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CC^{∞}_{C} and its Meta Theory

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by

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$CC^{\infty}_{\subset}$ and Its Meta Theory

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1 Introduction

The Calculus of Constructions (CC for short) [CH88] is a typed higher-order functional calculus which provides a nice basic framework for formalizing mathematics and developing proofs. The meta theory for CC was first studied by Th.Coquand in his PhD thesis [Coq85][Coq86b]. Later on, Th.Coquand proposed an extension of CC with an infinite type hierarchy (we will call it CC^{∞}) in [Coq86a], where the consistency of CC^{∞} is claimed.

$\Sigma CC_{\subset}^{\infty}$ [Luo88] is a higher-order calculus which can be seen as an extension of CC^{∞} by adding strong sum types (only to the type level) and including propositions as types (informally, $Prop \subset Type$). Strong sum types in $\Sigma CC_{\subset}^{\infty}$ provides a useful module mechanism so that abstract structures can be naturally expressed and theories can be thoroughly abstracted, leading to a comprehensive structuring of mathematical texts in proof development. Including propositions as types solves the technical difficulty that adding strong sums to the proposition level of CC results in inconsistency [Coq86a]. The type hierarchy increases the expressiveness of the calculus. For example, according to the research results about the models of the calculus, it seems that, without type hierarchy, one can only formalize recursive mathematics in the calculus of constructions (say, using *Prop* as the 'set universe'). With type hierarchy, one may take one of the type universes as set universes so that abstract mathematics can be formalized.[1] See [Luo88] for more discussions about this.

The theme of this paper is to study the meta theory of $CC_{\subset}^{\infty}$ — the subsystem of $\Sigma CC_{\subset}^{\infty}$ with Σ-types removed. The main result is the strong normalization theorem. We also discuss how the results can be extended to the whole system $\Sigma CC_{\subset}^{\infty}$. This work is based on the work of Th.Coquand [Coq86b] and the work of G.Pottinger and J.Seldin [Pot87][PS86] in their attempts to give a proof of strong normalization theorem for CC. The main contributions of this paper are summarized as follows:

1. As far as we know, this is the first attempt to study the meta theory and prove the strong normalization (henceforth the proof-theoretic consistency) of a system which is an extension of CC with infinite type hierarchy. The results and proofs given here also applies to the system CC^{∞} .

   The key difficulty of seeking a proof of SN theorem of such a system is that, not only propositions but also *proper types* (types that are not convertible to any proposition) can be values. As a consequence, a term of application form, MN, can be a proper type. This makes it very difficult to define a

[1] The proof-theoretic power of the calculus is unknown (as a matter of fact, the power of CC is unknown). The model construction given in [Luo88] can be extended to a model of $\Sigma CC_{\subset}^{\infty}$ by interpreting type universes as set universes. (This is shown to be true by S.Hayashi in a recent joint work with the author.) But we do not know whether the proof-theoretic power of the type hierarchy is really as strong as we expected.

complexity measure of types like the complexity measure β in [Coq86b] which is essential for the proof to work according to the insight of Th.Coquand. This difficulty is overcome by extending the way of using a measure adopted by G.Pottinger and J.Seldin in their attempt to prove the SN theorem for CC [Pot87][PS86] to first prove a *quasi-normalization* result which shows that every proper type can be reduced to a head normal form (see section 4).

2. With inclusion of propositions as types in $CC^{\infty}_{\subset}$, a sort of weak impredicativity is introduced to the type levels, which seems to be questionable concerned about the consistency of the calculus, as people generally believe that, for such a system which has an infinite type hierarchy to be logically consistent, its type levels should be predicative (see [Coq86a], for example). However, the result of strong normalization shows that $CC^{\infty}_{\subset}$ is still logically consistent (theorem 5.11). In other words, the impredicativity at the type levels of $CC^{\infty}_{\subset}$ is weak enough to retain the consistency.

3. Some results which show the basic properties of $CC^{\infty}_{\subset}$ are proved, which are important for people to properly understand the calculus. For example, unlike CC, type uniqueness fails for $CC^{\infty}_{\subset}$ (and CC^{∞}) because of the type universe inclusions. But we will prove a theorem of *type uniqueness upto kinds* (theorem 3.15) and show the existence of a unique *minimum type* (theorem 3.18). This result shows one of the crucial properties of the calculus and also plays an important role in proving the quasi/strong normalization theorem.

4. In $\Sigma CC^{\infty}_{\subset}$, with strong sum types, type uniqueness upto kinds fails. However, following the same pattern as we do for $CC^{\infty}_{\subset}$ and using the notion of *minimal types*, we are still able to prove the SN theorem, which shows the consistency of $\Sigma CC^{\infty}_{\subset}$ and establishes the theoretical soundness of using strong sums to express abstract structures and structure theories in proof development as described in [Luo88].

Section 2 presents the calculus $CC^{\infty}_{\subset}$ and its subsystems $CC^{n}_{\subset}$ 'stopping' at the nth level of types by which we will prove the quasi-normalization result. In section 3, we prove the basic properties of $CC^{\infty}_{\subset}$ (and $CC^{n}_{\subset}$). The quasi-normalization result is proved in section 4 and the strong normalization and consistency in section 5. In section 6, we discuss $\Sigma CC^{\infty}_{\subset}$ and its properties.

2 The Calculus $CC^{\infty}_{\subset}$

2.1 $CC^{\infty}_{\subset}$

$CC^{\infty}_{\subset}$ consists of an underlying (untyped) term calculus and a set of inference rules.

The basic expressions, called *terms*, are inductively defined by the following clauses:

- The constants *Prop* and $Type_i$ ($i \in \omega$), called *kinds*, are terms;
- Variables (x,y,...) are terms;
- If M and N are terms, so are the following:
  - $\Pi x{:}M.N$ (product)
  - $\lambda x{:}M.N$ (λ-abstraction)
  - MN (application)

In $\Pi x{:}M.N$ and $\lambda x{:}M.N$, the free occurences of variable x in N are bound by the binding operators Πx and λx respectively, and nothing in term M is bound by them. Terms which are the same up to changes of bound variables are identified (we will use $\equiv$ for identity). A term of the form $(\lambda x{:}A.M)N$ is a *β-redex* with $[N/x]M$ as its contractum, where $[N/x]M$, the substitution of term N for the free occurrences of variable x in M, is defined as usual with possible changes of bound variables. β-reduction ($\rhd_\beta$) and β-conversion ($\simeq_\beta$) are defined as usual.

Notation From this section to section 5, we only consider β-reduction. So, in these sections, $\rhd$ and $\simeq$ will refer to $\rhd_\beta$ and $\simeq_\beta$, respectively. And, $\rhd_1$ will refer to one step β-reduction (*i.e.*, a single replacement of a redex by its contractum). □

Church-Rosser property holds for the term calculus described so far:[2]

Theorem 2.1 (Church-Rosser theorem) *If $M_1 \simeq M_2$, then there exists M such that $M_1 \rhd M$ and $M_2 \rhd M$.* □

Contexts are finite lists of expressions of the form $x{:}M$, where x is a variable and M is a term. *Formulas* are the expressions of the form $M{:}N$, where M and N are terms. *Judgements* are of the form $\Gamma \vdash F$, where Γ is a context and F is a formula. The set of free variables in a formula $M{:}N$, context $F_1, ..., F_n$ and judgement $\Gamma \vdash F$ are defined as $FV(M) \cup FV(N)$, $\bigcup_{1 \le i \le n} FV(F_i)$ and $FV(\Gamma) \cup FV(F)$, respectively.

The following are the inference rules of CC^ω_C (where ω is the set of natural numbers):

(Ax) $$\frac{}{\vdash Prop{:}Type_0}$$

[2]As is well-known, Church-Rosser property does not hold for $\beta\eta$-conversion for the untyped calculus described above. For the $\beta\sigma$-conversion we will consider in section 6, Church-Rosser holds.

$$(C) \quad \frac{\Gamma \vdash A{:}Type_j}{\Gamma, x{:}A \vdash Prop{:}Type_0} \quad (x \notin FV(\Gamma), j \in \omega)$$

$$(T1) \quad \frac{\Gamma \vdash Prop{:}Type_0}{\Gamma \vdash Type_j{:}Type_{j+1}} \quad (j \in \omega)$$

$$(T2) \quad \frac{\Gamma \vdash A{:}Type_j}{\Gamma \vdash A{:}Type_{j+1}} \quad (j \in \omega)$$

$$(T3) \quad \frac{\Gamma \vdash P{:}Prop}{\Gamma \vdash P{:}Type_0}$$

$$(Tx) \quad \frac{\Gamma, x{:}A, \Gamma' \vdash Prop{:}Type_0}{\Gamma, x{:}A, \Gamma' \vdash x{:}A}$$

$$(\Pi 1) \quad \frac{\Gamma, x{:}A \vdash P{:}Prop}{\Gamma \vdash \Pi x{:}A.P{:}Prop}$$

$$(\Pi 2) \quad \frac{\Gamma \vdash A{:}Type_j \quad \Gamma, x{:}A \vdash B{:}Type_j}{\Gamma \vdash \Pi x{:}A.B{:}Type_j} \quad (j \in \omega)$$

$$(\lambda) \quad \frac{\Gamma, x{:}A \vdash M{:}B}{\Gamma \vdash \lambda x{:}A.M{:}\Pi x{:}A.B}$$

$$(app) \quad \frac{\Gamma \vdash M{:}\Pi x{:}A.B \quad \Gamma \vdash N{:}A}{\Gamma \vdash MN{:}[N/x]B}$$

$$(conv) \quad \frac{\Gamma \vdash M{:}A \quad \Gamma \vdash A'{:}Type_j}{\Gamma \vdash M{:}A'} \quad (A \simeq A', j \in \omega)$$

A *derivation* of a judgement J in $CC^{\infty}_{\subset}$ is a finite sequence of judgements $J_1, ..., J_n$ with $J_m \equiv J$ such that, for all $1 \leq i \leq m$, J_i is the conclusion of some instance of one of the rules above whose premises are among $\{ J_j \mid j < i \}$. We will write $\Gamma \vdash F$ for '$\Gamma \vdash F$ is derivable' and $\Gamma \nvdash F$ for '$\Gamma \vdash F$ is not derivable', when there is no confusion from the context. Specifically, we will sometimes say 'Γ *is a valid context*' to mean '$\Gamma \vdash Prop{:}Type_0$ is derivable'.

Remarks Several remarks are as follows.

1. $CC^{\infty}_{\subset}$ is the extension of CC^{∞} [Coq86a] by adding the rule (T3), which lifts propositions to the type levels. This is a further step of following the principle of 'propositions as types'. With this extension, strong sum types (see section 6) can be used as module mechanism to express abstract structures and structure theories in proof development. (See [Luo88] for more discussions about this.) Intuitively, for the type universes, we have

$$Prop \in Type_0 \in Type_1 \in ...$$

$$Prop \subset Type_0 \subset Type_1 \subset ...$$

As the level of propositions is impredicative, the universe inclusions propagate the impredicativity to the type levels. For example, suppose

$$P \equiv \Pi x{:}Type_0 \Pi B{:}Type_0 \rightarrow Prop.Bx$$

then we have $\vdash P{:}Type_0$ because P is a proposition. As we will show, $CC^{\infty}_{\subset}$ is still strongly normalizing and henceforth consistent, despite of this weak impredicativity.

2. The type universes (kinds) can be ordered as follows

$$Prop \prec Type_0 \prec ...$$

which can further induce a partial order between terms (see definition 3.16 in section 3.3).

3. Note that, in rule (conv), we regard '$A \simeq A'$' as a side condition instead of a part of derivation. When (conv) is applied, the side condition $A \simeq A'$ is checked when the derivations of the premises are completed. Although the untyped conversion relation is in general undecidable, it is decidable for the well-typed terms, by the strong normalization theorem we will prove. □

2.2 $CC^{n}_{\subset}$ and its relationship with $CC^{\infty}_{\subset}$

As we mentioned in the introduction, before proving the strong normalization theorem of $CC^{\infty}_{\subset}$, we shall first prove a quasi-normalization result, which is proved by considering the sunsystems $CC^{n}_{\subset}$ ($n \in \omega$), which are presented below.

