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Laboratory for Foundations of Computer Science
Department of Computer Science - University of Edinburgh

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by

C. Barry Jay

LFCS Report Series

ECS-LFCS-89-92
(also published as CSR-310-89)

LFCS
Department of Computer Science
University of Edinburgh
The King's Buildings
Edinburgh EH9 3JZ

September 1989

The structure of free closed categories[1]

C. Barry Jay
LFCS
Dept. of Comp. Sci.
University of Edinburgh
Edinburgh, U.K. EH9 3JZ

12th September, 1989

Abstract A new and simple method of describing all canonical natural transformations on closed categories is given by using internal languages to determine the structure of free closed categories.

0. Introduction

Coherence questions for symmetric, monoidal closed categories (hereafter called closed categories) can be resolved by examining the free closed categories on (finite) generating sets of objects. To describe each homset it suffices to list, perhaps with repetitions, its morphisms and then to decide when two such are equal. Using a different formulation of the problem, Kelly - Mac Lane [6] gave such a list with conditions under which the homset has at most one element. Then Voreadou [21] showed, in principle, how to reduce the list to the homset. Here is presented a radically simpler description, including an elegant new algorithm for determining equality of morphisms. The main result of [6] is recovered as a corollary.

Every morphism $f:X \to Y$ of a free closed category $\mathcal{V}$ can be decomposed as gh where $h:X \to Z$ has a definite and $g:Z \to Y$ an indefinite form. Roughly speaking, definite forms are built from evaluations and indefinite forms are evaluation free. The decomposition above is far from unique, but the choices for Z can be restricted sufficiently to use it in listing the morphisms of the homsets.

The kernel of the proof of decomposition can be traced to Gentzen's technique of cut-elimination [18], which was first applied to categorical coherence problems by Lambek [7,8], where it asserts, roughly, the redundancy of unfettered composition of morphisms. It appeared again in [6], in the work of Minc [15,16] for relevance logic (which indirectly settled the coherence question for closed categories), and also in [21], the complexity of which arises because cut-elimination was designed to establish the existence of morphisms (i.e., deductions), not their equality. For more detailed accounts of the influence of cut-elimination see [9,14] and the related [11,19,20].

The decision procedure for equality of morphisms given here is quite independent of the listing of the morphisms. Instead, it is employs a typed language $\mathcal{L}''(A)$ for the free closed category on a set A of objects, which is closely related to the language $\mathcal{L}'(\mathcal{V})$ for a monoidal category $\mathcal{V}$ as developed in [3,4,5,17] and similar in spirit to those for cartesian closed categories [10] and toposes [1]. If $f,f':X \to Y$ are morphisms and $x \in X$ is a variable then

$$f=f' \text{ in } \mathcal{V} \text{ iff } f(x) \equiv f'(x) \text{ in } \mathcal{L}'(\mathcal{V})$$

[1]To appear in the Journal of Pure and Applied Algebra

where $\equiv$ is an equivalence relation on the terms. This result also holds in $\mathcal{L}''(A)$ with the added feature, due to freeness, that equivalence of terms is decidable in linear time. First, we need only consider those terms whose type (an object of $\mathcal{V}$) is either a generating object, I or a hom-object, which allows the suppression of the canonical monoidal morphisms, and second, definite and indefinite terms are distinguished. Definite terms are equivalent iff they are built from equivalent terms: indefinite terms are equivalent iff they remain so upon evaluation (see Example 2.2).

1. Free closed categories

Let $(\mathcal{V},\otimes,I,a,l,r,c)$ be a symmetric monoidal category. The set of *expansions* of a set S of morphisms of $\mathcal{V}$ is its closure under identities, and tensoring with identity morphisms. The set of *iterates* of S is its closure under identities, tensoring and composing. Every iterate of S is equivalent to a composite of its expansions [12], e.g. if $f:X\to Y$ and $g:Z\to T$ are in S then

$$f\otimes g=(f\otimes 1)(1\otimes g):X\otimes Z\to Y\otimes T$$

Recall [2] that $\mathcal{V}$ is *closed* if, for each of its objects X, the functor $(-)\otimes X$ has a right adjoint

$$[X,-]:\mathcal{V}\to\mathcal{V}$$

Thus, there are morphisms, natural in Y

$$d_{Y,X} : Y\to[X,Y\otimes X] \qquad \text{(placemarker)}$$
$$e_{X,Y} : [X,Y]\otimes X\to Y \qquad \text{(evaluation)}$$

which satisfy the triangle laws for an adjunction

$$1 = e(d\otimes 1):Y\otimes X\to[X,Y\otimes X]\otimes X\to Y\otimes X$$
$$1 = [1,e]d:[X,Y]\to[X,[X,Y]\otimes X]\to[X,Y]$$

and also, *hom* $[-,-]:\mathcal{V}^{op}\times\mathcal{V}\to\mathcal{V}$ is functorial in both positions. Corresponding to morphisms $f:X\otimes Y\to Z$ and $g:X\to[Y,Z]$ are their *transposes* under the tensor-hom adjunction

$$\check{f}= [1,f]d_{X,Y}:X\to[Y,Z]$$
$$\hat{g}=e(g\otimes 1):X\otimes Y\to Z$$

A functor $F:\mathcal{V}\to\mathcal{W}$ between closed categories is a *strict closed functor* if it preserves the tensor, unit and hom strictly, e.g.

$$F(X\otimes Y)=FX\otimes FY$$

The category **Cl** consists of the small, closed categories and the strict functors.

