 $\neg(A \supset B) \equiv A \wedge \neg B$

Rule of Inference:

$$\frac{A \qquad A \supset B}{B}$$

Note: The first nine axioms provide a standard axiomatization of classical positive logic.

Theorem: $A_1, \ldots, A_n \vdash_{Be} B_1, \ldots, B_n$ iff $A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$ is a theorem of HBe.

Proof: It is a standard matter to show the equivalence of the Gentzen-type system for $\vdash_{Be}$ and HBe.

Theorems on extensions:

1. If we add either $\neg A \vee A$ or $(A \supset B) \supset (\neg A \supset B) \supset B$ to HBe we get a sound and complete Hilbert-type axiomatization of $\vdash_{Pac}$.

2. If we add either $\neg A \supset (A \supset B)$ or $(B \supset A) \supset (B \supset \neg A) \supset \neg B$ to HBe we get a sound and complete Hilbert-type axiomatization of $\vdash_{Kl}$.

3. By adding both $\neg A \vee A$ and $\neg A \supset (A \supset B)$ (say) to HBe we get classical logic.

We use now the Hilbert-type formulations for proving the following important theorem:

Compactness theorem: Let T,S be sets of sentences such that every valuation in $KB4$ which makes all the sentences in T true does the same to at least one sentence of S. Then there exist *finite* sets $\Gamma \subseteq \mathrm{T}, \Delta \subseteq \mathrm{S}$ such that $\Gamma \vdash_{Be} \Delta$. Similar results obtain for $\vdash_{Kl}, \vdash_{Pac}$ and classical logic.

Proof: We give the proof in the case of $\vdash_{Be}$. The other cases need only minor modifications.

Suppose that T and S are sets of sentences such that no such Γ and Δ exist. We may assume, without a loss in generality, that S is closed under finite disjunctions. It easily follows from the completeness theorem that $\mathrm{T} \nvdash_{HBe} A$ for every A in S. Let T_0 be a maximal extension of T with this property. We construct now a valuation v such that $v(B)$ is designated iff $\mathrm{T}_0 \vdash_{HBe} B$. The theorem will follow then immediately.

By the deduction theorem (which obviously obtains here) and the maximality of T_0, $\mathrm{T}_0 \nvdash_{HBe} C$ iff $\mathrm{T}_0 \vdash_{HBe} C \supset A$ for some A in S. Moreover, we have:

(†) $\mathrm{T}_0 \nvdash_{HBe} C$ iff $\mathrm{T}_0 \vdash_{HBe} C \supset B$ for every B.

Indeed, suppose There exist C and B such that both C and $C \supset B$ are not provable in T_0 . Then there exist sentences D and E in S such that $\mathrm{T}_0 \vdash_{HBe} C \supset E$ and $\mathrm{T}_0 \vdash_{HBe} (C \supset B) \supset D$. The two last sentence entail $E \vee D$ by a classical *positive* tautology. Every positive tautology is provable also in HBe, and so we get that $\mathrm{T}_0 \vdash_{HBe} E \vee D$. This last sentence, however, is in S. A contradiction.

Define now a valuation v as follows:

If $\mathrm{T}_0 \vdash_{HBe} C$, $\mathrm{T}_0 \nvdash_{HBe} \neg C$ then $v(C) = T$.
If $\mathrm{T}_0 \vdash_{HBe} C$, $\mathrm{T}_0 \vdash_{HBe} \neg C$ then $v(C) = \perp_B$.
If $\mathrm{T}_0 \nvdash_{HBe} C$, $\mathrm{T}_0 \vdash_{HBe} \neg C$ then $v(C) = F$.
If $\mathrm{T}_0 \nvdash_{HBe} C$, $\mathrm{T}_0 \nvdash_{HBe} \neg C$ then $v(C) = \perp_N$.