The underlying untyped term calculus of $CC^{n}_{\subset}$ is the same as that of $CC^{\infty}_{\subset}$ except that the constants $Type_{n+k+1}$ ($k \in \omega$) are removed. The rules for $CC^{n}_{\subset}$ are listed as follows (the names of the rules are the same as those for $CC^{\infty}_{\subset}$ for ease of comparison):

(Ax) $$\frac{}{\vdash Prop{:}Type_0}$$

(C) $$\frac{\Gamma \vdash A{:}Type_j}{\Gamma, x{:}A \vdash Prop{:}Type_0} \quad (x \notin FV(\Gamma), 0 \leq j \leq n)$$

(T1) $$\frac{\Gamma \vdash Prop{:}Type_0}{\Gamma \vdash Type_j{:}Type_{j+1}} \quad (0 \leq j < n)$$

(T2) $$\frac{\Gamma \vdash A{:}Type_j}{\Gamma \vdash A{:}Type_{j+1}} \quad (0 \leq j < n)$$

$$(T3)\qquad \frac{\Gamma \vdash P{:}Prop}{\Gamma \vdash P{:}Type_0}$$

$$(Tx)\qquad \frac{\Gamma, x{:}A, \Gamma' \vdash Prop{:}Type_0}{\Gamma, x{:}A, \Gamma' \vdash x{:}A}$$

$$(\Pi 1)\qquad \frac{\Gamma, x{:}A \vdash P{:}Prop}{\Gamma \vdash \Pi x{:}A.P{:}Prop}$$

$$(\Pi 2)\qquad \frac{\Gamma \vdash A{:}Type_j \quad \Gamma, x{:}A \vdash B{:}Type_j}{\Gamma \vdash \Pi x{:}A.B{:}Type_j} \quad (0 \leq j \leq n)$$

$$(\lambda)\qquad \frac{\Gamma, x{:}A \vdash M{:}B}{\Gamma \vdash \lambda x{:}A.M{:}\Pi x{:}A.B} \quad (B \not\equiv Type_n)$$

$$(app)\qquad \frac{\Gamma \vdash M{:}\Pi x{:}A.B \quad \Gamma \vdash N{:}A}{\Gamma \vdash MN{:}[N/x]B}$$

$$(conv)\qquad \frac{\Gamma \vdash M{:}A \quad \Gamma \vdash A'{:}Type_j}{\Gamma \vdash M{:}A'} \quad (A \simeq A', 0 \leq j \leq n)$$

Informally, we may describe the relationship of $\mathrm{CC}^n_{\mathsf{C}}$ and $\mathrm{CC}^\infty_{\mathsf{C}}$ as follows:

$$\mathrm{CC}^\infty_{\mathsf{C}} = \bigcup_{n\in\omega} \mathrm{CC}^n_{\mathsf{C}}$$

As any derivation is finite, the following theorem can be easily proved by induction on derivations.[3]

Theorem 2.2 *D is a derivation in $\mathrm{CC}^\infty_{\mathsf{C}}$ if, and only if, there is an $n \in \omega$ such that D is a derivation in $\mathrm{CC}^n_{\mathsf{C}}$.* □

Notation From the next section on, **n** stands for a fixed (arbitrary) natural number for which $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$ is under consideration. □

3 Basic Properties of $\mathrm{CC}^\infty_{\mathsf{C}}$ (and $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$)

Some basic properties of the calculus $\mathrm{CC}^\infty_{\mathsf{C}}$ (and $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$) are proved in this section. If not explicitly indicated, the results proved in this section hold for both $\mathrm{CC}^\infty_{\mathsf{C}}$ and $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$. The proofs are to be given for $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$ and can be easily modified for $\mathrm{CC}^\infty_{\mathsf{C}}$.

[3]We will say 'by induction on derivations of ...' to mean 'by induction on the length of derivations of ...'.

3.1 Some basic lemmas

Lemma 3.1 *Let K and K' be kinds and Γ be a valid context. Then, $\Gamma \vdash K{:}K'$ if, and only if, $K \prec K'$.*

Proof Sufficiency is obvious by using rules (T1)(T2), as Γ is valid. For proving necessity, we can prove the following stronger result:

$$\Gamma \vdash K{:}A \ \wedge \ A \simeq K' \ \Rightarrow \ K \prec K'$$

by induction on derivations of $\Gamma \vdash K{:}A$. □

Lemma 3.2 *Any derivation D of $\Gamma, x{:}A, \Gamma' \vdash F$ has a sub-derivation of $\Gamma \vdash A{:}K$ for some kind K.*

Proof Induction on D. □

Lemma 3.3 *If $\Gamma \vdash F$, then $FV(F) \subseteq FV(\Gamma)$.*

Proof By induction on derivations of $\Gamma \vdash F$ and use lemma 3.2 when considering the rules (Π1) and (λ). □

Lemma 3.4 *If $\Gamma \vdash F$, then Γ has the form $x_1{:}A_1, ..., x_m{:}A_m$ such that*

1. *$x_1, ..., x_m$ are distinctive;*
2. *$FV(A_i) \subseteq \{\, x_j \mid 1 \leq j \leq i-1 \,\}$ for $i = 1, ..., m$.*

Proof Induction on derivations of $\Gamma \vdash F$ and use lemma 3.3 when considering the rule (C). □

Lemma 3.5 *In $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$, if $\Gamma \vdash M{:}A$, then $Type_{\mathbf{n}}$ does not occur in M or Γ.*

Proof Induction on derivations of $\Gamma \vdash M{:}A$. □

Lemma 3.6 *In $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$, if $\Gamma \vdash M{:}A$, then either $A \equiv Type_{\mathbf{n}}$ or $Type_{\mathbf{n}}$ does not occur in A.*

Proof By induction on derivations of $\Gamma \vdash M{:}A$ and use lemma 3.5 when considering (Tx)(λ)(app)(conv). □

Lemma 3.7 *Any derivation of $\Gamma, \Gamma' \vdash F$ has a sub-derivation of $\Gamma \vdash Prop{:}Type_0$.*

Proof Induction on derivations of $\Gamma, \Gamma' \vdash F$. □

Remark The above lemma implies that Γ is a valid context whenever $\Gamma \vdash F$. □

Lemma 3.8 *If* $\Gamma \vdash F$ *and* Γ' *is a valid context which contains every component of* Γ, *then* $\Gamma' \vdash F$.

Proof By induction on derivations of $\Gamma \vdash F$. For the rules other than $(\Pi 1)(\lambda)(\Pi 2)$, applying induction hypothesis and the same rule suffices. For rule $(\Pi 2)$,

$$\frac{\Gamma \vdash A{:}Type_j \quad \Gamma, x{:}A \vdash B{:}Type_j}{\Gamma \vdash \Pi x{:}A.B{:}Type_j} \quad (0 \leq j \leq \mathbf{n})$$

by induction hypothesis, $\Gamma' \vdash A{:}Type_j$, and, assuming that $x \notin FV(\Gamma')$, $\Gamma', x{:}A \vdash B{:}Type_j$. Then, using rule $(\Pi 2)$, we have $\Gamma' \vdash \Pi x{:}A.B{:}Type_j$. For rules $(\Pi 1)(\lambda)$, using lemma 3.2 and similar arguments as that for $(\Pi 2)$ suffice. □

3.2 Some admissibility results

In this subsection, we study some admissibility results about the calculus. We will show that

1. type-preserving substitutions preserve derivability (theorem 3.9);
2. every type (other than $Type_{\mathbf{n}}$ for $\mathrm{CC}^{\mathbf{n}}_{\mathrm{C}}$) is typed by some kind (theorem 3.10);
3. replacing a predicate in a context by a convertible term preserves derivability (lemma 3.11);
4. subject reduction holds, *i.e.*, β-reduction preserves typing relations (theorem 3.12);
5. removing redundant components in a context preserves derivability (lemma 3.13).

Theorem 3.9 *If* $\Gamma, x{:}N, \Gamma' \vdash F$ *and* $\Gamma \vdash M{:}N$, *then* $\Gamma, [M/x]\Gamma' \vdash [M/x]F$.

Proof By induction on derivations of $\Gamma, x{:}N, \Gamma' \vdash F$. We only check rules (Tx) and $(\Pi 1)$. The other rules can be similarly checked.

For (Tx),

$$\frac{\Gamma_1, y{:}A, \Gamma_2 \vdash Prop{:}Type_0}{\Gamma_1, y{:}A, \Gamma_2 \vdash y{:}A}$$

there are two cases:

1. $x{:}N \equiv y{:}A$. We have to show $\Gamma, [M/x]\Gamma' \vdash M{:}N$. This is true by induction hypothesis and lemma 3.8, as $\Gamma \vdash M{:}N$.
2. $x{:}N$ occurs in Γ_1 or Γ_2. By induction hypothesis, $\Gamma, [M/x]\Gamma' \vdash Prop{:}Type_0$. As $x \not\equiv y$ by lemma 3.4, $\Gamma, [M/x]\Gamma'$ contains the component $y{:}[M/x]A$. So, application of (Tx) yields the result.

For (Π1),

$$\frac{\Gamma, x{:}N, \Gamma', y{:}A \vdash P{:}Prop}{\Gamma, x{:}N, \Gamma' \vdash \Pi y{:}A.P{:}Prop}$$

as $x \not\equiv y$ by lemma 3.4, we have, by induction hypothesis, $\Gamma, [M/x]\Gamma', y{:}[M/x]A \vdash [M/x]P{:}Prop$. Applying (Π1), $\Gamma, [M/x]\Gamma' \vdash \Pi y{:}[M/x]A.[M/x]P{:}Prop$. By lemma 3.3, $y \notin FV(M)$, as $\Gamma \vdash M{:}N$. So, $\Gamma, [M/x]\Gamma' \vdash [M/x]\Pi y{:}A.P{:}Prop$. □

Theorem 3.10 [4] *In* CC^{n}_{C} *, if* $\Gamma \vdash M{:}A$ *and* $A \not\equiv Type_{n}$*, then* $\Gamma \vdash A{:}K$ *for some kind* K*.*

Proof By induction on derivations of $\Gamma \vdash M{:}A$. We check the case for rule (app): with $M \equiv M'N$ and $A \equiv [N/x]B$:

$$\frac{\Gamma \vdash M'{:}\Pi x{:}A'.B \quad \Gamma \vdash N{:}A'}{\Gamma \vdash M'N{:}[N/x]B}$$

By induction hypothesis, $\Gamma \vdash \Pi x{:}A'.B{:}K$ and $\Gamma \vdash A'{:}K'$ for some kinds K and K'. A derivation of $\Gamma \vdash \Pi x{:}A'.B{:}K$ must end with (Π1),(Π2),(conv),(T2) or (T3). If it ends with (Π1) or (Π2), using Theorem 3.9 will suffice to get the required result. If (conv),(T2) or (T3), as the form of $\Pi x{:}A'.B$ does not change in these rules, there must be a sub-derivation which ends with rule (Π1) or (Π2) and whose conclusion is $\Gamma \vdash \Pi x{:}A'.B{:}K''$ for some kind $K'' \preceq K$. Then, using theorem 3.9 will also yield the desired result.

The other cases are easier. For (Tx), use lemma 3.2 and lemma 3.8. For (λ), use lemma 3.2. □

Lemma 3.11 *If* $\Gamma, x{:}A, \Gamma' \vdash F$*,* $\Gamma \vdash B{:}K$ *for some kind* K*, and* $A \simeq B$*, then* $\Gamma, x{:}B, \Gamma' \vdash F$*.*

Proof By induction on derivations of $\Gamma, x{:}A, \Gamma' \vdash F$. □

Theorem 3.12 (subject reduction) *If* $\Gamma \vdash M{:}A$ *and* $M \rhd N$*, then* $\Gamma \vdash N{:}A$*.*

Proof We only need to show that

$$\Gamma \vdash M{:}A \;\wedge\; M \rhd_1 N \;\Rightarrow\; \Gamma \vdash N{:}A$$

This is proved by induction on derivations of $\Gamma \vdash M{:}A$. We only check the cases (Π1) and (app) here, with other easier cases omitted.

For (Π1), with $M \equiv \Pi x{:}A'.P$ and $A \equiv Prop$:

$$\frac{\Gamma, x{:}A' \vdash P{:}Prop}{\Gamma \vdash \Pi x{:}A'.P{:}Prop}$$

[4] For CC^{∞}_{C} , this theorem is formulated as: if $\Gamma \vdash M{:}A$, then $\Gamma \vdash A{:}K$ for some kind K.

as $M \rhd_1 N$, N must have form $\Pi x{:}A''.P'$ such that either

$$A' \rhd_1 A'' \ \wedge \ P \equiv P'$$

or

$$A' \equiv A'' \ \wedge \ P \rhd_1 P'$$

In the first case, by lemma 3.2 and induction hypothesis, $\Gamma \vdash A''{:}K$ for some kind K, and then, by lemma 3.11, $\Gamma, x{:}A'' \vdash P'{:}Prop$. In the second case, by induction hypothesis, $\Gamma, x{:}A'' \vdash P'{:}Prop$. So, in both cases, an application of (Π1) yields the required result.

For (Π2), it is similar to (Π1).

For (λ), use (conv) and induction hypothesis.

