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Part III: Indexed Categories

by

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Some Fundamental Algebraic Tools for the Semantics of Computation
Part III: Indexed Categories

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Abstract

We present the concept of *indexed category*, a technical tool to model families of categories defined in a uniform way. We show how any indexed category gives rise to a single *flat* category, a disjoint union of the components with some additional morphisms between them. Similarly, any *indexed functor* (a family of functors between component categories) induces a flat functor between the corresponding flat categories. We prove that under some technical conditions flat categories are complete (resp. cocomplete) if all their components are so; flat functors have left adjoints if all their components do. A few examples illustrate the usefulness of these concepts and results.

1 Introduction

Even a brief overview of recent and not-so-recent developments in theoretical computer science has to indicate a fundamental role of many notions, techniques and results of category theory (and universal algebra). They have been used to clarify, formalise, appropriately generalise and sometimes even develop a number of practically important concepts and methods of computer programming.

Some leading examples may be found in the area of algebraic specification, from the very beginning based on the notion of initiality to explicate the very concept of abstract data type (cf. [Goguen, Thatcher & Wagner 76]) and then full of terminal objects (e.g. [Wand 79]), left adjoints (e.g. [Thatcher, Wagner & Wright 82], [Ehrich 82]), colimits (e.g. [Burstall & Goguen 77]), comma categories (e.g. [Goguen & Burstall 84]) etc.

Another example of a heavy use of category theory is a somewhat separate work, grown on the grounds of algebraic specification around a certain formalisation of the concept of logical system, the notion of institution introduced in [Goguen & Burstall 85] and further developed in [Goguen & Burstall 86]. The topics studied here (so far) include specification languages (CLEAR [Burstall & Goguen 80], ASL [Sannella & Tarlecki 84], Extended ML [Sannella & Tarlecki 86]), implementations ([Beierle & Voss 85], [Sannella & Tarlecki 87]), observational equivalence ([Sannella & Tarlecki 85]), free constructions ([Tarlecki 85, 87]), elements of model theory ([Tarlecki 86]). We are deeply convinced that most of these very important (in our view) topics cannot be adequately treated without categorial tools.

The main goal of this paper is to add to the equipment of "the working computer scientist" one more categorial tool, the notion of indexed category. The standard mathematical reference for the basic definitions and some deep mathematics around this notion is [Johnstone & Paré 78]. The underlying idea is very simple: quite often we define and deal with families of categories indexed by a collection of indices, rather than just single categories. Moreover, the components of such a family are defined in a uniform way; that is, any change (a morphism — indices form a category) from one index to another induces (contravariantly) a smooth translation (a functor) between the corresponding component categories. An indexed category is such a uniformly defined family of categories.

A prime example of indexed category the reader may be familiar with is the indexed category of many-sorted algebras (our Example 3 below). For each many-sorted algebraic signature Σ, we have a category $\mathbf{ALG}(\Sigma)$ of Σ-algebras defined in a well-known way. Thus, we indeed deal here with a collection of categories, indexed by many-sorted algebraic signatures. Moreover, many-sorted algebraic signatures form a category (with algebraic signature morphisms defined in a rather obvious way) and indeed, for each algebraic signature morphism $\sigma\colon \Sigma \to \Sigma'$, we have a "$\sigma$-reduct" functor $\mathbf{ALG}(\sigma)\colon \mathbf{ALG}(\Sigma') \to \mathbf{ALG}(\Sigma)$. (See, for example, [Burstall & Goguen 82] for an expository presentation of these concepts.)

Any indexed category in a natural way gives rise to a single "flat" category, formed, roughly, as a disjoint union of the component categories with some morphisms between them defined using the translations induced by index morphisms. It turns out that many categories studied in computer science are of this form, that is, may be built by "flattening" an indexed category. We introduce these basic definitions and illustrate them by a few rather simple examples in Section 2.

Let us point out here that such a category, constructed by "flattening" an indexed category, may be equipped in an obvious way with a projection functor that maps any object of this category to the index of the component category the objects "comes from". Thus, it forms what is known as a "fibred category" (concept introduced in [Grothendieck 63]) and the indexed category we have started with may be viewed as a presentation of this fibred category. It may be argued that fibred categories incorporate the same intuitive idea as indexed categories, but are conceptually simpler and easier to work with (cf. e.g. [Benabou 85]). Even if this was the case, we believe that the particular form of presentation of fibred categories via indexed categories has much intuitive appeal and is very close to the way we both construct and think about many categories that arise in applications of category theory. In particular, we believe that this is the case with applications of category theory in theoretical computer science, as we hope to illustrate in this paper.

One of the elementary concepts of category theory, the limit (and dually, colimit) construction, has been used in computer science to "put together" structures of all possible kinds (general systems in [Goguen 71], [Goguen & Ginali 78], theories in [Burstall & Goguen 77, 80], labelled graphs in [Ehrig *et al* 81] etc.). To use these constructions freely we have to show that in a given category the limit (resp. colimit) of any diagram exists, that is, the category is complete (resp. cocomplete). In Section 3 we prove that under some technical conditions, if all the component categories are complete (resp. cocomplete) then the flat category formed out of them is so as well. This gives a useful tool to structure and, in a sense, to localise proofs of (co)completeness of some categories.

Another basic notion is generalised to indexed categories in Section 4. Given two indexed categories (over the same indices) an indexed functor between them is just a family of functors between the corresponding component categories consistent with the translations induced by index morphisms. As with indexed categories, indexed functors can be "flattened" to obtain functors between flattened categories.

In the theory of algebraic specifications one of the basic technical tools is the notion of free functor (left adjoint to a usually obvious "forgetful" functor — cf. the concept of parameterised specification in [Thatcher, Wagner & Wright 82], to take just one example). Adjoint situations between flattened categories may be built "locally" on the components of the indexed categories: families of adjunctions can be flattened, as can indexed categories and functors. More exactly, we prove that if all the components of an indexed functor have left adjoints then so does the flattened functor. This offers a way to structure proofs that some functors have left adjoints and, in fact, gives another possible proof of the cocompleteness result for flattened categories mentioned above (details in Section 4).

Finally, let us stress once more that we advocate just the use of indexed categories as a simple and convenient technical tool. We are not interested here in any deep foundational problems; in particular, we are not taking any side in the controversy on whether indexed or fibred categories are more appropriate as a foundation of category theory (cf. [Benabou 85]). We remain (comfortably) on the grounds of what [Benabou 85] calls "naive category theory".

In fact, this allows us to considerably simplify the concept of indexed category and "naively" (but technically sound) work "up to equality" rather than "up to canonical isomorphisms" or "coherences" (cf. [Johnstone & Paré 78]). "Canonical isomorphisms" seem necessary if one is interested in foundational issues, but also lead to some technical difficulties (as pointed out in [Benabou 85]).

The technical results presented here seem new; at least we have not seen them published in this form. Even if they were known in the folklore of the area (with which, admittingly, we are not too familiar) we believe that they deserve an expository presentation which would be available to the "users" of category theory.

Throughout the paper we assume some familiarity with basic category theory and universal algebra, although not necessarily with any deep results. We refer to, for example, [MacLane 71], [Herrlich & Strecker 73], [Arbib & Manes 75] and [Burstall & Goguen 82] for some notation, terminology and definitions we omit here.

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2 Indexed Categories

It is perhaps quite surprising to realise how often we talk not just about a single category, but rather about a family of categories, "parameterised" by a certain collection of indices. Here is a very simple, but nevertheless quite typical example.

Example 1 *Many-sorted sets*

For any set S, we have a category $\mathbf{SSET}(S)$ of S-sorted sets, with S-sorted (i.e. indexed by S) families of sets as objects and S-sorted families of functions as morphisms. Formally, we define

$$\mathbf{SSET}(S) = [S \rightarrow \mathbf{Set}],$$

where for any two categories $\mathbf{A}$ and $\mathbf{B}$, $[\mathbf{A} \rightarrow \mathbf{B}]$ denotes the category of functors from $\mathbf{A}$ to $\mathbf{B}$ with natural transformations as morphisms and the obvious — vertical — composition (cf. [MacLane 71, II.4, p.40], where this category is written as $\mathbf{B}^{\mathbf{A}}$); $\mathbf{Set}$ is the category of all sets; and we identify any set (S here) with the corresponding discrete category.

For notational convenience, we write $X: S \rightarrow \mathbf{Set}$ in the form $\langle X_s \rangle_{s \in S}$, where $X_s = X(s), s \in S$. Similarly, $g: X \rightarrow Y$ in $\mathbf{SSET}(S)$ may be written as $g = \langle g_s: X_s \rightarrow Y_s \rangle_{s \in S}$.

□ (Ex. 1)

Of course, it wouldn't be of much interest to consider just any families of arbitrarily collected categories. It is only natural to assume that all the categories in a family are defined uniformly in the same way. Semantically, this means that any change of an index induces a smooth translation between the corresponding component categories. In the examples we look at, the translation goes in the opposite direction than the change of index.

Example 1 *Many-sorted sets* (continued)

The collection of indices, which are sets here, comes naturally equipped with index morphisms, functions between sets. Any function $f: S1 \rightarrow S2$ induces a functor

$$\mathbf{SSET}(f): \mathbf{SSET}(S2) \rightarrow \mathbf{SSET}(S1).$$

More exactly, this functor is defined as follows:

on objects: for any object $X \in |\mathbf{SSET}(S2)|$, $\mathbf{SSET}(f)(X) = f;X: S1 \to \mathbf{Set}$ (recall that $X: S2 \to \mathbf{Set}$), i.e. for $s1 \in S1$, $(\mathbf{SSET}(f)(X))_{s1} = X_{f(s1)}$;

on morphisms: for any morphism $g: X \to Y$ in $\mathbf{SSET}(S2)$, where $X, Y \in |\mathbf{SSET}(S2)|$ and $g = \langle g_{s2}: X_{s2} \to Y_{s2}\rangle_{s2 \in S2}$, $\mathbf{SSET}(f)(g) = \langle g_{f(s1)}: X_{f(s1)} \to Y_{f(s1)}\rangle_{s1 \in S1}: f;X \to f;Y$.

Moreover, the indices and their morphisms form a category, **Set**, the category of all sets and total functions between them. A crucial property is that the functors induced by index morphisms do not depend on any decomposition of index morphisms. More formally, **SSET** is a (contravariant) functor from **Set** to **Cat**, the category of "all" categories[1],

$$\mathbf{SSET}: \mathbf{Set}^{op} \to \mathbf{Cat}$$

□ (Ex. 1)

Notational remark

Throughout the paper, just as above, composition in any category is denoted by ; (semicolon) and written in the diagrammatical order. Identities are denoted by *id* (with some subscripts, if necessary).

Definition 1

An *indexed category* **C** over an *index category* **Ind** is a functor from $\mathbf{Ind}^{op}$ to **Cat**, the category of all categories. Thus, for each index $i \in |\mathbf{Ind}|$ there is a category $\mathbf{C}(i)$ and for each index morphism $\sigma: i \to j$ in **Ind** there is a functor $\mathbf{C}(\sigma): \mathbf{C}(j) \to \mathbf{C}(i)$.

□ (Def. 1)

We often write $\mathbf{C}_i$ and $\mathbf{C}_\sigma$ for $\mathbf{C}(i)$ and $\mathbf{C}(\sigma)$, respectively. We refer to $\mathbf{C}_i$, $i \in |\mathbf{Ind}|$, as *component categories* of **C**; we call $\mathbf{C}_\sigma$ the *translation functor* induced by σ.

Although in an indexed category each component is a separate category in itself, they are uniform enough that sometimes we want to consider them all together, in one single, usually rather "large" category. In other words, we may want to "flatten" the indexed category and consider, roughly, a disjoint union of all its components, with some additional morphisms between them built on index morphisms.[2]

Example 1 *Many-sorted sets* (continued).

We can flatten the indexed category $\mathbf{SSET}: \mathbf{Set}^{op} \to \mathbf{Cat}$ and get a category $\mathbf{SSet} = \mathbf{Flat}(\mathbf{SSET})$ of all many-sorted sets. More explicitly, **SSet** may be defined as follows:

objects: The objects of **SSet** are many-sorted sets with explicitly indicated set of sorts, i.e. pairs $\langle S, X\rangle$, where S is a set (of sorts) and $X: S \to \mathbf{Set}$ is an S-sorted set.

[1]Of course, some foundational difficulties are connected with the use of this (very) large category. We do not discuss this point here, and we disregard other such foundational issues in this paper.

[2]This is known as "the Grothendieck construction" — a frightening name for a simple idea.

morphisms: Given two such many-sorted sets, $\langle S, X\rangle$ and $\langle S', X'\rangle$ (possibly with different sets of sorts), a morphism between them is again a pair $\langle f, g\rangle: \langle S, X\rangle \to \langle S', X'\rangle$, where f is a function between their sets of sorts, $f: S \to S'$, and $g: X \to f; X'$ is a many-sorted function, $g = \langle g_s: X_s \to X'_{f(s)}\rangle_{s\in S}$.

composition: The composition in **SSet** is componentwise — we have to re-index the second component of morphisms, though. More exactly, for any two morphisms $\langle f, g\rangle: \langle S, X\rangle \to \langle S', X'\rangle$ and $\langle f', g'\rangle: \langle S', X'\rangle \to \langle S'', X''\rangle$, $\langle f, g\rangle; \langle f', g'\rangle = \langle \bar{f}, \bar{g}\rangle: \langle S, X\rangle \to \langle S'', X''\rangle$, where $\bar{f} = f; f': S \to S''$ and $\bar{g} = g; \mathbf{SSET}(f)(g'): X \to f; f'; X''$, i.e. $g = \langle g_s; g'_{f(s)}: X_s \to X''_{f'(f(s))}\rangle_{s\in S}$.