Let $|\ |:\mathbf{Cl}\to\mathbf{Set}$ be the forgetful functor mapping a closed category to its set of objects. The existence of its left adjoint F is guaranteed by the general adjoint functor theorem; categories in the image of F are called *free closed categories* which we will now study in greater detail.

Fix a set A of *generating objects*. The set of objects $|FA|$ of FA is defined inductively by

(i) $A \subset |FA|$
(ii) $I \in |FA|$
(iii) If $X, Y \in |FA|$ then $X \otimes Y \in |FA|$ and $[X,Y] \in |FA|$.

The closure of I under (iii) is the set of *constant* objects. The *forms* for the morphisms of FA are generated by

(1.1) All identities are forms.
(1.2) All components of associativities, units, symmetries, placemarkers and evaluations are forms, called *generating* forms.
(1.3) If $f: X \to Y$ and $g: Z \to T$ are forms then $f \otimes g : X \otimes Z \to Y \otimes T$ and $[f,g]: [Y,Z] \to [X,T]$ are, too. Also, if $Y = Z$ then $gf : X \to T$ is a form.

The closure of the components of a, l, r and c under tensoring and composition are the *central* forms. If two forms can be shown to represent the same morphism of FA merely by applying the axioms for the monoidal structure then they are equal. Expansions and iterates of forms are defined just as for morphisms. The *length* of a given construction is the number of applications of (1.3) it requires. Whenever induction on the length is performed, a given construction is assumed. The *morphisms* of FA are equivalence classes of forms for the smallest equivalence relation ($\equiv$) generated by the axioms for a closed category.

There is a (2-)category which is the theory of monoidal categories [3]. This is not so for closed categories since placemarker and evaluation are examples of dinatural transformations [13], whose composition is not well-defined. Instead it will suffice to examine the free closed categories.

Each instance of a generator appearing in the construction of an object X of FA may be given a *sign* in the following manner

(i) If X is a generator it has positive sign.
(ii) If $X = Y \otimes Z$ then the generators in Y and Z keep their sign in X.
(iii) If $X = [Y,Z]$ then the generators in Y change their sign in X while the generators in Z keep theirs.

X is *balanced* if each generator in X appears exactly once with each sign.

A pair (X,Y) of objects of FA is a *graph* if $[X,Y]$ is balanced. Replacing each pair of instances of a generating object in (X,Y) by a pair of linked, unlabelled nodes yields a graph in the sense of [6]. Also, note the similarity of these graphs to the *scope* or *generality* of [7,8]. Then let $f \in FA(X,Y)$ be a morphism and $\mathcal{V}$ be a closed category. The morphisms $\xi(f)$ as $\xi : FA \to \mathcal{V}$ ranges over the strict, closed functors are the components of a dinatural transformation, called an *allowable natural transformation* [6]. Thus, f may be thought of as the generic component of the transformation: two allowable transformations are equal in every closed category iff their corresponding generic components are equal as morphisms. Define the set of *canonical transformations with graph*

(X,Y) to be $FA(X,Y)$. Their description is the coherence problem for closed categories.

The *prime objects* of FA (so called because they are not products of others) are the generating objects, I and the hom-objects. Obviously, every object X has a unique *prime factorisation* $X = \bigotimes X_i$ as a multiple (bracketed) tensor of the X_i which are prime, the *prime factors* of X. If I is a prime factor (and $X \neq I$) then it is *trivial*. The *derived factors* of an object is a set of prime objects given by

(i) Generating objects and I have only themselves as derived factors.
(ii) The derived factors of $X \otimes Y$ are those of X and those of Y.
(iii) The derived factors of $[X,Y]$ are those of Y and $[X,Y]$ itself.

The *multiplicity* $\mu(P,X)$ (respectively, *derived multiplicity* $\nu(P,X)$) of a prime object P in X is the number of times P occurs in X as a prime (respectively, derived) factor. The *multiplicity* $\mu(X)$ of X is the total number of its prime factors.