For (app), with $M \equiv M'N'$ and $A \equiv [N'/x]B$:

$$\frac{\Gamma \vdash M'{:}\Pi x{:}A'.B \quad \Gamma \vdash N'{:}A'}{\Gamma \vdash M'N'{:}[N'/x]B}$$

There are two cases:

1. $N \equiv M''N''$, and, either $M' \rhd_1 M''$ and $N' \equiv N''$, or $M' \equiv M''$ and $N' \rhd_1 N''$. In this case, by induction hypothesis, $\Gamma \vdash M''{:}\Pi x{:}A'.B$ and $\Gamma \vdash N''{:}A'$. So, applying (app) yields $\Gamma \vdash N{:}[N''/x]B$. Since $[N''/x]B \simeq [N'/x]B$ (as $N' \simeq N''$) and, by theorem 3.10, $\Gamma \vdash [N'/x]B{:}K$ for some kind K, using (conv) yields $\Gamma \vdash N{:}[N'/x]B$.

2. $M \equiv M'N' \equiv (\lambda x{:}A''.C)N' \rhd_1 [N'/x]C \equiv N$. A derivation of $\Gamma \vdash M'{:}\Pi x{:}A'.B$ must end with (λ) or (conv). If it ends with (λ), applying theorem 3.9 suffices. If it ends with (conv), we have

$$\frac{\Gamma \vdash \lambda x{:}A''.C{:}X \quad \Gamma \vdash X{:}Type_j}{\Gamma \vdash \lambda x{:}A''.C{:}\Pi x{:}A'.B} \quad (X \simeq \Pi x{:}A'.B)$$

   We may assume that the last rule used to derive $\Gamma \vdash \lambda x{:}A'.C{:}X$ is not (conv), then it must be (λ), *i.e.*,

$$\frac{\Gamma, x{:}A'' \vdash C{:}B'}{\Gamma \vdash \lambda x{:}A''.C{:}\Pi x{:}A''.B'}$$

   where $X \equiv \Pi x{:}A''.B'$. By Church-Rosser theorem, we have $X \equiv \Pi x{:}A''.B' \rhd \Pi x{:}A_0.B_0$ and $\Pi x{:}A'.B \rhd \Pi x{:}A_0.B_0$ for some A_0, B_0 such that $A' \rhd A_0$, $A'' \rhd A_0$, $B' \rhd B_0$ and $B \rhd B_0$. So, $A'' \simeq A'$ and $B' \simeq B$. By theorem 3.10 and lemma 3.11 and applying (conv), we have $\Gamma, x{:}A' \vdash C{:}B$. Then, by theorem 3.9, we have $\Gamma \vdash [N'/x]C{:}[N'x]B$, *i.e.*, $\Gamma \vdash N{:}[N'x]B$.

This completes the proof of the theorem. □

Remark Unlike the pure calculus of constructions (CC) [CH88], although subject reduction holds, the following rule is in general *not admissible*:

$$(**) \qquad \frac{\Gamma \vdash M{:}A \quad \Gamma \vdash N{:}B}{\Gamma \vdash M{:}B} \quad (M \rhd N)$$

An example which shows this is as follows:

$$M \equiv (\lambda x{:}Type_1.x)Prop \rhd Prop \equiv N$$

We have $\vdash Prop{:}Type_0$, but $\not\vdash M{:}Type_0$. In fact, we only have $\vdash M{:}Type_i$ for $i \geq 1$. □

Lemma 3.13 *If* $\Gamma, y{:}Y, \Gamma' \vdash M{:}A$ *and* $y \notin FV(M{:}A) \cup FV(\Gamma')$, *then* $\Gamma, \Gamma' \vdash M{:}A$.

Proof First we prove, by induction on derivations of $\Gamma, y{:}Y, \Gamma' \vdash M{:}A$, the following statement:

$$\Gamma, y{:}Y, \Gamma' \vdash M{:}A \ \wedge \ y \notin FV(M) \cup FV(\Gamma') \ \Rightarrow \ \exists A' \simeq A.\ \Gamma, \Gamma' \vdash M{:}A'$$

We only check the rule (app) here. The other cases are either obvious or easier. For (app), with $M \equiv M_1M_2$, $A \equiv \Pi x{:}A_1.A_2$,

$$\frac{\Gamma, y{:}Y, \Gamma' \vdash M_1{:}\Pi x{:}A_1.A_2 \quad \Gamma, y{:}Y, \Gamma' \vdash M_2{:}A_1}{\Gamma, y{:}Y, \Gamma' \vdash M_1M_2{:}[M_2/x]A_2}$$

By induction hypothesis, there exists $A_1' \simeq A_1$ such that

$$\Gamma, \Gamma' \vdash M_2{:}A_1'$$

and there exists $A'' \simeq \Pi x{:}A_1.A_2$ such that $\Gamma, \Gamma' \vdash M_1{:}A''$. By lemma 3.3, $y \notin FV(A_1') \cup FV(A'')$. By Church-Rosser theorem, $A'' \rhd \Pi x{:}A_1''.A_2''$, where $A_i \rhd A_i''$ $(i = 1, 2)$. As reduction does not produce new free variables, $y \notin \Pi x{:}A_1''.A_2''$. By subject reduction,

$$\Gamma, \Gamma' \vdash M_1{:}\Pi x{:}A_1''.A_2''$$

As $A_1'' \simeq A_1 \simeq A_1'$, by theorem 3.11 and applying rule (conv), we have

$$\Gamma, \Gamma' \vdash M_2{:}A_1''$$

Now, applying rule (app), we have

$$\Gamma, \Gamma' \vdash M_1M_2{:}[M_2/x]A_2''$$

where $y \notin [M_2/x]A_2'' \simeq [M_2/x]A_2$ as required.

Now, suppose $\Gamma, y{:}Y, \Gamma' \vdash M{:}A$ and $y \notin FV(M{:}A) \cup FV(\Gamma')$. Then, by the statement just proved above, there exists $A' \simeq A$ such that

$$\Gamma, \Gamma' \vdash M{:}A'$$

and $y \notin FV(A')$ by lemma 3.3. To get $\Gamma, \Gamma' \vdash M{:}A$ by applying (conv), we only have to show that $\Gamma, \Gamma' \vdash A{:}K$ for some kind K. By theorem 3.11, $\Gamma, y{:}Y, \Gamma' \vdash A{:}K$ for some kind K. As $y \notin FV(A) \cup FV(\Gamma')$, using the above statement, we have, there is $P \simeq K$ such that $\Gamma, \Gamma' \vdash A{:}P$. If $Type_{\mathbf{n}}$ does not occur in P, then $K \not\equiv Type_{\mathbf{n}}$ by Church-Rosser theorem. So, $\Gamma, \Gamma' \vdash K{:}K'$ for some kind K'. By using (conv), we have $\Gamma, \Gamma' \vdash A{:}K$. If $Type_{\mathbf{n}}$ occurs in P, then $P \equiv Type_{\mathbf{n}}$ by lemma 3.6. Then, by Church-Rosser theorem, $K \equiv Type_{\mathbf{n}} \equiv P$. We also have $\Gamma, \Gamma' \vdash A{:}K$. So, we can apply (conv) to get $\Gamma, \Gamma' \vdash M{:}A$.

This completes the proof of the lemma. □

3.3 Typing properties

In this subsection, we study the properties of typing in the calculus. Although the type uniqueness fails because of the inclusions between type universes, we show that types of a term are 'unique upto kinds' (theorem 3.15). Furthermore, the type universes induces an ordering under which there exists a unique minimum type (upto conversion) for every well-typed term (theorem 3.18). We also define a classification of types according to their levels and show that the types at the highest level have specific forms.

Definition 3.14 (Γ-types) *Let Γ be a valid context. Then, For $\mathrm{CC}^{\infty}_{\mathrm{C}}$, A is a* Γ-type *iff $\Gamma \vdash A{:}K$ for some kind K. For $\mathrm{CC}^{\mathbf{n}}_{\mathrm{C}}$, A is a* Γ-type *iff $\Gamma \vdash A{:}K$ for some kind K or $A \equiv Type_{\mathbf{n}}$.* □

Theorem 3.15 (type uniqueness upto kinds) *If $\Gamma \vdash M{:}A$ and $\Gamma \vdash M{:}B$, then either*

$$A \simeq B$$

or,

$$A \simeq C \quad and \quad B \simeq C'$$

for some Γ-types

$$C \equiv \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K$$

$$C' \equiv \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K'$$

where $m \geq 0$ and K, K' are different kinds.

Proof By Church-Rosser theorem and theorem 3.12, we only have to show that, if $\Gamma \vdash M{:}A$ and $\Gamma \vdash M{:}B$, then either $A \simeq B$, or $A \simeq \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K$ and $B \simeq \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K'$ for some $C_1, ..., C_m$ and some kinds K and K'.

We prove this by induction on the sum of the lengths of derivations D_A and D_B of $\Gamma \vdash M{:}A$ and $\Gamma \vdash M{:}B$. In view of the forms of $\Gamma \vdash M{:}A$ and $\Gamma \vdash M{:}B$, we have the following table, which shows possible combinations of rules as the last ones for D_A and D_B:

	(conv)	(Ax)	(C)	(T1)	(T2)	(T3)	(Tx)	(Π1)	(Π2)	(λ)	(app)
(conv)	√	√	√	√	√	√	√	√	√	√	√
(Ax)		√	−	−	√	√	−	−	−	−	−
(C)			√	−	√	√	−	−	−	−	−
(T1)				√	√	√	−	−	−	−	−
(T2)					√	√	√	√	√	−	√
(T3)						√	√	√	√	−	√
(Tx)							√	−	−	−	−
(Π1)								√	√	−	−
(Π2)									√	−	−
(λ)										√	−
(app)											√

where − means that the situation is impossible and √ means that the result holds in that case (only half of the table is shown, due to the duality of D_A and D_B).

Let's check the case when both D_A and D_B end with (app). Suppose

$$\frac{\Gamma \vdash M_1{:}\Pi x{:}A_1.B_1 \quad \Gamma \vdash M_2{:}A_1}{\Gamma \vdash M_1M_2{:}[M_2/x]B_1}$$

$$\frac{\Gamma \vdash M_1{:}\Pi x{:}A_2.B_2 \quad \Gamma \vdash M_2{:}A_2}{\Gamma \vdash M_1M_2{:}[M_2/x]B_2}$$

with $M \equiv M_1M_2$, $A \equiv [M_2/x]B_1$ and $B \equiv [M_2/x]B_2$. By induction hypothesis, either

$$\Pi x{:}A_1.B_1 \simeq \Pi x{:}A_2.B_2$$

or, for some kinds K and K',

$$\Pi x{:}A_1.B_1 \simeq \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K$$

$$\Pi x{:}A_2.B_2 \simeq \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K'$$

In the first case, by Church-Rosser theorem, $B_1 \simeq B_2$. So,

$$A \equiv [M_2/x]B_1 \simeq [M_2/x]B_2 \equiv B$$

In the second case, $B_1 \simeq \Pi x_2{:}C_2 ... \Pi x_m{:}C_m.K$ and $B_2 \simeq \Pi x_2{:}C_2 ... \Pi x_m{:}C_m.K'$. We may assume that $x_2, ..., x_m \notin FV(M_2) \cup \{x\}$. So,

$$A \equiv [M_2/x]B_1 \simeq [M_2/x]\Pi x_2{:}C_2 ... \Pi x_m{:}C_m.K \equiv \Pi x_2{:}[M_2/x]C_2 ... \Pi x_m{:}[M_2/x]C_m.K$$

$$B \equiv [M_2/x]B_2 \simeq [M_2/x]\Pi x_2{:}C_2 ... \Pi x_m{:}C_m.K' \equiv \Pi x_2{:}[M_2/x]C_2 ... \Pi x_m{:}[M_2/x]C_m.K'$$

The other cases are simpler and omitted here. □

The order $\prec$ over kinds can be extended to a partial order over terms, under which a notion of minimal/minimum type will be introduced.

