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by

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On Functors Expressible in the Polymorphic Typed Lambda Calculus

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Abstract

Given a model of the polymorphic typed lambda calculus based upon a Cartesian closed category K, there will be functors from K to K whose action on objects can be expressed by type expressions and whose action on morphisms can be expressed by ordinary expressions. We show that if T is such a functor then there is a weak initial T-algebra and if, in addition, K possesses equalizers of all subsets of its morphism sets, then there is an initial T-algebra. It follows that there is no model of the polymorphic typed lambda calculus in which types denote sets and $S \rightarrow S'$ denotes the set of all functions from S to S'.

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The polymorphic, or second-order, typed lambda calculus [7,5,19] is an extension of the typed lambda calculus in which polymorphic functions can be defined by abstraction on type variables, and such functions can be applied to type expressions. It is known that all expressions of this language are normalizable [7,5], indeed strongly normalizable [17]. It is also known that the elements of any free many-sorted anarchic algebra are isomorphic to the closed normal expressions of a type that is determined by the signature of the algebra [10,3]. (This result was anticipated in [22, Proposition 3.15.18].) These facts led to the conjecture in [20] that the polymorphic typed lambda calculus should possess a set-theoretic model, in which types denote sets and $S \to S'$ denotes the set of all functions from S to S'.

However, Reynolds [18] later showed that no such model exists. Shortly thereafter, Plotkin [16] generalized this proof by considering, for models based upon arbitrary Cartesian closed categories, the behavior of functors that can be expressed in the calculus. In this joint paper, we give an exposition of this generalization, and show why it precludes the existence of a set-theoretic model.

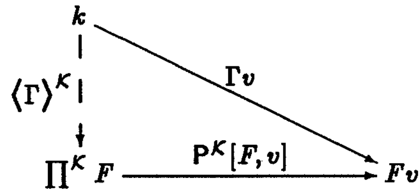
1. Mathematical Preliminaries

When f is a function, we write $\mathrm{dom}\, f$ for the domain of f, $f\rceil S$ for the restriction of f to $S \subseteq \mathrm{dom}\, f$, and $f x$ (often without parentheses) for the application of f to an argument x. We assume that application is left-associative, so that $f x\, y = (f x) y$.

We write $[\, f \mid x{:}\, x' \,]$ to denote the function with domain $\mathrm{dom}\, f \cup \{x\}$ such that $[\, f \mid x{:}\, x' \,] y = \textbf{if}\ y = x\ \textbf{then}\ x'\ \textbf{else}\ f y$, and also $[\, x_1{:}\, y_1 \mid \ldots \mid x_n{:}\, y_n \,]$ (where the x_i's are distinct) to denote the function with domain $\{x_1, \ldots, x_n\}$ that maps each x_i into y_i. As a special case, $[\,]$ denotes the empty function. We also write $\langle y_1, y_2 \rangle$ for the pair $[\, 1{:}\, y_1 \mid 2{:}\, y_2 \,]$.

When $\mathcal{K}$ is a category, we write $|\mathcal{K}|$ for the collection of objects of $\mathcal{K}$, $k \xrightarrow[\mathcal{K}]{} k'$ for the set of morphisms from $k \in |\mathcal{K}|$ to $k' \in |\mathcal{K}|$, $\alpha \,;_{\mathcal{K}}\, \alpha'$ for the composition (in diagrammatic order) of $\alpha \in k \xrightarrow[\mathcal{K}]{} k'$ with $\alpha' \in k' \xrightarrow[\mathcal{K}]{} k''$, and $I_k^{\mathcal{K}}$ for the identity morphism in $k \xrightarrow[\mathcal{K}]{} k$. (In these and later notations, we will frequently elide subscripts or superscripts denoting categories or other entities that are evident from context.) We also write $\mathcal{K}^{\mathrm{op}}$ for the dual of $\mathcal{K}$.

Let F be a function from some (finite) set $\mathrm{dom}\, F$ to $|\mathcal{K}|$. Then a (*finite*) *product* of F in $\mathcal{K}$ consists of an object $\prod^{\mathcal{K}} F$ and, for each $v \in \mathrm{dom}\, F$, a morphism $\mathsf{P}^{\mathcal{K}}[F, v] \in \prod^{\mathcal{K}} F \to Fv$, such that, if $k \in |\mathcal{K}|$ and Γ is a function with the same domain as F that maps each $v \in \mathrm{dom}\, F$ into a morphism in $k \to Fv$, then there is a unique morphism, denoted by $\langle \Gamma \rangle^{\mathcal{K}}$, in $k \to \prod^{\mathcal{K}} F$ such that



(1)

commutes in $\mathcal{K}$ for all $v \in \mathrm{dom}\, F$.

It is easily shown that, when $\Gamma v = \mathsf{P}^{\mathcal{K}}[F, v]$ for all $v \in \mathrm{dom}\, F$,

$$\langle \Gamma \rangle^{\mathcal{K}} = I_{\prod^{\mathcal{K}} F} \tag{2}$$

and, when $\beta \in k_0 \to k$,

$$\beta \,;\, \langle \Gamma \rangle^{\mathcal{K}} = \langle \Gamma' \rangle^{\mathcal{K}}, \tag{3}$$

where Γ' is the function with the same domain as Γ such that $\Gamma' v = \beta \,;\, \Gamma v$ for all $v \in \mathrm{dom}\, \Gamma$.

We will frequently use the abbreviations

$$\langle \Gamma \mid v \colon \varphi \rangle^{\mathcal{K}} \stackrel{\mathrm{def}}{=} \langle [\, \Gamma \mid v \colon \varphi \,] \rangle^{\mathcal{K}}$$

and

$$\langle v_1 \colon \varphi_1 \mid \ldots \mid v_n \colon \varphi_n \rangle^{\mathcal{K}} \stackrel{\mathrm{def}}{=} \langle [\, v_1 \colon \varphi_1 \mid \ldots \mid v_n \colon \varphi_n \,] \rangle^{\mathcal{K}} .$$

Thus Equation 3 implies

$$\beta \,;\, \langle v_1 \colon \varphi_1 \mid \ldots \mid v_n \colon \varphi_n \rangle^{\mathcal{K}} = \langle v_1 \colon \beta \,;\, \varphi_1 \mid \ldots \mid v_n \colon \beta \,;\, \varphi_n \rangle^{\mathcal{K}} . \tag{4}$$

An important special case of the product occurs when F is the empty function. Then its product in $\mathcal{K}$ is an object $\prod^{\mathcal{K}}[\,]$, called a *terminal object*, which we will denote more succinctly by $\top^{\mathcal{K}}$. It has the property that, for each $k \in |\mathcal{K}|$, the set $k \to \top^{\mathcal{K}}$ contains exactly one member, namely $\langle\rangle^{\mathcal{K}}$. (Note that k is determined by context.) The corresponding special case of Equation 4 is that, for $\beta \in k_0 \to k$,

$$\beta\,;\langle\rangle^{\mathcal{K}} = \langle\rangle^{\mathcal{K}}\,. \tag{5}$$

Another important special case occurs when $\mathrm{dom}\,F = \{1,2\}$. Here we write $k_1 \times_{\mathcal{K}} k_2$ for $\prod^{\mathcal{K}}[\,1{:}\,k_1 \mid 2{:}\,k_2\,]$, $\mathsf{p}^{i,\mathcal{K}}_{k_1 \times k_2}$ for $\mathsf{P}^{\mathcal{K}}[[\,1{:}\,k_1 \mid 2{:}\,k_2\,], i]$, and, when $\alpha_1 \in k \to k_1$ and $\alpha_2 \in k \to k_2$, $\langle\alpha_1, \alpha_2\rangle^{\mathcal{K}}$ for $\langle\,1{:}\,\alpha_1 \mid 2{:}\,\alpha_2\,\rangle^{\mathcal{K}}$. The corresponding special cases of Equations 1, 2, and 4 are that, for $\alpha_1 \in k \to k_1$, $\alpha_2 \in k \to k_2$, and $\beta \in k_0 \to k$,

$$\langle\alpha_1, \alpha_2\rangle\,;\mathsf{p}^{i}_{k_1 \times k_2} = \alpha_i\,, \tag{6}$$

$$\langle\mathsf{p}^{1}_{k_1 \times k_2}, \mathsf{p}^{2}_{k_1 \times k_2}\rangle = I_{k_1 \times k_2}\,, \tag{7}$$

$$\beta\,;\langle\alpha_1, \alpha_2\rangle = \langle\beta\,;\alpha_1, \beta\,;\alpha_2\rangle\,. \tag{8}$$

For $\gamma_1 \in k_1 \to k'_1$, and $\gamma_2 \in k_2 \to k'_2$, we define the morphism

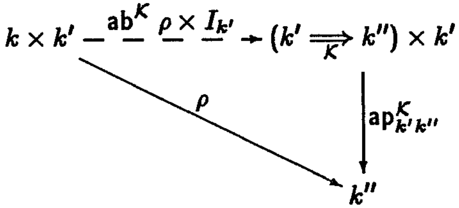
$$\gamma_1 \times_{\mathcal{K}} \gamma_2 \stackrel{\mathrm{def}}{=} \langle(\mathsf{p}^{1,\mathcal{K}}_{k_1 \times k_2}\,;\gamma_1), (\mathsf{p}^{2,\mathcal{K}}_{k_1 \times k_2}\,;\gamma_2)\rangle^{\mathcal{K}}$$

in $k_1 \times k_2 \to k'_1 \times k'_2$. (The use of $\times$ as an operation on both objects and morphisms reflects the fact that $\times$ is actually a bifunctor.) From Equations 8 and 6 it follows that, for $\alpha_1 \in k \to k_1$, $\alpha_2 \in k \to k_2$, $\gamma_1 \in k_1 \to k'_1$, and $\gamma_2 \in k_2 \to k'_2$,

$$\langle\alpha_1, \alpha_2\rangle\,;(\gamma_1 \times \gamma_2) = \langle\alpha_1\,;\gamma_1, \alpha_2\,;\gamma_2\rangle\,. \tag{9}$$

Let $\mathcal{K}$ be a category with finite products, and k', $k'' \in |\mathcal{K}|$. Then an *exponentiation* of k'' by k' consists of an object $k' \underset{\mathcal{K}}{\Longrightarrow} k''$ and a morphism $\mathsf{ap}^{\mathcal{K}}_{k'k''} \in (k' \underset{\mathcal{K}}{\Longrightarrow} k'') \times k' \to k''$ such that, for each $k \in |\mathcal{K}|$ and $\rho \in k \times k' \to$

k'', there is a unique morphism, denoted by $\mathsf{ab}^{\mathcal{K}}\,\rho$, in $k \to (k' \underset{\mathcal{K}}{\Rightarrow} k'')$ such that



(10)

commutes in $\mathcal{K}$.