Example 1.1 The derived factors of $X' = ([W,X] \otimes X) \otimes [X,Y]$ are $[W,X]$, X, $[X,Y]$ and Y which are all of derived multiplicity 1 except that $\nu(X,X') = 2$. □

Let $f:X \to Y$ be a form. Its *rank* $\rho(f)$ is the number of placemarkers and homs appearing in its construction, e.g.

$$\rho([[e,d],c]d) = 4$$

The forms are *ordered* by $f \geq g$ (respectively, $f > g$) if $f \equiv g$ and $\rho(f) \geq \rho(g)$ (respectively, $\rho(f) > \rho(g)$).

The *generating indefinite forms* are the placemarkers and homs. Their iterates are the *indefinite forms*, which are *prime* if their codomain is, and *trivial* if the identity. Let $g:Z \to X$ be an indefinite form and Y be an object. Then

$$e_{X,Y}(1 \otimes g) : [X,Y] \otimes Z \to Y$$

is a *generating definite form*. Iterates of these and central forms are called *definite forms* (since, like definite integrals, they are evaluated). It isn't meaningful to speak of indefinite morphisms of FA since any form $f:X \to [Y,Z]$ is equivalent to an indefinite representing $(\hat{f})\check{}$

$$f \equiv [1,e(f \otimes 1)]d \tag{1.4}$$

of higher rank. Conversely, if $g:X \to [Y,Z]$ is a non-trivial indefinite form then its *transpose* $\hat{g}:X \otimes Y \to Z$ is defined so as to have lower rank by

$$\hat{g} = \begin{cases} 1 \otimes g' & \text{if } g = dg' \\ ke(g' \otimes h) & \text{if } g = [h,k]g' \end{cases} \tag{1.5}$$

Otherwise the transposes of forms are defined just as for morphisms.

Lemma 1.2 Let $g:X\to Y$ be an indefinite form.

(a) If Y is a generating object or I then g is a trivial indefinite.

(b) If $Y = Y_1 \otimes Y_2$ then $X = X_1 \otimes X_2$ and $g = g_1 \otimes g_2$ where each $g_i : X_i \to Y_i$ is indefinite.

(c) If $\alpha : Y \to Y'$ is a central form then there is another such $\alpha' : X \to X'$ and an indefinite form $g' : X' \to Y'$ satisfying $\alpha g = g'\alpha'$.

Proof Trivial inductions yield the results, with (a) and (b) used to prove (c). □

Theorem 1.3 Given a form $f:X\to Y$ of FA there is an object Z chosen from a finite set of objects determined by X and Y and forms $h:X\to Z$ definite and $g:Z\to Y$ indefinite such that

$$f \equiv gh \tag{1.6}$$

Proof We begin by proving that $f \geq gh$ for some g and h as above (but without restrictions on Z). The induction is firstly on $\rho(f)$ and secondly on the length of its construction. If f is an identity, generating form, tensor or hom then the result is immediate. If it is a composite

$$f = h'g' : X \to Z' \to Y$$

then we may assume the result for g' and h' and, without loss of generality, that g' is an expansion of a generating indefinite and h' is a central form or generating definite (no expansion is here required by Lemma 1.2(b)). If the latter is central then apply Lemma 1.2(c). Otherwise, $h' = e(1\otimes g'')$ is some generating definite form and

$$h'g' = \begin{cases} e(1\otimes g'')(1\otimes k) = e(1\otimes g''k) & \text{if } g' = 1\otimes k \\ e(1\otimes g'')(k\otimes 1) \geq \hat{k}(1\otimes g'') & \text{if } g' = k\otimes 1 \end{cases}$$

The latter case is essentially cut-elimination. Thus (1.6) is satisfied for *some* object Z.

The prime factors of Z are all derived factors of X. More precisely, for any prime object $P \neq I$ we have

$$\mu(P,Z) \leq \nu(P,X)$$

There are only finitely many Z which satisfy this condition and also

$$\mu(I,Z) \leq \mu(Y)$$

which latter we force as follows. Let $Y = \otimes Y_i$ be a prime factorisation and let

$$g = \otimes g_i : \otimes Z_i \to \otimes Y_i$$

Choose a central form $\lambda = \otimes \lambda_i : \otimes Z'_i \to \otimes Z_i$ such that each Z'_i has no trivial factors. Then $g\lambda = \otimes g_i \lambda_i$ is equivalent to some indefinite g' by (1.4). Note that, in general, this may cause an increase in the rank, e.g.

$$g\lambda = dr^{-1} \leq [1, r^{-1}\otimes 1]d : Z' \to [U,(Z'\otimes I)\otimes U]$$

By defining $h' = \lambda^{-1}h$ we obtain $f \equiv g'h'$ where $\mu(I,Z') \leq \mu(Y)$ as required. □

Given an object X of FA let its *size* be the number $\sigma(X)$ of hom-objects employed in its construction.

Theorem 1.4 *Given objects* X *and* Y *in* FA *there is an algorithm for constructing a finite list* $FA°(X,Y)$ *of forms containing at least one representative for each morphism in* $FA(X,Y)$.