□ (Ex. 1)

Definition 2

For any indexed category $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$, $\mathbf{Flat}(\mathbf{C})$ is a category defined as follows:

objects: Objects in $\mathbf{Flat}(\mathbf{C})$ are pairs $\langle i, a\rangle$, where $i \in |\mathbf{Ind}|$ and $a \in |\mathbf{C}_i|$.

morphisms: Morphisms in $\mathbf{Flat}(\mathbf{C})$ from $\langle i, a\rangle$ to $\langle j, b\rangle$ are pairs $\langle \sigma, f\rangle$, where $\sigma: i \to j$ is a morphism in $\mathbf{Ind}$ and $f: a \to \mathbf{C}_\sigma(b)$ is a morphism in $\mathbf{C}_i$.

composition: For any morphisms $\langle \sigma, f\rangle: \langle i, a\rangle \to \langle j, b\rangle$ and $\langle \rho, g\rangle: \langle j, b\rangle \to \langle k, c\rangle$ in $\mathbf{Flat}(\mathbf{C})$, their composition is defined by

$$\langle \sigma, f\rangle; \langle \rho, g\rangle = \langle \sigma; \rho, f; \mathbf{C}_\sigma(g)\rangle: \langle i, a\rangle \to \langle k, c\rangle.$$

□ (Def. 2)

Just one more technical definition: every flattened category comes equipped with the projection on the first component of the pairs. This projection is a functor which may be used to reconstruct the original cleavage of the (indexed) category.

Definition 3

Let $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$ be an arbitrary indexed category. We define a functor $\mathbf{Cleave}: \mathbf{Flat}(\mathbf{C}) \to \mathbf{Ind}$ as follows:

on objects: for any object $\langle i, a\rangle$ of $\mathbf{Flat}(\mathbf{C})$, $\mathbf{Cleave}(\langle i, a\rangle) = i$, and

on morphisms: for any morphism $\langle \sigma, f\rangle$ in $\mathbf{Flat}(\mathbf{C})$, $\mathbf{Cleave}(\langle \sigma, f\rangle) = \sigma$.

□ (Def. 3)

To complete this section, let us have a look at a few more simple examples which illustrate the introduced notions.

Example 2 *Many-sorted algebraic signatures*

For any set S, a category $\mathbf{ALGSIG}(S)$ of S-sorted algebraic signatures is defined as the functor category

$$\mathbf{ALGSIG}(S) = [S^+ \to \mathbf{Set}]$$

(i.e. $\mathbf{ALGSIG}(S) = \mathbf{SSET}(S^+)$) where for any set S, S^+ is the set of all finite nonempty sequences of elements of S.

An S-sorted algebraic signature is just a family of sets (of operation names), one set for each finite, nonempty sequence of elements of S (rank, i.e. arity and result sort, of the operations named in the set); an S-sorted algebraic signature morphism is a renaming of operation names preserving their rank.

The map $S \mapsto S^+$ extends in the obvious way to a functor $(\_)^+\colon \mathbf{Set} \to \mathbf{Set}$. The indexed category of algebraic signatures is

$$\mathbf{ALGSIG} = (\_)^+; \mathbf{SSET}\colon \mathbf{Set}^{op} \to \mathbf{Cat}$$

(There is a slight technical inaccuracy in the above definition: we have identified the functor $(\_)^+\colon \mathbf{Set} \to \mathbf{Set}$ with its opposite, $((\_)^+)^{op}\colon \mathbf{Set}^{op} \to \mathbf{Set}^{op}$, which although identical as a map, formally is a different functor (as a morphism in $\mathbf{Cat}$).)

For any function $f\colon S \to S'$, the translation functor

$$\mathbf{ALGSIG}(f)\colon \mathbf{ALGSIG}(S') \to \mathbf{ALGSIG}(S)$$

"extracts" an S-sorted algebraic signature from any S'-sorted algebraic signature using the sort renaming f. For any S'-sorted algebraic signature Σ', for any sequence $s_1...s_n \in S^+$, the operation names of rank $s_1...s_n$ in the S-sorted algebraic signature $\mathbf{ALGSIG}(f)(\Sigma')$ are exactly the operation names of rank $f(s_1)...f(s_n) \in (S')^+$ in Σ'.

Finally, if we flatten the indexed category **ALGSIG**, we get what is usually meant by the category of algebraic signatures (cf. e.g. [Burstall & Goguen 82]):

$$\mathbf{AlgSig} = \mathbf{Flat}(\mathbf{ALGSIG})$$

Algebraic signatures (objects of **AlgSig**) are pairs of the form $\langle S, \langle \Sigma_r \rangle_{r \in S^+} \rangle$, where S is a set (of sorts) and for any $r \in S^+$, Σ_r is a set (of operation names of rank r). An algebraic signature morphism from $\Sigma = \langle S, \langle \Sigma_r \rangle_{r \in S^+} \rangle$ to $\Sigma' = \langle S', \langle \Sigma'_r \rangle_{r \in (S')^+} \rangle$ consists of a renaming of sorts $f\colon S \to S'$ and a renaming of operation names preserving their ranks (modified by f), i.e. a family of maps $g = \langle g_r\colon \Sigma_r \to \Sigma'_{f^+(r)} \rangle_{r \in S^+}$.

□ (Ex. 2)

Example 3 *Many-sorted algebras*

The category **AlgSig** of algebraic signatures is itself an index category in a prime example of an indexed category, the indexed category of many-sorted algebras

$$\mathbf{ALG}\colon \mathbf{AlgSig}^{op} \to \mathbf{Cat}$$

For any algebraic signature Σ, $\mathbf{ALG}(\Sigma)$ is the category of all Σ-algebras as objects and all Σ-homomorphisms as morphisms; for any algebraic signature morphism $\sigma\colon \Sigma \to \Sigma'$, $\mathbf{ALG}(\sigma)$ is the usual "forgetful" functor

$$\_|_\sigma\colon \mathbf{ALG}(\Sigma') \to \mathbf{ALG}(\Sigma)$$

(see e.g. [Burstall & Goguen 82] for the standard definitions).

In the flattened category of many-sorted algebras, $\mathbf{Flat(ALG)}$, the objects are many-sorted algebras with explicitly indicated signatures; a morphism between such algebras, say from $\langle\Sigma, A\rangle$ to $\langle\Sigma', B'\rangle$, consists of an algebraic signature morphism $\sigma\colon\Sigma \to \Sigma'$ and a Σ-homomorphism $h\colon A \to B|_\sigma$. A similar concept of a "generalised homomorphism" has been used in some work on algebraic specification, cf. e.g. [Kamin & Archer 84].

□ (Ex. 3)

Example 4 *Diagrams*

For any "target" category $\mathbf{T}$ and "source" category $\mathbf{G}$, we have a category

$$\mathbf{FUNC(T)(G)} = [\mathbf{G} \to \mathbf{T}]$$

of functors from $\mathbf{G}$ into $\mathbf{T}$.

It is often convenient to define a diagram in a category just as a functor with a small source category. This is essentially equivalent to the more standard definition of a diagram as a graph with nodes labelled by objects of the considered category and edges labelled by morphisms with the appropriate domain and codomain (cf. e.g. [Goguen & Burstall 84]). Thus, the category $\mathbf{FUNC(T)(G)}$ may be referred to as the category of diagrams of the shape $\mathbf{G}$ in the category $\mathbf{T}$.

As in the previous examples, it is easy to see that for any target category $\mathbf{T}$, $\mathbf{FUNC(T)}$ forms in fact an indexed category

$$\mathbf{FUNC(T)}\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$$

where we define:

component categories: for $\mathbf{G} \in |\mathbf{Cat}|$, $\mathbf{FUNC(T)(G)} = [\mathbf{G} \to \mathbf{T}]$ as above, and

translation functors: for $\mathbf{\Phi}\colon \mathbf{G} \to \mathbf{G}'$, $\mathbf{FUNC(T)}(\mathbf{\Phi})\colon [\mathbf{G}' \to \mathbf{T}] \to [\mathbf{G} \to \mathbf{T}]$ is the obvious functor defined on objects by $\mathbf{FUNC(T)}(\mathbf{\Phi})(\mathbf{D}') = \mathbf{\Phi};\mathbf{D}'$ for all $\mathbf{D}'\colon \mathbf{G}' \to \mathbf{T}$ (i.e. $\mathbf{D}' \in [\mathbf{G}' \to \mathbf{T}]$).

By flattening $\mathbf{FUNC(T)}$ we get the category $\mathbf{Func(T)} = \mathbf{Flat(FUNC(T))}$ of functors into $\mathbf{T}$ (or diagrams in $\mathbf{T}$ — cf. [Goguen 71] where a slightly different definition is used). A morphism from $\mathbf{D}\colon \mathbf{G} \to \mathbf{T}$ to $\mathbf{D}'\colon \mathbf{G}' \to \mathbf{T}$ in $\mathbf{Func(T)}$ consists of a functor $\mathbf{\Phi}\colon \mathbf{G} \to \mathbf{G}'$ and a natural transformation $\alpha\colon \mathbf{D} \to \mathbf{\Phi};\mathbf{D}'$ (between functors in $[\mathbf{G} \to \mathbf{T}]$).

□ (Ex. 4)

Example 5 *Theories*

We want to discuss theories built in an arbitrary logical system. The notion of institution introduced in [Goguen & Burstall 85] provides an appropriate framework.

An *institution* $\mathbf{I}$ consists of:

- a category $\mathbf{Sign}$ (of *signatures*);
- functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$ (giving for each signature $\Sigma \in |\mathbf{Sign}|$ a category $\mathbf{Mod}(\Sigma)$ of Σ-*models*);

- a functor $\mathbf{Sen}: \mathbf{Sign} \to \mathbf{Cat}$ (giving for each signature $\Sigma \in |\mathbf{Sign}|$ a *discrete* category $\mathbf{Sen}(\Sigma)$ of Σ-*sentences*);
- for each $\Sigma \in |\mathbf{Sign}|$, a (*satisfaction*) relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$

such that the following *satisfaction condition* holds:
for each $\sigma: \Sigma \to \Sigma'$ in $\mathbf{Sign}, m' \in |\mathbf{Mod}(\Sigma')|$ and $\varphi \in \mathbf{Sen}(\Sigma)$,

$$m' \models_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi) \iff \mathbf{Mod}(\sigma)(m') \models_\Sigma \varphi.$$

For any signature morphism $\sigma: \Sigma \to \Sigma'$, $\mathbf{Sen}(\sigma)$ will be written simply as σ and $\mathbf{Mod}(\sigma)$ as $\_|_\sigma$.

First, let us remark that this definition explicitly involves two indexed categories: **Mod** (indexed by **Sign**) and **Sen** (indexed by $\mathbf{Sign}^{op}$). In this example we want to discuss yet another indexed category: the indexed category **TH** of theories in **I**. It should be stressed that this category naturally arises in the study of specifications built over **I**.

For any signature $\Sigma \in |\mathbf{Sign}|$, a Σ-*presentation* is any set of Σ-sentences $\Psi \subseteq \mathbf{Sen}(\Sigma)$. Each Σ-presentation Ψ generates the set of its *logical consequences*,

$$Cl_\Sigma(\Psi) = \{\varphi \in \mathbf{Sen}(\Sigma) \mid \text{for all } m \in \mathbf{Mod}(\Sigma),\ m \models \varphi \text{ whenever } m \models \Psi\}.$$

By a Σ-*theory* we mean any Σ-presentation T that is closed under semantical consequence, i.e. $T = Cl_\Sigma(T)$.

Let $\mathbf{TH}(\Sigma)$ denote the poset category of Σ-theories ordered by inclusion. This extends to an indexed category

$$\mathbf{TH}: \mathbf{Sign}^{op} \to \mathbf{Cat},$$

where additionally we define for any signature morphism $\sigma: \Sigma \to \Sigma'$ in **Sign**, for any Σ'-theory T',

$$\mathbf{TH}(\sigma)(T') = \{\varphi \in \mathbf{Sen}(\Sigma) \mid \sigma(\varphi) \in T'\}.$$

The satisfaction condition implies that this is indeed a Σ-theory; it is obvious that $\mathbf{TH}(\sigma)$ is a functor, i.e. a monotone map.

We can flatten the indexed category of theories and get the category $\mathbf{Th} = \mathbf{Flat}(\mathbf{TH})$, the usual category of theories in the institution **I** (cf. [Goguen & Burstall 85]). **Th** has pairs of the form $\langle \Sigma, T \rangle$, where Σ is a signature and T is a Σ-theory, as objects; a theory morphism from $\langle \Sigma, T \rangle$ to $\langle \Sigma', T' \rangle$ in **Th** is just any signature morphism $\sigma: \Sigma \to \Sigma'$ such that for all $\varphi \in T$, $\sigma(\varphi) \in T'$.

Just for fun (and to get yet another example) notice that in a similar way we can define a somewhat larger indexed category of *presentations* in **I**. For any signature Σ, let $\mathbf{PRES}(\Sigma)$ be the poset category of Σ-presentations in **I**. This yields an indexed category

$$\mathbf{PRES}: \mathbf{Sign}^{op} \to \mathbf{Cat},$$

where for any $\sigma: \Sigma \to \Sigma'$ in **Sign**, for any $\Psi' \subseteq \mathbf{Sen}(\Sigma')$, $\mathbf{PRES}(\sigma)(\Psi') = \{\varphi \in \mathbf{Sen}(\Sigma) \mid \sigma(\varphi) \in \Psi'\}$.