Obviously $v(B)$ is designated iff $\mathrm{T}_0 \vdash_{HBe} B$. It remains to show that v is actually a valuation, i.e. : that it respects the operations. Now axiom N3 insures that for every A, $v(\neg A) = \neg v(A)$. We show next that $v(A \supset B) = v(A) \supset v(B)$ for every A and B.

case 1: Suppose $v(A) \in \{\perp_N, F\}$. This means that $\mathrm{T}_0 \nvdash_{HBe} A$. By (†) above and axiom N4 we have therefore that for every B, $\mathrm{T}_0 \vdash_{HBe} A \supset B$ but $\mathrm{T}_0 \nvdash_{HBe} \neg(A \supset B)$. Hence $v(A \supset B) = T$.

case 2: Suppose that $v(A) \in \{\perp_B, T\}$, i.e.: $\mathrm{T}_0 \vdash_{HBe} A$. In this case $\mathrm{T}_0 \vdash_{HBe} A \supset B$ iff $\mathrm{T}_0 \vdash_{HBe} B$, and by N4 also $\mathrm{T}_0 \vdash_{HBe} \neg(A \supset B)$ iff $\mathrm{T}_0 \vdash_{HBe} \neg B$. It follows that $v(A \supset B) = v(B)$.

We show now that $v(A \vee B) = v(A) \vee v(B)$. For this we note that D1–D3, together with the characterization above of the non-theorems of T_0, imply that $T_0 \vdash_{HBe} A \vee B$ iff either $T_0 \vdash_{HBe} A$ or $T_0 \vdash_{HBe} B$. On the other hand N1 entails that $T_0 \vdash_{HBe} \neg(A \vee B)$ iff both $T_0 \vdash_{HBe} \neg A$ and $T_0 \vdash_{HBe} \neg B$. From these facts the desired equation easily follows.

Similarly, by the above property of disjunction and N2 we have that $T_0 \vdash_{HBe} \neg(A \wedge B)$ if either $T_0 \vdash_{HBe} \neg A$ or $T_0 \vdash_{HBe} \neg B$. C1–C3 on the other hand, insure that $T_0 \vdash_{HBe} A \wedge B$ if both $T_0 \vdash_{HBe} A$ and $T_0 \vdash_{HBe} B$. These facts entail that $v(A \wedge B) = v(A) \wedge v(B)$.

5 Proof-theory of Lukasiewicz 3-valued Logic

5.1 A Hilbert-type formulation

A Hilbert-type formulation of Lukasiewicz 3-valued logic was first given in [Wa]. An axiomatization of the implicational fragment of this logic was provided in [MM]. The completeness proofs given in both cases are quite complicated, though. For the sake of completeness, and since we shall need the Hilbert-type formulations later, I include here a new formulation and a completeness proof for it which is much simpler than any other I was able to find in the literature. I took a special care to provide a *well-axiomatization*. This means that any fragment of the logic which contains $\rightarrow$ is completely axiomatized by those axioms below which mention just the connectives of that fragment. This includes the implicational fragment itself. I believe, by the way, that the present axiomatization is simpler and more transparent than the one given in [MM]. It is certainly shorter, since axiom I3 below is easily seen to be derivable from I1,I2 and I4 (we still prefer to include it as an axiom, since together with I1-I2 it provides a very natural subsystem).

5.1.1 The system HLuk

Axioms:

I1 $A \rightarrow B \rightarrow A$

I2 $(A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$

I3 $(A \rightarrow B \rightarrow C) \rightarrow B \rightarrow A \rightarrow C$

I4 $((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow A) \rightarrow A$

I5 $((((A \rightarrow B) \rightarrow A) \rightarrow A) \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow C)$

C1 $A \wedge B \rightarrow A$

C2 $A \wedge B \rightarrow B$

C3 $(A \rightarrow B) \rightarrow (A \rightarrow C) \rightarrow A \rightarrow B \wedge C$

D1 $A \rightarrow A \vee B$

D2 $B \rightarrow A \vee B$

D3 $(A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow A \vee B \rightarrow C$

N1 $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

Rule of Inference:

$$\frac{A \quad A \rightarrow B}{B}$$

Theorem: HLuk is sound and complete for Lukasiewicz 3-valued logic. Moreover, T $\vdash_{HLuk} A$ iff $v(A) = T$ for any valuation v which assigns T to all the sentences in T.