Proof Clearly, if $f \equiv gh$ is constructed as in Theorem 1.3 then

$$\sigma(Z) \leq \min\{\sigma(X), \sigma(Y)\}$$

By induction on $\sigma(X) + \sigma(Y)$ it suffices to list merely the definite and indefinite forms. Consider f a definite form. If it is central then the number of possibilities is determined by counting appropriate permutations of the prime factors of X. Otherwise, it is constructed using a generating definite form, i.e.

$$f \equiv h'h\alpha$$

where $\alpha : X \to X'$ is central, $h : X' \to X''$ is an expansion of a generating definite

$$e(1 \otimes g) : [U,V] \otimes W \to [U,V] \otimes U \to V$$

and $h' : X'' \to Z$ is a form. By choosing α appropriately, we may further assume

$$\mu(I,X') = \mu(I,W) \leq \mu(U)$$

as in Theorem 1.3. Thus, there are finitely many possible choices of X' and α and for each such, $FA°(W,U)$ and $FA°(X'',Y)$ exist by induction.

Alternatively, if f is indefinite we may assume that $Y = [U,V]$ is a hom. Then the indefinites in $FA°(X,[U,V])$ are the transposes of the forms in $FA°(X \otimes U, V)$, which exists by induction. □

Example 1.5 For $A = \{W,X,Y\}$ we will represent all the morphisms

$$X' = ([X,W] \otimes X) \otimes [X,Y] \quad \to \quad [X, W \otimes Y] = Y'$$

The derived factors of X' are given in Example 1.1. The forms that result are

(a) $[1,(1 \otimes e)a]d(e \otimes 1) : X' \to W \otimes [X,Y] \to Y'$
(b) $[1,(e \otimes 1)a^{-1}(1 \otimes c)a]d(1 \otimes ec)a : X' \to [X,W] \otimes Y \to Y'$
(c) $[1,(e \otimes e)a]d : X' \to X' \to Y'$
(d) $[1,(e \otimes 1)a^{-1}(1 \otimes c)a((1 \otimes ec)a \otimes 1)]d : X' \to X' \to Y'$

In fact, there are only two distinct morphisms represented, depending upon which the copies of X are used to evaluate each of the two homs (see Example 2.2). □

Example 1.6 Let $A = \{X\}$. Then the forms in $FA^\circ(X^{***} \otimes X^{**}, I)$ (where X^* denotes $[X,I]$) can be separated into those where X^{***} is 'evaluated'

$$X^{***} \otimes X^{**} \xrightarrow{e(1 \otimes g)} I$$

and $g : X^{**} \to X^{**}$ is either the identity or given by transposition, i.e. $[1,e]d$ or $[1,e(1 \otimes [1,e]d)]d$; or that where X^{**} is 'evaluated'

$$X^{***} \otimes X^{**} \xrightarrow{c} X^{**} \otimes X^{***} \xrightarrow{e(1 \otimes k^*)} I$$

with $k = [1,ec]d : X \to X^{**}$. Thus $FA^\circ(X^{***}, X^{***})$ consists of the transposes of the forms above and the identity. □

Clearly, more efficient algorithms can be found for listing the representatives of the morphisms. Indeed, it seems likely that the morphisms can be given unique normal forms. This is not our main purpose here, however, since the chief problem in practice is to decide the issue of equality of forms, which we tackle directly.

2. The Form Language

In order to complete the description of the homsets of FA we must be able to determine when forms are equivalent, which is done by comparing the corresponding terms in the *form language* $\mathcal{L}''(A)$ for FA wherein equivalence of terms is a decidable property. The *types* of $\mathcal{L}''(A)$ are the objects X of FA whose corresponding *terms* are denoted $t \in X$. Terms of type I are called *scalars*. The general definition of terms will also specify the *definite* and *indefinite* terms, which are constructed by analogy with the corresponding forms, and the *unitary* terms, which are the closure under tensoring of the indefinite terms and the unitary, definite terms, i.e. those definite terms which are not scalars, and the sole unitary scalar. An arbitrary term will be constructed as a scalar multiple of a unitary term, which in general will be neither definite nor indefinite. The terms form the smallest set closed under the following conditions.

(2.1) To each prime type $X \neq I$ is associated countably many variables, which are unitary, definite terms.

(2.2) $* \in I$ is a unitary, definite scalar.

(2.3) Let $X = \bigotimes X_i$ be a prime factorisation and for each i let $t_i \in X_i$ be a unitary term. Then $\bigotimes t_i \in \bigotimes X_i$ is a unitary term, definite (respectively, indefinite) if each t_i is.

(2.4) Let $g : X \to Y$ be a prime indefinite form and $t \in X$ be a definite term. Then $g(t) \in Y$ is an indefinite term ($1(t) = t$).