We can "enlarge" this even further by adding some new morphisms in the component categories. For any signature Σ, let $\mathbf{PRES}_\models(\Sigma)$ be the pre-order category of Σ-presentations preordered by semantical consequence $\models_\Sigma$ (for any Σ-presentations Ψ and Ψ', $\Psi' \models_\Sigma \Psi$ if $\Psi \subseteq Cl_\Sigma(\Psi')$). This yields an indexed category

$$\mathbf{PRES}_\models : \mathbf{Sign}^{op} \to \mathbf{Cat}.$$

The satisfaction condition implies that $\mathbf{PRES}_\models(\sigma): \mathbf{PRES}_\models(\Sigma') \to \mathbf{PRES}_\models(\Sigma)$, defined for any signature morphism $\sigma: \Sigma \to \Sigma'$ just as $\mathbf{PRES}(\sigma)$ above, preserves the semantical consequence.

Notice that **TH** is an *indexed subcategory* of **PRES**, and that **PRES** is an indexed subcategory of $\mathbf{PRES}_\models$ — we will formalise this concept later (Example 8 in Section 4).

□ (Ex. 5)

Example 6 *Institutions*

Let us start by recalling the notion of an institution morphism (cf. [Goguen & Burstall 85]).

Let $\mathbf{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \langle \models_\Sigma \rangle_{\Sigma \in |\mathbf{Sign}|} \rangle$ and $\mathbf{I}' = \langle \mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \langle \models'_{\Sigma'} \rangle_{\Sigma' \in |\mathbf{Sign}'|} \rangle$ be arbitrary institutions. An *institution morphism* from $\mathbf{I}$ to $\mathbf{I}'$ consists of:

- a functor $\mathbf{\Phi}: \mathbf{Sign} \to \mathbf{Sign}'$,
- a natural transformation $\beta: \mathbf{Mod} \to \mathbf{\Phi}; \mathbf{Mod}'$, and
- a natural transformation $\alpha: \mathbf{\Phi}; \mathbf{Sen}' \to \mathbf{Sen}$

such that the following *satisfaction condition* holds:
for each $\Sigma \in |\mathbf{Sign}|$, $m \in |\mathbf{Mod}(\Sigma)|$ and $\varphi' \in \mathbf{Sen}'(\mathbf{\Phi}(\Sigma))$,

$$m \models_\Sigma \alpha_\Sigma(\varphi') \iff \beta_\Sigma(m) \models'_{\mathbf{\Phi}(\Sigma)} \varphi'.$$

Intuitively, $\mathbf{I}$ is a "richer" institution than a "more primitive" $\mathbf{I}'$. $\mathbf{\Phi}$ extracts simpler $\mathbf{I}'$-signatures out of more complex $\mathbf{I}$-signatures; β extracts simpler $\mathbf{I}'$-models out of more complex $\mathbf{I}$-models; α translates $\mathbf{I}'$-sentences to $\mathbf{I}$-sentences, which is possible since $\mathbf{I}$ is "more powerful".

Now, institutions and institution morphisms with composition defined in a rather straightforward componentwise manner form a category (cf. [Goguen & Burstall 85]). We aim at presenting it using the indexed-category machinery.

It turns out, however, that technically it costs nothing to generalise the concept of institution to cover not just logical systems, where the meanings of sentences in models are logical values (**true** or **false**), but also arbitrary semantic systems, where the meanings of sentences ("syntactic phrases") in models ("semantic structures") are taken from an arbitrary "semantic" category ("of denotations").

Let $\mathbf{V}$ be an arbitrary category. The category $\mathbf{Room}(\mathbf{V})$ of $\mathbf{V}$-*rooms* (cf. [Mayoh 85]) is defined as a comma category

$$\mathbf{Room}(\mathbf{V}) = (|\_|/\mathbf{FUNC}_{Disc}(\mathbf{V})),$$

where $|\_|: \mathbf{Cat} \to \mathbf{Cat}$ is the discretization functor and $\mathbf{FUNC}_{Disc}(\mathbf{V}): \mathbf{DCat}^{op} \to \mathbf{Cat}$ is the indexed category of functors into $\mathbf{V}$ restricted to discrete categories $\mathbf{DCat}$ (cf. Example 4). Thus, a $\mathbf{V}$-room is a triple $\langle \mathbf{M}, \mathbf{R}, S \rangle$, where $\mathbf{M}$ is a category, S is a discrete category, and $\mathbf{R}: |\mathbf{M}| \to [S \to \mathbf{V}]$. A $\mathbf{V}$-room morphism $\langle \mathbf{f}, g \rangle: \langle \mathbf{M}, \mathbf{R}, S \rangle \to \langle \mathbf{M}', \mathbf{R}', S' \rangle$ consists of a functor $\mathbf{f}: \mathbf{M} \to \mathbf{M}'$ and of a function $g: S' \to S$ such that the following diagram (in $\mathbf{Cat}$) commutes.

$$\begin{array}{ccc} |\mathbf{M}| & \xrightarrow{\mathbf{R}} & [S \to \mathbf{V}] \\ \Big\uparrow {\scriptstyle |\mathbf{f}|} & & \Big\uparrow {\scriptstyle g;(\_)} \\ |\mathbf{M}'| & \xrightarrow[\mathbf{R}']{} & [S' \to \mathbf{V}] \end{array}$$

Consequently, for all $m \in |\mathbf{M}|$, $\mathbf{R}'(\mathbf{f}(m)) = g;\mathbf{R}(m)$, i.e. for all $m \in |\mathbf{M}|$ and $s' \in S'$,

$$\mathbf{R}'(\mathbf{f}(m))(s') = \mathbf{R}(m)(g(s'))$$

(notice a ghost of the satisfaction condition here).

The category of *generalised institutions* with signatures $\mathbf{Sign} \in |\mathbf{Cat}|$ is defined as the functor category

$$\mathbf{INS}(\mathbf{Sign}) = [\mathbf{Sign}^{op} \to \mathbf{Room}(\mathbf{V})].$$

This extends to an indexed category

$$\mathbf{INS}\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$$

where for any functor $\mathbf{\Phi}\colon \mathbf{Sign} \to \mathbf{Sign}'$, the translation functor $\mathbf{INS}(\mathbf{\Phi})\colon \mathbf{INS}(\mathbf{Sign}') \to \mathbf{INS}(\mathbf{Sign})$ is defined on objects by:

$$\mathbf{INS}(\mathbf{\Phi})(\mathbf{I}') = \mathbf{\Phi}^{op};\mathbf{I}'$$

for any $\mathbf{I}'\colon (\mathbf{Sign}')^{op} \to \mathbf{Room}(\mathbf{V})$, which naturally extends to morphisms in $\mathbf{INS}(\mathbf{Sign}')$ as well.

Finally, the category of generalised institutions is

$$\mathbf{Ins} = \mathbf{Flat}(\mathbf{INS}).$$

The reader is advised to check that if $\mathbf{V}$ is $\mathbf{Bool}$, the category with exactly two elements, then this definition coincides with the explicit definitions of institution and institution morphism spelled out above.

Let us point out that the above technicalities are slightly different from Definition 14 and Proposition 16 of [Goguen & Burstall 86], where a technical inaccuracy occurred.[3]

□ (Ex. 6)

[3] In [Goguen & Burstall 86, Prop.16] the category of $\mathbf{V}$-rooms was defined as the comma category $(|\_|^{op}/\mathbf{V}^{-})$, where $|\_|^{op}\colon \mathbf{Cat}^{op} \to \mathbf{Cat}^{op}$ is the opposite of the discretization functor and $\mathbf{V}^{-}\colon \mathbf{DCat} \to \mathbf{Cat}^{op}$ is the opposite of our $\mathbf{FUNC}_{Disc}(\mathbf{V})\colon \mathbf{DCat}^{op} \to \mathbf{Cat}$. Consequently, a $\mathbf{V}$-room is a triple $\langle \mathbf{M}, \mathbf{R}, S\rangle$, where $\mathbf{M}$ is a category, S is a discrete category, and $\mathbf{R}\colon \mathbf{M} \to [S \to \mathbf{V}]$ is a morphism in $\mathbf{Cat}^{op}$, that is, $\mathbf{R}$ is a functor from $[S \to \mathbf{V}]$ to $\mathbf{M}$, unlike in the more explicit definition. Correcting this inaccuracy leads to the definition we present here.

3 Completeness and Cocompleteness of Flattened Categories

In this section we study how (and if) limits and colimits in flattened categories may be constructed using the corresponding constructions in the index and in the component categories.

First, about limits:

Of course, we cannot hope to construct limits in a flattened category unless the corresponding limits exist in the index and in the component categories. It turns out that the only additional condition needed is that the translation functors induced by index morphisms preserve limits.

Theorem 1

Let $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$ be an indexed category such that

1. $\mathbf{Ind}$ is complete,
2. for all indices $i \in |\mathbf{Ind}|$, $\mathbf{C}_i$ is complete, and
3. for all index morphisms $\sigma: i \to j$, the translation functor $\mathbf{C}_\sigma: \mathbf{C}_j \to \mathbf{C}_i$ is continuous.

Then the category $\mathbf{Flat}(\mathbf{C})$ is complete.

Proof

It is sufficient to prove that $\mathbf{Flat}(\mathbf{C})$ has all products and equalisers (cf. [MacLane 71, Th.V.2.1, p.109]).

products:

Consider any family of objects in $\mathbf{Flat}(\mathbf{C})$, $\langle i_n, a_n \rangle$, $n \in N$ (N is an arbitrary set).

Let i with projections $\pi_n: i \to i_n$, $n \in N$, be a product of i_n, $n \in N$, in $\mathbf{Ind}$. Then, let a with projections $f_n: a \to \mathbf{C}_{\pi_n}(a_n)$, $n \in N$, be a product of $\mathbf{C}_{\pi_n}(a_n)$, $n \in N$, in $\mathbf{C}_i$. We claim that $\langle i, a \rangle$ with projections $\langle \pi_n, f_n \rangle: \langle i, a \rangle \to \langle i_n, a_n \rangle$, $n \in N$, is a product of $\langle i_n, a_n \rangle$, $n \in N$, in $\mathbf{Flat}(\mathbf{C})$.

Consider an object $\langle j, b \rangle \in |\mathbf{Flat}(\mathbf{C})|$ with morphisms $\langle \sigma_n, g_n \rangle: \langle j, b \rangle \to \langle i_n, a_n \rangle$, $n \in N$, in $\mathbf{Flat}(\mathbf{C})$. By the construction, there exists a unique index morphism $\sigma: j \to i$ such that for $n \in N$, $\sigma; \pi_n = \sigma_n$ in $\mathbf{Ind}$. Moreover, the continuity of $\mathbf{C}_\sigma$ guarantees that $\mathbf{C}_\sigma(a)$ with morphisms $\mathbf{C}_\sigma(f_n): \mathbf{C}_\sigma(a) \to \mathbf{C}_\sigma(\mathbf{C}_{\pi_n}(a_n))$, $n \in N$, is a product of $\mathbf{C}_\sigma(\mathbf{C}_{\pi_n}(a_n)) = \mathbf{C}_{\sigma_n}(a_n)$, $n \in N$, in $\mathbf{C}_j$. Hence, there exists a unique morphism $g: b \to \mathbf{C}_\sigma(a)$ such that for $n \in N$, $g; \mathbf{C}_\sigma(f_n) = g_n$ in $\mathbf{C}_j$.

Then, $\langle \sigma, g \rangle: \langle j, b \rangle \to \langle i, a \rangle$ is a unique morphism in $\mathbf{Flat}(\mathbf{C})$ such that $\langle \sigma, g \rangle; \langle \pi_n, f_n \rangle = \langle \sigma_n, g_n \rangle$ for $n \in N$.

equalisers:

Consider any two "parallel" morphisms in $\mathbf{Flat}(\mathbf{C})$, $\langle \sigma 1, f1 \rangle, \langle \sigma 2, f2 \rangle: \langle i, a \rangle \to \langle j, b \rangle$.

Let $\sigma: k \to i$ be an equaliser of $\sigma 1, \sigma 2: i \to j$ in $\mathbf{Ind}$. Notice that we have $\mathbf{C}_\sigma(\mathbf{C}_{\sigma 1}(b)) = \mathbf{C}_{\sigma;\sigma 1}(b) = \mathbf{C}_{\sigma;\sigma 2}(b) = \mathbf{C}_\sigma(\mathbf{C}_{\sigma 2}(b))$. Let $f: c \to \mathbf{C}_\sigma(a)$ be an equaliser of $\mathbf{C}_\sigma(f1), \mathbf{C}_\sigma(f2): \mathbf{C}_\sigma(a) \to \mathbf{C}_\sigma(\mathbf{C}_{\sigma 1}(b))$ in $\mathbf{C}_k$.

We claim that $\langle \sigma, f \rangle: \langle k, c \rangle \to \langle i, a \rangle$ is an equaliser of $\langle \sigma 1, f1 \rangle, \langle \sigma 2, f2 \rangle$ in $\mathbf{Flat}(\mathbf{C})$.

