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Equations, Dependent Equations and Quasi-dependent Equations - on their Unification

by

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Equations, Dependent Equations and Quasi-dependent Equations — on their unification*

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Abstract

This paper concentrates on many-sorted calculi for equations, equational implications and conditional equations (or quasi-equations). It attempts to unify the three of them not only on their names, say dependent equations and quasi-dependent equations for the equational implications and conditional equations (or quasi-equations) respectively, but aslo on their semantical results which will be totally presented in Birkhoff's approach. Their deduction systems are written as $\tilde{D}$, $\tilde{D}^d$ *and* $\tilde{D}^q$ *respectively. The completeness of* $\tilde{D}$ *is not new, but a proof of it (to be presented) is very unique and deserves an special attention. A new concept of cross-fully invariant congruences is introduced to capture the completeness by Birkhoff's method, which does not have its place in single-sorted case. The completeness of* $\tilde{D}^d$ *is achieved with a pay-off on total derivability. The calculi for* $\tilde{D}^d$ *and* $\tilde{D}^q$ *presented in [14] and in [11] respectively will be shown unsound. The right one for* $\tilde{D}^d$ *is found with a pay-off on total derivability. Although the calculus in [11] can be served as an alternative for the right one, a sound and complete* $\tilde{D}^q$ *is remained open. Nevertheless, the results, especially the ones related to* $\tilde{D}^q$*, suggest us to look for an universal equational form to unite the three equational forms. One candidate is briefly proposed.*

Contents

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1 Introduction

Various equational theories, including their corresponding software system environments, have been developed dramatically in recent years and still on their fast growing. These theories are based on calculus of equations (or equational logic), calculus of equational implications or/and calculus of conditional equations (or quasi-equations). Interestingly, those calculi were developed individually as their names suggest. Little effort has been made to unify the three of them (not just two of them). One reason for this may be the fact that the theoretical work on the calculi seems unrelated. For example, a completeness proof of one calculus is irrelevant to another. It is my attempt in this paper to establish the close relationship among the three calculi, i.e. to provide their soundness and completeness in one approach, the traditional Birkhoff's approach. As a result of this, I suggest the names of *dependent equations* and *quasi-dependent equations* for the equational implications and the conditional equations (or the quasi-equations) respectively as an emblematic efford, which is demonstrated in the title of this paper. Further on the unification, an universal equational form to unite the three equational forms is proposed and will be named as the *universal equation*.

An equation $t \simeq_{\vec{X}} u$ is presented with a variable indexed $\simeq$ rather than an universal quantifier over the equation $\forall X.t \simeq u$ as is presented in [4, 8]. Let Σ be a signature, $\vec{V}$ be the collection of variables indexed by $Sort(\Sigma)$, $\mathbf{T}_\Sigma(\vec{X})$ be the term Σ-algebra with variables in $\vec{X} \subseteq \vec{V}$, t, u be terms in $\mathbf{T}_\Sigma(\vec{X})$, $\mathbf{A}$ be a Σ-algebra and α be a Σ-homomorphism from $\mathbf{T}_\Sigma(\vec{X})$ to $\mathbf{A}$, written as $\alpha : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{A}$. Sometimes, we will omit the signature Σ in our terminology for simplicity reason. For equation $t \simeq_{\vec{X}} u$, we define that

1. $\mathbf{A}, \alpha \models t \simeq_{\vec{X}} u$ iff $\alpha(t) = \alpha(u)$; and

2. $\mathbf{A} \models t \simeq_{\vec{X}} u$ iff $\mathbf{A}, \alpha \models t \simeq_{\vec{X}} u$, for every $\alpha : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{A}$.

We should be aware of that $\mathbf{A} \models t \simeq_{\vec{X}} u$ is a *semantical* property (or meta-property) of algebra $\mathbf{A}$. Intuitively, it is the indistinguishability of $\mathbf{A}$. Obviously, we will have more interest on the indistinguishability if it is universal. More precisely, t and u are *indistinguishable* iff for every algebra $\mathbf{A}$, $\mathbf{A} \models t \simeq_{\vec{X}} u$. It is very important and convenient if we have an inference-rule system, $\tilde{D}$, which can deduce all indistinguishable terms.

For example, given a collection Γ of equations, $\Gamma \vdash t \simeq_{\vec{X}} u$ implies $\Gamma \models t \simeq_{\vec{X}} u$, where $\Gamma \vdash t \simeq_{\vec{X}} u$ expresses $t \simeq_{\vec{X}} u \in \tilde{D}(\Gamma)$ and $\Gamma \models t \simeq_{\vec{X}} u$ means that for every algebra $\mathbf{A}$, $\mathbf{A} \models t \simeq_{\vec{X}} u$ if $\mathbf{A} \models \Gamma$; and vice versa. This is the commonly named soundness and completeness of the calculus $\tilde{D}$.