A category is said to be *Cartesian closed* if it possesses all finite products (including a terminal element) and all exponentiations. (For a given category, there may be several definitions of $\prod$, $\Rightarrow$, and their associated morphisms that meet the definitions given above. However, when we speak of a category as Cartesian closed, we will assume that these entities have unambiguous meanings, i.e. that a Cartesian closed category is a category with *distinguished* finite products and exponentiations.)

For $\alpha \in k_0 \to (k' \Rightarrow k'')$ and $\alpha' \in k_0 \to k'$ we define

$$\alpha \rhd_{\mathcal{K}} \alpha' \stackrel{\text{def}}{=} \langle \alpha, \alpha' \rangle^{\mathcal{K}} \,;\, \mathsf{ap}^{\mathcal{K}}_{k'k''} \,.$$

From Equation 8, it follows that, for $\beta \in k_1 \to k_0$,

$$\beta \,;\, (\alpha \rhd \alpha') = \beta \,;\, \alpha \rhd \beta \,;\, \alpha' \,. \tag{11}$$

For $\rho \in k \times k' \to k''$, $\delta \in k_0 \to k$, and $\theta \in k_0 \to k'$, the definition of $\rhd$ and Equation 9 give

$$\delta \,;\, \mathsf{ab}\,\rho \rhd \theta = \langle \delta, \theta \rangle \,;\, (\mathsf{ab}\,\rho \times I_{k'}) \,;\, \mathsf{ap}_{k'k''} \,,$$

so that Diagram 10 gives

$$\delta \,;\, \mathsf{ab}\,\rho \rhd \theta = \langle \delta, \theta \rangle \,;\, \rho \,. \tag{12}$$

On the other hand, suppose 12 holds for all $\rho \in k \times k' \to k''$, $\delta \in k_0 \to k$, and $\theta \in k_0 \to k'$. Taking $k_0 = k \times k'$, $\delta = \mathsf{p}^1_{k \times k'}$, and $\theta = \mathsf{p}^2_{k \times k'}$, the definition of $\rhd$ and Equation 9 give

$$\langle \mathsf{p}^1_{k\times k'}, \mathsf{p}^2_{k\times k'} \rangle \,;\, (\mathsf{ab}\,\rho \times I_{k'}) \,;\, \mathsf{ap}_{k'k''} = \langle \mathsf{p}^1_{k\times k'}, \mathsf{p}^2_{k\times k'} \rangle \,;\, \rho \,,$$

so that Equation 7 gives Diagram 10. Thus, for $\rho \in k \times k' \to k''$, $\mathsf{ab}\,\rho$ is the unique morphism in $k \to (k' \Rightarrow k'')$ such that Equation 12 holds for all $k_0 \in |\mathcal{K}|$, $\delta \in k_0 \to k$ and $\theta \in k_0 \to k'$.

In a category with a distinguished terminal element, a morphism in $\top \to k$ is called an *element* of k. When the category is Cartesian closed, there is an isomorphism between the elements of $k' \Rightarrow k''$ and the morphisms in $k' \to k''$. To see this, suppose $\alpha \in k' \to k''$ and take $k = \top$, $k_0 = k'$, $\delta = \langle\rangle$, $\theta = I_{k'}$, and $\rho = \mathsf{p}^2_{\top \times k'} ; \alpha$ in Equation 12. Then, by Equation 6, $\mathsf{ab}(\mathsf{p}^2_{\top \times k'} ; \alpha)$ is the unique solution of

$$\langle\rangle \,;\, \mathsf{ab}(\mathsf{p}^2_{\top \times k'} \,;\, \alpha) \;\triangleright I_{k'} = \alpha\,.$$

Thus, if we define the functions $\phi^{\mathcal{K}}_{k'k''}$ from $\top \to (k' \Rightarrow k'')$ to $k' \to k''$ and $\psi^{\mathcal{K}}_{k'k''}$ from $k' \to k''$ to $\top \to (k' \Rightarrow k'')$ by

$$\phi^{\mathcal{K}}_{k'k''}\gamma \stackrel{\text{def}}{=} \langle\rangle \,;\, \gamma \triangleright I_{k'}\,, \tag{13}$$

and

$$\psi^{\mathcal{K}}_{k'k''}\alpha \stackrel{\text{def}}{=} \mathsf{ab}(\mathsf{p}^2_{\top \times k'} \,;\, \alpha)\,,$$

then

$$\phi_{k'k''}(\psi_{k'k''}\alpha) = \alpha\,, \tag{14}$$

and

$$\psi_{k'k''}(\phi_{k'k''}\gamma) = \gamma\,.$$

For any object B of a Cartesian closed category $\mathcal{K}$, there is a functor $Q^{\mathcal{K}}_B$ from $\mathcal{K}$ to $\mathcal{K}^{\mathrm{op}}$ such that $Q^{\mathcal{K}}_B(k) = k \underset{\mathcal{K}}{\Rightarrow} B$ for all $k \in |\mathcal{K}|$. A characterization of the action of $Q^{\mathcal{K}}_B$ on morphisms can be obtained from Equation 12 by replacing k by $k' \Rightarrow B$, k' by k, and k'' by B, to find that, for $\rho \in (k' \Rightarrow B) \times k \to B$, $\mathsf{ab}\,\rho$ is the unique morphism in $(k' \Rightarrow B) \to (k \Rightarrow B)$ such that 12 holds for all $k_0 \in |\mathcal{K}|$, $\delta \in k_0 \to (k' \Rightarrow B)$, and $\theta \in k_0 \to k$. Next, for any $\alpha \in k \to k'$, take $\rho = (I_{k' \Rightarrow B} \times \alpha) ; \mathsf{ap}_{k'B}$, so that $\langle\delta, \theta\rangle ; \rho = \delta \triangleright \theta ; \alpha$ by Equation 9 and the definition of $\triangleright$, and define $Q_B\alpha$ to be $\mathsf{ab}\,\rho$. Then $Q_B\alpha$ is the unique morphism in $(k' \Rightarrow B) \to (k \Rightarrow B)$ such that

$$\delta \,;\, Q_B\alpha \triangleright \theta = \delta \triangleright \theta \,;\, \alpha \tag{15}$$

holds for all $k_0 \in |\mathcal{K}|$, $\delta \in k_0 \to (k' \Rightarrow B)$, and $\theta \in k_0 \to k$.

It is immediately evident that $Q_B I_k = I_{k \Rightarrow B}$. To see that Q_B satisfies the composition law for functors, suppose $\alpha \in k \rightarrow k'$, $\alpha' \in k' \rightarrow k''$, $\delta' \in k_0 \rightarrow (k'' \Rightarrow B)$, and $\theta \in k_0 \rightarrow k$. Substituting $\delta' ; Q_B\alpha'$ for δ in Equation 15 and $\theta ; \alpha$ for θ' in the analogous equation with primed variables gives

$$\delta' ; Q_B\alpha' ; Q_B\alpha \triangleright \theta = \delta' ; Q_B\alpha' \triangleright \theta ; \alpha = \delta' \triangleright \theta ; \alpha ; \alpha' ,$$

which establishes that $Q_B(\alpha ; \alpha') = Q_B\alpha' ; Q_B\alpha$.

2. The Polymorphic Typed Lambda Calculus

The following syntactic description is somewhat unusual, since we wish to avoid assumptions that are stronger than necessary to obtain our results. In particular, we wish to encompass extensions of the polymorphic typed lambda calculus involving, for example, additional type and expression constructors.

We assume that the language is built from infinite sets $\mathcal{T}$ of *type variables* and $\mathcal{V}$ of *ordinary variables*. For each finite set N of type variables, there is a set Ω_N of *type expressions* over the type variables in N. These sets must satisfy:

1. If $\tau \in N$ then $\tau \in \Omega_N$,
2. If $\omega, \omega' \in \Omega_N$ then $\omega \rightarrow \omega' \in \Omega_N$,
3. If $\tau \in \mathcal{T}$ and $\omega \in \Omega_{N \cup \{\tau\}}$ then $\Delta\tau.\ \omega \in \Omega_N$,
4. If $N \subseteq N'$ then $\Omega_N \subseteq \Omega_{N'}$.

For example,

$$\mathbf{s} \in \Omega_{\{\mathbf{s}\}} \subseteq \Omega_{\{\mathbf{s},\mathbf{t}\}} ,$$

$$\mathbf{s} \rightarrow \mathbf{t} \in \Omega_{\{\mathbf{s},\mathbf{t}\}} ,$$

$$\Delta\mathbf{s}.\ \mathbf{s} \rightarrow \mathbf{t} \in \Omega_{\{\mathbf{t}\}} \subseteq \Omega_{\{\mathbf{s},\mathbf{t}\}} .$$

We will not need to make any assumptions about equality of type expressions (although it is usual to regard as equal type expressions that are alpha variants with respect to the binding structure induced by Δ).

A *type assignment* π over N is a function from some finite set $\text{dom}\,\pi$ of ordinary variables to Ω_N; we write Ω_N^* for the set of type assignments over N. For example,

$$[\mathbf{x}{:}\,\mathbf{s} \mid \mathbf{f}{:}\,\mathbf{s} \rightarrow \mathbf{t} \mid \mathbf{p}{:}\,\Delta \mathbf{s}.\ \mathbf{s} \rightarrow \mathbf{t}] \in \Omega^*_{\{\mathbf{s},\mathbf{t}\}}\ .$$

From Condition 4, we have

5. If $N \subseteq N'$ then $\Omega_N^* \subseteq \Omega_{N'}^*$.

Finally, we must define ordinary expressions. For each finite set N of type variables and finite set V of ordinary variables, there is a set E_V^N of *ordinary expressions* over the variables in N and V. These sets must satisfy:

6. If $v \in V$ then $v \in E_V^N$,
7. If $e_1, e_2 \in E_V^N$ then $e_1 e_2 \in E_V^N$,
8. If $v \in \mathcal{V}$, $\omega \in \Omega_N$, and $e \in E^N_{V \cup \{v\}}$ then $\lambda v_\omega .\ e \in E_V^N$,
9. If $e \in E_V^N$ and $\omega \in \Omega_N$ then $e[\omega] \in E_V^N$,
10. If $\tau \in \mathcal{T}$ and $e \in E_V^{N \cup \{\tau\}}$ then $\Lambda\tau.\ e \in E_V^N$,
11. If $N \subseteq N'$ and $V \subseteq V'$ then $E_V^N \subseteq E_{V'}^{N'}$.