Definition 3.16 (partial order induced by universes) *Define $\preceq$ as the smallest partial order over terms which is congruent w.r.t. $\simeq$ such that*

1. *$Prop \preceq Type_0 \preceq Type_1 \preceq ...$;*
2. *if $A \simeq A'$ and $B \preceq B'$, then $\Pi x{:}A.B \preceq \Pi x{:}A'.B'$.*

$A \prec B$ if, and only if, $A \preceq B$ and $A \not\simeq B$. □

Definition 3.17 (minimal/minimum type) *Let $\Gamma \vdash M{:}A$. Then,*

1. *A is called a* minimal type *of M (under Γ) if, for all A' such that $\Gamma \vdash M{:}A'$, $A' \not\prec A$;*
2. *A is called a* minimum type *of M (under Γ) if, for all A' such that $\Gamma \vdash M{:}A'$, $A \preceq A'$.* □

Remark By the definition of $\preceq$ (definition 3.16), there is no infinite decreasing sequence $A_1 \succ A_2 \succ ...$. So, every well-typed term has minimal types. A minimum type of a term (if it exists) is unique upto conversion. For the system ΣCC^{∞}_{C} , not every well-typed term has a minimum type (see section 6 for more discussions). But, for CC^{n}_{C} (CC^{∞}_{C} and CC^{∞}), by theorem 3.15, we have that every well-typed term has a minimum type. □

Theorem 3.18 *In CC^{n}_{C} (CC^{∞}_{C}), every well-typed term has a minimum type, i.e., if $\Gamma \vdash M{:}A$, then, there exists a minimum type of M.*

Proof We show that all minimal types of M are convertible, which implies that every minimal type is a minimum type. Let C be the set defined as follows:

$$C =_{\mathrm{df}} \{\, [B]_{\simeq} \mid B \textit{ is a minimal type of } M \,\}$$

where $[B]_{\simeq} =_{\mathrm{df}} \{\, B' \mid \Gamma \vdash M{:}B' \wedge B \simeq B' \,\}$. We show that $\#C = 1$.[5] First, as $\Gamma \vdash M{:}A$, $C \neq \emptyset$. If $\#C > 1$, let $[B]_{\simeq}, [B']_{\simeq} \in C$ be different. By theorem 3.15, we have, for some kinds $K \not\equiv K'$ (say, $K \prec K'$), $B \simeq \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K$ and $B' \simeq \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K'$ (as $B \not\simeq B'$). But this implies that $B \prec B'$, which contradicts the assumption that B' is a minimal type of M. So, we have $\#C = 1$,

[5] We use $\#S$ to express the cardinality of a set S.

i.e., all minimal types of M are convertible. As minimal types of M always exist, so does a minimum type of M. □

Remark Note that the minimum type is not the 'most general' type. In other words, the following rule may be not admissible:

$$\frac{\Gamma \vdash M{:}A \quad \Gamma \vdash A'{:}K}{\Gamma \vdash M{:}A'} \quad (A \preceq A')$$

For example, we may have

$$x{:}Type_0 \rightarrow Type_0 \not\vdash x{:}Type_0 \rightarrow Type_1$$

If we had subject reduction also for η-reduction, than the above rule would become admissible. But, as well-known, untyped η-conversion destroys Church-Rosser property of the underlying untyped term calculus. Further research is needed to see whether restricting conversions to well-typed terms can make the above rule be admissible. □

Definition 3.19 (categories of Γ-types) *Let A be a Γ-type. Then, for $j \geq 0$, (where $Type_{-1} \equiv Prop$),*

1. *A is a Γ-proposition iff $\exists A' \simeq A.\Gamma \vdash A'{:}Prop$;*
2. *A is a Γ-j-type iff $\exists A' \simeq A.\Gamma \vdash A'{:}Type_j$;*
3. *A is a proper Γ-j-type iff A is a Γ-j-type and A is not a Γ-$(j-1)$-type.*

A is called a proper Γ-type *if it is a proper Γ-j-type for some $j \geq 0$.*

Remark The fact that the rule (**) in the remark in section 3.2 is not admissible makes the above definition more complicated than expected, as we have to regard a type which can be reduced to a Γ-j-type but it itself can not be typed by $Type_j$ as residing in the jth level. For example, $(\lambda x{:}Type_1.x)Prop$ is a proper $<>$-0-type, but $\not\vdash (\lambda x{:}Type_1.x)Prop{:}Type_0$. □

The next lemma shows that the above classification of types are 'exhaustive' and 'exclusive'.

Lemma 3.20 *In $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$, if $\Gamma \vdash M{:}A$, then exactly one of the following holds:*

1. *$A \equiv Type_{\mathbf{n}}$;*
2. *A is a Γ-proposition;*
3. *A is a proper Γ-j-type for exactly one $j \in \omega$.*

Proof Exhaustiveness is by theorem 3.10, lemmas 3.5 and 3.6. Exclusiveness is obvious from the definition. □

Lemma 3.21 *In* $CC^{\mathbf{n}}_{C}$ *, Let* A *be a proper* Γ-$\mathbf{n}$-*type. Then,* A *has one of the following forms:*

- $Type_{\mathbf{n}-1}$ *(Prop, when* $\mathbf{n}=0$*)*
- $\Pi x{:}A_1.A_2$

Proof By induction on derivations of $\Gamma \vdash A{:}Type_{\mathbf{n}}$. For the rules (Ax),(C),(T1) and (Π2), it is obvious. The other rules are not applicable as the last rule, either because of the form of $A{:}Type_{\mathbf{n}}$ or because of the condition that A does not convert to any Γ-($\mathbf{n}-1$)-type. For example, (app) is not applicable because, otherwise, either $B \equiv Type_{\mathbf{n}}$ or $N \equiv Type_{\mathbf{n}}$ but the former is impossible by the side-condition of (λ) and the later impossible by lemma 3.5. □

Lemma 3.22 *In* $CC^{\mathbf{n}}_{C}$ *, let* $A \equiv \Pi x{:}A_1.A_2$*. Then,* A *is a proper* Γ-$\mathbf{n}$-*type if, and only if,*

1. *either* A_1 *is a proper* Γ-$\mathbf{n}$-*type or* A_2 *is a proper* $(\Gamma, x{:}A_1)$-$\mathbf{n}$-*type, and*
2. A_2 *is a proper* $(\Gamma, x{:}A_1)$-j-*type for some* $0 \leq j \leq \mathbf{n}$.

Proof Sufficiency is by theorem 3.10, applying (Π2) and inspecting the rules. For necessity, suppose $\Pi x{:}A_1.A_2$ is a proper Γ-$\mathbf{n}$-type. Then, A_1 is a Γ-type and A_2 is a $(\Gamma, x{:}A_1)$-type. If A_1 is not a proper Γ-$\mathbf{n}$-type and A_2 is not a proper $(\Gamma, x{:}A_1)$-$\mathbf{n}$-type, then, by lemma 3.20, A_1 is a Γ-proposition or a Γ-j-type for some $j < \mathbf{n}$ and A_2 is a $(\Gamma, x{:}A_1)$-proposition or a $(\Gamma, x{:}A_1)$-j'-type for some $j' < \mathbf{n}$. By applying rules (T2)(T3)(Π1)(Π2), we have $\Pi x{:}A_1.A_2$ is a Γ-proposition or a Γ-j''-type for some $j'' < \mathbf{n}$, which contradicts the assumption. If A_2 is not a proper Γ-j-type for every $0 \leq j \leq \mathbf{n}$, then it is a Γ-proposition by lemma 3.20. So, A is a Γ-proposition by (Π1), which also contradicts the assumption. □

Remark All of the results proved in the above three subsections hold for the system CC^{∞} [Coq86a] with some of them slightly modified. For example, the theorem of type uniqueness upto kinds for CC^{∞} can be stated as follows:

Theorem In CC^{∞} , if $\Gamma \vdash M{:}A$ and $\Gamma \vdash M{:}B$, then either

$$A \simeq B$$

or,

$$A \simeq C \ \ and \ \ B \simeq C'$$

for some Γ-types

$$C \equiv \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.Type_i$$
$$C' \equiv \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.Type_j$$

□

4 Quasi Normalization of $\mathrm{CC}^{\mathrm{n}}_{\mathrm{C}}$

In this section, we prove the quasi-normalization result, which says that every proper type can be reduced to a head normal form (theorems 4.21 and 4.23). Section 4.1 introduces the notion of environment — 'infinite universal context', following [Pot87]. Section 4.2 consists of an inductive proof of the quasi-normalization theorem. In section 4.3, we summarize the result of quasi-normalization.

4.1 Environments

Definition 4.1 (Environment) *An* environment *is an infinite list*

$$\mathcal{E} \equiv x_1{:}M_1, x_2{:}M_2, ...$$

where x_i is a variable and M_i is a term, such that

1. *for all $i \in \omega$, $\mathcal{E}^i \vdash Prop{:}Type_0$, and*
2. *for any $i \in \omega$, for any kind K, if $\mathcal{E}^i \vdash A{:}K$, then the set $\{\, j \mid \mathcal{E}_j \equiv x_j{:}A \,\}$ is infinite*

where $\mathcal{E}^i \equiv x_1{:}M_1, ..., x_i{:}M_i$ and $\mathcal{E}_j \equiv x_i{:}M_j$. □

Every valid context can be extended to an environment, as the following lemma shows:

Lemma 4.2 *If Γ is a valid context, then there exists an environment $\mathcal{E}$ such that $\mathcal{E}^i \equiv \Gamma$ for some $i \in \omega$.*

For an environment $\mathcal{E}$, we write $\mathcal{E} \vdash M{:}A$ if, and only if, $\mathcal{E}^i \vdash M{:}A$ for some $i \in \omega$. If $\mathcal{E} \vdash M{:}A$, $M{:}A$ is called an *$\mathcal{E}$-formula* and, M and A are called *$\mathcal{E}$-terms.* A will also be called an *$\mathcal{E}$-type.* $M \rhd_{\mathcal{E}} N$ ($M \simeq_{\mathcal{E}} N$) if, and only if, M and N are both $\mathcal{E}$-terms and $M \rhd N$ ($M \simeq N$).

Lemma 4.3 *Let $\mathcal{E}$ be an environment.*

1. *If $\mathcal{E}^i \vdash F$, then $\mathcal{E}^j \vdash F$ for all $j \geq i$.*
2. *If $\lambda x{:}A.M$ ($\Pi x{:}A.M$) is an $\mathcal{E}$-term, then $\lambda x{:}A.M \equiv \lambda x'{:}A.M'$ ($\Pi x{:}A.M \equiv \Pi x'{:}A.M'$) for some $\mathcal{E}$-terms x' and M'.*

Proof By definition of environment and lemma 3.8. □

Lemma 4.4 *Let $\mathcal{E}$ be an environment and $\mathcal{E}^j \equiv x_1{:}A_1, ..., x_j{:}A_j$. If $\mathcal{E}^j \vdash M{:}A$ and, for all $i \leq j$, $\mathcal{E} \vdash N_i{:}[N_{i-1}/x_{i-1}]...[N_1/x_1]A_i$, then $\mathcal{E} \vdash [N_{i-1}/x_{i-1}]...[N_1/x_1](M{:}A)$.*

Proof Consider the form of A.

1. $A \equiv Type_{\mathbf{n}}$. As $\mathcal{E}^j \vdash M{:}A$, by repeated applications of (Π1) and/or (Π2), we have

$$\vdash \Pi x_1{:}A_1...\Pi x_j{:}A_j.M : Type_{\mathbf{n}}$$

As $\mathcal{E}$ is an environment, we have that, for some variable y,

$$\mathcal{E} \vdash y{:}\Pi x_1{:}A_1...\Pi x_j{:}A_j.M$$

So, by repeated application of (app), we have that

$$\mathcal{E} \vdash yN_1...N_j : [N_{i-1}/x_{i-1}]...[N_1/x_1]M$$

By theorem 3.10, for some kind K,

$$\mathcal{E} \vdash [N_{i-1}/x_{i-1}]...[N_1/x_1]M : K$$

Then, using (T2) and/or (T3) several (maybe zero) times yields the result.