Similarly, we can give definitions for dependent indistinguishability and quasi dependent indistinguishability as follows, and their inference systems as $\tilde{D}^d$ and $\tilde{D}^q$ respectively.

1. dependent equation $\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$:

   (a) $\mathbf{A} \models \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$ iff either not $\mathbf{A} \models \gamma_{\vec{x}}$ or $\mathbf{A} \models \Delta_{\vec{x}}$; where $\gamma_{\vec{x}}$ ranges over collections of equations, $\Delta_{\vec{x}}$ ranges over equations, and $\mathbf{A} \models \gamma_{\vec{x}}$ means $\mathbf{A} \models \Delta'_{\vec{x}}$ for each $\Delta'_{\vec{x}} \in \gamma_{\vec{x}}$.

   (b) $\Gamma^d \models \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$ iff $\mathbf{A} \models \Gamma^d$ implies $\mathbf{A} \models \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$ for every algebra $\mathbf{A}$; where Γ^d range over collections of dependent equations and $\mathbf{A} \models \Gamma^d$ means for each $\gamma'_{\vec{X}} \mapsto \Delta'_{\vec{X}} \in \Gamma^d$, $\mathbf{A} \models \gamma'_{\vec{X}} \mapsto \Delta'_{\vec{X}}$.

2. quasi-dependent equation $\gamma_{\vec{x}} \hookrightarrow \Delta_{\vec{x}}$:

   (a) $\mathbf{A} \models \gamma_{\vec{x}} \hookrightarrow \Delta_{\vec{x}}$ iff $\mathbf{A}, \alpha \models \gamma_{\vec{x}}$ implies $\mathbf{A}, \alpha \models \Delta_{\vec{x}}$ for every $\alpha : \mathbf{T}_\sigma(\vec{x}) \to \mathbf{T}_\sigma(\vec{x})$; where $\mathbf{A}, \alpha \models \gamma_{\vec{x}}$ means $\mathbf{A}, \alpha \models \Delta'_{\vec{x}}$ for each $\Delta'_{\vec{x}} \in \gamma_{\vec{x}}$.

   (b) $\Gamma^q \models \gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$ iff $\mathbf{A} \models \Gamma^d$ implies $\mathbf{A} \models \gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$ for every algebra $\mathbf{A}$; where Γ^q range over collections of quasi-dependent equations and $\mathbf{A} \models \Gamma^q$ means for each $\gamma'_{\vec{X}} \hookrightarrow \Delta'_{\vec{X}} \in \Gamma^q$, $\mathbf{A} \models \gamma'_{\vec{X}} \hookrightarrow \Delta'_{\vec{X}}$.

Note that when $\gamma_{\vec{x}} = \emptyset$, either a dependent equation $\gamma_{\vec{x}} \mapsto \Delta_{\vec{x}}$ or a quasi-dependent equation $\gamma_{\vec{x}} \hookrightarrow \Delta_{\vec{x}}$ is the equation $\Delta_{\vec{x}}$.

The soundness and completeness of many-sorted equational calculus $\tilde{D}$, once being believed a trivial extension of single-sorted one, was first claimed by Goguen and Meseguer in [6] and demonstrated that the naive belief did not hold. The full version of the proof appears in [8]. Their proof involves building higher-order function spaces from term algebras and verify that these higher spaces form clones (see [3] for basic properties of clones). Since they only allow arbitrary finite quantification over equations, they have to borrow the co-limit result from category theory [12] in eliminating quantification over equations. Ehrig and Mahr follows the outline of their proof [6] and provide a proof in [4]. There has been some confusion between these two proofs, see [5] and [9].

Different from the above two, the proof of soundness and completeness of $\tilde{D}$, to be presented, is totally along the usual Birkhoff's approach (see [1, 3, 10] for usual Birkhoff's approach in single-sorted case). It introduces a new concept of *cross-fully invariant congruences* to capture the completeness, instead of building higher order function spaces from term algebras and verifying them as clones as [8] does, and unlike [4] excluding possible empty carriers. The new concept does not have its place in single-sorted case, and I believe that this is its first appearance in literature. The necessary and sufficient condition for variable index free $\tilde{D}$ is derived unlike [8] using the co-limit result of category theory [12]. This also helps to clarify the confusion between [5] and [9]. This approach is conceptually simpler and more coherent in extending to include dependent equations and quasi-dependent equations. It also provides information about models, say what is an equationally definable, dependent equationally definable and/or quasi-dependent equationally definable classes of algebras (see [2, 13] as examples about models of quasi-dependent equations), which can not be easily derived in other approach such as [8] if it is not impossible, especially in the case of quasi-dependent equations.