The relationship between ordinary and type expressions is expressed by formulas called *typings*. If $\pi \in \Omega_N^*$, $\omega \in \Omega_N$, and e is an ordinary expression then $\pi \vdash_N e{:}\,\omega$ is a typing that asserts that e belongs to $E^N_{\text{dom}\,\pi}$ and takes on type ω when its free ordinary variables are assigned types by π. We assume that the following inference rules for typings are valid:

12. For $\pi \in \Omega_N^*$ and $v \in \operatorname{dom} \pi$:

$$\frac{}{\pi \vdash_N v: \pi v\,,}$$

13. For $\pi \in \Omega_N^*$ and $\omega, \omega' \in \Omega_N$:

$$\frac{\pi \vdash_N e_1: \omega \to \omega' \qquad \pi \vdash_N e_2: \omega}{\pi \vdash_N e_1 e_2: \omega'\,,}$$

14. For $\pi \in \Omega_N^*$ and $\omega, \omega' \in \Omega_N$:

$$\frac{[\,\pi \mid v: \omega\,] \vdash_N e: \omega'}{\pi \vdash_N \lambda v_\omega.\ e: \omega \to \omega'\,,}$$

15. For $\pi \in \Omega_N^*$, $\omega \in \Omega_N$, and $\tau \in N$:

$$\frac{\pi \vdash_N e: \Delta\tau.\ \omega}{\pi \vdash_N e[\tau]: \omega\,,}$$

16. For $\pi \in \Omega_{N-\{\tau\}}^*$ and $\omega \in \Omega_{N\cup\{\tau\}}$:

$$\frac{\pi \vdash_{N\cup\{\tau\}} e: \omega}{\pi \vdash_N \Lambda\tau.\ e: \Delta\tau.\ \omega\,,}$$

17. For $N \subseteq N'$, $\pi \in \Omega_N^*$, and $\omega \in \Omega_N$:

$$\frac{\pi \vdash_N e: \omega}{\pi \vdash_{N'} e: \omega\,,}$$

18. For $\pi, \pi' \in \Omega_N^*$ such that $\pi = \pi' \rceil \operatorname{dom} \pi$, and $\omega \in \Omega_N$:

$$\frac{\pi \vdash_N e: \omega}{\pi' \vdash_N e: \omega\,.}$$

For example, the following are valid typings:

$$[\mathbf{f}{:}\,\mathbf{t} \to \mathbf{t} \mid \mathbf{x}{:}\,\mathbf{t}] \vdash_{\{\mathbf{t}\}} \mathbf{f}{:}\,\mathbf{t} \to \mathbf{t} \qquad \text{by 12}$$

$$[\mathbf{f}{:}\,\mathbf{t} \to \mathbf{t} \mid \mathbf{x}{:}\,\mathbf{t}] \vdash_{\{\mathbf{t}\}} \mathbf{x}{:}\,\mathbf{t} \qquad \text{by 12}$$

$$[\mathbf{f}{:}\,\mathbf{t} \to \mathbf{t} \mid \mathbf{x}{:}\,\mathbf{t}] \vdash_{\{\mathbf{t}\}} \mathbf{f\,x}{:}\,\mathbf{t} \qquad \text{by 13}$$

$$[\mathbf{f}{:}\,\mathbf{t} \to \mathbf{t} \mid \mathbf{x}{:}\,\mathbf{t}] \vdash_{\{\mathbf{t}\}} \mathbf{f(f\,x)}{:}\,\mathbf{t} \qquad \text{by 13}$$

$$[\mathbf{f}{:}\,\mathbf{t} \to \mathbf{t}] \vdash_{\{\mathbf{t}\}} \lambda\mathbf{x_t}.\, \mathbf{f(f\,x)}{:}\,\mathbf{t} \to \mathbf{t} \qquad \text{by 14}$$

$$[\,] \vdash_{\{\mathbf{t}\}} \lambda\mathbf{f_{t\to t}}.\, \lambda\mathbf{x_t}.\, \mathbf{f(f\,x)}{:}\,(\mathbf{t} \to \mathbf{t}) \to (\mathbf{t} \to \mathbf{t}) \qquad \text{by 14}$$

$$[\,] \vdash_{\{\}} \Lambda\mathbf{t}.\, \lambda\mathbf{f_{t\to t}}.\, \lambda\mathbf{x_t}.\, \mathbf{f(f\,x)}{:}\,\Delta\mathbf{t}.\,(\mathbf{t} \to \mathbf{t}) \to (\mathbf{t} \to \mathbf{t}) \qquad \text{by 16}$$

$$[\,] \vdash_{\{\mathbf{t}\}} \Lambda\mathbf{t}.\, \lambda\mathbf{f_{t\to t}}.\, \lambda\mathbf{x_t}.\, \mathbf{f(f\,x)}{:}\,\Delta\mathbf{t}.\,(\mathbf{t} \to \mathbf{t}) \to (\mathbf{t} \to \mathbf{t}) \qquad \text{by 17}$$

$$[\,] \vdash_{\{\mathbf{t}\}} \big(\Lambda\mathbf{t}.\, \lambda\mathbf{f_{t\to t}}.\, \lambda\mathbf{x_t}.\, \mathbf{f(f\,x)}\big)[\mathbf{t}]{:}\,(\mathbf{t} \to \mathbf{t}) \to (\mathbf{t} \to \mathbf{t}) \qquad \text{by 15}$$

$$[\mathbf{g}{:}\,\mathbf{t} \to \mathbf{t}] \vdash_{\{\mathbf{t}\}} \big(\Lambda\mathbf{t}.\, \lambda\mathbf{f_{t\to t}}.\, \lambda\mathbf{x_t}.\, \mathbf{f(f\,x)}\big)[\mathbf{t}]{:}\,(\mathbf{t} \to \mathbf{t}) \to (\mathbf{t} \to \mathbf{t})\,. \qquad \text{by 18}$$

Actually, for the ordinary polymorphic typed lambda calculus, Inference Rule 15 is subsumed by the more general rule

15′. For $\pi \in \Omega_N^*$, $\omega \in \Omega_{N \cup \{\tau\}}$, and $\omega' \in \Omega_N$:

$$\frac{\pi \vdash_N e{:}\,\Delta\tau.\,\omega}{\pi \vdash_N e[\omega']{:}\,(\omega/\tau \to \omega')}\,,$$

where $(\omega/\tau \to \omega')$ denotes the result of substituting ω' for τ in ω. However, Rule 15 is sufficient for our needs, and we wish to avoid the difficulty of defining substitution (with renaming) in a way that would not circumscribe possible extensions of the language.

The notion of typing is prerequisite to any semantics of the polymorphic typed lambda calculus; ordinary expressions will possess meanings only when they satisfy typings, which will determine the kind of meanings they will possess. Specifically, for each $\pi \in \Omega_N^*$ and $\omega \in \Omega_N$, the set

$$E_{\pi\omega}^N \stackrel{\text{def}}{=} \{\, e \mid e \in E_{\operatorname{dom}\pi}^N \text{ and } \pi \vdash_N e{:}\,\omega \,\}\,,$$

of expressions that take on type ω under the type assignment π, must be mapped into meanings appropriate to π and ω.

3. $\mathcal{K}$-Models

It is well known that Cartesian closed categories provide models of the ordinary typed lambda calculus. In this section, we formalize the idea of extending such models to the polymorphic case. As with syntax, the properties that we postulate for such extensions are weaker than those one would normally require of a model; our intent is to assume only those properties needed to obtain the results of this paper.

(We believe that these properties hold for any general category-theoretic definition of the concept of a model. For example, given a PL category $(\mathbf{G}, \mathbf{S})$ in the sense of Seely [21], one can take $\mathcal{K}$ to be the Cartesian closed category $\mathbf{G}(1)$, where 1 is the terminal object of $\mathbf{S}$.)

Given a category $\mathcal{K}$, a function from a finite set of type variables to $|\mathcal{K}|$ is called an object assignment. Then, a *$\mathcal{K}$-model* of the polymorphic typed lambda calculus consists of:

1. A Cartesian closed category $\mathcal{K}$.
2. For each object assignment O with domain N, a semantic function $\mathcal{M}O$ from Ω_N to $|\mathcal{K}|$. These functions must satisfy:
   (a) If $\tau \in N$ then
   $$\mathcal{M}O\tau = O\tau\,, \qquad (16)$$
   (b) If $\omega, \omega' \in \Omega_N$ then
   $$\mathcal{M}O(\omega \to \omega') = \mathcal{M}O\omega \underset{\mathcal{K}}{\Longrightarrow} \mathcal{M}O\omega'\,, \qquad (17)$$
   (c) If $O = O' \rceil N$ and $\omega \in \Omega_N$ then
   $$\mathcal{M}O'\omega = \mathcal{M}O\omega\,. \qquad (18)$$

3. For each object assignment O with domain N, $\pi \in \Omega_N^*$, and $\omega \in \Omega_N$, a semantic function $\mu^O_{\pi\omega}$ from $E^N_{\pi\omega}$ to $\prod^K(\mathcal{M}O \cdot \pi) \xrightarrow[K]{} \mathcal{M}O\omega$, where $\mathcal{M}O \cdot \pi$ denotes the function from $\mathrm{dom}\,\pi$ to $|\mathcal{K}|$ such that $(\mathcal{M}O \cdot \pi)v = \mathcal{M}O(\pi v)$ for all $v \in \mathrm{dom}\,\pi$. These functions must satisfy:

   (a) If $\pi \in \Omega_N^*$ and $v \in \mathrm{dom}\,\pi$ then

   $$\mu^O_{\pi,\pi v}[\![v]\!] = \mathsf{P}[\mathcal{M}O \cdot \pi, v] \in \prod(\mathcal{M}O \cdot \pi) \xrightarrow[K]{} \mathcal{M}O(\pi v)\,,$$

   (b) If $\pi \in \Omega_N^*$, $\omega, \omega' \in \Omega_N$, $\pi \vdash_N e_1 : \omega \to \omega'$, and $\pi \vdash_N e_2 : \omega$ then

   $$\mu^O_{\pi\omega'}[\![e_1 e_2]\!] = \mu^O_{\pi,\omega\to\omega'}[\![e_1]\!] \rhd \mu^O_{\pi\omega}[\![e_2]\!] \in \prod(\mathcal{M}O \cdot \pi) \xrightarrow[K]{} \mathcal{M}O\omega'\,,$$

   (c) If $\pi \in \Omega_N^*$, $\omega, \omega' \in \Omega_N$, and $[\,\pi \mid v{:}\,\omega\,] \vdash_N e : \omega'$ then

   $$\mu^O_{\pi,\omega\to\omega'}[\![\lambda v_\omega.\ e]\!] = \mathsf{ab}\big(\big\langle\, \Xi \mid v{:}\, \mathsf{p}^2_{\Pi(\mathcal{M}O\cdot\pi)\times\mathcal{M}O\omega} \big\rangle\, ;\, \mu^O_{[\pi\mid v:\omega],\omega'}[\![e]\!]\big)\,,$$

   where Ξ is the function with the same domain as π such that

   $$\Xi v' = \mathsf{p}^1_{\Pi(\mathcal{M}O\cdot\pi)\times\mathcal{M}O\omega}\, ;\, \mathsf{P}[\mathcal{M}O \cdot \pi, v']$$

   for all $v' \in \mathrm{dom}\,\pi$; in other words, $\mu^O_{\pi,\omega\to\omega'}[\![\lambda v_\omega.\ e]\!]$ is the unique morphism in $\prod(\mathcal{M}O \cdot \pi) \xrightarrow[K]{} (\mathcal{M}O\omega \Rightarrow \mathcal{M}O\omega')$ such that

   $$\begin{array}{ccc}
   \prod(\mathcal{M}O\cdot\pi)\times\mathcal{M}O\omega & \xrightarrow{\ \mu^O[\![\lambda v_\omega.\ e]\!]\times I_{\mathcal{M}O\omega}\ } & (\mathcal{M}O\omega \Rightarrow \mathcal{M}O\omega')\times\mathcal{M}O\omega \\
   \Big\downarrow \langle\, \Xi \mid v{:}\, \mathsf{p}^2_{\Pi(\mathcal{M}O\cdot\pi)\times\mathcal{M}O\omega}\rangle & & \Big\downarrow \mathsf{ap}_{\mathcal{M}O\omega,\mathcal{M}O\omega'} \\
   \prod(\mathcal{M}O\cdot[\,\pi \mid v{:}\,\omega\,]) & \xrightarrow{\ \mu^O[\![e]\!]\ } & \mathcal{M}O\omega'
   \end{array}$$

   commutes in $\mathcal{K}$, where

   $$\begin{array}{ccc}
   \prod(\mathcal{M}O\cdot\pi)\times\mathcal{M}O\omega & & \\
   \Big\downarrow \mathsf{p}^1_{\Pi(\mathcal{M}O\cdot\pi)\times\mathcal{M}O\omega} & \searrow^{\ \Xi v'} & \\
   \prod(\mathcal{M}O\cdot\pi) & \xrightarrow{\ \mathsf{P}[\mathcal{M}O\cdot\pi, v']\ } & \mathcal{M}O(\pi v')
   \end{array}$$

   commutes for all $v' \in \mathrm{dom}\,\pi$.

(d) If $O = O' \rceil N$, $\pi \in \Omega_N^*$, $\omega \in \Omega_N$, and $\pi \vdash_N e\colon \omega$ then

$$\mu^{O'}_{\pi\omega}[\![e]\!] = \mu^{O}_{\pi\omega}[\![e]\!], \tag{19}$$

(e) If π, $\pi' \in \Omega_N^*$, $\pi = \pi' \rceil \operatorname{dom}\pi$, $\omega \in \Omega_N$, and $\pi \vdash_N e\colon \omega$ then

$$\mu^{O}_{\pi'\omega}[\![e]\!] = \langle \Upsilon \rceil \operatorname{dom}\pi \rangle \,;\, \mu^{O}_{\pi\omega}[\![e]\!],$$

where Υ is the function with the same domain as π' such that

$$\Upsilon v' = \mathsf{P}[\mathcal{M}O \cdot \pi', v']$$

for all $v' \in \operatorname{dom}\pi'$,

(f) If $\pi \in \Omega^*_{N-\{\tau\}}$, $\omega \in \Omega_N$, $\tau \in N$, and $\pi \vdash_N e\colon \omega$ then

$$\mu^{O}_{\pi\omega}[\![(\lambda v_{\Delta\tau.\,\omega}.\, v[\tau])(\Lambda\tau.\, e)]\!] = \mu^{O}_{\pi\omega}[\![e]\!]. \tag{20}$$

Conditions 2a, 2b, 3a, 3b, and 3c stipulate that the semantics of the ordinary typed lambda calculus, which is a sublanguage of the polymorphic typed lambda calculus, is the standard semantics given by the Cartesian closed category $\mathcal{K}$. Conditions 2c and 3d stipulate that the meanings of type and ordinary expressions are independent of irrelevant type variables, while Condition 3e stipulates that the meanings of ordinary expressions are independent of irrelevant ordinary variables. Condition 3f stipulates the soundness of the following combination of an ordinary and type beta-reduction:

$$(\lambda v_{\Delta\tau.\,\omega}.\, v[\tau])(\Lambda\tau.\, e) \Longrightarrow (\Lambda\tau.\, e)[\tau] \Longrightarrow e.$$

Conditions 3a, 3b, 3c, and 3e can be recast in forms more suitable for analyzing the meanings of specific expressions. In the following, suppose Γ is a function with the same domain as π such that $\Gamma v \in k_0 \to \mathcal{M}O(\pi v)$ for all $v \in \operatorname{dom}\pi$, Γ' bears a similar relation to π', and $\varphi \in k_0 \to \mathcal{M}O\omega$. If $\pi \in \Omega_N^*$ and $v \in \operatorname{dom}\pi$ then Condition 3a and Equation 1 give

$$\langle \Gamma \rangle \,;\, \mu^{O}_{\pi,\pi v}[\![v]\!] = \Gamma v. \tag{21}$$

If $\pi \in \Omega_N^*$, ω, $\omega' \in \Omega_N$, $\pi \vdash_N e_1\colon \omega \to \omega'$, and $\pi \vdash_N e_2\colon \omega$ then 3b and 11 give

$$\langle \Gamma \rangle \,;\, \mu^{O}_{\pi\omega'}[\![e_1 e_2]\!] = \langle \Gamma \rangle \,;\, \mu^{O}_{\pi,\omega\to\omega'}[\![e_1]\!] \rhd \langle \Gamma \rangle \,;\, \mu^{O}_{\pi\omega}[\![e_2]\!]. \tag{22}$$

If $\pi \in \Omega_N^*$, ω, $\omega' \in \Omega_N$, and $[\,\pi \mid v{:}\,\omega\,] \vdash_N e{:}\,\omega'$ then 3c, 12, 3, 6, and 1 give

$$\langle \Gamma \rangle\,;\, \mu^{O}_{\pi,\omega\to\omega'}[\![\lambda v_\omega.\, e]\!] \triangleright \varphi = \langle\, \Gamma \mid v{:}\,\varphi \,\rangle\,;\, \mu^{O}_{[\pi \mid v:\omega],\omega'}[\![e]\!]\,. \tag{23}$$

If π, $\pi' \in \Omega_N^*$, $\pi = \pi'\rceil\, \mathrm{dom}\,\pi$, $\omega \in \Omega_N$, and $\pi \vdash_N e{:}\,\omega$ then 3e, 3, and 1 give

$$\langle \Gamma' \rangle\,;\, \mu^{O}_{\pi'\omega}[\![e]\!] = \langle \Gamma' \rceil\, \mathrm{dom}\,\pi \rangle\,;\, \mu^{O}_{\pi\omega}[\![e]\!]\,. \tag{24}$$

The use of these equations is illustrated by the following proposition, which shows that a nontrivial $\mathcal{K}$ leads to a nontrivial $\mathcal{K}$-model:

Proposition 1 *Let* $\mathbf{B} \stackrel{\mathrm{def}}{=} \Delta\mathbf{k}.\, \mathbf{k} \to (\mathbf{k} \to \mathbf{k}) \in \Omega_{\{\}}$ *and, in some* $\mathcal{K}$*-model,* $B \stackrel{\mathrm{def}}{=} \mathcal{M}[\,]\mathbf{B} \in |\mathcal{K}|$. *If any morphism set of* $\mathcal{K}$ *has more than one member, then* B *has more than one element.*

Proof: It suffices to show that if the elements of B are all equal then any pair of members of any morphism set are equal. Consider the ordinary expressions

$$\Lambda\mathbf{k}.\, \lambda\mathbf{x}_{\mathbf{k}}.\, \lambda\mathbf{y}_{\mathbf{k}}.\, \mathbf{x}\,, \quad \Lambda\mathbf{k}.\, \lambda\mathbf{x}_{\mathbf{k}}.\, \lambda\mathbf{y}_{\mathbf{k}}.\, \mathbf{y} \in E^{\{\}}_{[\,],\mathbf{B}}\,,$$

which are mapped by $\mu^{[\,]}$ into elements of B. If these elements are equal then, for any k_0, $k \in |\mathcal{K}|$ and α, $\beta \in k_0 \to k$,

$$\begin{aligned} &((\langle\rangle\,;\, \mu^{[\mathbf{k}:k]}[\![\lambda \mathbf{b}_{\mathbf{B}}.\, \mathbf{b}[\mathbf{k}]]\!] \triangleright \langle\rangle\,;\, \mu^{[\,]}[\![\Lambda\mathbf{k}.\, \lambda\mathbf{x}_{\mathbf{k}}.\, \lambda\mathbf{y}_{\mathbf{k}}.\, \mathbf{x}]\!]) \triangleright \alpha) \triangleright \beta \\ =\; &((\langle\rangle\,;\, \mu^{[\mathbf{k}:k]}[\![\lambda \mathbf{b}_{\mathbf{B}}.\, \mathbf{b}[\mathbf{k}]]\!] \triangleright \langle\rangle\,;\, \mu^{[\,]}[\![\Lambda\mathbf{k}.\, \lambda\mathbf{x}_{\mathbf{k}}.\, \lambda\mathbf{y}_{\mathbf{k}}.\, \mathbf{y}]\!]) \triangleright \alpha) \triangleright \beta\,. \end{aligned}$$