(2.5) Let $t \in X$ be a unitary term and $\varphi \in [X, \bigotimes Y_i]$ be a definite term where each Y_i is prime. Then each $\varphi i(t) \in Y_i$ is a definite term, unitary iff $Y_i \neq I$.

(2.6) If $\{u_j | j \in J\}$ is a finite (unordered) set of scalars which are not unitary then $\prod u_j$ is a scalar ($\prod$ denotes the multiplication of the canonical monoidal structure on I).

(2.7) If u is a scalar and $t \in X$ is a unitary term then $u.t \in X$ is a term ($*.t = t$ and $u.* = u$).

A general term thus has *standard form*

$$t = u.\otimes(g_i(s_i)) \qquad (2.8)$$

where u is a scalar, each $s_i \in X_i$ is a unitary, definite term and each $g_i: X_i \rightarrow Y_i$ is a prime, indefinite form (perhaps trivial). For example, if $x \in X$, $z \in Z$ and

$$\varphi \in [X \otimes [Y, Z \otimes Y], I \otimes W]$$

are variables and $g: X \rightarrow X'$ is a prime indefinite form then there is the term

$$t' = \varphi^1(x \otimes d(z)).(g(x) \otimes \varphi^2(x \otimes d(z))) \in X' \otimes W$$

Each $g_i(s_i)$ is a *prime factor* of t. Its *factors* are all the terms $\otimes t_j$ where the t_j's form a set of prime factors of t.Unlike some earlier languages in this style (e.g. [4]), the terms t_i of (2.3) (which may include variables) are not required to be distinct. However, the *basic terms*, which are the definite terms constructed by tensoring copies of $*$ and some distinct variables, remain unchanged. Two terms $t \in X$ and $t' \in X'$ are *orthogonal* if they are constructed using different variables. The terms $\varphi^i(t)$ of (2.5) are the *components* of the *evaluation* $\varphi(t)$ of φ at t which is defined below.

The *rank* $\rho(t)$ of a term $t \in X$ is the number of generating indefinites occuring in its construction, though the indefinites employed in an evaluation should only be counted once, no matter how many of its components arise, e.g. the term t' above has rank $\varphi(g)+1$. If $s \equiv t$ and $\rho(s) \geq \rho(t)$ then $s \geq t$.

Tensors and scalar multiples of terms are defined as follows. With t as in (2.8) and $t' = v.\otimes(h_j(s'_j))$ another standard form then

$$t \otimes t' = (u.v).(\otimes g_i(s_i)) \otimes (\otimes h_j(s'_j))$$
$$v.t \;\; = (v.u) \otimes g_i(s_i)$$

where $u.v = \prod(\{u_i\} \cup \{v_j\})$. Hence, if $t \in X \otimes Y$ is unitary then $t = x \otimes y$ for a unique pair of terms $x \in X$ and $y \in Y$ by (2.8). Now consider the application of function symbols. Let $f: X \rightarrow Y$ be a form and $u.t \in X$ be a term in which t is unitary. Define

$$f(u.t) = u.f(t)$$

where $f(t)$ is given by induction, first on $\rho(f) + \rho(t)$, and second on the length of f's construction. Each case below considers one possibility for f and expresses t as a tensor or standard form, as appropriate.

(i) $1(t) = t$

(ii) $a((x_1 \otimes x_2) \otimes x_3) = x_1 \otimes (x_2 \otimes x_3)$

$a^{-1}(x_1 \otimes (x_2 \otimes x_3)) = (x_1 \otimes x_2) \otimes x_3$

(iii) $l(* \otimes x) = x$; $l^{-1}(t) = (* \otimes t)$

(iv) $r(x \otimes *) = x$; $r^{-1}(t) = (t \otimes *)$

(v) $c(x_1 \otimes x_2) = x_2 \otimes x_1$

(vi) $e(\varphi \otimes z) = \varphi(z) = \begin{cases} \otimes \varphi^i(z) & \text{if } \varphi \text{ is definite} \\ \hat{g}(s \otimes z) & \text{if } \varphi = g(s) \text{ is a standard form} \end{cases}$

(vii) $g.(\otimes g_i(s_i)) = (g.\otimes g_i)(s)$ if $g: X \to Y$ is a prime indefinite.

(viii) $(kh)(t) = k(h(t))$ provided kh is not a prime indefinite.

(ix) $(h \otimes k)(x_1 \otimes x_2) = h(x_1) \otimes k(x_2)$.

That (vi) and (ix) are well-defined follows by induction on $\rho(f) + \rho(t)$ and the length, respectively.

Note that if f is a definite form and t is basic then $f(t)$ is not in general definite since its standard form is $u'.t' \in Y$ where $t' \in Y$ is definite and u' is a scalar.

This language is not dependent on the results of Section 1 but has been constructed in parallel with them. In particular, (vi) echoes cut-elimination.