For, first observe that by the construction we have indeed:

$$\begin{aligned}\langle\sigma, f\rangle;\langle\sigma 1, f1\rangle &= \langle\sigma;\sigma 1, f;\mathbf{C}_\sigma(f1)\rangle\\ &= \langle\sigma;\sigma 2, f;\mathbf{C}_\sigma(f2)\rangle\\ &= \langle\sigma, f\rangle;\langle\sigma 2, f2\rangle.\end{aligned}$$

Then, consider any morphism $\langle\rho, g\rangle: \langle m, d\rangle \rightarrow \langle i, a\rangle$ such that in $\mathbf{Flat}(\mathbf{C})$

$$\langle\rho, g\rangle;\langle\sigma 1, f1\rangle = \langle\rho, g\rangle;\langle\sigma 2, f2\rangle,$$

i.e. $\rho;\sigma 1 = \rho;\sigma 2$ in **Ind** and $g;\mathbf{C}_\rho(f1) = g;\mathbf{C}_\rho(f2)$ in $\mathbf{C}_m$. By the construction, there exists a unique index morphism $\theta: m \rightarrow k$ such that $\theta;\sigma = \rho$ in **Ind**. Moreover, $\mathbf{C}_\theta$ is continuous and so $\mathbf{C}_\theta(f): \mathbf{C}_\theta(c) \rightarrow \mathbf{C}_\theta(C_\sigma(a)) = \mathbf{C}_\rho(a)$ is an equaliser of $\mathbf{C}_\theta(\mathbf{C}_\sigma(f1)) = \mathbf{C}_\rho(f1)$ and $\mathbf{C}_\theta(\mathbf{C}_\sigma(f2)) = \mathbf{C}_\rho(f2): \mathbf{C}_\rho(a) \rightarrow \mathbf{C}_{\theta;\sigma;\sigma 1}(b)$ in $\mathbf{C}_m$. Hence, there is a unique morphism $h: d \rightarrow \mathbf{C}_\theta(c)$ such that $h;\mathbf{C}_\theta(f) = g$ in $\mathbf{C}_m$.

Then, $\langle\theta, h\rangle: \langle m, d\rangle \rightarrow \langle k, c\rangle$ is a unique morphism in $\mathbf{Flat}(\mathbf{C})$ such that $\langle\theta, h\rangle;\langle\sigma, f\rangle = \langle\rho, g\rangle$.

□ (Th. 1)

It is easy to see that in fact a sharper result may be proved in a similar way. Namely, a diagram $\mathbf{D}: \mathbf{G} \rightarrow \mathbf{Flat}(\mathbf{C})$ has a limit in $\mathbf{Flat}(\mathbf{C})$ provided that $\mathbf{D};\mathbf{Cleave}: \mathbf{G} \rightarrow \mathbf{Ind}$ has a limit in **Ind** such that the component category corresponding to the limit index is **G**-complete and the translation functors induced by index morphisms into the limit index are **G**-continuous. (We say that a category **K** is **G**-complete if any diagram of the shape **G** has a limit in **K**; a functor is **G**-continuous if it preserves the limits of all diagrams of the shape **G**.)

A construction of colimits in a flattened category is not quite so simple. The proof of Theorem 1 cannot be directly dualised. Roughly, the problem is that in the construction of limits we had to translate objects (and morphisms) of component categories *against* index morphisms, which was easy using the translation functors of the indexed category. In the analogous construction of colimits that we present below, it is necessary to translate objects and morphisms of component categories *along* index morphisms, which requires the translation functors to be, in a sense, reversible.

Definition 4

An indexed category $\mathbf{C}: \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$ is *locally reversible* if for every index morphism $\sigma: i \rightarrow j$ in **Ind** the translation functor $\mathbf{C}_\sigma: \mathbf{C}_j \rightarrow \mathbf{C}_i$ has a left adjoint.

For any $\sigma: i \rightarrow j$ in **Ind**, we will denote the left adjoint (an arbitrary but fixed left adjoint) to $\mathbf{C}_\sigma: \mathbf{C}_j \rightarrow \mathbf{C}_i$ by $\mathbf{F}_\sigma: \mathbf{C}_i \rightarrow \mathbf{C}_j$ and the unit of the adjunction by $\eta^\sigma: \mathbf{id}_{\mathbf{C}_i} \rightarrow \mathbf{F}_\sigma;\mathbf{C}_\sigma$.

□ (Def. 4)

Notice that we do not require that **C** be "globally" reversible, i.e. that the family of left adjoints forms an indexed (by $\mathbf{Ind}^{op}$) category. In general, $\mathbf{F}_{\sigma;\rho} \neq \mathbf{F}_\sigma;\mathbf{F}_\rho$. However:

Fact 1

Let $\mathbf{C}: \mathbf{Ind}^{op} \rightarrow \mathbf{Cat}$ be a locally reversible indexed category. For any index morphisms $\sigma: i \rightarrow j$ and $\rho: j \rightarrow k$ in **Ind**, there is a natural isomorphism

$$\iota_{\sigma,\rho}: \mathbf{F}_{\sigma;\rho} \rightarrow \mathbf{F}_\sigma;\mathbf{F}_\rho.$$

Proof

Obvious, since $\mathbf{F}_\sigma; \mathbf{F}_\rho$ is left adjoint to $\mathbf{C}_{\sigma;\rho} = \mathbf{C}_\rho; \mathbf{C}_\sigma$ (cf. [MacLane 71, Th. IV.8.1, p.101]) and any two left adjoints to the same functor are naturally isomorphic (cf. [MacLane 71, Cor. IV.1.1, p.83]).

In fact, for any object $a \in |\mathbf{C}_i|$, $\iota_{\sigma,\rho}(a): \mathbf{F}_{\sigma;\rho}(a) \to \mathbf{F}_\rho(\mathbf{F}_\sigma(a))$ is given by

$$\iota_{\sigma,\rho}(a) = (\eta^\sigma(a); \mathbf{C}_\sigma(\eta^\rho(\mathbf{F}_\sigma(a))))^\#$$

and its inverse

$$\iota_{\sigma,\rho}^{-1}(a) = ((\eta^{\sigma;\rho}(a))^\#)^\#: \mathbf{F}_\rho(\mathbf{F}_\sigma(a)) \to \mathbf{F}_{\sigma;\rho}(a)$$

(we leave it as an exercise for the reader to indicate which adjunctions the sharps "#" in the above formulae refer to).

□ (Fact 1)

Definition 5

Let $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$ be a locally reversible category. For any index morphism $\rho: i \to j$, we can "lift along ρ" any morphism $\langle \sigma, g\rangle: \langle k, a\rangle \to \langle i, b\rangle$ (the same i) in $\mathbf{Flat}(\mathbf{C})$ to a morphism in $\mathbf{C}_j$:

$$L_\rho(\langle \sigma, g\rangle) = \iota_{\sigma,\rho}(a); \mathbf{F}_\rho(g^\#): \mathbf{F}_{\sigma;\rho}(a) \to \mathbf{F}_\rho(b)$$

□ (Def. 5)

Lemma 1

Under the notation and assumptions of Definition 5, for any index morphism $\theta: j \to m$ in $\mathbf{Ind}$ and morphism $\langle \rho;\theta, f\rangle: \langle i, b\rangle \to \langle m, c\rangle$ in $\mathbf{Flat}(\mathbf{C})$, $f^\#: \mathbf{F}_\sigma(b) \to \mathbf{C}_\theta(c)$ is a morphism in $\mathbf{C}_j$ such that in $\mathbf{Flat}(\mathbf{C})$

$$\langle \sigma;\rho, \eta^{\sigma;\rho}(a)\rangle; \langle \theta, L_\rho(\langle \sigma, g\rangle); f^\#\rangle = \langle \sigma, g\rangle; \langle \rho;\theta, f\rangle: \langle k, a\rangle \to \langle m, c\rangle.$$

Proof

We verify that in $\mathbf{C}_k$

$$\eta^{\sigma;\rho}(a); \mathbf{C}_{\sigma;\rho}(L_\rho(\langle \sigma, g\rangle); f^\#) = g; \mathbf{C}_\sigma(f): a \to \mathbf{C}_{\sigma;\rho;\theta}(c).$$

$\eta^{\sigma;\rho}(a); \mathbf{C}_{\sigma;\rho}(L_\rho(\langle \sigma, g\rangle); f^\#)$	(Def. 5)
$= \eta^{\sigma;\rho}(c); \mathbf{C}_{\sigma;\rho}(\iota_{\sigma,\rho}(a)); \mathbf{C}_{\sigma;\rho}(\mathbf{F}_\rho(g^\#); f^\#)$	(proof of Fact 1)
$= \eta^\sigma(a); \mathbf{C}_\sigma(\eta^\rho(\mathbf{F}_\sigma(a)); \mathbf{C}_{\sigma;\rho}(\mathbf{F}_\rho(g^\#); f^\#)$	($\mathbf{C}_{\sigma;\rho} = \mathbf{C}_\rho; \mathbf{C}_\sigma$)
$= \eta^\sigma(a); \mathbf{C}_\sigma(\eta^\rho(\mathbf{F}_\sigma(a)); \mathbf{C}_\rho(F_\rho(g^\#)); \mathbf{C}_\rho(f^\#))$	(η^ρ is natural)
$= \eta^\sigma(a); \mathbf{C}_\sigma(g^\#; \eta^\rho(b); \mathbf{C}_\rho(f^\#))$	($f = \eta^\rho(b); \mathbf{C}_\rho(f^\#)$)
$= \eta^\sigma(a); \mathbf{C}_\sigma(g^\#); \mathbf{C}_\sigma(f)$	($g = \eta^\sigma(a); \mathbf{C}_\sigma(g^\#)$)
$= g; \mathbf{C}_\sigma(f)$	

□ (Lemma 1)

Corollary 1

Under the notation and assumptions of Definition 5

$$\eta^{\sigma;\rho}(a); \mathbf{C}_{\sigma;\rho}(L_\rho(\langle \sigma, g\rangle)) = g; \mathbf{C}_\sigma(\eta^\rho(b))$$

Proof

Obvious by Lemma 1, since $\eta^\rho(b)^\# = id_{\mathbf{F}_\rho(b)}$.

□ (Cor. 1)

We are now ready to state our main theorem:

Theorem 2

Let $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$ be an indexed category such that

1. $\mathbf{Ind}$ is cocomplete,
2. for all $i \in |\mathbf{Ind}|, \mathbf{C}_i$ is cocomplete, and
3. $\mathbf{C}$ is locally reversible.

Then $\mathbf{Flat}(\mathbf{C})$ is cocomplete.

Proof

Dually to the proof of Theorem 1, it is sufficient to prove that $\mathbf{Flat}(\mathbf{C})$ has all coproducts and coequalisers.

coproducts:

Consider any family of objects in $\mathbf{Flat}(\mathbf{C})$, $\langle i_n, a_n\rangle$, $n \in N$ (where N is an arbitrary set).

Let i with injections $\rho_n: i_n \to i$ be a coproduct of i_n, $n \in N$, in $\mathbf{Ind}$. Then, let a with injections $f_n^\#: \mathbf{F}_{\rho_n}(a_n) \to a$ be a coproduct of $\mathbf{F}_{\rho_n}(a_n)$, $n \in N$, in $\mathbf{C}_i$. Finally, let for $n \in N$, $f_n = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(f_n^\#): a_n \to \mathbf{C}_{\rho_n}(a)$.

We claim that $\langle i, a\rangle$ with injections $\langle \rho_n, f_n\rangle: \langle i_n, a_n\rangle \to \langle i, a\rangle$, $n \in N$, is a coproduct of the family $\langle i_n, a_n\rangle$, $n \in N$, in $\mathbf{Flat}(\mathbf{C})$.

For, consider an object $\langle j, b\rangle$ with morphisms $\langle \sigma_n, g_n\rangle: \langle i_n, a_n\rangle \to \langle j, b\rangle$, $n \in N$, in $\mathbf{Flat}(\mathbf{C})$.

By the construction, there exists a unique index morphism $\sigma: i \to j$ such that in $\mathbf{Ind}$ for $n \in N$, $\rho_n; \sigma = \sigma_n$. Moreover, there is a unique morphism $g: a \to \mathbf{C}_\sigma(b)$ such that for $n \in N$, $f_n^\#; g = g_n^\#: \mathbf{F}_{\rho_n}(a_n) \to \mathbf{C}_\sigma(b)$ ($g_n^\#$ is well-defined since $g_n: a_n \to \mathbf{C}_{\rho_n}(\mathbf{C}_\sigma(b))$).

Then, since in $\mathbf{C}_{i_n}$:

$$\begin{aligned} f_n; \mathbf{C}_{\rho_n}(g) &= \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(f_n^\#); \mathbf{C}_{\rho_n}(g) \\ &= \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(f_n^\#; g) \\ &= \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(g_n^\#) \\ &= g_n, \end{aligned}$$

$\langle \sigma, g\rangle: \langle i, a\rangle \to \langle j, b\rangle$ satisfies $\langle \rho_n, f_n\rangle; \langle \sigma, g\rangle = \langle \sigma_n, g_n\rangle$, $n \in N$, in $\mathbf{Flat}(\mathbf{C})$. Moreover, $\langle \sigma, g\rangle$ is the only morphism in $\mathbf{Flat}(\mathbf{C})$ with this property: the uniqueness of σ is obvious; the uniqueness of g follows by its construction from the fact that if for some $g': a \to \mathbf{C}_\sigma(b)$, $f_n; \mathbf{C}_{\rho_n}(g') = g_n$ for $n \in N$, then $f_n^\#; g' = g_n^\#$ for $n \in N$, and thus $g = g'$.

coequalisers:

Consider any two "parallel" morphisms in $\mathbf{Flat}(\mathbf{C})$, $\langle \sigma 1, f1\rangle, \langle \sigma 2, f2\rangle: \langle i, a\rangle \to \langle j, b\rangle$.