But the left side of this equation equals

$$\begin{aligned} &(\langle\rangle\,;\, \mu^{[\mathbf{k}:k]}[\![(\lambda \mathbf{b}_{\mathbf{B}}.\, \mathbf{b}[\mathbf{k}])(\Lambda\mathbf{k}.\, \lambda\mathbf{x}_{\mathbf{k}}.\, \lambda\mathbf{y}_{\mathbf{k}}.\, \mathbf{x})]\!] \triangleright \alpha) \triangleright \beta && \text{by 19, 22} \\ &= (\langle\rangle\,;\, \mu^{[\mathbf{k}:k]}[\![\lambda\mathbf{x}_{\mathbf{k}}.\, \lambda\mathbf{y}_{\mathbf{k}}.\, \mathbf{x}]\!] \triangleright \alpha) \triangleright \beta && \text{by 20} \\ &= \langle\, \mathbf{x}{:}\, \alpha \,\rangle\,;\, \mu^{[\mathbf{k}:k]}[\![\lambda\mathbf{y}_{\mathbf{k}}.\, \mathbf{x}]\!] \triangleright \beta && \text{by 23} \\ &= \langle\, \mathbf{x}{:}\, \alpha \mid \mathbf{y}{:}\, \beta \,\rangle\,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{x}]\!] && \text{by 23} \\ &= \alpha\,, && \text{by 21} \end{aligned}$$

and by a similar argument the right side equals β. (*End of Proof*)

4. SET-Models

An important special case of a $\mathcal{K}$-model arises when $\mathcal{K}$ is the Cartesian closed category SET, for which:

1. $|\mathrm{SET}|$ is the class of sets, and
   (a) $k \xrightarrow[\mathrm{SET}]{} k'$ is the set of all functions from k to k',
   (b) Composition is functional composition,
   (c) I_k^{SET} is the identity function on k.
2. $\prod^{\mathrm{SET}}$ is the general Cartesian product, and
   (a) If $v \in \mathrm{dom}\, F$ then $\mathsf{P}[F, v] \in \prod F \to Fv$ is the function such that

   $$\mathsf{P}[F, v]\eta = \eta v$$

   for all $\eta \in \prod F$,
   (b) If, for all $v \in \mathrm{dom}\, F$, $\Gamma v \in k \to Fv$, then $\big\langle \Gamma \big\rangle \in k \to \prod F$ is the function such that

   $$\big\langle \Gamma \big\rangle x\, v = \Gamma v x$$

   for all $x \in k$ and $v \in \mathrm{dom}\, F$.
3. $\times_{\mathrm{SET}}$ is the binary Cartesian product, and
   (a) $\mathsf{p}^i_{k_1 \times k_2} \in k_1 \times k_2 \to k_i$ is the function such that

   $$\mathsf{p}^i_{k_1 \times k_2} \langle x_1, x_2 \rangle = x_i$$

   for all $x_1 \in k_1$ and $x_2 \in k_2$,
   (b) If $\alpha_1 \in k \to k_1$ and $\alpha_2 \in k \to k_2$ then $\big\langle \alpha_1, \alpha_2 \big\rangle^{\mathrm{SET}} \in k \to k_1 \times k_2$ is the function such that

   $$\big\langle \alpha_1, \alpha_2 \big\rangle^{\mathrm{SET}} x = \langle \alpha_1 x, \alpha_2 x \rangle$$

   for all $x \in k$.

4. $k' \xrightarrow[\text{SET}]{} k''$ is the set $k' \to k''$, and

   (a) $\mathsf{ap}_{k'k''} \in (k' \to k'') \times k' \to k''$ is the function such that

   $$\mathsf{ap}_{k'k''}\langle f', x'\rangle = f'x'$$

   for all $f' \in k' \to k''$ and $x' \in k'$,

   (b) If $\rho \in k \times k' \to k''$ then $\mathsf{ab}\,\rho \in k \to (k' \to k'')$ is the function such that

   $$\mathsf{ab}\,\rho\, x\, x' = \rho\langle x, x'\rangle$$

   for all $x \in k$ and $x' \in k'$.

By substituting these equations into the general definition of a $\mathcal{K}$-model, we find that a SET-model consists of:

1. The Cartesian closed category SET.

2. For each set assignment O with domain N, a semantic function $\mathcal{M}O$ from Ω_N to $|\text{SET}|$, such that:

   (a) If $\tau \in N$ then

   $$\mathcal{M}O\tau = O\tau\,,$$

   (b) If $\omega, \omega' \in \Omega_N$ then

   $$\mathcal{M}O(\omega \to \omega') = \mathcal{M}O\omega \to \mathcal{M}O\omega'\,,$$

   (c) If $O = O'\rceil N$ and $\omega \in \Omega_N$ then

   $$\mathcal{M}O'\omega = \mathcal{M}O\omega\,.$$

3. For each set assignment O with domain N, $\pi \in \Omega_N^*$, and $\omega \in \Omega_N$, a semantic function $\mu^O_{\pi\omega}$ from $E^N_{\pi\omega}$ to $\prod^{\text{SET}}(\mathcal{M}O \cdot \pi) \to \mathcal{M}O\omega$, such that

   (a) If $\pi \in \Omega_N^*$ and $v \in \operatorname{dom}\pi$ then, for all $\eta \in \prod(\mathcal{M}O \cdot \pi)$,

   $$\mu^O_{\pi,\pi v}[\![v]\!]\eta = \eta v\,,$$

(b) If $\pi \in \Omega_N^*$, $\omega, \omega' \in \Omega_N$, $\pi \vdash_N e_1 : \omega \to \omega'$, and $\pi \vdash_N e_2 : \omega$ then, for all $\eta \in \prod(\mathcal{M}O \cdot \pi)$,

$$\mu^O_{\pi\omega'}[\![e_1 e_2]\!]\eta = (\mu^O_{\pi,\omega\to\omega'}[\![e_1]\!]\eta)(\mu^O_{\pi\omega}[\![e_2]\!]\eta)\,,$$

(c) If $\pi \in \Omega_N^*$, $\omega, \omega' \in \Omega_N$, and $[\pi \mid v{:}\omega] \vdash_N e : \omega'$ then, for all $\eta \in \prod(\mathcal{M}O \cdot \pi)$ and $a \in \mathcal{M}O\omega$,

$$\mu^O_{\pi,\omega\to\omega'}[\![\lambda v_\omega.\ e]\!]\eta a = \mu^O_{[\pi|v:\omega],\omega'}[\![e]\!][\,\eta \mid v{:}\,a\,]\,,$$

(d) If $O = O'\rceil N$, $\pi \in \Omega_N^*$, $\omega \in \Omega_N$, and $\pi \vdash_N e : \omega$ then

$$\mu^{O'}_{\pi\omega}[\![e]\!] = \mu^O_{\pi\omega}[\![e]\!]\,,$$

(e) If $\pi, \pi' \in \Omega_N^*$, $\pi = \pi'\rceil \operatorname{dom}\pi$, $\omega \in \Omega_N$, and $\pi \vdash_N e : \omega$ then, for all $\eta' \in \prod(\mathcal{M}O \cdot \pi')$,

$$\mu^O_{\pi'\omega}[\![e]\!]\eta' = \mu^O_{\pi\omega}[\![e]\!](\eta'\rceil \operatorname{dom}\pi)\,,$$

(f) If $\pi \in \Omega^*_{N-\{\tau\}}$, $\omega \in \Omega_N$, $\tau \in N$, and $\pi \vdash_N e : \omega$ then

$$\mu^O_{\pi\omega}[\![(\lambda v_{\Delta\tau.\ \omega}.\ v[\tau])(\Lambda\tau.\ e)]\!] = \mu^O_{\pi\omega}[\![e]\!]\,.$$

Note that 2a, 2b, 3a, 3b, and 3c stipulate the "classical" set-theoretic semantics of the ordinary typed lambda calculus.

5. Expressible Functors

Let T be a functor from $\mathcal{K}$ to $\mathcal{K}$. Roughly speaking, we say that T is *expressible* in a $\mathcal{K}$-model when its action on objects can be expressed by type expressions and its action on morphisms can be expressed by ordinary expressions. More precisely, T is *expressible* in a $\mathcal{K}$-model if and only if:

1. For each $\omega \in \Omega_N$, there is a type expression $\mathbf{T}[\omega] \in \Omega_N$ such that, for all object assignments O with domain N,

$$\mathcal{M}O(\mathbf{T}[\omega]) = T(\mathcal{M}O\omega)\,, \tag{25}$$

2. For each $\omega, \omega' \in \Omega_N$, $\pi \in \Omega_N^*$, and e satisfying $\pi \vdash_N e: \omega \to \omega'$, there is an ordinary expression $\mathbf{T}_{\omega\omega'}[e]$ satisfying $\pi \vdash_N \mathbf{T}_{\omega\omega'}[e]: \mathbf{T}[\omega] \to \mathbf{T}[\omega']$ such that, for all object assignments O with domain N and elements η of $\prod(\mathcal{M}O \cdot \pi)$,

$$\phi(\eta\,;\mu^O[\![\mathbf{T}_{\omega\omega'}[e]]\!]) = T(\phi(\eta\,;\mu^O[\![e]\!]))\,, \tag{26}$$

where ϕ is the isomorphism defined by Equation 13.