We now introduce an extremely simple relation $\equiv$ on the terms which will be used to determine whether forms are equivalent or not. It is the smallest relation satisfying

(2.9) $x \equiv x$ if $x \in X$ is a variable or $*$.

(2.10) If $\varphi \equiv \psi \in [X,Y]$ are definite terms and $s \equiv t \in X$ are unitary terms then

$\varphi^i(s) \equiv \psi^i(t) \in Y_i$ for any prime factor Y_i of Y.

(2.11) If $s,t \in [X,Y]$ are unitary terms, of which at least one is indefinite, and $x \in X$ is a basic term orthogonal to s and t then $s(x) \equiv t(x)$ implies $s \equiv t$.

(2.12) If $s_i \equiv t_i \in X_i$ are unitary terms of prime type for $1 \le i \le m$ and $u_j \equiv v_j$ are non-unitary, definite scalars for $j \in J$ then $(\prod u_j).\otimes s_i \equiv (\prod v_j).\otimes t_i$.

Theorem 2.1 *Let* A *be a set. Then* $\equiv$ *is a decidable equivalence relation on the terms of* $\mathcal{L}''(A)$.

Proof No equivalence is the conclusion of two distinct clauses above. Thus, every equivalence has a unique proof, the converses of (2.10) - (2.12) follow and it suffices to consider definite and indefinite terms of prime type.

Reflexivity and symmetry of the relation are immediate. Let $r \equiv s \equiv t \in X$ be terms. The proof of transitivity is by induction on $\rho(r) + \rho(s) + \rho(t)$. If they are all definite then $r \equiv t$ follows by (2.9) or (2.10). Otherwise, one of them is indefinite and $X = [Y,Z]$ is a hom. Let $y \in Y$ be a basic term orthogonal to r, s and t. Then $r(y) \equiv s(y) \equiv t(y)$ which implies $r(y) \equiv t(y)$ by induction. Consequently $r \equiv t$ as required.

For decidability, note that if an equivalence $s \equiv t$ of terms follows from some other such $s_i \equiv t_i$ by one of (2.10) - (2.12) then $\rho(s_i) + \rho(t_i) \le \rho(s) + \rho(t)$ with equality only if s_i and t_i have shorter constructions than s and t, while variables are equivalent iff equal. □

Example 2.2 Let $\varphi \in [X,W]$, $\psi \in [X,Y]$ and $x, x' \in X$ be variables and apply the forms of Example 1.5 to $((\varphi \otimes x) \otimes \psi) \in X'$ and then evaluate the resulting terms at x' to obtain terms of type $W \otimes Y$.

$$\text{(a) } [1,(1\otimes e)a]d(e\otimes 1)((\varphi\otimes x)\otimes\psi)(x') = [1,(1\otimes e)a]d(\varphi(x)\otimes\psi)(x')$$
$$\equiv (1\otimes e)a(d(\varphi(x)\otimes\psi))(x')$$
$$\equiv (1\otimes e)a((\varphi(x)\otimes\psi)\otimes x')$$
$$\equiv (1\otimes e)(\varphi(x)\otimes(\psi\otimes x'))$$
$$\equiv \varphi(x)\otimes\psi(x')$$

Similarly, the other forms yield (b) $\varphi(x')\otimes\psi(x)$ (c) $\varphi(x)\otimes\psi(x')$ and (d) $\varphi(x')\otimes\psi(x)$. Thus, the terms of (a) and (c) are equivalent, as other those of (b) and (d), but neither pair is equivalent to the other. Hence

$$[1,(1\otimes e)a]d(e\otimes 1)((\varphi\otimes x)\otimes\psi) \equiv [1,(e\otimes e)a]d((\varphi\otimes x)\otimes\psi)$$

and similarly for the other pair. It will follow from Theorem 2.6 that the corresponding forms are equivalent. □

Example 2.3 Let $\varphi \in [X,Y]$ be a variable. Then

$$[1,e]d(\varphi)(x) \equiv e(d(\varphi)(x)) \equiv e(\varphi\otimes x) \equiv \varphi(x)$$

Thus, $[1,e]d(\varphi) \equiv \varphi$ by (2.11) which is desirable since $[1,e]d \equiv 1$ as forms. From this it will follow that applying each of the three forms in which X^{***} is 'evaluated' in Example 1.6 to $\varphi\otimes\psi \in X^{***}\otimes X^{**}$ yields $\varphi(\psi)$. By contrast, applying the form in which X^{**} is evaluated to $\varphi\otimes\psi$ yields

$$\psi([1,g]d(\varphi)) \not\equiv \varphi(\psi) \qquad □$$

Proposition 2.4 *If* $f \equiv f' : X \to Y$ *are forms and* $x \equiv x' \in X$ *are terms then* $f(x) \equiv f'(x')$.