2. $A \not\equiv Type_{\mathbf{n}}$. As $\mathcal{E}^j \vdash M{:}A$, by repeated applications of (λ), we have

$$\mathcal{E} \vdash \lambda x_1{:}A_1...\lambda x_j{:}A_j.M : \Pi x_1{:}A_1...\Pi x_j{:}A_j.A$$

Then, by repeated application of (app), we have

$$\mathcal{E} \vdash (\lambda x_1{:}A_1...\lambda x_j{:}A_j.M)N_1...N_j : [N_{i-1}/x_{i-1}]...[N_1/x_1]A$$

Now, by subject reduction, we have the result. □

Definition 4.5 (Categories of $\mathcal{E}$-types) *Let A be an $\mathcal{E}$-type. Then, where $Type_{-1} \equiv Prop$,*

1. *A is an $\mathcal{E}$-*proposition *iff* $\exists A' \simeq A.\ \mathcal{E} \vdash A'{:}Prop$;
2. *A is an $\mathcal{E}$-j-*type *iff* $\exists A' \simeq A.\ \mathcal{E} \vdash A'{:}Type_j$;
3. *A is a* proper $\mathcal{E}$-j-type *iff A is an $\mathcal{E}$-j-type and A is not an $\mathcal{E}$-$(j-1)$-type.*

A is called a proper $\mathcal{E}$-type *if it is a proper $\mathcal{E}$-j-type for some $j \geq 0$.* □

Remark Similar to the classification of Γ-types, the above categories of $\mathcal{E}$-types are exhaustive and exclusive in the senese that, if $\mathcal{E} \vdash M{:}A$, then exactly one of the following holds (for $\mathrm{CC}^{\mathbf{n}}_{\mathrm{C}}$):

1. $A \equiv Type_{\mathbf{n}}$;

2. A is a $\mathcal{E}$-proposition;

3. A is a proper $\mathcal{E}$-j-type for exactly one $0 \leq j \leq \mathbf{n}$.

We also have that, if $A \simeq_{\mathcal{E}} B$, then A is an $\mathcal{E}$-proposition (proper $\mathcal{E}$-j-type) if, and only if, B is an $\mathcal{E}$-proposition (proper $\mathcal{E}$-j-type). □

Notation For a term M, we use the following notation:

$$\mathcal{T}_{\mathcal{E}}M =_{\mathrm{df}} \{ A \mid \mathcal{E} \vdash M{:}A \ \wedge \ \forall A' \prec A. \mathcal{E} \not\vdash M{:}A' \}$$

i.e., $\mathcal{T}_{\mathcal{E}}M$ is the set of minimal types of M under $\mathcal{E}$. By theorem 3.15, $\mathcal{T}_{\mathcal{E}}M$ is in fact the set of the minimum types and we have

$$A \in \mathcal{T}_{\mathcal{E}}M \ \Rightarrow \ \mathcal{T}_{\mathcal{E}}M = [A]_{\mathcal{E}}$$

where $[A]_{\mathcal{E}} =_{\mathrm{df}} \{ B \mid A \simeq_{\mathcal{E}} B \}$. □

4.2 Quasi normalization of $\mathrm{CC}^{\mathbf{n}}_{\subset}$ I : an inductive argument

The quasi-normalization result for $\mathrm{CC}^{\mathbf{n}}_{\subset}$ is proved by induction from $\mathbf{n}$ to 0. The following definitions, lemmas and theorems are inductively defined and proved for $j = \mathbf{n}, \mathbf{n}-1, ..., 1, 0$. We first define the $\mathbf{n}$-degree of $\mathcal{E}$-terms, which is well-defined by lemma 3.21 and then prove the quasi-normalization result (theorem 4.14 and theorem 4.18) for $j = \mathbf{n}$. Then, we define the $(\mathbf{n}-1)$-degree, which is well-defined by theorem 4.18 (for $j = \mathbf{n}$) and then prove the quasi-normalization result for $j = \mathbf{n} - 1$. And we go on until $j = 1$. At the last step, based on the quasi-normalization result for $j = \mathbf{n}, ..., 1$, we define 0-degree of $\mathcal{E}$-terms.

Definition 4.6 **(δ_j, j-degree of $\mathcal{E}$-terms)** *Let $A \rhd_{\mathcal{E}} A^{\circ}$, where, if $j = \mathbf{n}$, $A^{\circ} \equiv A$, and if $j < \mathbf{n}$, A° is an i-Q-normal term for every i such that $j < i \leq \mathbf{n}$. Then, define $\delta_j A$, the* j-degree of A, *as follows:*

- *If A° is not a proper $\mathcal{E}$-j-type, $\delta_j A =_{\mathrm{df}} 0$;*
- *$\delta_j Type_{j-1} =_{\mathrm{df}} 1$ (when $j = 0$, $\delta_0 Prop =_{\mathrm{df}} 1$);*
- *If $A^{\circ} \equiv xA_1...A_m$ is a proper $\mathcal{E}$-j-type, $\delta_j A =_{\mathrm{df}} 1$.*
- *If $A^{\circ} \equiv \Pi x{:}A_1.A_2$ is a proper $\mathcal{E}$-j-type, $\delta_j A =_{\mathrm{df}} max\{\delta_j A_1, \delta_j A_2\} + 1$.* □

Remark For $j = \mathbf{n}$, δ_j is well-defined by Church-Rosser theorem and lemma 3.21. For, $j < \mathbf{n}$, it is well-defined by Church-Rosser theorem and theorem 4.14 below. That is, δ_j is a function from $\mathcal{E}$-terms to natural numbers. □

Lemma 4.7 *If $A \simeq_{\mathcal{E}} B$, then $\delta_j A = \delta_j B$.*

Proof For $j = \mathbf{n}$, by Church-Rosser theorem and lemma 3.21. For $j < \mathbf{n}$, by Church-Rosser theorem and theorem 4.14 below. □

Remark The above lemma implies that, for all $A, B \in \mathcal{T}_{\mathcal{E}}M$, $\delta_j A = \delta_j B$. □

Lemma 4.8 *A is a proper $\mathcal{E}$-j-type iff $\delta_j A \geq 1$.*

Proof Obvious from the definition. □

Notations We introduce the following notations:

1. Let $A \in \mathcal{T}_{\mathcal{E}}M$. Then,

$$\delta_j(\mathcal{T}_{\mathcal{E}}M) =_{\mathrm{df}} \delta_j(A)$$

2. Let $\mathcal{E}$-term $R \equiv R_1R_2$ be a redex. Then,

$$\delta_j^*(R) =_{\mathrm{df}} \delta_j(\mathcal{T}_{\mathcal{E}}R_1)$$

3. Let M be an $\mathcal{E}$-term. Then,

$$\gamma_j M =_{\mathrm{df}} max\{\, \delta_j^*(R) \mid R \text{ is a redex in } M\,\}$$

$$\rho_j M =_{\mathrm{df}} \#\{\, R \mid \delta_j^*(R) = \gamma_j M \geq 1\,\}$$

□

Remark These measures are extensions of the measures used by G.Pottinger in [Pot87]. $\delta_j(\mathcal{T}_{\mathcal{E}}M)$ is the j-degree of the minimum type of M; δ_j^* assigns every $\mathcal{E}$-term R_1R_2 of redex form a measure value, *i.e.*, the j-degree of the minimum type of R_1; $\gamma_j M$ is the largest δ_j^*-value of the redexes occurring in M; and, $\rho_j M$ is the number of redexes occurring in M whose δ_j^*-value is equal to $\gamma_j M$ and greater than 0. □

Definition 4.9 (j-Q-normal $\mathcal{E}$-terms) *An $\mathcal{E}$-term M is* j-quasinormal (j-Q-normal) *if, and only if, $\gamma_j M = 0$.* □

Remark An $\mathcal{E}$-term M is j-Q-normal if, and only if, it does not contain any redex R_1R_2 such that the minimum type of R_1 is a proper $\mathcal{E}$-j-type. □

The next step is to show that, every $\mathcal{E}$-term can be reduced to an $\mathcal{E}$-term which is i-Q-normal for all $i \geq j$ (theorem 4.14 and corollary 4.15). First, we prove several lemmas which are needed to prove this result.

Lemma 4.10 *Let* $\mathcal{E}^{k+1} \equiv \mathcal{E}^k, x{:}A$. *If*

- $\mathcal{E}^k \vdash N{:}A$,
- $\mathcal{E}^{k+1} \vdash B{:}Type_{\mathbf{n}}$,
- B *is not a proper* $\mathcal{E}$-i-*type for any* $i > j$,

then, if $[N/x]B$ *is a proper* $\mathcal{E}$-j-*type, then so is* B.

Proof Suppose B is not a proper $\mathcal{E}$-j-type. Then, it is an $\mathcal{E}$-proposition or a proper $\mathcal{E}$-j'-type for some $j' < j$, *i.e.*, there is $B' \simeq B$ such that $\mathcal{E} \vdash B'{:}K$ for some $K \prec Type_j$. By lemma 3.13, we may assume $\mathcal{E}^k, x{:}A \vdash B'{:}K$. So, by theorem 3.9, $\mathcal{E}^k \vdash [N/x]B'{:}K$. As $[N/x]B' \simeq_{\mathcal{E}} [N/x]B$, $[N/x]B$ is not a proper $\mathcal{E}$-j-type. This contradicts the assumption. So, B is a proper $\mathcal{E}$-j-type if $[N/x]B$ is. □

Lemma 4.11 *Let* $\mathcal{E}^{k+1} \equiv \mathcal{E}^k, x{:}A$. *If*

- $\mathcal{E}^k \vdash N{:}A$,
- A *is not a proper* $\mathcal{E}$-i-*type for any* $i > j$,
- $\mathcal{E}^{k+1} \vdash B{:}Type_{\mathbf{n}}$,
- B *is not a proper* $\mathcal{E}$-i-*type for any* $i > j$,

then,

$$\delta_j([N/x]B) \leq \delta_j B$$

Proof By theorem 4.14 for $i > j$, $B \rhd B^\circ$ for some B° which is i-Q-normal for all $i > j$ (when $j = \mathbf{n}$, $B^\circ \equiv B$). As $\delta_j B = \delta_{B^\circ}$ and $\delta_j([N/x]B) = \delta_j([N/x]B^\circ)$ by lemma 4.7, we only have to show that $\delta_j([N/x]B^\circ) \leq \delta_j B^\circ$.

By induction on the structure of B°. By theorem 4.18 for $j+1$, we only have to consider the following cases:

1. $B^\circ \equiv Type_{j-1}$. It is obvious.
2. $B^\circ \equiv yB_1...B_m$. If $y \not\equiv x$, then

   $$\delta_j([N/x]B^\circ) = 1 = \delta_j B^\circ$$

   But it can not be the case that $y \equiv x$, for otherwise, we would have that $B^\circ \equiv xB_1...B_m$ is a proper $\mathcal{E}$-j-type, which contradicts that A is not proper $\mathcal{E}$-i-type for any $i > j$.

3. $B^\circ \equiv \Pi y{:}B_1.B_2$. If $[N/x]B^\circ$ is not a proper $\mathcal{E}$-j-type,

$$\delta_j([N/x]B^\circ) = 0 \leq \delta_j B^\circ$$

If $[N/x]B^\circ$ is a proper $\mathcal{E}$-j-type, then so is B° by the above lemma. So, by induction hypothesis, we have

$$\begin{aligned} \delta_j([N/x]B^\circ) &= max\{\delta_j([N/x]B_1), \delta_j([N/x]B_2)\} + 1 \\ &\leq max\{\delta_j B_1, \delta_j B_2\} + 1 \\ &= \delta_j B^\circ \end{aligned}$$

□

Lemma 4.12 *Let $\mathcal{E}^{k+1} \equiv \mathcal{E}^k, x{:}A$. If*

- $\mathcal{E}^k \vdash N{:}A$,
- $A \in \mathcal{T}_\mathcal{E} N$ *is not a proper $\mathcal{E}$-i-type for any $i > j$,*
- $\mathcal{E}^{k+1} \vdash M{:}B$, *and*
- M *is i-Q-normal for all i such that $j < i \leq \mathbf{n}$,*

then,

$$\gamma_j([N/x]M) \leq max\{\gamma_j M, \gamma_j N, \delta_j A\}$$

Proof By induction on the structure of M.

1. If M is a constant or a variable, it is obvious as either $[N/x]M \equiv M$ or $[N/x]M \equiv N$.

2. $M \equiv \Pi y{:}M_1.M_2$ or $M \equiv \lambda y{:}M_1.M_2$. Then, by induction hypothesis,

$$\begin{aligned} \gamma_j([N/x]M) &= max\{\, \delta_j^*(R) \mid R \in redexes([N/x]M)\,\} \\ &= max\{\, \delta_j^*(R) \mid R \in redexes([N/x]M_1) \cup redexes([N/x]M_2)\,\} \\ &= max\{\gamma_j([N/x]M_1), \gamma_j([N/x]M_2)\} \\ &\leq max\{max\{\gamma_j(M_1), \gamma_j(N), \delta_j(A)\}, max\{\gamma_j(M_2), \gamma_j(N), \delta_j(A)\}\} \\ &= max\{\gamma_j(M_1), \gamma_j(M_2), \gamma_j(N), \delta_j(A)\} \\ &= max\{\gamma_j M, \gamma_j N, \delta_j A\} \end{aligned}$$

3. $M \equiv M_1 M_2$. There are two sub-cases.

   (a) $[N/x]M$ is not a redex such that $\delta_j^*([N/x]M) > 0$. Then, a similar argument to that for the above case suffices.