Trivially, the identity function can be expressed by $\mathbf{T}[\omega] = \omega$ and $\mathbf{T}_{\omega\omega'}[e] = e$. A less trivial family of expressible functors is provided by the following proposition:

Proposition 2 *Suppose* $\mathbf{B} \in \Omega_{\{\}}$ *and, in some* $\mathcal{K}$*-model,* $B = \mathcal{M}[\,]\mathbf{B} \in |\mathcal{K}|$*. Then there is an expressible functor* T*, from* $\mathcal{K}$ *to* $\mathcal{K}$*, such that* $Tk = (k \Rightarrow B) \Rightarrow B$ *for all* $k \in |\mathcal{K}|$*.*

Proof: We take T to be the composition of the functor Q_B, from $\mathcal{K}$ to $\mathcal{K}^{\mathrm{op}}$, with itself, so that $Tk = Q_B(Q_B k) = (k \Rightarrow B) \Rightarrow B$.

Our main task is to show that, roughly speaking (since it is a functor from $\mathcal{K}$ to $\mathcal{K}^{\mathrm{op}}$ rather than $\mathcal{K}$ to $\mathcal{K}$), Q_B is expressible. For $\omega \in \Omega_N$, let

$$\mathbf{Q}[\omega] \stackrel{\mathrm{def}}{=} \omega \to \mathbf{B} \in \Omega_N\,.$$

Then, for any object assignment O with domain N,

$$\mathcal{M}O(\mathbf{Q}[\omega]) = \mathcal{M}O\omega \Rightarrow B = Q_B(\mathcal{M}O\omega)\,. \tag{27}$$

Next, for $\omega, \omega' \in \Omega_N$, $\pi \in \Omega_N^*$, and e satisfying $\pi \vdash_N e: \omega \to \omega'$, let

$$\mathbf{Q}_{\omega\omega'}[e] \stackrel{\mathrm{def}}{=} (\lambda \mathbf{f}_{\omega\to\omega'}.\, \lambda \mathbf{g}_{\omega'\to\mathbf{B}}.\, \lambda \mathbf{x}_\omega.\, \mathbf{g}(\mathbf{f}\,\mathbf{x}))e\,,$$

which satisfies $\pi \vdash_N \mathbf{Q}_{\omega\omega'}[e]: \mathbf{Q}[\omega'] \to \mathbf{Q}[\omega]$. Then, for any object assignment O with domain N, element η of $\prod(\mathcal{M}O \cdot \pi)$, object $k_0 \in |\mathcal{K}|$, and morphisms $\delta \in k_0 \to (\mathcal{M}O\omega' \Rightarrow B)$ and $\theta \in k_0 \to \mathcal{M}O\omega$,

$$
\begin{aligned}
&\delta\,;\phi(\eta\,;\mu^{O}[\![\mathbf{Q}_{\omega\omega'}[e]]\!]) \rhd \theta \\
&= (\langle\rangle\,;\eta\,;\mu^{O}[\![\mathbf{Q}_{\omega\omega'}[e]]\!] \rhd \delta) \rhd \theta && \text{by 13, 11, 5} \\
&= ((\langle\rangle\,;\mu^{O}[\![\lambda\mathbf{f}_{\omega\to\omega'}.\ \lambda\mathbf{g}_{\omega'\to\mathbf{B}}.\ \lambda\mathbf{x}_{\omega}.\ \mathbf{g}(\mathbf{f}\,\mathbf{x})]\!] \rhd \langle\rangle\,;\eta\,;\mu^{O}[\![e]\!]) \\
&\qquad \rhd\delta) \rhd \theta && \text{by 22, 24} \\
&= \langle\, \mathbf{f}\colon \langle\rangle\,;\eta\,;\mu^{O}[\![e]\!] \mid \mathbf{g}\colon\delta \mid \mathbf{x}\colon\theta \,\rangle\,;\mu^{O}[\![\mathbf{g}(\mathbf{f}\,\mathbf{x})]\!] && \text{by 23} \\
&= \delta \rhd (\langle\rangle\,;\eta\,;\mu^{O}[\![e]\!] \rhd \theta) && \text{by 22, 21} \\
&= \delta \rhd \theta\,;\phi(\eta\,;\mu^{O}[\![e]\!])\,. && \text{by 5, 11, 13}
\end{aligned}
$$

Thus, by the uniqueness property of Equation 15,

$$\phi(\eta\,;\mu^{O}[\![\mathbf{Q}_{\omega\omega'}[e]]\!]) = Q_{B}\big(\phi(\eta\,;\mu^{O}[\![e]\!])\big)\,. \tag{28}$$

Finally, for $\omega \in \Omega_N$, let

$$\mathbf{T}[\omega] \stackrel{\text{def}}{=} \mathbf{Q}\big[\mathbf{Q}[\omega]\big] \in \Omega_N\,,$$

and, for ω, $\omega' \in \Omega_N$, $\pi \in \Omega_N^{*}$, and e satisfying $\pi \vdash_N e\colon \omega \to \omega'$, let

$$\mathbf{T}_{\omega\omega'}[e] \stackrel{\text{def}}{=} \mathbf{Q}_{\mathbf{Q}[\omega'],\mathbf{Q}[\omega]}\big[\mathbf{Q}_{\omega\omega'}[e]\big]\,,$$

which satisfies $\pi \vdash_N \mathbf{T}_{\omega\omega'}[e]\colon \mathbf{T}[\omega] \to \mathbf{T}[\omega']$. Then Equations 27 and 28 imply that T is expressed by $\mathbf{T}[\omega]$ and $\mathbf{T}_{\omega\omega'}[e]$. *(End of Proof)*

We can now establish our main result about expressible functors:

Proposition 3 *Suppose T is a functor from $\mathcal{K}$ to $\mathcal{K}$ that is expressible in a $\mathcal{K}$-model. Then there is an object $P \in |\mathcal{K}|$ and a morphism $H \in TP \to P$ such that, for all $k \in |\mathcal{K}|$ and $\alpha \in Tk \to k$, there is a morphism $M \in P \to k$ making the diagram*

$$
\begin{array}{ccc}
TP & \xrightarrow{\;TM\;} & Tk \\
\Big\downarrow{\scriptstyle H} & & \Big\downarrow{\scriptstyle \alpha} \\
P & \xrightarrow{\;M\;} & k
\end{array}
$$

commute in $\mathcal{K}$.

Proof: Let

$$\begin{aligned} \mathbf{P} &\stackrel{\text{def}}{=} \Delta\mathbf{k}.\ (\mathbf{T}[\mathbf{k}] \to \mathbf{k}) \to \mathbf{k} \in \Omega_{\{\}}\ , \\ \mathbf{M} &\stackrel{\text{def}}{=} \lambda\mathbf{p}_{\mathbf{P}}.\ \mathbf{p}[\mathbf{k}]\mathbf{f}\ , \\ \mathbf{H} &\stackrel{\text{def}}{=} \lambda\mathbf{q}_{\mathbf{T}[\mathbf{P}]}.\ \Lambda\mathbf{k}.\ \lambda\mathbf{f}_{\mathbf{T}[\mathbf{k}]\to\mathbf{k}}.\ \mathbf{f}(\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\mathbf{q})\ , \end{aligned}$$

so that

$$[\,\mathbf{f}: \mathbf{T}[\mathbf{k}] \to \mathbf{k}\,] \vdash_{\{\mathbf{k}\}} \mathbf{M}: \mathbf{P} \to \mathbf{k}\ ,$$

$$[\,] \vdash_{\{\}} \mathbf{H}: \mathbf{T}[\mathbf{P}] \to \mathbf{P}\ .$$

Intuitively, our proof is based on the fact that the diagram

$$\begin{array}{ccc} \mathbf{T}[\mathbf{P}] & \xrightarrow{\ \mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\ } & \mathbf{T}[\mathbf{k}] \\ \Big\downarrow \mathbf{H} & & \Big\downarrow \mathbf{f} \\ \mathbf{P} & \xrightarrow{\quad \mathbf{M} \quad} & \mathbf{k} \end{array}$$

commutes syntactically, i.e. by expressing composition as usual in the lambda calculus, and using beta reduction and type beta reduction. To formalize this intuition, we must work through the semantics of the expressions in this diagram.

Let $P \stackrel{\text{def}}{=} \mathcal{M}[\,]\mathbf{P}$. Since $\mu^{[\,]}[\![\mathbf{H}]\!]$ is an element of $\mathcal{M}[\,](\mathbf{T}[\mathbf{P}] \to \mathbf{P})$, and by Equations 17 and 25, $\mathcal{M}[\,](\mathbf{T}[\mathbf{P}] \to \mathbf{P}) = TP \Rightarrow P$, we may define

$$H \stackrel{\text{def}}{=} \phi(\mu^{[\,]}[\![\mathbf{H}]\!]) \in TP \to P\ . \tag{29}$$

Then, for any $k \in |\mathcal{K}|$ and $\alpha \in Tk \to k$, by Equations 17, 25, and 16,

$$\psi\alpha \in \top \to (Tk \Rightarrow k) = \top \to \mathcal{M}[\,\mathbf{k}: k\,](\mathbf{T}[\mathbf{k}] \to \mathbf{k})\ ,$$

so that

$$\langle\,\mathbf{f}: \psi\alpha\,\rangle\ ;\ \mu^{[\mathbf{k}:k]}[\![\mathbf{M}]\!] \in \top \to \mathcal{M}[\,\mathbf{k}: k\,](\mathbf{P} \to \mathbf{k})\ ,$$

and by Equations 17, 18, and 16, $\mathcal{M}[\,\mathbf{k}: k\,](\mathbf{P} \to \mathbf{k}) = P \Rightarrow k$, so that we may define

$$M \stackrel{\text{def}}{=} \phi(\langle\,\mathbf{f}: \psi\alpha\,\rangle\ ;\ \mu^{[\mathbf{k}:k]}[\![\mathbf{M}]\!]) \in P \to k\ . \tag{30}$$