Let $\sigma: j \to k$ be a coequaliser of $\sigma 1, \sigma 2: i \to j$ in $\mathbf{Ind}$. Then in $\mathbf{C}_k$ there are two "parallel" morphisms (cf. Definition 5)

$$L_\sigma(\langle \sigma 1, f1\rangle), L_\sigma(\langle \sigma 2, f2\rangle): \mathbf{F}_{\sigma 1;\sigma}(a) \to \mathbf{F}_\sigma(b).$$

Let $f^{\#}: \mathbf{F}_\sigma(b) \to c$ be their coequaliser in $\mathbf{C}_k$. Consider $f = \eta^\sigma(b); \mathbf{C}_\sigma(f^{\#}): b \to \mathbf{C}_\sigma(c)$ in $\mathbf{C}_j$.

We claim that $\langle \sigma, f \rangle: \langle j, b \rangle \to \langle k, c \rangle$ is a coequaliser of $\langle \sigma 1, f1 \rangle, \langle \sigma 2, f2 \rangle: \langle i, a \rangle \to \langle j, b \rangle$ in $\mathbf{Flat}(\mathbf{C})$.

First notice that indeed in $\mathbf{Flat}(\mathbf{C})$ by Lemma 1 we have:

$$\begin{aligned}\langle \sigma 1, f1 \rangle; \langle \sigma, f \rangle &= \langle \sigma 1; \sigma, \eta^{\sigma 1;\sigma}(a) \rangle; \langle id_k, L_\sigma(\langle \sigma 1, f1 \rangle); f^{\#} \rangle \\ &= \langle \sigma 2; \sigma, \eta^{\sigma 2;\sigma}(a) \rangle; \langle id_k, L_\sigma(\langle \sigma 2, f2 \rangle); f^{\#} \rangle \\ &= \langle \sigma 2, f2 \rangle; \langle \sigma, f \rangle\end{aligned}$$

Then, consider any morphism $\langle \rho, g \rangle: \langle j, b \rangle \to \langle m, d \rangle$ such that in $\mathbf{Flat}(\mathbf{C})$

$$\langle \sigma 1, f1 \rangle; \langle \rho, g \rangle = \langle \sigma 2, f2 \rangle; \langle \rho, g \rangle,$$

i.e. $\sigma 1; \rho = \sigma 2; \rho$ in **Ind** and $f1; \mathbf{C}_{\sigma 1}(g) = f2; \mathbf{C}_{\sigma 2}(g)$ in $\mathbf{C}_i$.

By the construction, there exists a unique index morphism $\theta: k \to m$ such that $\sigma; \theta = \rho$ in **Ind**.

Moreover, by Lemma 1, in $\mathbf{C}_i$

$$\begin{aligned}\eta^{\sigma 1;\sigma}(a); \mathbf{C}_{\sigma 1;\sigma}(L_\sigma(\langle \sigma 1, f1 \rangle); g^{\#}) &= f1; \mathbf{C}_{\sigma 1}(g) \\ &= f2; \mathbf{C}_{\sigma 2}(g) \\ &= \eta^{\sigma 2;\sigma}(a); \mathbf{C}_{\sigma 2;\sigma}(L_\sigma(\langle \sigma 2, f2 \rangle; g^{\#})\end{aligned}$$

(recall that $\sigma 1; \sigma = \sigma 2; \sigma$ and that $g^{\#}: \mathbf{F}_\sigma(\sigma) \to \mathbf{C}_\theta(d)$). Hence, the properties of adjunction imply $L_\sigma(\langle \sigma 2, f2 \rangle); g^{\#} = L_\sigma(\langle \sigma 1, f1 \rangle); g^{\#}$. Thus, there exists a unique morphism $h: c \to \mathbf{C}_\theta(d)$ such that $f^{\#}; h = g^{\#}$ in $\mathbf{C}_k$.

Now, $\langle \theta, h \rangle: \langle k, c \rangle \to \langle m, d \rangle$ satisfies $\langle \sigma, f \rangle; \langle \theta, h \rangle = \langle \rho, g \rangle$ in $\mathbf{Flat}(\mathbf{C})$ (since in $\mathbf{C}_j$ we have: $f; \mathbf{C}_\sigma(h) = \eta^\sigma(b); C_\sigma(f^{\#}; h) = \eta^\sigma(b); C_\sigma(g^{\#}) = g$).

Moreover, $\langle \theta, h \rangle$ is the only morphism in $\mathbf{Flat}(\mathbf{C})$ with this property: the uniqueness of θ is obvious; the uniqueness of h follows from its construction (if for some $h': c \to \mathbf{C}_\theta(d)$, $f; \mathbf{C}_\sigma(h') = g$ then $f^{\#}; h' = g^{\#}$, and thus $h = h'$).

□ (Th. 2)

A sharper result may be proved in a similar manner: a diagram $\mathbf{D}: \mathbf{G} \to \mathbf{Flat}(\mathbf{C})$ has a colimit in $\mathbf{Flat}(\mathbf{C})$ provided that $\mathbf{D}; \mathbf{Cleave}: \mathbf{G} \to \mathbf{Ind}$ has a colimit in **Ind** such that the component category corresponding to the colimit index is **G**-cocomplete and all the translation functors induced by the index morphisms in the colimit cocone have left adjoints.

To complete this section, let us illustrate that the above results may indeed be used to prove completeness and/or cocompleteness of some interesting categories. The sample results of this form that we present below are known. Theorems 1 and 2, however, may be used to essentially simplify their standard, rather laborious proofs.

Example 1 *Many-sorted sets*

Consider the indexed category

$$\mathbf{SSET}: \mathbf{Set}^{op} \to \mathbf{Cat}$$

of many-sorted sets. It is well-known that for any set S, the category $\mathbf{SSET}(S)$ of S-sorted sets is both complete and cocomplete. The index category, **Set**, is both complete and cocomplete as well. Moreover, it is trivial to see that for any index morphism (a function) $f: S \to S'$, $\mathbf{SSET}(f): \mathbf{SSET}(S') \to \mathbf{SSET}(S)$ is continuous and has a left adjoint (which sends any S-sorted set $\langle X_s \rangle_{s \in S}$ to the S'-sorted set $\langle \biguplus \{X_s \mid f(s) = s'\} \rangle_{s' \in S'}$, where $\biguplus$ denotes disjoint union).

Thus, Theorems 1 and 2 directly imply that the (flattened) category of many-sorted sets **SSet** = **Flat**(**SSET**) is both complete and cocomplete.

□ (Ex. 1)

Example 2 *Algebraic signatures*
Consider the indexed category

$$\mathbf{ALGSIG}\colon \mathbf{Set}^{op} \to \mathbf{Cat}$$

of many-sorted algebraic signatures. Just as with sets, the index category and all the component categories are both complete and cocomplete, and the translation functors are continuous and have left adjoints (in fact, this directly follows from the definition **ALGSIG** = $(\_)^+$; **SSET**, since **SSET** has all these properties).

Thus, the category of algebraic signatures **AlgSig** = **Flat**(**ALGSIG**) is both complete and cocomplete.

□ (Ex. 2)

Example 3 *Many-sorted algebras*
Consider the indexed category

$$\mathbf{ALG}\colon \mathbf{AlgSig}^{op} \to \mathbf{Cat}$$

of many-sorted algebras. Again, we can repeat: the index category is complete and cocomplete (by Example 2 above), all the component categories are so as well, and the translation (forgetful) functors are continuous and have left adjoints. Let us point out, however, that the existence of left adjoints to the forgetful functors here is a non-trivial although well-known property (cf. e.g. [Burstall & Goguen 82] for an expository presentation). Similarly, the cocompleteness of the category of Σ-algebras, for any algebraic signature Σ, is not quite obvious (to form a coproduct of a family of Σ-algebras, one has to, roughly, consider their disjoint union and then complete it to a total Σ-algebra in a free way; coequalisers are easy).

Anyway, we can conclude that by Theorems 1 and 2 the category **Flat**(**ALG**) of many-sorted algebras is both complete and cocomplete.

Notice that this yields an appropriate framework to formulate operations like the amalgamated union of algebras over different signatures used in e.g. [Ehrig & Mahr 85].

□ (Ex. 3)

Example 4 *Diagrams*
Let **T** be any category. Consider the indexed category

$$\mathbf{FUNC}(\mathbf{T})\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$$

of functors into (or diagrams in) **T**.

Clearly, the index category, **Cat**, is both complete and cocomplete. Then, if **T** is complete, all the component categories are complete as well. Namely, for any $\mathbf{G} \in |\mathbf{Cat}|$ limits in $\mathbf{FUNCT}(\mathbf{T})(\mathbf{G}) = [\mathbf{G} \to \mathbf{T}]$ are constructed "pointwise", as limits in **T** "parameterised" by (the objects of) **G** (cf.

[MacLane 71, V.3, p.112]). Moreover, it is obvious that the translation functors in **FUNC(T)** preserve limits constructed in such a way.

Thus, we easily conclude that the category **Func(T)** = **Flat(FUNC(T))** of all diagrams in **T** is complete whenever **T** itself is complete. Dually, we know that if **T** is cocomplete then all the component categories are cocomplete and all the translation functors are cocontinuous. This is not enough, though, to apply Theorem 2: we would have to prove that the translation functors have left adjoints. Unfortunately, this need not be the case in general. (We were surprised to notice how close we came here to the famous notion of Kan extension, cf. [MacLane 71, X]).

□ (Ex. 4)

Example 5 *Theories*

Let **I** be an institution. Consider the indexed category

$$\mathbf{TH}: \mathbf{Sign}^{op} \to \mathbf{Cat}$$

of theories in **I**.

Clearly, for any signature $\Sigma \in |\mathbf{Sign}|$, $\mathbf{TH}_\Sigma$ forms a complete lattice, i.e. is complete and cocomplete as a category. Moreover, it is easy to see that for any signature morphism $\sigma: \Sigma \to \Sigma'$ in **Sign**, $\mathbf{TH}_\sigma: \mathbf{TH}_{\Sigma'} \to \mathbf{TH}_\Sigma$ has a left adjoint which maps any Σ-theory T to the Σ'-theory generated by the set $\{\sigma(\varphi) \mid \varphi \in T\}$ of Σ'-sentences. Thus, by Theorem 2, we directly conclude that the category **Th** = **Flat(TH)** of theories in **T** is cocomplete whenever the category **Sign** of signatures is cocomplete.

It is even easier to see that the categories **Pres** = **Flat(PRES)** and $\mathbf{Pres}_{\models} = \mathbf{Flat}(\mathbf{PRES}_{\models})$ are cocomplete whenever **Sign** is cocomplete.

About completeness: a similar result holds (all the component categories are complete, the translation functors are continuous) — it is not very interesting here though.

□ (Ex. 5)

Example 6 *Institutions*

Let **V** be an arbitrary category. Consider the indexed category

$$\mathbf{INS}: \mathbf{Cat}^{op} \to \mathbf{Cat}$$

of institutions. Recall that for any $\mathbf{Sign} \in |\mathbf{Cat}|$, $\mathbf{INS}(\mathbf{Sign}) = [\mathbf{Sign}^{op} \to \mathbf{Room}(V)]$. Arguing as in Example 4 above, it can be shown that the category **Ins** = **Flat(INS)** is complete provided that the category **Room(V)** is complete. To prove that this is indeed the case, we can use the following general result on comma categories (its dual was stated in [Beierle & Voss 85], proved in detail in [Tarlecki 86] — a slightly weaker result was given in [MacLane 71, Lemma in V.6] and [Goguen & Burstall 84, Prop. 2]).

Lemma 2

For any categories $\mathbf{A}, \mathbf{B}, \mathbf{K}$ and functors $\mathbf{F}: \mathbf{A} \to \mathbf{K}$, $\mathbf{G}: \mathbf{B} \to \mathbf{K}$, if the categories $\mathbf{A}$ and $\mathbf{B}$ are complete and the functor $\mathbf{G}: \mathbf{B} \to \mathbf{K}$ is continuous then the comma category $(\mathbf{F}/\mathbf{G})$ is complete.

□ (Lemma 2)

Now, recall that we have defined $\mathbf{Room}(\mathbf{V}) = (|\_|/\mathbf{FUNC}_{Disc}(\mathbf{V}))$, where $|\_|: \mathbf{Cat} \to \mathbf{Cat}$ and $\mathbf{FUNC}_{Disc}(\mathbf{V}): \mathbf{DCat}^{op} \to \mathbf{Cat}$. Since $\mathbf{Cat}$ is complete and $\mathbf{DCat}$, the category of discrete categories, is cocomplete (hence $\mathbf{DCat}^{op}$ is complete), the only thing to check is the continuity of $\mathbf{FUNC}_{Disc}(\mathbf{V})$. This, however, follows from the construction of colimits in $\mathbf{DCat}$ and limits in $\mathbf{Cat}$. The coproduct in $\mathbf{DCat}$ of any family of discrete categories $\mathbf{S}_n$, $n \in N$, is just their disjoint union $\mathbf{S} = \biguplus_{n \in N} \mathbf{S}_n$. It is easy to see that the functor category $[\mathbf{S} \to \mathbf{V}]$ is (isomorphic to) the product category of $[\mathbf{S}_n \to \mathbf{V}], n \in N$. Then, the coequaliser in $\mathbf{DCat}$ of any two "parallel" functors $\mathbf{F}, \mathbf{G}: \mathbf{S1} \to \mathbf{S2}$ is given as the natural quotient functor $\mathbf{H}: \mathbf{S2} \to \mathbf{S2}/\equiv$, where $\equiv$ is the least equivalence on (objects of) $\mathbf{S2}$ such that $\mathbf{F}(s) \equiv \mathbf{G}(s)$ for all $s \in \mathbf{S1}$; and $\mathbf{S2}/\equiv$ is the quotient (discrete) category. Again, it is easy to see that the functor category $[\mathbf{S2}/\equiv \to \mathbf{V}]$ is isomorphic to the subcategory of $[\mathbf{S2} \to \mathbf{V}]$ that contains as objects all functors $\mathbf{D}: \mathbf{S2} \to \mathbf{V}$ such that $\mathbf{F}; \mathbf{D} = \mathbf{G}; \mathbf{D}$, and similarly for morphisms. The isomorphism is given by the functor

$$\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{H}): [\mathbf{S2}/\equiv \to \mathbf{V}] \to [\mathbf{S2} \to \mathbf{V}].$$

This shows that $\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{H})$ is an equaliser of $\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{F})$ and $\mathbf{FUNC}_{Disc}(\mathbf{V})(\mathbf{G})$ in $\mathbf{Cat}$.