(b) $[N/x]M$ is a redex such that $\delta_j^*([N/x]M) > 0$. Then,

$$\begin{aligned}
\gamma_j([N/x](M_1M_2)) &= max\{\gamma_j([N/x]M_1), \gamma_j([N/x]M_2), \delta_j^*([N/x](M_1M_2))\} \\
&\leq max\{\gamma_j(M_1), \gamma_j(M_2), \gamma_j(N), \delta_j(A), \delta_j^*([N/x](M_1M_2))\} \\
&= max\{\gamma_j(M_1), \gamma_j(M_2), \gamma_j(N), \delta_j(A), \delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1))\} \\
&\leq max\{max\{\gamma_j(M_1M_2), \gamma_j(N), \delta_j(A)\}, \delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1))\}
\end{aligned}$$

So, we only have to show that

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) \leq max\{\gamma_j(M_1M_2), \gamma_j(N), \delta_j(A)\}$$

Note that, as $[N/x]M_1$ is of λ-abstraction form, either $M_1 \equiv x$ or $M_1 \equiv \lambda y{:}M_1'.M_1''$.

i. $M_1 \equiv x$. Then, as $A \in \mathcal{T}_\mathcal{E}N$, we have,

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) = \delta_j(\mathcal{T}_\mathcal{E}N) = \delta_j(A)$$

ii. $M_1 \equiv \lambda y{:}M_1'.M_1''$. We only have to show $\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) \leq \delta_j(\mathcal{T}_\mathcal{E}M_1)$, as then,

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) \leq \delta_j(\mathcal{T}_\mathcal{E}M_1) = \delta_j^*(M_1M_2) \leq \gamma_j(M_1M_2)$$

Let $A_1 \in \mathcal{T}_\mathcal{E}M_1$. (Note that A_1 is not a proper $\mathcal{E}$-i-type for any $i > j$ because, by assumption, $M \equiv M_1M_2$ is i-Q-normal for every $i > j$.) By lemma 4.4, $\mathcal{E} \vdash [N/x]M_1{:}[N/x]A_1$. Consider $[N/x]A_1$. There are two cases.

A. $[N/x]A_1 \in \mathcal{T}_\mathcal{E}([N/x]M_1)$. Then, by lemma 4.11,

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) = \delta_j([N/x]A_1) \leq \delta_j A_1 = \delta_j(\mathcal{T}_\mathcal{E}M_1)$$

B. $[N/x]A_1 \notin \mathcal{T}_\mathcal{E}([N/x]M_1)$. Let $A_1' \in \mathcal{T}_\mathcal{E}([N/x]M_1)$. Then, we have $A_1' \not\preceq [N/x]A_1$. By theorem 3.15 and Church-Rosser theorem, for some kinds $K \prec K'$ and $m \geq 1$ (as M_1 is of λ-abstraction form),

$$A_1' \rhd \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K$$

$$[N/x]A_1 \rhd \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K'$$

If $K \not\equiv Type_{j-1}$, we have, by the definition of δ_j and lemma 4.11,

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) = \delta_j(A_1') \leq \delta_j([N/x]A_1) \leq \delta_j A_1 = \delta_j(\mathcal{T}_\mathcal{E}M_1)$$

But K can not be $Type_{j-1}$, for otherwise, if $j = \mathbf{n}$, then $K' \equiv Type_\mathbf{n}$, which is impossible; if $j < \mathbf{n}$, then we have $[N/x]A_1$ is a proper $\mathcal{E}$-i-type for some $i > j$, which, by lemma 4.10, implies that A_1 is a proper $\mathcal{E}$-i-type for some $i > j$.

This completes the proof of the lemma. □

Lemma 4.13 *Let $MN \equiv (\lambda x{:}A.M_1)N$ be an $\mathcal{E}$-term. If*

- *MN is i-Q-normal for all i such that $j < i \leq \mathbf{n}$,*
- *$\rho_j(MN) = 1$, and*
- *$\delta_j^*(MN) = \gamma_j(MN)$,*

then,

$$\gamma_j([N/x]M_1) < \gamma_j(MN)$$

Furthermore, if $[N/x]M_1$ is of the form $\lambda y{:}X.Y$, then

$$\delta_j(\mathcal{T}_{\mathcal{E}}([N/x]M_1)) < \gamma_j(MN)$$

Proof Because MN is the only redex in MN such that $\delta_j^*(MN) = \gamma_j(MN)$, we have

$$\gamma_j M_1 < \gamma_j(MN)$$
$$\gamma_j N < \gamma_j(MN)$$
$$\delta_j A < \delta_j(\mathcal{T}_{\mathcal{E}} M) = \gamma_j(MN)$$

As $\mathcal{E} \vdash N{:}A$, we consider two cases according to A.

1. $A \in \mathcal{T}_{\mathcal{E}} N$. Then, by lemma 4.12,

   $$\gamma_j([N/x]M_1) \leq max\{\gamma_j M_1, \gamma_j N, \delta_j A\}$$

   So, $\gamma_j([N/x]M_1) < \gamma_j(MN)$.

2. $A \notin \mathcal{T}_{\mathcal{E}} N$. By theorem 3.15, there is $A' \equiv \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K \in \mathcal{T}_{\mathcal{E}} N$ such that $A \simeq_{\mathcal{E}} \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K'$ for some kinds $K \prec K'$. Then, by lemma 4.12,

   $$\gamma_j([N/x]M_1) \leq max\{\gamma_j M_1, \gamma_j N, \delta_j A'\}$$

   So, we only have to show $\delta_j A' \leq \delta_j A$. But $\delta_j A' > \delta_j A$ is impossible, for it implies that $K \equiv Type_{j-1} \prec K'$, which further implies that A is a proper $\mathcal{E}$-i-type for some $i > j$, *i.e.*, $\delta_i^*(MN) > 0$ for some $i > j$, contradicting the assumption that MN is i-Q-normal.

Now, we prove that, when $[N/x]M_1$ is of the form $\lambda y{:}X.Y$, then

$$\delta_j(\mathcal{T}_{\mathcal{E}}([N/x]M_1)) < \delta_j(\mathcal{T}_{\mathcal{E}} M) = \gamma_j(MN)$$

By the form of $[N/x]M_1$, there are only two possibilities to consider:

1. $M_1 \equiv x$ and $N \equiv \lambda y{:}X.Y$. We consider two sub-cases:

(a) $A \in \mathcal{T}_\mathcal{E} N$. Then, because $\mathcal{T}_\mathcal{E} M = [\Pi x{:}A.B]_\mathcal{E}$ for some B, we have

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) = \delta_j(\mathcal{T}_\mathcal{E} N) = \delta_j A < \delta_j(\mathcal{T}_\mathcal{E} M)$$

(b) $A \notin \mathcal{T}_\mathcal{E} N$. Let $A' \in \mathcal{T}_\mathcal{E} N$. Then, by theorem 3.15 and Church-Rosser theorem, for some kinds $K \prec K'$ and $m \geq 1$ (as M_1 is of λ-abstraction form),

$$A' \rhd \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K$$

$$[N/x]A \rhd \Pi x_1{:}C_1 ... \Pi x_m{:}C_m.K'$$

If $K \not\equiv Type_{j-1}$, we have

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) = \delta_j(\mathcal{T}_\mathcal{E} N) = \delta_j A' \leq \delta_j A < \delta_j(\mathcal{T}_\mathcal{E} M)$$

But K can not be $Type_{j-1}$, for otherwise, for $j = \mathbf{n}$, $K' \equiv Type_{\mathbf{n}}$, which is impossible, and, for $j < \mathbf{n}$, A is a proper $\mathcal{E}$-i-type for some $i > j$, which contradicts the assumption that MN is i-Q-normal.

2. $M_1 \equiv \lambda y{:}M_1'.M_1''$. Let $A_1 \in \mathcal{T}_\mathcal{E} M_1$. Then, A_1 is not a proper $\mathcal{E}$-i-type for any $i > j$, for otherwise, the types in $\mathcal{T}_\mathcal{E} M$ are not proper $\mathcal{E}$-j-type. So, by a similar argument to that in 3(b)ii in the proof of lemma 4.12, we have

$$\delta_j(\mathcal{T}_\mathcal{E}([N/x]M_1)) \leq \delta_j(\mathcal{T}_\mathcal{E} M_1) < \delta_j(\mathcal{T}_\mathcal{E} M)$$

□

Theorem 4.14 *If $\mathcal{E}$-term M is i-Q-normal for all $j < i \leq \mathbf{n}$, then $M \rhd_\mathcal{E} N$ for some N which is i-Q-normal for all $j \leq i \leq \mathbf{n}$.*

Proof First, for $j < \mathbf{n}$, by lemma 4.16 below (for $j+1$), β-contraction preserves i-Q-normalness for all $j < i \leq \mathbf{n}$. So, we only have to show that M can be reduced to a j-Q-normal term. This can be proved by a double ind on $\gamma_j M$ and $\rho_j M$.

We only indicate that, by lemma 4.13, for any $\mathcal{E}$-term M which is i-Q-normal for every $i > j$, reducing M by contracting any redex $R_1 R_2$ such that $\delta_j(\mathcal{T}_\mathcal{E} R_1) = \gamma_j M$ and, in R_1 or R_2, there is no redex whose δ_j^*-value is $\gamma_j M$ (*e.g.*, the *rightmost* of the *innermost* redexes whose δ_j^*-value are $\gamma_j M$) decreases the ρ_j-value of the term by one. □

Corollary 4.15 *If M is an $\mathcal{E}$-term, then $M \rhd_\mathcal{E} N$ for some N which is i-Q-normal for every $i \geq j$.* □

Lemma 4.16 *Let $M \rhd_\mathcal{E} N$. If M is i-Q-normal for every $i \geq j$, so is N.*

Proof By the 'global' induction argument on j, we only have to show that, if $M \rhd_1 N$ and M is i-Q-normal for every $i \geq j$, then N is j-Q-normal. Suppose

$$M \equiv ...((\lambda x{:}A.M_0)N_0)... \rhd_1 ...[N_0/x]M_0... \equiv N$$

By lemma 4.12, if N is not j-Q-normal, then $[N_0/x]M_0$ must be of the form $\lambda y{:}Y.Z$ such that $N \equiv ...((\lambda y{:}Y.Z)N')...$ and $\delta_j(\mathcal{T}_\mathcal{E}(\lambda y{:}Y.Z)) > 0$. But this is impossible because, if so, there are only two possibilities according to the form of $[N_0/x]M_0$:

1. $M_0 \equiv x$, $N_0 \equiv \lambda y{:}Y.Z$ and $\delta_j(\mathcal{T}_\mathcal{E} N_0) > 0$. This implies that $\delta_i(\mathcal{T}_\mathcal{E}(\lambda x{:}A.M_0)) > 0$ for some $i \geq j$.
2. $M_0 \equiv \lambda y{:}Y_1.Z_1$ and $\delta_j(\mathcal{T}_\mathcal{E}([N_0/x]M_0)) > 0$. This implies that $\delta_i(\mathcal{T}_\mathcal{E}(\lambda y{:}Y_1.Z_1)) > 0$.

So, both imply that M is not i-Q-normal for some $i \geq j$, contradicting the assumption. □

Lemma 4.17 *Let MN be an $\mathcal{E}$-term and the minimum type of MN (under $\mathcal{E}$) is a proper $\mathcal{E}$-j-type. Then, the minimum type of M under ($\mathcal{E}$) is a proper $\mathcal{E}$-k-type for some $k \geq j$.*

Proof As MN is an $\mathcal{E}$-term, the minimum type of M has Π-form, *i.e.*,

$$\mathcal{T}_\mathcal{E} M = [\Pi x{:}A.B]_\mathcal{E}$$

for some A and B. Then, $\mathcal{E} \vdash MN{:}[N/x]B$. By lemma 4.10, we have B is a proper $\mathcal{E}$-k'-type for some $k' \geq j$. So, $\Pi x{:}A.B \in \mathcal{T}_\mathcal{E} M$ is a proper $\mathcal{E}$-k-type for some $k \geq k' \geq j$. □

The following theorem is proved only for $j > 0$.

Theorem 4.18 *Let A be an i-Q-normal proper $\mathcal{E}$-$(j-1)$-type for every i such that $j \leq i \leq \mathbf{n}$. Then, A has one of the following forms:*

- $Type_{j-2}$
- $xA_1...A_m$
- $\Pi x{:}A_1.A_2$

where, when $j = 1$, $Type_{-1} \equiv Prop$.

Proof By induction on the structure of A.

1. If A is a constant (kind), $A \equiv Type_{j-2}$, because this is the only kind which is a proper $\mathcal{E}$-$(j-1)$-type.
2. If A is a variable or of the form $\Pi x{:}A_1.A_2$, it is obvious.

3. A is not of the form $\lambda x{:}A_1.A_2$, as A is a type.

4. $A \equiv A_1...A_m$ ($m > 1$ and A_1 is not of the form $A_{11}A_{12}$. In this case, as A is a proper $\mathcal{E}$-$(j-1)$-type, $\mathcal{T}_\mathcal{E}A = [Type_{j-1}]_\mathcal{E}$. By lemma above, for each $1 \le k \le m-1$, every type in $\mathcal{T}_\mathcal{E}(A_1...A_k)$ is a proper $\mathcal{E}$-j_k-type for some $j_k \ge j$. This implies that A_1 is not of the form $\lambda x{:}X.Y$ for otherwise A is not j_1-Q-normal (note that $j_1 \ge j$). So, A_1 must be a variable.