Proof Use induction on $\rho(f)+\rho(f')$ and secondarily, the length of their constructions. Consider all possible proofs of $f \equiv f'$. For most of the axioms for closed categories the desired equivalence follows directly, as is easily checked. Now consider the other axioms. Let $y \in Y'$ be a basic term orthogonal to x.

(i) Functoriality of hom: given $[h',k'][h,k] : [Y'',Z''] \to [Y',Z']$ then

$$\begin{aligned}[h',k'][h,k](x)(y) &= k'([h,k](x)(h'(y)) \\ &= k'k(x(hh'(y))) \\ &\equiv k'k(x'(hh'(y))) \\ &= [hh',k'k](x')(y)\end{aligned}$$

where the only equivalence holds by induction (with $f = f' = k'k$) since

$$x(hh'(y)) \equiv x'(hh'(y))$$

by definition. Thus $[h',k'][h,k](x) \equiv [hh',k'k](x')$ (similarly $[1,1](x) \equiv x'$).
(ii) Naturality of placemarkers: given $g : X \to X'$ then

$$\begin{aligned} ([1,g\otimes 1]d_{X,Y'}(x))(y) &= (g\otimes 1)(x\otimes y) \\ &\equiv (g\otimes 1)(x'\otimes y) && \text{(induction)} \\ &= (d_{X',Y'}g(x'))(y) \end{aligned}$$

(iii) Second adjunction law:

$$([1,e]d_{[Y',Z'],Y'})(x)(y) = e(d(x)(y)) = x(y) \equiv x'(y)$$

Alternatively, let f and f' be equivalent by construction, e.g.

$$f = [g,h] : [Y'',Z''] \to [Y',Z']$$

and $f' = [g',h']$ where $g \equiv g'$ and $h \equiv h'$. Then $g(y) \equiv g'(y)$ by induction and $x(g(y)) \equiv x'(g'(y))$ by definition whence

$$h(x(g(y))) \equiv h'(x'(g'(y)))$$

again by induction. If f and f' are composites or tensors of equivalent pairs of forms then the result follows similarly. □

Lemma 2.5 Let $\varphi \in [X,Z]$ and $x \in X$ be a terms where φ is definite and x is unitary. If $s \equiv t \in Y$ are terms in standard form and φ is evaluated in s at x then there is an $x' \equiv x$ at which φ is evaluated in t.
Proof Use induction on the length of the proof of $s \equiv t$. □

Lemma 2.6 Let $f : X_1 \otimes X_2 \to Y_1 \otimes Y_2$ be a form and $x = x_1 \otimes x_2 \in X_1 \otimes X_2$ be a definite term with

$$f(x_1 \otimes x_2) \equiv y_1 \otimes y_2$$

where the prime factors of x_i only appear in y_i. Then there are forms $f_i : X_i \to Y_i$ for $i = 1,2$ such that

$$f = f_1 \otimes f_2$$

Proof Set $f = \beta f' \alpha$ where $\alpha : X_1 \otimes X_2 \to Z$ and $\beta : T \to Y_1 \otimes Y_2$ are central forms and $f' : Z \to T$ is some form. The proof is by induction on the length of the construction of f'.
(i) If f' is a generating form or identity then the result holds trivially.
(ii) If $f' = gh$ for some $h : Z \to T$ and $g : T \to Y_1 \otimes Y_2$ of shorter construction than f' then there is a central $\gamma : Z \to Z_1 \otimes Z_2$ such that

$$\gamma h \alpha (x_1 \otimes x_2) \equiv t_1 \otimes t_2$$

where the prime factors of x_i only appear in t_i. Thus $\gamma h \alpha = h_1 \otimes h_2$ for som $h_i : X_i \rightarrow Z_i$ by induction. Similarly $\beta g \gamma^{-1} = g_1 \otimes g_2$ and

$$f \equiv g_1 h_1 \otimes g_2 h_2$$

(iii) If $f' = g_1 \otimes g_2 : Z_1 \otimes Z_2 \rightarrow Y_1 \otimes Y_2$ then $\alpha = \alpha_1 \otimes \alpha_2$ and $\beta = \beta_1 \otimes \beta$ by monoidal coherence.