Summing up, $\mathbf{FUNC}_{Disc}(\mathbf{V})$ maps coproducts in $\mathbf{DCat}$ to products in $\mathbf{Cat}$ and coequalisers in $\mathbf{DCat}$ to equalisers in $\mathbf{Cat}$. Hence, $\mathbf{FUNC}_{Disc}(\mathbf{V})$ is continuous as a functor from $\mathbf{DCat}^{op}$ to $\mathbf{Cat}$. Thus, by Lemma 2, $\mathbf{Room}(\mathbf{V})$ is complete.

Finally, we can conclude that the category $\mathbf{Ins}$ of institutions is complete.

Notice that since morphisms in $\mathbf{Ins}$ go from richer to more primitive institutions, the limit, not the colimit, construction may be appropriate for "putting institutions together" (and hence, the completeness, not the cocompleteness, of the category of institutions is important).

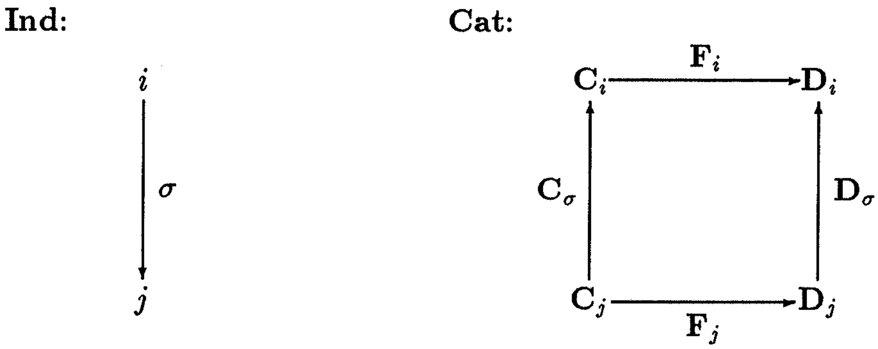
□ (Ex. 6)

4 Indexed Functors

Given the notion of an indexed category, it is only natural to generalise in a similar way category morphisms — functors.

Definition 6

An *indexed functor* $\mathbf{F}$ from an $\mathbf{Ind}$-indexed category $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$ to an $\mathbf{Ind}$-indexed category $\mathbf{D}: \mathbf{Ind}^{op} \to \mathbf{Cat}$ is a natural transformation $\mathbf{F}: \mathbf{C} \to \mathbf{D}$. Thus, for each index $i \in |\mathbf{Ind}|$ there is a functor $\mathbf{F}_i: \mathbf{C}_i \to \mathbf{D}_i$ such that for each index morphism $\sigma: i \to j$, $\mathbf{F}_j; \mathbf{D}_\sigma = \mathbf{C}_\sigma; \mathbf{F}_i$.



This yields the category **INDEXEDCAT(Ind)** of **Ind**-indexed categories (with the obvious vertical composition of morphisms).

□ (Def. 6)

A very simple example of an indexed functor is the many-sorted powerset construction:

Example 7 *Powerset functor*

For any set S, define the S-sorted powerset functor

$$\mathbf{P}_S : \mathbf{SSET}(S) \to \mathbf{SSET}(S)$$

in the obvious way. $\mathbf{P}_S$ maps any S-sorted set $\langle X_s \rangle_{s \in S}$ to the S-sorted set $\langle 2^{X_s} \rangle_{s \in S}$ of the powersets of its components; $\mathbf{P}_S$ maps any S-sorted function $\langle g_s : X_s \to Y_s \rangle_{s \in S}$ to the S-sorted family $\langle \vec{g_s} : 2^{X_s} \to 2^{Y_s} \rangle_{s \in S}$ of the corresponding image functions, $\vec{g_s}\,(A) = \{g_s(x) \mid x \in A\}$ for any $A \subseteq X_s$, $s \in S$.

It is trivial to see that the family $\mathbf{P} = \langle \mathbf{P}_S \rangle_{S \in |\mathbf{Set}|}$ actually forms an indexed functor

$$\mathbf{P} : \mathbf{SSET} \to \mathbf{SSET}.$$

□ (Ex. 7)

Example 8

Recall that in Example 5 we have defined three indexed categories

$$\begin{array}{ll} \mathbf{TH}: & \mathbf{Sign}^{op} \to \mathbf{Cat} \\ \mathbf{PRES}: & \mathbf{Sign}^{op} \to \mathbf{Cat} \\ \mathbf{PRES}_{\models}: & \mathbf{Sign}^{op} \to \mathbf{Cat} \end{array}$$

such for any signature $\Sigma \in |\mathbf{Sign}|$, $\mathbf{TH}_\Sigma$ is a subcategory of $\mathbf{PRES}_\Sigma$, which in turn is a subcategory of $(\mathbf{PRES}_{\models})_\Sigma$. Moreover, it is easy to see that the families of inclusion functors (from $\mathbf{TH}_\Sigma$ to $\mathbf{PRES}_\Sigma$ and, respectively, from $\mathbf{PRES}_\Sigma$ to $(\mathbf{PRES}_{\models})_\Sigma$) indexed by signatures $\Sigma \in |\mathbf{Sign}|$ form indexed functors (from **TH** to **PRES** and, respectively, from **PRES** to $\mathbf{PRES}_{\models}$).

This suggests a notion of an *indexed subcategory*: given two indexed categories $\mathbf{C} : \mathbf{Ind}^{op} \to \mathbf{Cat}$ and $\mathbf{D} : \mathbf{Ind}^{op} \to \mathbf{Cat}$ (over the same category of indices), we say that **D** is an indexed subcategory of **C** if for each $i \in |\mathbf{Ind}|$, $\mathbf{D}_i$ is a subcategory of $\mathbf{C}_i$, and the family of inclusion functors forms an indexed functor from **D** to **C**. This may be somewhat generalised in a rather obvious way by considering indexed subcategories **D** over a subcategory of indices of **C**.

□ (Ex. 8)

The operation of flattening of an indexed category may be extended to indexed functors as well.

Definition 7

Let **Ind** be any (index) category.

The *flatten functor* $\mathbf{Flat_{Ind}}: \mathbf{INDEXEDCAT(Ind)} \to \mathbf{Cat}$ is defined as follows

on objects: for any **Ind**-indexed category $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$, $\mathbf{Flat_{Ind}}(\mathbf{C})$ is the flattened category as defined in Definition 2, and

on morphisms: for any **Ind**-indexed functor $\mathbf{F}: \mathbf{C} \to \mathbf{D}$ (where $\mathbf{C}, \mathbf{D}: \mathbf{Ind}^{op} \to \mathbf{Cat}$) we define the functor $\mathbf{Flat_{Ind}}(\mathbf{F}): \mathbf{Flat_{Ind}}(\mathbf{C}) \to \mathbf{Flat_{Ind}}(\mathbf{D})$ by:

on objects: for any object $\langle i, a\rangle \in |\mathbf{Flat_{Ind}}(\mathbf{C})|$, $\mathbf{Flat_{Ind}}(\mathbf{F})(\langle i, a\rangle) = \langle i, \mathbf{F}_i(a)\rangle$, and

on morphisms: for any morphism $\langle \sigma, f\rangle: \langle i, a\rangle \to \langle j, \sigma\rangle$ in $\mathbf{Flat_{Ind}}(\mathbf{C})$, $\mathbf{Flat_{Ind}}(\mathbf{F})(\langle \sigma, f\rangle) = \langle \sigma, \mathbf{F}_i(f)\rangle: \langle i, \mathbf{F}_i(a)\rangle \to \langle j, \mathbf{F}_j(b)\rangle$ in $\mathbf{Flat_{Ind}}(\mathbf{D})$ (recall that $\mathbf{D}_\sigma(\mathbf{F}_j(b)) = \mathbf{F}_i(\mathbf{C}_\sigma(b))$).

It is straightforward to check that this indeed is a functor.

□ (Def. 7)

We often omit the subscript and write simply **Flat**, as before, instead of $\mathbf{Flat_{Ind}}$.

Intuitively, flattened indexed functors leave the first element of their arguments unchanged, but use it to select the appropriate component of the family the indexed functor is to operate on the second element of the arguments. In a sense, the flattening of an indexed functor may be viewed as forming the "disjoint union" of its components.

Notice the similarity between Definition 6 and the definitions given in Example 4 (of the category of functors into an arbitrary but fixed target category). In fact, Definition 4 yields yet another example of an indexed category: the indexed category of indexed categories.

Example 9 *Indexed categories*

The indexed category of indexed categories is defined by

$$\mathbf{INDEXEDCAT} = \mathbf{OP}; \mathbf{FUNC(Cat)}: \mathbf{Cat}^{op} \to \mathbf{Cat}$$

where $\mathbf{OP}: \mathbf{Cat}^{op} \to \mathbf{Cat}^{op}$ maps any category **K** to its opposite $\mathbf{K}^{op}$ and any functor $\mathbf{F}: \mathbf{K} \to \mathbf{M}$ to its opposite $\mathbf{F}^{op}: \mathbf{K}^{op} \to \mathbf{M}^{op}$. (It makes a nice puzzle to define $\mathbf{OP} = ((\_)^{op})^{op}$.)

Thus, for any $\mathbf{Ind} \in |\mathbf{Cat}|$, $\mathbf{INDEXEDCAT(Ind)} = [\mathbf{Ind}^{op} \to \mathbf{Cat}]$, as in Definition 6. For any functor $\mathbf{\Phi}: \mathbf{Ind} \to \mathbf{Ind}'$ and indexed category $\mathbf{C}': (\mathbf{Ind}')^{op} \to \mathbf{Cat}$, $\mathbf{INDEXEDCAT}(\mathbf{\Phi})(\mathbf{C}') = \mathbf{\Phi}^{op}; \mathbf{C}': \mathbf{Ind}^{op} \to \mathbf{Cat}$.

Flattening the indexed category of indexed categories yields the category

$$\mathbf{IndexedCat} = \mathbf{Flat(INDEXEDCAT)}$$

of indexed categories with objects that consist of an index category and an indexed category over this index category. In **IndexedCat**, an indexed category morphism from $\langle \mathbf{Ind1}, \mathbf{C1}: \mathbf{Ind1}^{op} \to \mathbf{Cat} \rangle$ to $\langle \mathbf{Ind2}, \mathbf{C2}: \mathbf{Ind2}^{op} \to \mathbf{Cat} \rangle$ is again a pair $\langle \mathbf{\Phi}, \mathbf{F} \rangle$ where $\mathbf{\Phi}: \mathbf{Ind1} \to \mathbf{Ind2}$ is a functor and $\mathbf{F}: \mathbf{C1} \to \mathbf{\Phi}^{op}; \mathbf{C2}$ is a natural transformation.

To make this perhaps a bit more readable, let us consider the relationship between the indexed categories of many-sorted algebras (Example 3) and of many-sorted sets (Example 1).

First, we have the obvious functor $\mathbf{Sorts}: \mathbf{AlgSig} \to \mathbf{Set}$ which maps any signature to its set of sorts (in fact, this is just the cleavage functor as defined in Definition 3).

Then, for any algebraic signature Σ, we have the obvious "forgetful" functor (cf. e.g. [Burstall & Goguen 82])

$$\mathbf{U}_\Sigma: \mathbf{Alg}(\Sigma) \to \mathbf{SSET}(\mathbf{Sorts}(\Sigma))$$

which maps any Σ-algebra to its many-sorted carrier. It is easy to check that the family $\mathbf{U} = \langle \mathbf{U}_\Sigma \rangle_{\Sigma \in |\mathbf{AlgSig}|}$ forms a natural transformation

$$\mathbf{U}: \mathbf{ALG} \to \mathbf{Sorts}^{op}; \mathbf{SSET}.$$

Thus, $\langle \mathbf{Sorts}, \mathbf{U} \rangle: \langle \mathbf{AlgSig}, \mathbf{ALG} \rangle \to \langle \mathbf{Set}, \mathbf{SSET} \rangle$ is an indexed category morphism in **Indexed-Cat**.

Let us also point out that $\mathbf{Flat} = \langle \mathbf{Flat}_{\mathbf{Ind}} \rangle_{\mathbf{Ind} \in |\mathbf{Cat}|}$ as defined in Definition 7 is an indexed functor as well. It goes from the **Cat**-indexed category **INDEXEDCAT** to the constant **Cat**-indexed category that assigns the category **Cat** to each index (and the identity functor on **Cat** to each index morphism.)

□ (Ex. 9)

Part of our original motivation for looking more carefully at indexed categories was that we sought some means to reduce a family of adjunctions (between component categories) to a single adjunction (between flattened categories) — a remote echo of this motive may be found in the process of "getting a charter out of a parchment" in [Goguen & Burstall 86].