Some misclaims about the soundness and completeness of $\tilde{D}^d$ and $\tilde{D}^q$ in literature are clarified, say in [11, 14].

[14] made a false claim of a sound and complete calculus for equational implications (his terminology for dependent equations) for single-sorted case. Essentially, his calculus

is $\tilde{D}^d_-$ (which is sound and complete with respect to equations only) except that his substitution is the generalized one (in the single-sorted situation). Because of this, his calculus is neither sound nor complete, counter-examples for his claim will be presented.

By extending $\tilde{D}^d_-$, we can still reach a sound and complete deduction map $\tilde{D}^d$. But the map $\tilde{D}^d$ is not known as a monotonic function. Because of this, we lose the ability of total symbolic manipulation of dependent equations, i.e. total derivability of valid dependent equations. Fortunately, we have $\tilde{D}^d_-$, which has the total valid derivability of dependent equations only with respect to equations.

For quasi-dependent equations, [11] made a wrong claim that its calculus of conditional equations (its terminology for quasi-dependent equations) is sound and complete. In principle, this calculus is the same as $\tilde{D}^q_+$. Therefore, it can not be sound but complete. An counter-example against soundness of $\tilde{D}^q_+$ will be presented. However, $\tilde{D}^q_+$ is sound and complete with respect to dependent equations. Also it is different from $\tilde{D}^q_d$. So, it does not fail to be a good alternative for $\tilde{D}^q$ (if it does exist, see Section 6).

Similar to the dependent equations, we do not have a total derivability for valid quasi-dependent equations either. But we do have a total valid derivability of quasi-dependent equations only with respect to equations. In general, from a syntactical characterization (or derivability) point of view, the situation for quasi-dependent equations is worse than the one for dependent equations, since we even do not know how to get a deduction map which totally captures the semantical quasi-dependent indistinguishability regardless of whether it is monotonic. There are some literature about the models of the quasi-dependent equations, see quasi-variety in [13, 2] for an example.

As a summary for derivabilities, we have sound and complete calculi $\tilde{D}$, $\tilde{D}^d_-$ and $\tilde{D}^q_-$ for equations, dependent equations and quasi-dependent equations only with respect to equations.

With regard to the valid deduction systems of $\tilde{D}^d$ and $\tilde{D}^q$, there are some open problems remained to be solved. For instance, whether is there a sound and complete deduction map $\tilde{D}^q$? However, their completeness of valid derivabilities are established but only with respect to equations.

The tremendous success of syntactical characterizations for each individual $\tilde{D}$, $\tilde{D}^d$ and $\tilde{D}^q$ in the traditional Birkhoff's approach shows the close relationship among equations, dependent equations and quasi-dependent equations. This leads us to think of a unification of the three equational forms syntactically. I discover that $\{\gamma_m \hookrightarrow \Delta_m | m \in M\} \mapsto (\gamma \hookrightarrow \Delta)$ can serve the purpose. For example, when $M = \emptyset$ it is a pure quasi-dependent equation, further if $\gamma = \emptyset$ it is a pure equation; when all γ's are empty it is a pure dependent equation, of course further if $M = \emptyset$ it is a pure equation. Therefore, we can name $\{\gamma_m \hookrightarrow \Delta_m | m \in M\} \mapsto (\gamma \hookrightarrow \Delta)$ as an *universal equation.* It semantical definition is omitted, which can be easily given as its notation suggests. Based on the work presented in this paper, the soundness and completeness of the deduction map $\tilde{D}^u$ for the universal equations can be developed analogously and they are deliberatively left out. Nevertheless, the interested readers can do it as an exercise.

Due to the space reason, I will present the main results with few proofs. However, full results with proofs will be collected in a chapter of my forthcoming thesis [15].

2 Equational Calculus with Variable Index

Upon equations, there is an essential difference between single-sorted algebras and many-sorted algebras, namely , the potential empty carriers. This fact can be demonstrate as follows :

Fact 2.1 (role of variable index) :

1. *$\mathbf{A} \models t \simeq_{\vec{X}\cup\vec{Y}} u$ implies $\mathbf{A} \models t \simeq_{\vec{X}} u$, provided that there is a (total and non-empty) map α from $\vec{X} \cup \vec{Y}$ to $\mathbf{A}$.*

2. *$\mathbf{A} \models t \simeq_{\vec{X}} u$ implies $\mathbf{A} \models t \simeq_{\vec{X}\cup\vec{Y}} u$.*

Following a quite standard procedure, we can get a theorem which is similar to Birkhoff's theorem, formally,

Theorem 2.2 (Birkhoff's Theorem) : *The following three statements are equivalent,*

1. *$\mathbf{T}_\Sigma(\vec{X})/Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$;*

2. *$\mathbf{A} \models t \simeq_{\vec{X}} u$;*

3. *$< t, u > \in Ker_{\vec{X}}(\mathbf{A})$;*

where $Ker_{\vec{X}}(\mathbf{A})$ is the kernel congruence of $\mathbf{A}$ (defined as $\bigcap_{\alpha:\mathbf{T}_\Sigma(\vec{X})\to\mathbf{A}} ker(\alpha)$, which is, in turn, defined as $ker(\alpha) =_{df} \{< t, u > | \alpha(t) = \alpha(u)\}$) and $\mathbf{T}_\Sigma(\vec{X})/Ker_{\vec{X}}(\mathbf{A})$ is the quotient algebra.