(iv) That $f' = [h,k]$ is a hom is impossible. □

Theorem 2.7 *Let* $x \in X$ *be a factor of* $h(z)$ *where* $z \in Z$ *is a basic term a* $h : Z \rightarrow X'$ *is a definite form (e.g.* x *is a basic term). If* $f, f' : X \rightarrow Y$ *are forms satisfyi* $f(x) \equiv f'(x)$ *then* $f \equiv f'$. *Hence equality of morphisms in* FA *is decidable.*

Proof The proof is by induction on $\rho(f) + \rho(f')$ and secondly, on the lengths of the constructions. Consider the case where one of them, say, f is constructed using a gene ating definite. Without loss of generality,

$$f = k(e(1 \otimes g) \otimes 1) : ([U,V] \otimes W) \otimes Z \rightarrow Y$$

where $g : W \rightarrow U$ is an indefinite form and $k : V \otimes Z \rightarrow Y$ is a form. Let

$$x = (\varphi \otimes w) \otimes z$$

Then φ is evaluated in $f(x)$ at $g(w)$ and so is evaluated at some $u \equiv g(w)$ in t standard form of $f'(x)$ by Lemma 2.5. Thus

$$f' = k'(e \otimes 1)h$$

for some forms

$$h : ([U,V] \otimes W) \otimes Z \rightarrow ([U,V] \otimes U) \otimes Z'$$

and $k' : V \otimes Z' \rightarrow Y$ satisfying $h(x) \equiv (\varphi \otimes u) \otimes z'$ for some $z' \in Z'$. Now

$$h \geq (1 \otimes g') \otimes h'$$

for some forms $h' : Z \rightarrow Z'$ and $g' : W \rightarrow U$ by two applications of Lemma 2.6 and we may assume that $h' = 1$. Hence $g \equiv g'$ by induction and without loss of generali they are equal. Let $s = (e(1 \otimes g) \otimes 1)(x)$. Consequently, $k(s) \equiv k'(s)$ and $k \equiv k'$ by induction, which yields the result.

Alternatively, both forms are constructed without the use of generating definite Then $f \geq g\alpha$ where $\alpha : X \rightarrow X'$ is central and we may assume f' is indefinite. Let

$$g \equiv \otimes g_i : \otimes X_i \rightarrow \otimes Y_i$$
$$f' \equiv \otimes f'_i : \otimes X'_i \rightarrow \otimes Y_i$$

where $Y = \otimes Y_i$ is a prime factorisation. If $x = \otimes x_i$ and $\alpha(x) = \otimes x'_i$ then $g_i(x'_i) \equiv f'_i(x_i)$ whence the definition of equivalence forces the variables of x_i and x'_i to agree, and thus the existence of central forms $\alpha_i : X \to X'$ satisfying

$$\alpha_i(x_i) \equiv x'_i$$

Thus $\alpha = \otimes \alpha_i$ by coherence and we may assume $Y = [Y', Z']$ is a hom since each x_i satisfies the hypothesis for x. Choose a basic term $y \in Y'$ orthogonal to x. Then

$$f^{\wedge}(x \otimes y) \equiv f(x)(y) \equiv f'(x)(y) \equiv (f')^{\wedge}(x \otimes y)$$

Thus $f^{\wedge} \equiv (f')^{\wedge}$ by induction and so $f \equiv f'$ by transposition.

Thus $f \equiv f'$ as required. Thus morphisms of FA are equal iff their application to a basic term yields equivalent terms, which is decidable. □

Corollary 2.8 Morphisms of FA which have definite forms are epimorphisms.
Proof Let the definite form be h of the theorem with $X' = X$. □

Corollary 2.9 (Voreadou) *There is an algorithm for constructing the set of distinct canonical transformations having a given graph.*
Proof Theorem 1.4 allows us to give a complete list of candidates for the canonical transformations (possibly with duplications) and Theorem 2.7 is used to pare the list back to one representative for each transformation. □

An object X is *proper* (respectively, *constant-free)* if its construction employs no objects of the form $[Z,C]$ where C is constant and Z is not (respectively, employs no constants, unless $X = I$). Clearly, every proper object is isomorphic to a constant-free object. A canonical transformation α with graph (X,Y) is *proper* if X and Y are.

Theorem 2.10 (Kelly-Mac Lane) *If* α and β *are proper canonical transformations with the same graph then* $\alpha = \beta$.
Proof Let $[X,Y]$ be balanced with X proper. Without loss of generality, X is constant-free. Let $s,t \in Y'$ be terms built using only the variables of some basic term $x \in X$. It suffices to prove $s \equiv t$ which is done by induction on $\rho(s) + \rho(t)$. Since X is constant-free, s and t are unitary and, without loss of generality, of prime type. If either s or t is indefinite then $Y' = [Z,T]$ is a hom and $s \equiv t$ since $s(z) \equiv t(z)$ for any basic term $z \in Z$ by induction. Alternatively, if $s = \varphi^i(s')$ and $t = \psi^j(t')$ are both definite then $i = j$ and φ and ψ have the same type since $[X,Y]$ is balanced. Consequently s' and t' also have the same type. Hence $\varphi \equiv \psi$ and $s' \equiv t'$ by induction, which yields the result. Finally, if s is a variable then so is t and they are equal. □

Acknowledgements The dependence of this paper on Lambek's pioneering work relating category theory and deductive systems is clear. Saunders Mac Lane has encouraged this work from the outset and made many helpful suggestions during his careful reading of many drafts of this manuscript. To them both go my deep thanks.

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