Finally, we must show that the diagram given in the proposition commutes, i.e. that $H \,;\, M = TM \,;\, \alpha$. We have

$$H \,;\, M$$

$$= H \,;\, (\langle\rangle \,;\, \langle\, \mathbf{f}{:}\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![M]\!] \rhd I_P) \qquad \text{by 30, 13}$$

$$= \langle\, \mathbf{f}{:}\, \langle\rangle \,;\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![M]\!] \rhd H \qquad \text{by 11, 5, 4}$$

$$= \langle\, \mathbf{f}{:}\, \langle\rangle \,;\, \psi\alpha \mid \mathbf{p}{:}\, H \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{p}[\mathbf{k}]\mathbf{f}]\!] \qquad \text{by 23}$$

$$= \langle\, \mathbf{p}{:}\, H \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{p}[\mathbf{k}]]\!] \rhd \langle\rangle \,;\, \psi\alpha \qquad \text{by 22, 24, 21}$$

$$= (\langle\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\lambda \mathbf{p}_{\mathbf{P}}.\, \mathbf{p}[\mathbf{k}]]\!] \rhd H) \rhd \langle\rangle \,;\, \psi\alpha \qquad \text{by 23}$$

$$= (\langle\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\lambda \mathbf{p}_{\mathbf{P}}.\, \mathbf{p}[\mathbf{k}]]\!] \rhd \langle\, \mathbf{q}{:}\, I_{TP} \,\rangle \,;\, \mu^{[\,]}[\![\Lambda \mathbf{k}.\, \lambda \mathbf{f}_{\mathbf{T}[\mathbf{k}]\to\mathbf{k}}.\, \mathbf{f}(\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\mathbf{q})]\!])$$
$$\rhd \langle\rangle \,;\, \psi\alpha \qquad \text{by 29, 13, 23}$$

$$= \langle\, \mathbf{q}{:}\, I_{TP} \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![(\lambda \mathbf{p}_{\mathbf{P}}.\, \mathbf{p}[\mathbf{k}])(\Lambda \mathbf{k}.\, \lambda \mathbf{f}_{\mathbf{T}[\mathbf{k}]\to\mathbf{k}}.\, \mathbf{f}(\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\mathbf{q}))]\!]$$
$$\rhd \langle\rangle \,;\, \psi\alpha \qquad \text{by 24, 19, 22}$$

$$= \langle\, \mathbf{q}{:}\, I_{TP} \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\lambda \mathbf{f}_{\mathbf{T}[\mathbf{k}]\to\mathbf{k}}.\, \mathbf{f}(\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\mathbf{q})]\!] \rhd \langle\rangle \,;\, \psi\alpha \qquad \text{by 20}$$

$$= \langle\rangle \,;\, \psi\alpha \rhd \langle\, \mathbf{q}{:}\, I_{TP} \mid \mathbf{f}{:}\, \langle\rangle \,;\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\mathbf{q}]\!] \qquad \text{by 23, 22, 21}$$

$$= \langle\, \mathbf{q}{:}\, I_{TP} \mid \mathbf{f}{:}\, \langle\rangle \,;\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\mathbf{q}]\!] \,;\, (\langle\rangle \,;\, \psi\alpha \rhd I_{Tk}) \qquad \text{by 11, 5}$$

$$= \langle\, \mathbf{q}{:}\, I_{TP} \mid \mathbf{f}{:}\, \langle\rangle \,;\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]\mathbf{q}]\!] \,;\, \alpha \qquad \text{by 13, 14}$$

$$= (\langle\, \mathbf{f}{:}\, \langle\rangle \,;\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]]\!] \rhd I_{TP}) \,;\, \alpha \qquad \text{by 22, 24, 21}$$

$$= \phi(\langle\, \mathbf{f}{:}\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{T}_{\mathbf{P}\mathbf{k}}[\mathbf{M}]]\!]) \,;\, \alpha \qquad \text{by 4, 13}$$

$$= T(\phi(\langle\, \mathbf{f}{:}\, \psi\alpha \,\rangle \,;\, \mu^{[\mathbf{k}:k]}[\![\mathbf{M}]\!])) \,;\, \alpha \qquad \text{by 26}$$

$$= TM \,;\, \alpha\,. \qquad \text{by 30}$$

(*End of Proof*)

6. T-algebras

Our result about expressible functors can be stated more succinctly by introducing the concepts of T-algebras and weak initiality.

If $\mathcal{K}$ is a category and T is a functor from $\mathcal{K}$ to $\mathcal{K}$, then Talg is the category such that

$$|T\text{alg}| \overset{\text{def}}{=} \{ \langle k, \alpha \rangle \mid k \in |\mathcal{K}| \text{ and } \alpha \in Tk \xrightarrow[\mathcal{K}]{} k \},$$

$$\langle k, \alpha \rangle \xrightarrow[T\text{alg}]{} \langle k', \alpha' \rangle \overset{\text{def}}{=} \{ \beta \mid \beta \in k \xrightarrow[\mathcal{K}]{} k' \text{ and } T\beta ;_{\mathcal{K}} \alpha' = \alpha ;_{\mathcal{K}} \beta \},$$

$$\beta ;_{T\text{alg}} \beta' \overset{\text{def}}{=} \beta ;_{\mathcal{K}} \beta',$$

$$I^{T\text{alg}}_{\langle k, \alpha \rangle} \overset{\text{def}}{=} I^{\mathcal{K}}_k .$$

The objects of Talg are called *T-algebras*, and the morphisms in $\langle k, \alpha \rangle \xrightarrow[T\text{alg}]{} \langle k', \alpha' \rangle$ are called *homomorphisms* from $\langle k, \alpha \rangle$ to $\langle k', \alpha' \rangle$.

An *initial* (*weak initial*) object of a category $\mathcal{K}$ is an object $v \in |\mathcal{K}|$ such that, for all $k \in |\mathcal{K}|$, the set $v \to k$ contains exactly one (at least one) morphism.

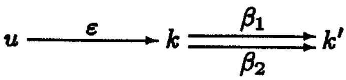
Then Proposition 3 can be restated as:

Proposition 4 *If a functor T from $\mathcal{K}$ to $\mathcal{K}$ is expressible in a $\mathcal{K}$-model then there is a weak initial T-algebra.*

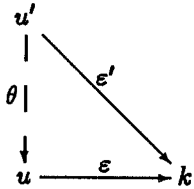
7. Equalizers and Initiality

Our next goal is to sharpen Proposition 4 by finding circumstances in which expressible functors will lead to initial, rather than just weak initial, T-algebras. We will find that a sufficient condition is the existence of enough equalizers in $\mathcal{K}$.

Suppose $\mathcal{K}$ is any category, k, $k' \in |\mathcal{K}|$, and $S \subseteq k \to k'$. If $u \in |\mathcal{K}|$ and $\varepsilon \in u \to k$ are such that



commutes for all β_1, $\beta_2 \in S$, then ε is said to be an *equalizing cone* of S. If $\varepsilon \in u \to k$ is an equalizing cone of S and, for all equalizing cones $\varepsilon' \in u' \to k$ of S, there is exactly one morphism $\theta \in u' \to u$ such that



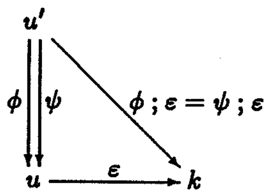
commutes, then ε is said to be an *equalizer* of S.

In the particular case where $\mathcal{K}$ is SET, it is easily seen that an equalizer of S is obtained by taking ε to be the identity injection from u to k, where

$$u = \left\{ x \mid x \in k \text{ and } (\forall \beta_1,\, \beta_2 \in S)\ \beta_1 x = \beta_2 x \right\}.$$

Thus SET possesses equalizers of all subsets of its morphism sets.

For any category $\mathcal{K}$, suppose $\varepsilon \in u \to k$ is an equalizer of some $S \subseteq k \to k'$, and ϕ, $\psi \in u' \to u$. Then $\phi\,;\varepsilon$ and $\psi\,;\varepsilon$ are both equalizing cones of S. Thus, if $\phi\,;\varepsilon = \psi\,;\varepsilon$ then the commutativity of



implies $\phi = \psi$. In other words, equalizers are right-cancellable or *monic*.

The connection between equalizers and initiality is established by the following proposition, which is a slight variation of Theorem V.6.1 in [11]:

Proposition 5 *In a category with a weak initial object w, there is an initial object v if and only if both:*

1. *$w \to w$ has an equalizer,*

2. *For all objects k and k', the morphism set $k \to k'$ has an equalizing cone.*

Proof: Suppose Conditions (1) and (2) hold, and let $\varepsilon \in v \to w$ be the equalizer of $w \to w$. For every object k, since w is weakly initial, there is a morphism $\phi \in w \to k$, so that $\varepsilon ; \phi \in v \to k$; thus v is also weakly initial. To see that it is actually initial, suppose $\beta_1, \beta_2 \in v \to k$. Let $\varepsilon' \in u \to v$ be an equalizing cone of $v \to k$, and let ρ be some morphism in $w \to u$, whose existence is insured by the weak initiality of w. Then

$$v \xrightarrow{\varepsilon} w \xrightarrow{\rho} u \xrightarrow{\varepsilon'} v \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} k$$

commutes, since ε' is an equalizing cone. But

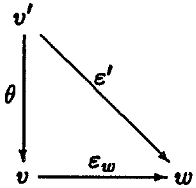
$$v \xrightarrow{\varepsilon} w \underset{I_w}{\overset{\rho ; \varepsilon' ; \varepsilon}{\rightrightarrows}} w$$

also commutes, since ε equalizes $w \to w$. Moreover, since ε is monic, $\varepsilon ; \rho ; \varepsilon' ; \varepsilon = \varepsilon$ implies $\varepsilon ; \rho ; \varepsilon' = I_v$. Thus

$$\beta_1 = \varepsilon ; \rho ; \varepsilon' ; \beta_1 = \varepsilon ; \rho ; \varepsilon' ; \beta_2 = \beta_2 .$$

On the other hand, suppose v is initial, with unique morphisms $\varepsilon_k \in v \to k$ for each object k. Then, for any morphism set $k \to k'$, ε_k is an equalizing cone of $k \to k'$ since, for $\beta_1, \beta_2 \in k \to k'$, initiality gives $\varepsilon_k;\beta_1 = \varepsilon_{k'} = \varepsilon_k;\beta_2$.