Notes:

1. $\vdash_{HLuk}$ corresponds to (the single-conclusioned fragment of) $\vdash_{Kl}$, not to that of $\vdash_{Luk}$. Thus $A \rightarrow A \rightarrow B, A \vdash_{HLuk} B$ though $A \rightarrow A \rightarrow B, A \not\vdash_{Luk} B$. Recall, however, that the two C.R.s have the same logical theorems!

2. It is a standard task to show that a sentence is derivable from I1–I3 alone (using M.P.) iff it has a proof *without contraction* in the intuitionistic Gentzen-type implicational calculus. Since the last criterion is very easy to apply, we shall feel free below to claim derivability using I1–I3 without giving the formal derivation.

3. Since $(A \rightarrow B) \rightarrow B$ is equivalent to $A \vee B$, Axioms I4 and I5 are just purely implicational formulations of, respectively, the more perspicuous propositions $A \vee B \rightarrow B \vee A$ and $A \vee (A \rightarrow B) \vee (B \rightarrow C)$.

Proof of the theorem: The soundness part is easy. The completeness is a special case of the second claim. It remains to show that if T$\not\vdash_{HLuk} \phi$ then there exists a valuation which assigns T to all the sentences in T but not to ϕ. Let T_0 be a maximal extension of T such that $\text{T}_0 \not\vdash_{HLuk} \phi$. The main property of T_0 is:

$$\text{T}_0 \not\vdash_{HLuk} A \quad \text{iff} \quad \text{T}_0, A \vdash_{HLuk} \phi$$

Define now:

$$v(A) = \begin{cases} T & \text{T}_0 \vdash_{HLuk} A \\ F & \text{T}_0 \vdash_{HLuk} A \rightarrow B \text{ for every } B \\ \bot & \text{otherwise} \end{cases}$$

Obviously $v(A) = T$ for every A in T while $v(\phi) \neq T$. It remains therefore to show that v is really a valuation, i.e., it respects the operations. For this we need first some lemmas.
Lemma 1: If $\mathrm{T}_0, A \vdash_{HLuk} B$ then $\mathrm{T}_0, (A \to C) \to C \vdash_{HLuk} (B \to C) \to C$.
Proof: By induction on the length of the proof of B from T_0, A. If B is in T_0 or it is an axiom then $\mathrm{T}_0 \vdash_{HLuk} (B \to C) \to C$ since $(B \to C) \to C$ is derivable from B using only I1–I3. The case $B = A$ is trivial. Finally, the induction step follows from the fact that $(B \to C) \to C$ is derivable from $((D \to B) \to C) \to C$ and $(D \to C) \to C$ using only I1–I3.
Lemma 2: If $\mathrm{T}_0, A \vdash_{HLuk} C$ and $\mathrm{T}_0, B \vdash_{HLuk} C$ then $\mathrm{T}_0, (A \to B) \to B \vdash_{HLuk} C$.
Proof: By Lemma 1 we have in this case that $\mathrm{T}_0, (A \to B) \to B \vdash_{HLuk} (C \to B) \to B$ and $\mathrm{T}_0, (B \to C) \to C \vdash_{HLuk} (C \to C) \to C$. Lemma 2 follows from these facts with the help of axiom I4 and the fact that $C \to C$ is derivable from I1–I3.
Lemma 3: If $\mathrm{T}_0 \vdash_{HLuk} (A \to B) \to B$ then either $\mathrm{T}_0 \vdash_{HLuk} A$ or $\mathrm{T}_0 \vdash_{HLuk} B$.
Proof: Immediate from Lemma 2 and the above main property of T_0.
Lemma 4: For every A and B, either $v(A) = T$ or $v(B) = F$ or $v(A \to B) = T$.
Proof: Applying Lemma 3 to axiom I5 we get that either $\mathrm{T}_0 \vdash_{HLuk} B \to C$ for every C or $\mathrm{T}_0 \vdash_{HLuk} ((A \to B) \to A) \to A$. In the first case $v(B) = F$ by definition of v. In the second case $v(A) = T$ or $v(A \to B) = T$ by another application of Lemma 3.
Lemma 5: For every A and B, either $v(A \to B) = T$ or $v(B \to A) = T$.
Proof: This follows from Lemma 4, the definition of v and axiom I1.
Lemma 6: If $v(A \to B) = v(B \to A) = T$ then $v(A) = v(B)$.
Proof: By definition of v and axiom I2.