Lemma 4.19 *Let $A \equiv \Pi x{:}A_1.A_2$ be an $\mathcal{E}$-$(j-1)$-type. Then, A is a proper Γ-$(j-1)$-type if, and only if,*

1. *either A_1 or A_2 is a proper $\mathcal{E}$-$(j-1)$-type, and*
2. *A_2 is a proper $\mathcal{E}$-k-type for some $0 \le k \le j-1$.*

Proof Similar to lemma 3.22. □

4.3 Quasi normalization of $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$ II

Definition 4.20 (Q-normal terms) *An $\mathcal{E}$-term M is Q-normal iff M is i-Q-normal for every i such that $0 \le i \le \mathbf{n}$.* □

Theorem 4.21 (Quasi normalization of $\mathrm{CC}^{\mathbf{n}}_{\mathsf{C}}$) *Every $\mathcal{E}$-term reduces to some Q-normal $\mathcal{E}$-term.*

Proof By theorem 4.14 (ind from $\mathbf{n}$ to 0). □

Definition 4.22 (head normal forms for types) *An $\mathcal{E}$-type A is in* head normal form *if, and only if, it has one of the following forms:*

- *a kind K*
- *$xA_1...A_m$, where x is a variable and $m \ge 0$*
- *$\Pi x{:}A.B$, where B is in head normal form*

□

Corollary 4.23 *A Q-normal $\mathcal{E}$-type which is not an $\mathcal{E}$-proposition is in head normal form. Henceforth, every $\mathcal{E}$-type can be reduced to some head normal form.*

Proof By lemma 3.21, theorem 4.18, theorem 4.21 and lemma 4.19. □

5 Strong Normalization of $\mathrm{CC}^{\infty}_{\mathsf{C}}$

In this section, the strong normalization theorem of $\mathrm{CC}^{\infty}_{\mathsf{C}}$ is proved.

5.1 A complexity measure for $\mathcal{E}$-types

Based on the result of quasi-normalization in the last section, we are now able to define a complexity measure of types satisfying the required property as stated in lemma 5.2 below.

Definition 5.1 (complexity of $\mathcal{E}$-types, β) *Let A be an $\mathcal{E}$-type. Then define the complexity of A, βA, as follows:*

$$\beta A =_{\mathrm{df}} \begin{cases} (0,0) & \textit{if } A \textit{ is an } \mathcal{E}\textit{-proposition} \\ (j+1, \delta_j A) & \textit{if } A \textit{ is a proper } \mathcal{E}\textit{-}j\textit{-type} \end{cases}$$

where δ_j is defined in definition 4.6. β-values of $\mathcal{E}$-types are ordered by the lexicographic ordering. □

Remark The above definition is well-defined because δ_j is well-defined. Furthermore, it is obvious that, if $A \simeq_{\mathcal{E}} B$, then $\beta(A) = \beta(B)$ by the properties we stated in the remark after definition 4.5. The following lemma shows the most important property of the complexity measure. □

Lemma 5.2 *If $\mathcal{E}$-type A is not an $\mathcal{E}$-proposition and reduces to a Q-normal $\mathcal{E}$-type $\Pi x{:}A_1.A_2$, then we have*

$$\beta(A_1) < \beta(A)$$

and, for every N such that $\mathcal{E} \vdash N{:}A_1$,

$$\beta([N/x]A_2) < \beta(A)$$

Proof As A is not an $\mathcal{E}$-prop, if A_1 is a proper $\mathcal{E}$-j-type, then A is a proper $\mathcal{E}$-j'-type for some $j' \geq j$. So, by definition,

$$\beta A_1 < \beta A$$

Consider $[N/x]A_2$. There are two cases:

1. $[N/x]A_2$ is an $\mathcal{E}$-proposition, then it is obvious that $\beta([N/x]A_2) = (0,0) < \beta A$, as A is not an $\mathcal{E}$-proposition.

2. $[N/x]A_2$ is a proper $\mathcal{E}$-j-type. Then, by lemma 4.10, A_2 is a proper $\mathcal{E}$-i-type for some $i \geq j$, and henceforth A is a proper $\mathcal{E}$-i'-type for some $i' \geq i$.

   (a) $i > j$. Then, we have

   $$\beta([N/x]A_2) = (j+1, \delta_j([N/x]A_2)) < (i+1, \delta_i A_2) \leq \beta A$$

   (b) $i = j$. By lemma 4.11, either A_1 is a proper $\mathcal{E}$-k-type for some $k > j$, or $\delta_j([N/x]A_2) \leq \delta_j A_2$. For these two cases, we respectively have

   $$\beta([N/x]A_2) = (j+1, \delta_j([N/x]A_2)) < (k+1, \delta_k A_1) = \beta A_1 \leq \beta A$$
   $$\beta([N/x]A_2) = (j+1, \delta_j([N/x]A_2)) \leq (j+1, \delta_j A_2) < \beta A$$

□

5.2 Values of $\mathcal{E}$-terms

Definition 5.3 *Let A be an $\mathcal{E}$-type. Then, define*

$$SN_{\mathcal{E}}(A) =_{\mathrm{df}} \{\, M \mid M \textit{ is strongly normalizable and } \mathcal{E} \vdash M{:}A \,\}$$

□

Definition 5.4 (saturated sets) *Let A be an $\mathcal{E}$-type. S is an A-*saturated set *if, and only if,*

1. $S \subseteq SN_{\mathcal{E}}(A)$;
2. *If* $xM_1...M_m \in SN_{\mathcal{E}}(A)$, *then* $xM_1...M_m \in S$, *where* $m \geq 0$ *and* x *is a variable or constant;*
3. *If* $(\lambda x{:}B.M)NN_1...N_m \in SN_{\mathcal{E}}(A)$ *and* $([N/x]M)N_1...N_m \in S$, *then* $(\lambda x{:}B.M)NN_1...N_m \in S$, *where* $m \geq 0$.

$Sat_{\mathcal{E}}(A)$ *is defined to be the set of A-saturated sets.* □

Remark $Sat_{\mathcal{E}}(A)$ is not empty. In fact, $SN_{\mathcal{E}}(A) \in Sat_{\mathcal{E}}(A)$. □

Definition 5.5 (possible values of $\mathcal{E}$-terms) *The set of* (possible) values *of an $\mathcal{E}$-term M, $V(M)$, is defined as follows.*

1. *If M is an $\mathcal{E}$-type, then* $V(M) =_{\mathrm{df}} Sat_{\mathcal{E}}(M)$;
2. *If M is not an $\mathcal{E}$-type, then suppose that* $\mathcal{E} \vdash M{:}A$ *and define $V(M)$ by induction on the complexity $\beta(A)$ as follows:*
   (a) *if A is an $\mathcal{E}$-proposition, then* $V(M) =_{\mathrm{df}} \{\kappa\}$, *where κ is a fixed arbitrary symbol;*
   (b) *if A is not an $\mathcal{E}$-proposition and A reduces to a Q-normal term* $xA_1...A_m$, *then* $V(M) =_{\mathrm{df}} \{\kappa\}$;
   (c) *if A is not an $\mathcal{E}$-proposition and A reduces to a Q-normal term* $\Pi x{:}A_1.A_2$, *then define $V(M)$ as consisting of the functions f such that*
      i. $dom(f) = \{\, (N, v) \mid \mathcal{E} \vdash N{:}A_1, v \in V(N) \,\}$,
      ii. $f(N, v) \in V(MN)$ *for all* $(N, v) \in dom(f)$, *and*
      iii. *if* $N \simeq_{\mathcal{E}} N'$, *then* $f(N, v) = f(N', v)$ *for all* $v \in V(N) \cap V(N')$.

□

Remark By lemma 5.2, the above definition is well-defined. Furthermore, by Church-Rosser theorem, theorems 3.15 and 3.12, we can prove by induction on the β-complexity of types of M and N that, if $M \simeq_{\mathcal{E}} N$, then $V(M) = V(N)$. For every $\mathcal{E}$-term M, $V(M)$ is not empty, as the following definition shows. □

Definition 5.6 (canonical value of $\mathcal{E}$-terms) *Define the* canonical value *of an $\mathcal{E}$-term M, v_M, as follows.*

1. *If M is an $\mathcal{E}$-type, $v_M =_{\mathrm{df}} SN_{\mathcal{E}}(M)$;*
2. *If M is not an $\mathcal{E}$-type, then suppose that $\mathcal{E} \vdash M{:}A$ and define v_M by induction on the complexity $\beta(A)$ as follows:*
   (a) *if A is an $\mathcal{E}$-proposition, then $v_M =_{\mathrm{df}} \kappa$;*
   (b) *if A is not an $\mathcal{E}$-proposition and A reduces to a Q-normal term $xA_1...A_m$, then $v_M =_{\mathrm{df}} \kappa$;*
   (c) *if A is not an $\mathcal{E}$-proposition and A reduces to a Q-normal term $\Pi x{:}A_1.A_2$, then define v_M as the function f such that $f \in V(M)$ and $f(N, v) = v_{MN}$ for all $(N, v) \in dom(f)$.*

□

5.3 Evaluation of $\mathcal{E}$-terms

An *$\mathcal{E}$-assignment* is a function ϕ with $FV(\mathcal{E}^j)$ as domain for some $j \in \omega$ and terms as images such that, for each $0 \le i \le j$, $\mathcal{E}^i \vdash \phi(x_i){:}\phi(A_i)$, where $x_i{:}A_i \equiv \mathcal{E}_i$. (We also write ϕ for the substitution determined by ϕ.)

An *$\mathcal{E}$-valuation* is a pair of functions

$$\rho = (\phi, val)$$

such that ϕ is an $\mathcal{E}$-assignment and val is a function with $dom(\phi)$ as domain and terms as images such that, for each $x_i \in dom(\phi)$, $val(x_i) \in V(\phi(x_i))$. The domain of ρ, $dom(\rho)$, is $dom(\phi)$. An $\mathcal{E}$-valuation ρ *$\mathcal{E}$-covers* an $\mathcal{E}$-formula $M{:}N$ if, and only if, $\mathcal{E}^j \vdash M{:}N$, where $dom(\rho) = FV(\mathcal{E}^j)$. In this case, we also say that ρ $\mathcal{E}$-covers M and N.

Now, we define the *evaluation* function of $\mathcal{E}$-terms.

Definition 5.7 (Evaluation $Eval_\rho$) *Let $\rho = (\phi, val)$ be an $\mathcal{E}$-valuation. The* evaluation function $Eval_\rho$ *of $\mathcal{E}$-terms which are $\mathcal{E}$-covered by ρ are defined by induction on the structure of $\mathcal{E}$-terms as follows:*

1. *For M being an $\mathcal{E}$-proof (i.e., $\mathcal{E} \vdash M{:}P$ for some $\mathcal{E}$-proposition P),*

$$Eval_\rho(M) =_{\mathrm{df}} \kappa$$

2. *For M being not an $\mathcal{E}$-proof, $Eval_\rho(M)$ is defined by induction on the structure of M:*

   (a) *M is a variable, then*

   $$Eval_\rho(M) =_{\mathrm{df}} val(M)$$

   (b) *M is a constant (i.e., a kind), then*

   $$Eval_\rho(M) =_{\mathrm{df}} SN_{\mathcal{E}}(M)$$

   (c) *$M \equiv M_1M_2$, then*

   $$Eval_\rho(M) =_{\mathrm{df}} Eval_\rho(M_1)(\phi(M_2), Eval_\rho(M_2))$$

   (d) *$M \equiv \lambda x{:}M_1.M_2$. We may assume that $x \notin dom(\rho)$. Then, $Eval_\rho(M)$ is defined to be the function f such that*

      i. *$dom(f) = \{\,(N, v) \mid \mathcal{E} \vdash N{:}\phi(M_1), v \in V(N)\,\}$, and*

      ii. *$f(N, v) = Eval_{\rho'}(M_2)$ for all $(N, v) \in dom(f)$, where ρ' extends ρ such that $\rho'(x) = (N, v)$.*

   (e) *$M \equiv \Pi x{:}M_1.M_2$. We may assume that $x \notin dom(\rho)$. Then, $Eval_\rho(M)$ is defined to be the set of the terms F such that*

      i. *$\mathcal{E} \vdash F{:}\phi(M)$, and*

      ii. *for all $N \in Eval_\rho(M_1)$ and $v \in V(N)$, $FN \in Eval_{\rho'}(M_2)$, where, ρ' extends ρ such that $\rho'(x) = (N, v)$.*

□

Remark Note that, if ρ_1 and ρ_2 agree on the free variables of an $\mathcal{E}$-term M, then $Eval_{\rho_1}(M) = Eval_{\rho_2}(M)$. □

Lemma 5.8 (substitution property) *Suppose*

1. *$\rho = (\phi, val)$ is an $\mathcal{E}$-valuation which $\mathcal{E}$-covers N and $[N/x]M$ and $x \notin dom(\rho)$;*

2. *$\rho' = (\phi', val')$ is an $\mathcal{E}$-valuation which $\mathcal{E}$-covers x and M and extends ρ such that $\rho'(x) = (\phi(N), Eval_\rho(N))$.*

Then,

$$Eval_\rho([N/x]M) = Eval_{\rho'}(M)$$

Proof By induction on the structure of M. Here, we only check the case when $M \equiv \Pi y{:}M_1.M_2$. We may assume that $y \notin dom(\rho')$. Then, we have

1. $\phi([N/x]M) \equiv \phi'(M)$;

2. By induction hypothesis,

$$Eval_\rho([N/x]M_1) = Eval_{\rho'}(M_1)$$

3. For all $N' \in Eval_\rho([N/x]M_1) = Eval_{\rho'}(M_1)$ and $v' \in V(N')$, as $y \notin dom(\rho')$, there is an $\mathcal{E}$-valuation ρ'' which extends ρ' (and ρ) such that $\rho''(y) = (N', v')$.