Definition 8

Let $\mathbf{U}: \mathbf{C} \to \mathbf{D}$ be an **Ind**-indexed functor. We say that **U** has locally a left adjoint, if for each index $i \in |\mathbf{Ind}|$, $\mathbf{U}_i: \mathbf{C}_i \to \mathbf{D}_i$ has a left adjoint.

□ (Def. 8)

Theorem 3

For any **Ind**-indexed functor $\mathbf{U}: \mathbf{C} \to \mathbf{D}$, if **U** has locally a left adjoint, then $\mathbf{Flat}(\mathbf{U}): \mathbf{Flat}(\mathbf{C}) \to \mathbf{Flat}(\mathbf{D})$ has a left adjoint.

Proof

Consider any object $\langle i, a \rangle$ in $\mathbf{Flat}(\mathbf{C})$. By the assumptions, $\mathbf{U}_i: \mathbf{C}_i \to \mathbf{D}_i$ has left adjoint $\mathbf{F}_i: \mathbf{D}_i \to \mathbf{C}_i$ with a unit $\eta_i: \mathbf{id}_{\mathbf{C}_i} \to \mathbf{F}_i; \mathbf{U}_i$.

Now, we claim that $\langle i, \mathbf{F}_i(a) \rangle$ is a free object in $\mathbf{Flat}(\mathbf{D})$ over $\langle i, a \rangle$ w.r.t. $\mathbf{Flat}(\mathbf{U})$ with unit $\langle id_i, \eta_i(a) \rangle: \langle i, a \rangle \to \langle i, \mathbf{U}_i(\mathbf{F}_i(a)) \rangle = \mathbf{Flat}(\mathbf{U})(\langle i, \mathbf{F}_i(a) \rangle)$.

For, let $\langle j\,,\,b\rangle$ be an object in $\mathbf{Flat(D)}$ and $\langle\sigma\,,\,f\rangle\colon\langle i\,,\,a\rangle \to \mathbf{Flat(U)}(\langle j\,,\,b\rangle) = \langle j\,,\,\mathbf{U}_i(b)\rangle$ be a morphism in $\mathbf{Flat(C)}$. Moreover, let $f^{\#}\colon\mathbf{F}_i(c) \to b$ be the unique morphism in $\mathbf{D}_i$ such that $\eta_i(a);\mathbf{U}_i(f^{\#}) = f$ in $\mathbf{C}_i$. Then $\langle\sigma\,,\,f^{\#}\rangle\colon\langle i\,,\,\mathbf{F}_i(a)\rangle \to \langle j\,,\,b\rangle$ is the only morphism in $\mathbf{Flat(D)}$ such that in $\mathbf{Flat(C)}$ $\langle id_i\,,\,\eta_i(a)\rangle;\langle\sigma\,,\,f^{\#}\rangle = \langle\sigma\,,\,f\rangle$.

□ (Th. 3)

Example 10

Recall that in Example 9 we have defined the **AlgSig**-indexed "forgetful" functor

$$\mathbf{U}\colon\mathbf{ALG} \to \mathbf{Sorts}^{op};\mathbf{SSET}$$

It is well-known that the forgetful functor $\mathbf{U}_\Sigma\colon\mathbf{ALG}(\Sigma) \to \mathbf{SSET}(\mathbf{Sorts}(\Sigma))$ has a left adjoint for each algebraic signature Σ. Theorem 3 allows us to conclude that the "disjoint union" of this forgetful functors,

$$\mathbf{Flat(U)}\colon\mathbf{Flat(ALG)} \to \mathbf{Flat}(\mathbf{Sorts}^{op};\mathbf{SSET})$$

has a left adjoint which again is formed as a "disjoint union" of the "local" left adjoints.

□ (Ex. 10)

Example 11

Recall that in Example 8 we have considered the **Sign**-indexed inclusion functor from the indexed category **TH** of theories to the indexed category **PRES** of presentations in an arbitrary institution **I**. It is clear from the definitions in Example 5 (where these categories where defined) that for each signature $\Sigma \in |\mathbf{Sign}|$, the inclusion functor from $\mathbf{TH}_\Sigma$ to $\mathbf{PRES}_\Sigma$ has a left adjoint (i.e., that $\mathbf{TH}_\Sigma$ is a reflexive subcategory of $\mathbf{PRES}_\Sigma$ — cf. [MacLane 71, V.3, p.88/89]). In fact, the left adjoint is just the closure operator $Cl_\Sigma\colon\mathbf{PRES}_\Sigma \to \mathbf{TH}_\Sigma$, as defined in Example 5. Now, by Theorem 3 we can conclude that the category $\mathbf{Th} = \mathbf{Flat(TH)}$ of theories in **I** is a reflective subcategory of $\mathbf{Pres} = \mathbf{Flat(PRES)}$, the category of presentations in **I**.

□ (Ex. 11)

Theorem 3 suggests a different, rather neat way of proving Theorem 2 (the cocompleteness of flattened categories).

Recall that for any "shape" category **G** and any "target" category **T** the *diagonal functor*

$$\Delta^{\mathbf{G}}_{\mathbf{T}}\colon\mathbf{T} \to [\mathbf{G} \to \mathbf{T}]$$

is defined as follows:

on objects: for any object $t \in |\mathbf{T}|$, $\Delta^{\mathbf{G}}_{\mathbf{T}}(t)$ is the obvious "constant" diagram, i.e. the functor that maps each object of **G** to t and each morphism in **G** to the identity on t, and

on morphisms: for any morphism $f\colon t1 \to t2$ in **T**, $\Delta^{\mathbf{G}}_{\mathbf{T}}(f)\colon\Delta^{\mathbf{G}}_{\mathbf{T}}(t1) \to \Delta^{\mathbf{G}}_{\mathbf{T}}(t2)$ is the obvious "constant" natural transformation, i.e. $\Delta^{\mathbf{G}}_{\mathbf{T}}(f)_n = f$ for each $n \in |\mathbf{G}|$.

Fact 2

For any categories $\mathbf{G}$ and $\mathbf{T}$, $\mathbf{T}$ is $\mathbf{G}$-cocomplete (i.e., any diagram of the shape $\mathbf{G}$ has a colimit in $\mathbf{T}$) if and only if the diagonal functor $\Delta^{\mathbf{G}}_{\mathbf{T}}: \mathbf{T} \to [\mathbf{G} \to \mathbf{T}]$ has a left adjoint.

Proof

Well known: for any diagram $\mathbf{D}: \mathbf{G} \to \mathbf{T}$, the free object over $\mathbf{D}$ w.r.t. $\Delta^{\mathbf{G}}_{\mathbf{T}}$ is a colimit of $\mathbf{D}$; the unit is the colimiting cocone on $\mathbf{D}$, and vice versa, the colimit of $\mathbf{D}$ is a free object over $\mathbf{D}$ w.r.t. $\Delta^{\mathbf{G}}_{\mathbf{T}}$.

□ (Fact 2)

We will try to follow this hint in a (new) proof of a slightly stronger formulation of Theorem 2.

Theorem 2′

Consider any category $\mathbf{G}$. Let $\mathbf{C}: \mathbf{Ind}^{op} \to \mathbf{Cat}$ be an indexed category such that

1. $\mathbf{Ind}$ is $\mathbf{G}$-cocomplete,
2. for all $i \in |\mathbf{Ind}|, \mathbf{C}_i$ is $\mathbf{G}$-cocomplete, and
3. $\mathbf{G}$ is locally reversible.

Then $\mathbf{Flat}(\mathbf{C})$ is $\mathbf{G}$-cocomplete.

Proof

$\mathbf{C}$ gives rise in a rather natural way to an $\mathbf{Ind}$-indexed category $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}$ of $\mathbf{G}$-diagrams in $\mathbf{C}$. Namely:

component categories: for $i \in |\mathbf{Ind}|$, $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}(i) = [\mathbf{G} \to \mathbf{C}_i]$, and

translation functors: for $\sigma: i \to j$ in $\mathbf{Ind}$, the functor $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}(\sigma): [\mathbf{G} \to \mathbf{C}_j] \to [\mathbf{G} \to \mathbf{C}_i]$ is defined on objects by $\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}(\sigma)(\mathbf{D}) = \mathbf{D}; \mathbf{C}_\sigma$ for $\mathbf{D}: \mathbf{G} \to \mathbf{C}_j$; this extends to morphisms in $[\mathbf{G} \to \mathbf{C}_j]$ in the obvious way.

Now, we have the diagonal $\mathbf{Ind}$-indexed functor

$$\Delta^{\mathbf{G}}_{\mathbf{C}}: \mathbf{C} \to \mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}$$

defined by: $(\Delta^{\mathbf{G}}_{\mathbf{C}})_i = \Delta^{\mathbf{G}}_{\mathbf{C}_i}: \mathbf{C}_i \to [\mathbf{G} \to \mathbf{C}_i]$ for $i \in |\mathbf{Ind}|$. (It is easy to check that this is indeed an indexed functor.) Moreover, by (2) and Fact 2, for each $i \in |\mathbf{Ind}|$, $\Delta^{\mathbf{G}}_{\mathbf{C}_i}$ has a left adjoint. Hence, by Theorem 3,

$$\mathbf{Flat}(\Delta^{\mathbf{G}}_{\mathbf{C}}): \mathbf{Flat}(\mathbf{C}) \to \mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$$

has a left adjoint.

Notice that we can identify $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ with a certain subcategory of $[\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$. Roughly, it contains such $\mathbf{G}$-diagrams in $\mathbf{Flat}(\mathbf{C})$ that fit entirely into one of the component categories of $\mathbf{C}$. A diagram $\mathbf{D}: \mathbf{G} \to \mathbf{Flat}(\mathbf{C})$ is "in" $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ if and only if $\mathbf{D}; \mathbf{Cleave}: \mathbf{G} \to \mathbf{Ind}$ is a constant functor; similarly, a diagram morphism δ is "in" $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ if and only if δ "horizontally" composed with $\mathbf{Cleave}$ yields a constant natural transformation.

The corresponding faithful functor $\mathbf{J}: \mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}}) \to [\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$ may be defined as follows:

on objects: for any object $\langle i, \mathbf{D}\rangle \in |\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})|$ (i.e. $i \in |\mathbf{Ind}|$ and $\mathbf{D}: \mathbf{G} \to \mathbf{C}_i$) the $\mathbf{G}$-diagram $\mathbf{J}(\langle i, \mathbf{D}\rangle): \mathbf{G} \to \mathbf{Flat}(\mathbf{C})$ is defined thus:

on objects: $\mathbf{J}(\langle i, \mathbf{D}\rangle)(n) = \langle i, \mathbf{D}(n)\rangle$ for $n \in |\mathbf{G}|$, and

on morphisms: $\mathbf{J}(\langle i, \mathbf{D}\rangle)(e) = \langle id_i, \mathbf{D}(e)\rangle$ for any morphism e in $\mathbf{G}$;

on morphisms: for any morphism $\langle \gamma, \alpha\rangle: \langle i, \mathbf{D}\rangle \to \langle j, \mathbf{E}\rangle$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ (i.e. $\gamma: i \to j$ is a morphism in $\mathbf{Ind}$ and $\alpha: \mathbf{D} \to \mathbf{E}; \mathbf{C}_\gamma$ is a natural transformation, a morphism in $[\mathbf{G} \to \mathbf{C}_i]$), $\mathbf{J}(\langle \gamma, \alpha\rangle): \mathbf{J}(\langle i, \mathbf{D}\rangle) \to \mathbf{J}(\langle j, \mathbf{E}\rangle)$ is the natural transformation (a morphism in $[\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$) defined by $\mathbf{J}(\langle \gamma, \alpha\rangle)(n) = \langle \gamma, \alpha(n)\rangle: \langle i, \mathbf{D}(n)\rangle \to \langle j, \mathbf{E}(n)\rangle$ for $n \in |\mathbf{G}|$.

It is easy to see that $\mathbf{J}(\langle \gamma, \alpha\rangle)$ is indeed a natural transformation, that $\mathbf{J}$ is indeed a functor and that it is faithful.

In the following we will identify $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ with its image under $\mathbf{J}$ in $[\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$ and refer to $\mathbf{J}$ as the inclusion functor.

Unfortunately, in general $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ is a *proper* subcategory of $[\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$ (and so the proof of Theorem 2′ is not finished yet).

One can directly check that the diagonal functor $\Delta^{\mathbf{G}}_{\mathbf{Flat}(\mathbf{C})}: \mathbf{Flat}(\mathbf{C}) \to [\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$ satisfies

$$\Delta^{\mathbf{G}}_{\mathbf{Flat}(\mathbf{C})} = \mathbf{Flat}(\Delta^{\mathbf{G}}_{\mathbf{C}}); \mathbf{J}.$$

Since we know already that $\mathbf{Flat}(\Delta^{\mathbf{G}}_{\mathbf{C}})$ has a left adjoint, to show that $\Delta^{\mathbf{G}}_{\mathbf{Flat}(\mathbf{C})}$ has a left adjoint it is enough to prove that $\mathbf{J}$ has a left adjoint (cf. [MacLane 71, Th. V.8.1., p.101]). Thus, to complete the proof of the theorem we need the following lemma.

Lemma 2

The inclusion functor $\mathbf{J}$ has a left adjoint, i.e. $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ is a reflexive subcategory of $[\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$ (cf. [MacLane 71, V.3, p.88/89] for the definition and basic facts about reflexive subcategories).