We are ready now to prove that v respects the various operations:

$v(A \to B) = v(A) \to v(B)$.

1. If $v(A) = F$ then $v(A \to B) = T$ by definition of v.
2. If $v(A) = T$ then $(A \to B) \to B$ and $B \to (A \to B)$ are both theorems of T_0, by I1–I3. Hence $v(A \to B) = v(B)$ in this case, by lemma 6.
3. If $v(B) = T$ then $v(A \to B) = T$ by axiom I1.
4. If $v(A) = v(B) = \perp$ then $v(A \to B) = T$ by Lemma 4.
5. Suppose $v(A) = \perp$ and $v(B) = F$. Then there exists D such that $\mathrm{T}_0 \not\vdash_{HLuk} A \to D$, while $\mathrm{T}_0 \vdash_{HLuk} B \to D$. Hence, by I2, $\mathrm{T}_0 \not\vdash_{HLuk} A \to B$ and so $v(A \to B) \neq T$. Since neither A nor B are theorems of T_0, it follows by Lemma 3 that $\mathrm{T}_0 \not\vdash_{HLuk} (A \to B) \to B$ and so $v(A \to B) \neq F$. Hence $v(A \to B) = \perp$ in this case.

These 5 cases cover all possibilities and in each of them we got the required value for $v(A \to B)$. Hence the equation.

$v(A \vee B) = v(A) \vee v(B)$.

1. Axioms D1–D3, I2 and the definition of v immediately imply that $v(A \vee B) = F$ iff both $v(A) = F$ and $v(B) = F$.
2. D1 and D2 implies that if either $v(A) = T$ or $v(B) = T$ then $v(A \vee B) = T$. Conversely, if $v(A \vee B) = T$ then Lemma 3 implies that either $v(A) = T$ or $v(B) = T$, since it is easy to show that $A \vee B \to (A \to B) \to B$ is derivable from D3 using I1-I3.

These two facts suffice for establishing the required equation.

$v(A \wedge B) = v(A) \wedge v(B)$.

1. From C1–C3 follows easily that $v(A \wedge B) = T$ iff both $v(A) = T$ and $v(B) = T$.
2. C1 and C2 imply that if either $v(A) = F$ or $v(B) = F$ then $v(A \wedge B) = F$. The converse follows from the fact that $v(A \wedge B)$ is always equal to either $v(A)$ or to $v(B)$. Indeed, if $v(A \to B) = T$ then C3, C1 and Lemma 6 entail that $v(A) = v(A \wedge B)$. Similarly, if $v(B \to A) = T$ then $v(B) = v(A \wedge B)$. The claimed fact follows, therefore, from Lemma 5.

$v(\neg A) = \neg v(A)$.