Moreover, if w is weakly initial then ε_w is an equalizer of $w \to w$. To see this, suppose $\varepsilon' \in v' \to w$ is an equalizing cone of $w \to w$, and let ρ be some morphism in $w \to v$, whose existence is guaranteed by the weak initiality of w. Then $\rho ; \varepsilon_w \in w \to w$, so that $\varepsilon' ; \rho ; \varepsilon_w = \varepsilon' ; I_w$ since ε' is an equalizing cone. Thus taking $\theta = \varepsilon' \,;\, \rho$ makes



commute. On the other hand, the initiality of v gives $I_v = \varepsilon_w \,;\, \rho$. Thus, if θ is any morphism making the above diagram commute, then $\theta = \theta \,;\, \varepsilon_w \,;\, \rho = \varepsilon' \,;\, \rho$.
(*End of Proof*)

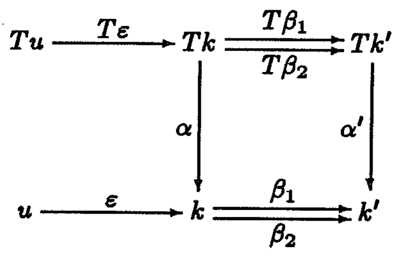
Next, to apply the above proposition to the existence of initial T-algebras, we must relate equalizers in Talg to equalizers in the underlying category $\mathcal{K}$:

Proposition 6 *Suppose T is a functor from $\mathcal{K}$ to $\mathcal{K}$ and, for some T-algebras $\langle k, \alpha \rangle$ and $\langle k', \alpha' \rangle$,*

$$S \subseteq \langle k, \alpha \rangle \xrightarrow[\mathrm{Talg}]{} \langle k', \alpha' \rangle \subseteq k \xrightarrow[\mathcal{K}]{} k' .$$

If S has an equalizer in $\mathcal{K}$ then S has an equalizer in Talg.

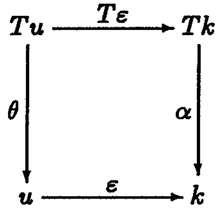
Proof: Let $\varepsilon \in u \to k$ be the equalizer of S in $\mathcal{K}$. For any β_1, $\beta_2 \in S$, consider the diagram



in $\mathcal{K}$. Since ε is an equalizer, $\varepsilon \,;\, \beta_1 = \varepsilon \,;\, \beta_2$, and since T is a functor, $T\varepsilon \,;\, T\beta_1 = T\varepsilon \,;\, T\beta_2$. Then, since β_1 and β_2 are morphisms of T-algebras,

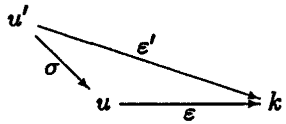
$$T\varepsilon \,;\, \alpha \,;\, \beta_1 = T\varepsilon \,;\, T\beta_1 \,;\, \alpha' = T\varepsilon \,;\, T\beta_2 \,;\, \alpha' = T\varepsilon \,;\, \alpha \,;\, \beta_2 \,.$$

Thus $T\varepsilon\,;\alpha$ is an equalizing cone of S in $\mathcal{K}$, so that there is a unique $\theta \in Tu \to u$ such that

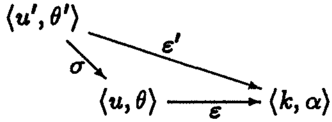


commutes. This implies that $\varepsilon \in \langle u,\theta\rangle \xrightarrow[T\mathrm{alg}]{} \langle k,\alpha\rangle$. Moreover, for any β_1, $\beta_2 \in S$, since composition is the same in Talg as in $\mathcal{K}$, we have $\varepsilon\,;_{T\mathrm{alg}}\,\beta_1 = \varepsilon\,;_{T\mathrm{alg}}\,\beta_2$. Thus ε is an equalizing cone of S in Talg.

Now suppose $\varepsilon' \in \langle u',\theta'\rangle \xrightarrow[T\mathrm{alg}]{} \langle k,\alpha\rangle$ is any equalizing cone of S in Talg. Since composition is the same in Talg as in $\mathcal{K}$, ε' is also an equalizing cone of S in $\mathcal{K}$, so that there is a unique σ such that

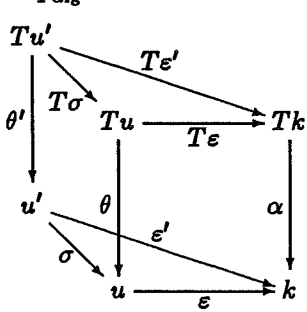


commutes in $\mathcal{K}$. Then σ will also be the unique morphism such that



commutes in Talg, providing it is a morphism of T-algebras.

To see that $\sigma \in \langle u',\theta'\rangle \xrightarrow[T\mathrm{alg}]{} \langle u,\theta\rangle$, consider the diagram



in $\mathcal{K}$. The lower triangle commutes since ε is an equalizer and ε' is an equalizing cone, and the upper triangle then commutes since T is a functor. The square commutes since $\varepsilon \in \langle u, \theta \rangle \xrightarrow[\text{Talg}]{} \langle k, \alpha \rangle$, and the rear parallelogram commutes since $\varepsilon' \in \langle u', \theta' \rangle \xrightarrow[\text{Talg}]{} \langle k, \alpha \rangle$. Thus

$$T\sigma \,;\, \theta \,;\, \varepsilon = T\sigma \,;\, T\varepsilon \,;\, \alpha = T\varepsilon' \,;\, \alpha = \theta' \,;\, \varepsilon' = \theta' \,;\, \sigma \,;\, \varepsilon\,,$$

and since ε is monic, $T\sigma \,;\, \theta = \theta' \,;\, \sigma$. Thus $\sigma \in \langle u', \theta' \rangle \xrightarrow[\text{Talg}]{} \langle u, \theta \rangle$.

(*End of Proof*)

From Propositions 4, 5, and 6, it follows that:

Proposition 7 *If a functor T from $\mathcal{K}$ to $\mathcal{K}$ is expressible in a $\mathcal{K}$-model and all subsets of the morphism sets of $\mathcal{K}$ have equalizers, then there is an initial T-algebra.*

8. Initial T-algebras and Isomorphisms

To complete our development, we use the fact that the morphism parts of initial T-algebras are isomorphisms. The following proposition is given in [2], where it is attributed to J. Lambek:

Proposition 8 *If $\langle u, \theta \rangle$ is an initial T-algebra, then θ is an isomorphism from Tu to u in $\mathcal{K}$.*

Proof: From the obviously commuting diagram

$$\begin{array}{ccc} T(Tu) & \xrightarrow{T\theta} & Tu \\ \downarrow{\scriptstyle T\theta} & & \downarrow{\scriptstyle \theta} \\ Tu & \xrightarrow{\theta} & u \end{array}$$

it is evident that $\langle Tu, T\theta \rangle$ is a T-algebra and $\theta \in \langle Tu, T\theta \rangle \xrightarrow[\text{Talg}]{} \langle u, \theta \rangle$.

Let η be the unique morphism in $\langle u, \theta\rangle \xrightarrow[\mathbf{Talg}]{} \langle Tu, T\theta\rangle$. Then $\eta\,;\theta$ and I_u are both morphisms belonging to $\langle u, \theta\rangle \xrightarrow[\mathbf{Talg}]{} \langle u, \theta\rangle$, so that the initiality of $\langle u, \theta\rangle$ gives $\eta\,;\theta = I_u$. Moreover, since $\eta \in \langle u, \theta\rangle \xrightarrow[\mathbf{Talg}]{} \langle Tu, T\theta\rangle$ and T is a functor,

$$\theta\,;\eta = T\eta\,;T\theta = T(\eta\,;\theta) = T(I_u) = I_{Tu}\,.$$

(*End of Proof*)

Propositions 7 and 8 imply

Proposition 9 *If a functor T from $\mathcal{K}$ to $\mathcal{K}$ is expressible in a $\mathcal{K}$-model, and all subsets of the morphism sets of $\mathcal{K}$ have equalizers, then there is a $u \in |\mathcal{K}|$ such that Tu is isomorphic to u.*

9. The Impossibility of SET-Models

Suppose that there is a SET-model. Let $B = \mathcal{M}[\,]\mathbf{B}$, where $\mathbf{B} = \Delta\mathbf{k}.\ \mathbf{k} \rightarrow (\mathbf{k} \rightarrow \mathbf{k})$. Since SET has morphism sets with more than one member, Proposition 1 shows that B has more than one element. Moreover, since the terminal element of SET is (some) singleton set, the elements of B are in one-to-one correspondence with the members of B, so that B has more than one member.

By Proposition 2 there is an expressible functor T from SET to SET such that $Tk = (k \Rightarrow B) \Rightarrow B$, which in SET is the set of functions $(k \rightarrow B) \rightarrow B$. Thus by Proposition 9, since all subsets of the morphism sets of SET have equalizers, there is a set u such that $(u \rightarrow B) \rightarrow B$ is isomorphic to u. But it is well known that, when B has more than one member, $(u \rightarrow B) \rightarrow B$ has higher cardinality than u, and thus cannot be isomorphic to u. Therefore:

Proposition 10 *There is no* SET-*model.*

10. Application to Known Models

In several models of the polymorphic typed lambda calculus, the meaning of a type is (the set of equivalence classes of) a partial equivalence relation on a model of the untyped lambda calculus [5,23,14,15,4,9]. The underlying Cartesian closed categories of such models possess the equalizers needed to apply Propositions 7 and 9. An important open question for these models, however, is whether the equalizer construction is necessary, or whether $\langle P, H \rangle$, as defined in the proof of Proposition 3, is already an initial (rather than just weakly initial) T-algebra.

Other models, such as [13], [12], [1], and [6], have an underlying Cartesian closed category that is a subcategory of the category of complete partial orders (with a least element) and continuous functions. Unfortunately, these subcategories have a paucity of equalizers. For example, if w is the two-point c.p.o.

$$\begin{array}{c} \top \\ | \\ \perp \end{array}$$

and $\beta_1, \beta_2 \in w \to w$ are the constant functions yielding $\perp$ and $\top$, then $\{\beta_1, \beta_2\}$ has no equalizer.

Indeed, there are few T-algebras for the category of complete partial orders and continuous functions; the usual notion of continuous algebra [8] is equivalent to that of a T-algebra for the category of complete partial orders and *strict* continuous functions, which possesses equalizers of all subsets of its morphism sets, but is not Cartesian closed.

There seems to be a connection between the weak T-algebras obtained for these models and continuous algebras based on the category of complete partial orders and strict continuous functions. However, we have been unable to formulate a precise description of this connection.

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