Proof (of Lemma 2)

For any $\mathbf{G}$-diagram $\mathbf{D}: \mathbf{G} \to \mathbf{Flat}(\mathbf{C})$ we are to find its reflection in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$, that is, a $\mathbf{G}$-diagram $\mathbf{R}(\mathbf{D}): \mathbf{G} \to \mathbf{Flat}(\mathbf{C})$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ together with a diagram morphism $\eta_{\mathbf{D}}: \mathbf{D} \to \mathbf{R}(\mathbf{D})$ such that for any diagram $\mathbf{D}'$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ and a morphism $\delta: \mathbf{D} \to \mathbf{D}'$ there exists a unique morphism $\delta^{\#}: \mathbf{R}(\mathbf{D}) \to \mathbf{D}'$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ such that $\eta_{\mathbf{D}}; \delta^{\#} = \delta$ (in $[\mathbf{G} \to \mathbf{Flat}(\mathbf{C})]$).

So, consider an arbitrary diagram $\mathbf{D}: \mathbf{G} \to \mathbf{Flat}(\mathbf{C})$. Let $\mathbf{D}(n) = \langle i_n, a_n\rangle$ for $n \in |\mathbf{G}|$, and $\mathbf{D}(e) = \langle \sigma_e, f_e\rangle: \langle i_n, a_n\rangle \to \langle i_m, a_m\rangle$ for $e: n \to m$ in $\mathbf{G}$.

Then, let i with injections $\rho_n: i_n \to i$, $n \in |\mathbf{G}|$, be a colimit of $\mathbf{D}; \mathbf{Cleave}: \mathbf{G} \to \mathbf{Ind}$ in $\mathbf{Ind}$ ($\mathbf{Ind}$ is $\mathbf{G}$-cocomplete by (1)).

Define $\mathbf{R}(\mathbf{D}): \mathbf{G} \to \mathbf{Flat}(\mathbf{C})$ as follows:

on objects: $\mathbf{R}(\mathbf{D})(n) = \langle i, \mathbf{F}_{\rho_n}(a_n)\rangle$ for $n \in |G|$,

on morphisms: $\mathbf{R}(\mathbf{D})(e) = \langle id_i, L_{\rho_m}(\langle \sigma_e, f_e\rangle)\rangle: \langle i, \mathbf{F}_{\rho_n}(a_n)\rangle \to \langle i, \mathbf{F}_{\rho_m}(a_m)\rangle$ for $e: n \to m$ in $\mathbf{G}$.

Recall that indeed $L_{\rho_m}(\langle \sigma_e , f_e \rangle): \mathbf{F}_{\sigma_e;\rho_m}(a_n) = \mathbf{F}_{\rho_n}(a_n) \to \mathbf{F}_{\rho_m}(a_m)$ (Definition 5).

Let us check that $\mathbf{R}(\mathbf{D})$ is a functor, that is, preserves identities and composition. It is obvious that it preserves identities (Definition 5 implies that $L_{\rho_n}(\langle id_n , id_{a_n} \rangle) = \mathbf{F}_{\rho_n}(id_{a_n}) = id_{\mathbf{F}_{\rho_n}(a_n)}$.)

About composition:

For $e: n \to m$ and $d: m \to k$ in $\mathbf{G}$ we have to show that in $\mathbf{C}_i$

$$L_{\rho_m}(\langle \sigma_e , f_e \rangle); L_{\rho_k}(\langle \sigma_d , f_d \rangle) = L_{\rho_k}(\langle \sigma_e , f_e \rangle; \langle \sigma_d , f_d \rangle).$$

This may be checked by "going back" to $\mathbf{C}_{i_n}$:

On one hand, in $\mathbf{C}_{i_n}$:

$$\begin{aligned} &\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_k}(\langle \sigma_e , f_e \rangle; \langle \sigma_d , f_d \rangle)) \\ &\quad = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_k}(\langle \sigma_e; \sigma_d , f_e; \mathbf{C}_{\sigma_e}(f_d) \rangle)) && (\text{Cor. } 1,\ \rho_n = \sigma_e; \sigma_d; \rho_k) \\ &\quad = f_e; \mathbf{C}_{\sigma_e}(f_d); \mathbf{C}_{\sigma_e;\sigma_d}(\eta^{\rho_k}(a_k)). \end{aligned}$$

On the other hand, in $\mathbf{C}_{i_n}$:

$$\begin{aligned} &\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle \sigma_e , f_e \rangle); L_{\rho_k}(\langle \sigma_d , f_d \rangle)) && (\text{Cor. } 1,\ \rho_n = \sigma_e; \rho_m) \\ &\quad = f_e; \mathbf{C}_{\sigma_e}(\eta^{\rho_m}(a_m)); \mathbf{C}_{\sigma_e}(\mathbf{C}_{\rho_m}(L_{\rho_k}(\langle \sigma_d , f_d \rangle))) && (\text{Cor. } 1,\ \rho_m = \sigma_d; \rho_k) \\ &\quad = f_e; \mathbf{C}_{\sigma_e}(f_d); \mathbf{C}_{\sigma_e}(\mathbf{C}_{\sigma_d}(\eta^{\rho_k}(a_k))). \end{aligned}$$

Hence, in $\mathbf{C}_{i_n}$

$$\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle \sigma_e , f_e \rangle); L_{\rho_k}(\langle \sigma_d , f_d \rangle)) = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_k}(\langle \sigma_e , f_e \rangle; \langle \sigma_d , f_d \rangle)),$$

which by the properties of adjunction implies that indeed

$$L_{\rho_m}(\langle \sigma_e , f_e \rangle); L_{\rho_k}(\langle \sigma_d , f_d \rangle) = L_{\rho_k}(\langle \sigma_e , f_e \rangle; \langle \sigma_d , f_d \rangle).$$

Clearly, $\mathbf{R}(\mathbf{D})$ is in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$.

Having defined $\mathbf{R}(\mathbf{D})$ as above, there is an obvious way to define $\eta_{\mathbf{D}}: \mathbf{D} \to \mathbf{R}(\mathbf{D})$: for $n \in |\mathbf{G}|$, let $\eta_{\mathbf{D}}(n) = \langle \rho_n , \eta^{\rho_n}(a_n) \rangle : \langle i_n , a_n \rangle \to \langle i , \mathbf{F}_{\rho_n}(a_n) \rangle$. We have to check that $\eta_{\mathbf{D}}$ is a natural transformation.

Consider any $e: n \to m$ in $\mathbf{G}$.

We are to show that

$$\mathbf{D}(e); \eta_{\mathbf{D}}(m) = \eta_{\mathbf{D}}(n); \mathbf{R}(\mathbf{D})(e),$$

that is

$$\langle \sigma_e , f_e \rangle; \langle \rho_m , \eta^{\rho_m}(a_m) \rangle = \langle \rho_n , \eta^{\rho_n}(a_n) \rangle; \langle id_i , L_{\rho_m}(\langle \sigma_e , f_e \rangle) \rangle.$$

Since by the construction $\sigma_e; \rho_m = \rho_n$, the only thing to check is

$$f_e; \mathbf{C}_{\sigma_e}(\eta^{\rho_m}(a_m)) = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle \sigma_e , f_e \rangle)),$$

which follows directly from Corollary 1.

Now, we claim that $\mathbf{R}(\mathbf{D})$ with unit $\eta_{\mathbf{D}}: \mathbf{D} \to \mathbf{R}(\mathbf{D})$ is a reflection of $\mathbf{D}$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$.

For, consider any diagram $\mathbf{D}'$ in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$ and diagram morphism $\delta: \mathbf{D} \to \mathbf{D}'$. Let $\mathbf{D}'(n) = \langle j , b_n \rangle$ for $n \in |\mathbf{G}|$, and $\mathbf{D}'(e) = \langle id_j , g_e \rangle$ for $e: n \to m$ in $\mathbf{G}$, $g_e: b_n \to b_m$ in $\mathbf{C}_j$ (such an index $j \in |\mathbf{Ind}|$ exists since $\mathbf{D}'$ is in $\mathbf{Flat}(\mathbf{DIAG}^{\mathbf{G}}_{\mathbf{C}})$).

Further on, let for $n \in |\mathbf{G}|$, $\delta(n) = \langle \theta_n , h_n \rangle : \langle i_n , a_n \rangle \to \langle j , b_n \rangle$.

By the construction there exists a unique index morphism $\gamma: i \to j$ such that $\rho_n; \gamma = \theta_n$ for $n \in |\mathbf{G}|$.

Define a diagram morphism $\delta^{\#}: \mathbf{R}(\mathbf{D}) \to \mathbf{D}'$ by $\delta^{\#}(n) = \langle \gamma, h_n^{\#} \rangle : \langle i, \mathbf{F}_{\rho_n}(a_n) \rangle \to \langle j, b_n \rangle$ for $n \in |\mathbf{G}|$, where $h_n^{\#}: \mathbf{F}_{\rho_n}(a_n) \to \mathbf{C}_\gamma(b_n)$ is the unique morphism in $\mathbf{C}_i$ that satisfies $\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(h_n^{\#}) = h_n: a_n \to \mathbf{C}_{\rho_n}(\mathbf{C}_\gamma(b_n))$.

First, let us check that $\delta^{\#}$ is indeed a morphism in $\mathbf{Flat}(\mathbf{DIAG}_{\mathbf{C}}^{\mathbf{G}})$; the non-trivial part is to verify that $\delta^{\#}$ is a natural transformation, that is, for any $e: n \to m$ in $\mathbf{G}$

$$\delta^{\#}(n); \mathbf{D}'(e) = \mathbf{R}(\mathbf{D})(e); \delta^{\#}(m),$$

or equivalently

$$\langle \gamma, h_n^{\#} \rangle; \langle id_j, g_e \rangle = \langle id_i, L_{\rho m}(\langle \sigma_e, f_e \rangle) \rangle; \langle \gamma, h_m^{\#} \rangle.$$

We are to prove that in $\mathbf{C}_i$

$$h_n^{\#}; \mathbf{C}_\gamma(g_e) = L_{\rho_m}(\langle \sigma_e, f_e \rangle); h_m^{\#}.$$

To see this, notice that by the construction in $\mathbf{C}_{i_n}$

$$\eta^{\rho_n}(a_n); C_{\rho_n}(h_n^{\#}; \mathbf{C}_\gamma(g_e)) = h_n; \mathbf{C}_{\theta_n}(g_e)$$

and by Lemma 1 (since $\rho_n = \sigma_e; \rho_m$)

$$\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle \sigma_e, f_e \rangle); h_m^{\#}) = f_e; \mathbf{C}_{\sigma_e}(h_m).$$

However, since $\delta: \mathbf{D} \to \mathbf{D}'$ is a natural transformation,

$$\mathbf{D}(e); \delta(m) = \delta(n); \mathbf{D}'(e),$$

that is

$$\langle \sigma_e, f_e \rangle; \langle \theta_m, h_m \rangle = \langle \theta_n, h_n \rangle; \langle id_j, g_e \rangle,$$

which implies

$$f_e; \mathbf{C}_{\sigma_e}(h_m) = h_n; \mathbf{C}_{\theta_n}(g_e).$$

Hence, putting these equations together,

$$\eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(h_n^{\#}; \mathbf{C}_\gamma(g_e)) = \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(L_{\rho_m}(\langle \sigma_e, f_e \rangle); h_m^{\#}).$$

Thus indeed

$$h_n^{\#}; \mathbf{C}_\gamma(ge) = L_{\rho_m}(\langle \sigma_e, f_e \rangle); h_m^{\#}.$$

We claim that $\delta^{\#}: \mathbf{R}(\mathbf{D}) \to \mathbf{D}'$ is a unique morphism in $\mathbf{Flat}(\mathbf{DIAG}_{\mathbf{C}}^{\mathbf{G}})$ such that $\eta_{\mathbf{D}}; \delta^{\#} = \delta$.

First, we have to verify that for $n \in |\mathbf{G}|$, $\eta_{\mathbf{D}}(n); \delta^{\#}(n) = \delta(n)$, that is

$$\langle \rho_n, \eta^{\rho_n}(a_n) \rangle; \langle \gamma, h_n^{\#} \rangle = \langle \theta_n, h_n \rangle,$$

or equivalently

$$\langle \rho_n; \gamma, \eta^{\rho_n}(a_n); \mathbf{C}_{\rho_n}(h_n^{\#}) \rangle = \langle \theta_n, h_n \rangle,$$

which is clearly true.

Moreover, the construction guarantees that $\delta^{\#}(n)$ is the only morphism in $\mathbf{Flat}(\mathbf{C})$ such that $\mathbf{Cleave}(\delta^{\#}(n)) = \gamma$ and $\eta_{\mathbf{D}}(n); \delta^{\#}(n) = \delta(n)$. Then, since the uniqueness of γ is obvious, this implies the uniqueness of $\delta^{\#}$.

This completes the proof of Lemma 2 and so the proof of Theorem 2′ as well.

□ (Lemma 2)(Th. 2′)

We do not think we should apologise for providing a second proof of a theorem previously proved in the same paper. On the contrary, we feel that the details of this second proof are worth looking through. Especially the "reflection lemma" (Lemma 2) seems quite intriguing and perhaps worth exploring further.

5 Summary of Results

We have presented the concept of indexed category. We believe, and the examples we have given in this paper seem to support quite strongly the view that this is a very useful tool to structure and clarify some categorial definitions and proofs.

We have shown how any indexed category **C** may be used to produce a single flat category **Flat**(**C**) which contains all the components of **C**. Moreover, we have proven that this flattening construction preserves some important properties of component categories: completeness and cocompleteness (Theorems 1 and 2, respectively).

Then, the notion of indexed category comes naturally equipped with the notion of indexed functor. We have shown how the flattening construction applies to indexed functors as well. Moreover, we have proven that the flattening preserves existence of left adjoints: if all the components of an indexed functor have left adjoints then the flattened functor does as well (Theorem 3).

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