1. I1–I3 and N1 entail that $A \vdash_{HLuk} \neg A \to B$. It follows that if $v(A) = T$ then $v(\neg A) = F$. Conversely, if $v(\neg A) = F$ then $\mathrm{T}_0 \vdash_{HLuk} \neg A \to \neg(A \to A)$, and so by N1 and the provability of $A \to A$, $v(A) = T$.
2. I1–I3 and N1 entail also that $\neg\neg A \to A$ is a theorem. It follows that if $v(A) = F$ then $\mathrm{T}_0 \vdash_{HLuk} \neg\neg A \to B$ for every B. In particular. $\mathrm{T}_0 \vdash_{HLuk} \neg\neg A \to \neg(A \to A)$ and so, by N1 and the provability of $A \to A$, $v(\neg A) = T$ in this case. Conversely, if $v(\neg A) = T$ then $\mathrm{T}_0 \vdash_{HLuk} A \to B$ for every B and so $v(A) = F$.

Again these two facts suffice for establishing the required equation.

Note that in each case in the proof above we have used only the axioms concerning $\to$ and the connective under discussion. Hence the above system is indeed well axiomatized.

5.2 A Gentzen-type formulation

As was emphasized above the structural rule of contraction is not valid for $\vdash_{Luk}$. A natural first attempt to construct a Gentzen-type formalism for it would be, therefore, to delete this rule from (an appropriate version of) the corresponding classical system. The resulting formalism is equivalent to the Hilbert-type system which is obtained from HLuk above by dropping I4 and I5. To capture the whole system we need to employ a calculus of *Hypersequents* [10]. We start by recalling the definition of a Hypersequent in [Av5]:

Definition: Let L be a language. A hypersequent is a creature of the form:

$$\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2 | \ldots | \Gamma_n \Rightarrow \Delta_n$$

where Γ_i, Δ_i are finite sequences of formulas of L. The $\Gamma_i \Rightarrow \Delta_i$ -s will be called the *components* of the hypersequent. We shall use G, H as metavariables for (possibly empty, i.e., without components) hypersequents.

The intended semantics of hypersequents is given in the following natural generalization of the semantics of $\vdash_{Luk}$:

Definition: A hypersequent G is $\vdash_{Luk}$-valid if for every valuation v, there is a component of G which contains either a formula on its r.h.s. which gets T (under v), or a formula on its l.h.s. which gets F, or two different occurances of formulas which get $\bot$.

We next provide a corresponding (generalized) Gentzen-type formalism.

5.2.1 The system Gluk

Axioms:

$$A \Rightarrow A$$

External structural rules:

EW (External Weakening):

$$\frac{G}{G|H}$$

EC (External Contraction):

$$\frac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}$$

[10]Such a calculus was first introduce in [Pot] for the modal S5, and independently in [Av5] for the semi-relevant RM.

EP **(External Permutation):**

$$\frac{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2 \Rightarrow \Delta_2|H}{G|\Gamma_2 \Rightarrow \Delta_2|\Gamma_1 \Rightarrow \Delta_1|H}$$

Internal structural rules:

IW **(Internal Weakening):**

$$\frac{G|\Gamma \Rightarrow \Delta}{G|A, \Gamma \Rightarrow \Delta} \qquad \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta, A}$$

IP **(Internal Permutation):**

$$\frac{G|\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{G|\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta} \qquad \frac{G|\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{G|\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}$$

M **(Merging):**

$$\frac{G|\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3 \qquad G|\Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1', \Delta_2', \Delta_3'}{G|\Gamma_1, \Gamma_1' \Rightarrow \Delta_1, \Delta_1'|\Gamma_2, \Gamma_2' \Rightarrow \Delta_2, \Delta_2'|\Gamma_3, \Gamma_3' \Rightarrow \Delta_3, \Delta_3'}$$