So, by definition of *Eval*, $Eval_\rho([N/x]M) = Eval_{\rho'}(M)$. The other cases can be similarly verified. □

Lemma 5.9 *Let $\rho = (\phi, val)$ be an $\mathcal{E}$-valuation. Then,*

1. *for any $\mathcal{E}$-term M $\mathcal{E}$-covered by ρ, $Eval_\rho(M) \in V(\phi(M))$;*

2. *for any $\mathcal{E}$-terms M and N $\mathcal{E}$-covered by ρ, if $M \simeq_{\mathcal{E}} N$, then $Eval_\rho(M) = Eval_\rho(N)$.*

Proof The proofs of 1 and 2 are by mutual induction on the structure of M, using lemma 5.8 when considering the case $M \equiv \lambda x{:}M_1.M_2$ in proving 1 and the case $M \equiv M_1M_2$ in proving 2. □

Lemma 5.10 *Let $\rho = (\phi, val)$ be an $\mathcal{E}$-valuation such that*

- *$M{:}N$ is $\mathcal{E}$-covered by ρ;*
- *for all $x_j \in dom(\rho)$, $\phi(x_j) \in Eval_\rho(A_j)$, where $\mathcal{E}_j \equiv x_j{:}A_j$.*

Then, $\phi(M) \in Eval_\rho(N)$.

Proof By induction on the structure of M and using lemmas 5.9 and 5.8. □

5.4 Strong normalization and consistency of CC^∞_C

Theorem 5.11 (Strong normalization for CC^∞_C) *If $\Gamma \vdash M{:}N$, then M is strongly normalizable.*

Proof Take some environment $\mathcal{E}$ such that $\mathcal{E}^i \equiv \Gamma$. Let ρ_0 be an $\mathcal{E}$-valuation (ϕ_0,val) such that ϕ_0 is the identity function with $FV(\Gamma)$ as domain. Then, by lemma 5.9 and lemma 5.10, we have

$$M \equiv \phi_0(M) \in Eval_{\rho_0}(N) \in V(\phi_0(N)) = V(N)$$

As N is an $\mathcal{E}$-type, $V(N) = Sat_{\mathcal{E}}(N)$. So, $Eval_{\rho_0}(N) \subseteq SN_{\mathcal{E}}(N)$. So, $M \in SN_{\mathcal{E}}(N)$ is strongly normalizable. □

Corollary 5.12 (Consistency of CC^∞_C) *CC^∞_C is consistent in the sense that there is a proposition which is not inhabited by any term. Particularly, for any term M, $\not\vdash M{:}\Pi x{:}Prop.x$.*

Proof Suppose $\vdash M{:}\Pi x{:}Prop.x$. By SN theorem and subject reduction (theorem 3.12), we may assume that M is in normal form. So, M has the form $\lambda x{:}Prop.M'$. As M' is in normal form, it must have the form $yM_1...M_m$, and $x{:}Prop \vdash M'{:}x$. By lemma 3.3, $x \equiv y$. So, we have

$$x{:}Prop \vdash xM_1...M_m{:}x$$

If $m = 0$, then we have $x{:}Prop \vdash x{:}x$. But, by theorem 3.15 and CR-theorem, this implies that $x \rhd Prop$, which is impossible. If $m > 0$, then we have that $x{:}Prop \vdash x{:}\Pi x{:}A.B$ for some A and B. But, by theorem 3.15 and CR-theorem, this would imply that $\Pi y{:}A.B \rhd Prop$, which is impossible, either. So, $\not\vdash M{:}\Pi x{:}Prop.x$. □

Remark Based on the strong normalization theorem, we have that the problem of type-checking and type computation for CC^∞_C (and CC^∞) is decidable. □

6 ΣCC^∞_C and Its Strong Normalization

ΣCC^∞_C is the system presented in [Luo88], which extends CC^∞_C by adding strong sum types in the following way:

1. Adding the following term-forming clause:

   - if M and N are terms, so are the following:

   $$\Sigma x{:}M.N,\ (M,N),\ \pi_1(M),\ \pi_2(M)$$

2. For $j = 1,2$, $\pi_j(M_1, M_2)$ is called a *σ-redexes* with M_j as its contractum (*i.e.*, $\pi_j(M_1, M_2) \rhd_\sigma M_j$). $\rhd$ and $\simeq$ are correspondingly extended to $\beta\sigma$-reduction/conversion.

3. Adding the following inference rules:

$$(\Sigma)\qquad \frac{\Gamma \vdash A{:}Type_j \quad \Gamma, x{:}A \vdash B{:}Type_j}{\Gamma \vdash \Sigma x{:}A.B{:}Type_j}\quad (j \in \omega)$$

$$(pair)\qquad \frac{\Gamma \vdash M{:}A \quad \Gamma \vdash N{:}[M/x]B \quad \Gamma, x{:}A \vdash B{:}Type_j}{\Gamma \vdash (M,N){:}\Sigma x{:}A.B}\quad (j \in \omega)$$

$$(\pi 1)\qquad \frac{\Gamma \vdash M{:}\Sigma x{:}A.B}{\Gamma \vdash \pi_1(M){:}A}$$

$$(\pi 2)\qquad \frac{\Gamma \vdash M{:}\Sigma x{:}A.B}{\Gamma \vdash \pi_2(M){:}[\pi_1(M)/x]B}$$

The Church-Rosser theorem and the properties we proved for $CC_{\mathcal{C}}^{\infty}$ in sections 3.1 and 3.2 can also be proved for $\Sigma CC_{\mathcal{C}}^{\infty}$, which we summarize as the following theorem:

Theorem 6.1 *In $\Sigma CC_{\mathcal{C}}^{\infty}$, we have*

1. *(Church-Rosser) If $M_1 \simeq M_2$, then there exists M such that $M_1 \rhd M$ and $M_2 \rhd M$.*
2. *Any derivation D of $\Gamma, x{:}A, \Gamma' \vdash F$ has a sub-derivation of $\Gamma \vdash A{:}K$ for some kind K.*
3. *Any derivation of $\Gamma, \Gamma' \vdash F$ has a sub-derivation of $\Gamma \vdash Prop{:}Type_0$.*
4. *If $\Gamma \vdash F$ and Γ' is a valid context which contains every component of Γ, then $\Gamma' \vdash F$.*
5. *If $\Gamma, x{:}N, \Gamma' \vdash F$ and $\Gamma \vdash M{:}N$, then $\Gamma, [M/x]\Gamma' \vdash [M/x]F$.*
6. *If $\Gamma \vdash M{:}A$, then $\Gamma \vdash A{:}K$ for some kind K.*
7. *(subject reduction) If $\Gamma \vdash M{:}A$ and $M \rhd N$, then $\Gamma \vdash N{:}A$.*
8. *If $\Gamma, y{:}Y, \Gamma' \vdash M{:}A$ and $y \notin FV(M{:}A) \cup FV(\Gamma')$, then $\Gamma, \Gamma' \vdash M{:}A$.*

□

The partial order induced by universes are now extended to $\Sigma CC_{\mathcal{C}}^{\infty}$ by adding the following clause to definition 3.16:

- if $A \preceq A'$ and $B \preceq B'$, then $\Sigma x{:}A.B \preceq \Sigma x{:}A'.B'$.

Every well-typed term in $\Sigma CC_{\mathcal{C}}^{\infty}$ has minimal types under this ordering, although its minimum type might not exist. For example, let

$$A \equiv \Pi x{:}Type_0 \Pi y{:}x.x$$

$$a \equiv \lambda x{:}Type_0 \lambda y{:}x.y$$

then, we can derive, for $i, j \geq 1$,

$$\vdash (A, a) : \Sigma z{:}Type_i.z$$

$$\vdash (A, a) : \Sigma z{:}Type_j.A$$

We can easily see that, under the empty context, $\Sigma z{:}Type_1.z$ and $\Sigma z{:}Type_1.A$ are minimal types of (A, a) but they are not $\beta\sigma$-convertible. However, an interesting fact is that, if we extend the complexity measure δ_j by adding the following clauses to definition 4.6:

- If $A^\circ \equiv \pi_{i_1}...\pi_{i_j}(xA_1...A_m)A'_1...A'_{m'}$ is a proper $\mathcal{E}$-j-type, then $\delta_j A =_{\mathrm{df}} 1$;
- If $A^\circ \equiv \Sigma x{:}A_1.A_2$, then $\delta_j A =_{\mathrm{df}} max\{\delta_j A_1, \delta_j A_2\} + 1$.

then we have, for all j,

$$\delta_j(\Sigma z{:}Type_1.z) = \delta_j(\Sigma z{:}Type_1.A)$$

With this property, we can prove the quasi-normalization result for $\Sigma\mathrm{CC}^\infty_{\mathcal{C}}$ which says that every well-typed term can be reduced to a term which does not contain any σ-redex or β-redex R_1R_2 such that R_1 has a type which is a proper type. This further implies that every well-typed proper type can be reduced to one of the following forms:

$$K,\ \pi_{i_1}...\pi_{i_j}(xA_1...A_m)A'_1...A'_{m'},\ \Pi x{:}A_1.A_2,\ \Sigma x{:}A_1.A_2$$

where K is a kind, x is a variable, $j, m, m' \geq 0$ and $i_k \in \{1,2\}$. Then, the complexity measure β for the $\mathcal{E}$-types defined as in definition 5.1 has the property that, if an $\mathcal{E}$-type A is not an $\mathcal{E}$-proposition and reduces to a Q-normal $\mathcal{E}$-type $\Pi x{:}A_1.A_2$ or $\Sigma x{:}A_1.A_2$, then

$$\beta(A_1) < \beta(A)$$

and, for every N such that $\mathcal{E} \vdash N{:}A_1$,

$$\beta([N/x]A_2) < \beta(A)$$

With this, one can define the possible values and canonical value of $\mathcal{E}$-terms by extending definitions 5.5 and 5.6 by the following clause (with the definition of saturated set appropriately extended):

- If $\mathcal{E} \vdash M{:}A$ and A reduces to a Q-normal term $\Sigma x{:}A_1.A_2$, then

$$V(M) =_{\mathrm{df}} \{\, (v_1, v_2) \mid v_1 \in V(\pi_1(M)), v_2 \in V(\pi_2(M)) \,\}$$

$$v_M =_{\mathrm{df}} (v_{\pi_1(M)}, v_{\pi_2(M)})$$

Then, the definition of the evaluation function $Eval$ is extended by adding the following clauses:

- $Eval_\rho(\Sigma x{:}M_1.M_2)$ is defined to be the set of the terms P such that $\mathcal{E} \vdash P{:}\phi(M)$, $\pi_1(P) \in Eval_\rho(M_1)$ and $\pi_2(P) \in Eval_{\rho'}(M_2)$, where ρ' extends ρ such that $\rho'(x) = (\pi_1(P), v_{\pi_1(P)})$;
- $Eval_\rho((M_1, M_2)) =_{\mathrm{df}} (Eval_\rho(M_1), Eval_\rho(M_2))$;
- For $j = 1, 2$, $Eval_\rho(\pi_j(M)) =_{\mathrm{df}} v_j$, if $Eval_\rho(M) = (v_1, v_2)$.

The strong normalization theorem for ΣCC_C^∞ can then be proved following the same pattern as the proof of SN theorem for CC_C^∞ we give in section 5.

Remark As the type-uniqueness upto kinds fails, the author does not know whether type checking for ΣCC_C^∞ is decidable or not. The introduction rule (pair) for Σ-types 'loses' some type information which seems to be necessary to get a direct type-checking algorithm (*c.f.*, [Coq86b]). □

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