Logical Rules:

$$\frac{G|\Gamma, A \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta, \neg A} \quad \frac{G|\Gamma \Rightarrow \Delta, A}{G|\neg A, \Gamma \Rightarrow \Delta}$$

$$\frac{G|\Gamma, A \Rightarrow \Delta \quad G|\Gamma, B \Rightarrow \Delta}{G|\Gamma, A \vee B \Rightarrow \Delta} \quad \frac{G|\Gamma \Rightarrow \Delta, A}{G|\Gamma \Rightarrow \Delta, A \vee B} \quad \frac{G|\Gamma \Rightarrow \Delta, B}{G|\Gamma \Rightarrow \Delta, A \vee B}$$

$$\frac{G|\Gamma, A \Rightarrow \Delta}{G|\Gamma, A \wedge B \Rightarrow \Delta} \quad \frac{G|\Gamma, B \Rightarrow \Delta}{G|\Gamma, A \wedge B \Rightarrow \Delta} \quad \frac{G|\Gamma \Rightarrow \Delta, A \quad G|\Gamma \Rightarrow \Delta, B}{G|\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\frac{G|\Gamma_1 \Rightarrow \Delta_1, A \quad G|B, \Gamma_2 \Rightarrow \Delta_2}{G|\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2} \quad \frac{G|\Gamma, A \Rightarrow \Delta, B}{G|\Gamma \Rightarrow \Delta, A \rightarrow B}.$$

Note: We shall assume below that an empty component of a hypersequent is automatically omitted, unless it is the unique component of the hypersequent. This involves no loss of generality, since the effect of this convention can always be achieved by using internal weakenings and an external contraction (more generally it is always possible to derive $G|\Gamma, \Gamma_1, \Gamma_2 \Rightarrow \Delta, \Delta_1, \Delta_2$ from $G|\Gamma, \Gamma_1 \Rightarrow \Delta, \Delta_1|\Gamma, \Gamma_2 \Rightarrow \Delta, \Delta_2$). In practice this convention allows us, e.g., to have less components in the conclusion of rule M then its formulation suggests.

Soundness Theorem: Every Hypersequent which is derivable in Gluk is valid.

Proof: By checking that every rule leads from valid hypersequents to a valid hypersequent. The only non-standard case is rule M. Given a valuation v, there ia again only one intersting case to note: when both $\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3$ and $\Gamma_1', \Gamma_2', \Gamma_3' \Rightarrow \Delta_1', \Delta_2', \Delta_3'$ from the premises of the rule have two occurances of formulas which get $\perp$ under v. In this case, however, the pigeon-hole principle entails that one of the components of the conclusion will have this property as well.

We turn now to the problem of the completeness of Gluk. It turns out that showing completeness is relatively easy if we deal only with the $\{\neg, \wedge, \vee\}$ fragment. We start therfore with this fragment. We shall later attack the full system case with a completely different method.

5.2.2 Completeness of the fragment without implication

We start with the case of hypersequents which include only atomic formulas.

Definition: An atomic formula will be called *special* for a hypersequent G if it occures on the r.h.s. of some component of G and also on the l.h.s. of another (not necessarily distinct) such component.

Lemma: Let G be a hypersequent which contains only atomic formulas. The following conditions are equivalent:

1. G is provable in Gluk.
2. G is valid
3. Some component of G contains two occurances of (not necessarily distinct) special formulas for G.

Proof:

$(1) \Rightarrow (2)$: This follows from the soundness theorem.

$(2) \Rightarrow (3)$: Suppose that no component of G includes two occurances of special formulas. Define v as follows: $v(P) = \perp$ if P is special, $v(P) = T$ if P occures on the r.h.s. of some component of G but is not special, $v(P) = F$ otherwise. It is easy to see that none of the conditions for validity applies to G with respect to v, and so G is not valid.

$(3) \Rightarrow (1)$: Let P and Q be the (perhapse identical) special formulas for G which have (together) two occurances in some component of G. By applying rule

M to $P \Rightarrow P$ and $Q \Rightarrow Q$ in the appropriate way we can get a hypersequent from which G is derivable by using internal and external weakenings.

The Completeness Proof for the fragment without implication: We present in what follows a set of reduction steps. Each of these step produces from any hypersequent G to which it is applicable one or two hypersequents with the following properties:

1. They are valid if G is.
2. G is easily derivable from them.
3. The number of connectives occuring in each of them is less than the number of connectives in G.

Since at least one of the reduction steps will be applicable to any hypersequent (in the language under discussion) which contain a non-atomic formula, these reduction steps together with the previous Lemma suffice for establishing the desired completeness result.

The reduction steps are:

- Reduce $G|\Gamma \Rightarrow \Delta, \neg A$ to $G|\Gamma, A \Rightarrow \Delta$.
- Reduce $G|\Gamma, \neg A \Rightarrow \Delta$ to $G|\Gamma \Rightarrow \Delta, A$.
- Reduce $G|\Gamma \Rightarrow \Delta, A \wedge B$ to $G|\Gamma \Rightarrow \Delta, A$ and $G|\Gamma \Rightarrow \Delta, B$.
- Reduce $G|\Gamma, A \wedge B \Rightarrow \Delta$ to $G|\Gamma, A \Rightarrow \Delta|\Gamma, B \Rightarrow \Delta$.
- Reduce $G|\Gamma \Rightarrow \Delta, A \vee B$ to $G|\Gamma \Rightarrow \Delta, A|\Gamma \Rightarrow \Delta, B$.
- Reduce $G|\Gamma, A \vee B \Rightarrow \Delta$ to $G|\Gamma, A \Rightarrow \Delta$ and $G|\Gamma, B \Rightarrow \Delta$.

5.2.3 Completeness of the full system

The method of proof we use for the previous fragment does not work for the full system since no reduction step seems to be available for formulas of the form $A \rightarrow B$ which occure on a l.h.s. of a component. Here the lack of internal contraction is crucial. Because of it we cannot just assume the same side formulas in both premises of the corresponding rule. Instead of a direct reduction we shall rely here on the completeness of the Hilbert-type system which was proved in the previous subsection. First we need, however, the following theorem (which is of a major importance on its own right):

The Cut-elimination Theorem If $G|\Gamma_1 \Rightarrow \Delta_1, A$ and $G|A, \Gamma_2 \Rightarrow \Delta_2$ are both derivable in Gluk then so is also $G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

The proof of this theorem uses the "history" technique of [Av5]. Like in the case of the hypersequential formulation of *RM* which was investigated there, external contraction is the source of the main difficulties. The proof, however, closely follows that in [Av5] and since it is rather tedious we shall not repeat it here (the lack of the *internal* contraction rule somewhat simplifies the proof in the present case, though).

Note: In the case of the fragment without implication cut-elimination is an easy corollary of the completeness result we prove above.

Definition: Let G be a hypersequent, which is not the empty sequent. We define its translation, ϕ_G as follows:

- if G is of the form $A_1, \ldots, A_n \Rightarrow B$ the ϕ_G is $A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow B) \ldots)$.
- If G has a single nonempty component then ϕ_G is any translation of one of its single-conclusioned versions (recall that $\vdash_{Luk}$ is symmetric!).
- If G has the form $S_1|S_2|\ldots|S_n$, where the S_i's are ordinary sequents then ϕ_G is $\phi_{S_1} \vee \phi_{S_2} \vee \ldots \vee \phi_{S_n}$.

Lemma: G is provable in GLuk iff $\Rightarrow \phi_G$ is.

Proof: It is easy to see that if G is derivable so is ϕ_G. The converse is also not difficult, using the cut elimination theorem. The most significant step is to show that if $\Rightarrow A_1 \vee \ldots \vee A_n$ is provable then so is $\Rightarrow A_1|\ldots| \Rightarrow A_n$. For this it is enough to show that in general, if $G|\Gamma \Rightarrow \Delta, A \vee B$ is provable then so is $G|\Gamma \Rightarrow \Delta, A|\Gamma \Rightarrow \Delta, B$. This can be done by using two cuts (followed by external contractions), if we srart from the given provable hypersequent and $A \vee B \Rightarrow A|A \vee B \Rightarrow B$. The last hypersequent can be derived as follows: By applying rule M to $A \Rightarrow A$ and $B \Rightarrow B$ we can infer $A \Rightarrow B|B \Rightarrow A$. Two applications of the $\vee \Rightarrow$ rule to this sequent and to its two premises give then the desired result.

Proof of the completeness of GLuk: By the last Lemma and the completeness of the Hilbert-type system HLuk it is enough to show that every theorem of the later is derivable in GLuk. Since $A \rightarrow B, A \Rightarrow B$ is provable in GLuk, the cut elimination theorem entails that if $\Rightarrow A \rightarrow B$ and $\Rightarrow A$ are derivable in GLuk then so is B. It remains therefore to derive the axioms of HLuk. With the exception of I4 and I5 these axioms are all easily derivable in the classical Gentzen-type calculus (for ordinary sequents) without using either contraction or cut. Hence it is enough to probide proofs for I4 and I5. Now by applying rule M to $A \Rightarrow A$ and to $B \Rightarrow B$ we can obtain $B \Rightarrow \quad | \quad \Rightarrow A | A \Rightarrow B$. From each of the 3 components of this hypersequent one can easily derive both $\Rightarrow$ I4 and $\Rightarrow$ I5 in the classical system, *without using contractions or cuts* (for example: starting from $B \Rightarrow$ and the easily derived $\Rightarrow A, A \rightarrow B$ one can infer $(A \rightarrow B) \rightarrow B \Rightarrow A$ and then I4 by weakening and two applications of $\Rightarrow\rightarrow$). Since we can independently work with each component, we can use these three classical proofs and external contractions to obtain I4 and I5.

6 Proof-theory of RM_3

Hilbert-type representations of RM_3 and its various fragments were extensively investigated in the past. We refer the reader to [AB] and [Du] for details and references[11]. Gentzen-type formulations, on the other hand, were known so far only for the fragments without the combining connectives[12]. We remedy this now by intoducing a Gentzen-type formulation for the full system. Again we find it necessary to employ hypersequents in order to achieve this purpose. The discussion closely resembles that of the previous section, and so we shall make it as brief as possible.

The system GRM3

Axioms, external structural rules and logical rules: Like in GLuk.

Internal structural rules:

IC (Internal Contraction):

$$\frac{G|\Gamma, A, A \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta} \qquad \frac{G|\Gamma \Rightarrow \Delta, A, A}{G|\Gamma \Rightarrow \Delta, A}$$

IP (Internal Permutation): Like in GLuk.

Mi (Mingle):

$$\frac{G|\Gamma_1 \Rightarrow \Delta_1 \qquad G|\Gamma_2 \Rightarrow \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

[11] [Av6] includes an axoimatization of the pure implicational fragment which is more perspicuous than those mentioned in these two resources.

[12] Such a formulation appears, e.g., in [Av5], but was known long before.

WW (Weak Weakening):

$$\frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G|\Gamma_1 \Rightarrow \Delta_1|\Gamma_2, \Gamma' \Rightarrow \Delta_2, \Delta'}$$

provided $\Gamma_1 \Rightarrow \Delta_1$ is not empty.

The Soundness of GRM3 can easily be proved. As for completeness, the situation is exactly like in the case of GLuk. It can rather easily be proved for the fragment without implication, using the same method as before. The only change which is necessary is to replace the third condition in the Lemma from **5.2.2** by the condition:

Some nonempty component of G contains only special formulas.

We leave it to the reader to check that any hypersequent which satisfies this criterion is easily derivable using WW and external weakenings, and that a sequent which includes only atomic formulas is valid iff it satisfies this criterion. Other hypersequents can be reduced to such hypersequents by the same reduction steps we used above.

For dealing with the full system we should start, as before, by proving cut-elimination with the help of the history technique. Having done this we can use the same method of translation as before in order to rely on the completeness of the Hilbert-type formulations. The use of hypersequents is necessary for proving the distribution axiom $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$ [13] and the characteristic axiom of RM_3: $A \vee A \rightarrow B$. The proof of the last formula uses, of course, the WW rule. Other details are left to the reader.

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