This result can be easily extended to a theorem, formally,

Theorem 2.3 (Birkhoff's Theorem) : *Let $\mathcal{K}$ be a collection of Σ-algebras, then the following three statements are equivalent*

1. *$\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$;*

2. *$\mathbf{A} \models t \simeq_{\vec{X}} u$ for each $\mathbf{A} \in \mathcal{K}$;*

3. *$< t, u > \in \bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A})$;*

Remind you of that the above two theorem are related to the variable indexed $\vec{X}$. But the index $\vec{X}$ play two differrent roles. One is an index for equations and the other is the free generator of the quotient term algebra, which reflects the size of the quotient algebra. The below results are intended to clarify the impact of the different roles.

Theorem 2.4 (Relationship among Different Indexed Term Algebras) : *Let $t, u \in \mathbf{T}_\Sigma(\vec{X})$, we have the following :*

1. *$\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}\cup\vec{Y}} u$ implies $\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$, provided that there is a homomorphism $\alpha : \mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) \to \mathbf{T}_\Sigma(\vec{X})$.*

2. *$\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}\cup\vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X}\cup\vec{Y}} u$ implies $\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$, provided that there is a homomorphism $\alpha : \mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) \to \mathbf{T}_\Sigma(\vec{X})$;*

3. *$\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$ implies $\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}\cup\vec{Y}} u$;*

4. $\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X} \cup \vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X} \cup \vec{Y}} u$ *implies* $\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X} \cup \vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$;

5. $\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X} \cup \vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$ *implies* $\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X} \cup \vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X} \cup \vec{Y}} u$

6. $\mathbf{T}_\Sigma(\vec{X}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$ *implies* $\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X} \cup \vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X} \cup \vec{Y}} u$;

7. $\mathbf{T}_\Sigma(\vec{X}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$ *implies* $\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X} \cup \vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X}} u$;

8. $\mathbf{T}_\Sigma(\vec{X} \cup \vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X} \cup \vec{Y}}(\mathbf{A}) \models t \simeq_{\vec{X} \cup \vec{Y}} u$ *implies* $\mathbf{T}_\Sigma(\vec{X}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{X} \cup \vec{Y}} u$.

The first two items of Theorem 2.4 show the role of the variable index in equations and the last two are more interesting. They express that the quotient term algebras with larger. sizes preserve the equation which is satisfied by a quotient term algebra with a smaller size, and vice versa. Other items listed in Theorem 2.4 is an efford to help readers to understand the inter-relationship among the members of a *consequence family* (to be defined) in next section (Section 3). Also, the order among the items partly reflects a logical way in their proofs.

The key point in the proof of Theorem 2.4 is the very carefull manipulation of variable index and the fact that for every $\alpha : \mathbf{T}_\Sigma(\vec{X}) \rightarrow \mathbf{T}_\Sigma(\vec{Y}) / \bigcap_{\mathbf{A} \in \mathcal{K}} Ker_{\vec{Y}}(\mathbf{A})$, there is a homomorphism $\breve{\alpha}$ such that $\alpha = \nu_{\vec{Y}} \circ \breve{\alpha}$, where $\nu_{\vec{Y}}$ is the natural homomorphism from the term algebra $\mathbf{T}_\Sigma(\vec{Y})$ to its quotient algebra. This fact also leads us to introduce the *Cross-fully Invariant Congruence*, since $\breve{\alpha}$ can be viewed as a *substitution*.

Let $\tilde{\theta}$ be a family of fully invariant congruence on $\mathbf{A} \in \mathcal{K}$, $\tilde{\theta}$ is said to be cross-fully invariant over $\mathcal{K}$ iff for all $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, and for every $< a, a' > \in \vec{\theta}^{\mathbf{A}}$, we have that $< \alpha(a), \alpha(a') > \in \vec{\theta}^{\mathbf{B}}$ for each $\alpha : \mathbf{A} \rightarrow \mathbf{B}$. (Note : this definition coincides with the *fully invariant congruence* if $|\mathcal{K}| = 1$). Note that the concept of cross-fully invariant congruences does not appear in literature before, as far as I know.

With the introduction of cross-fully invariant congruence, we can establish a similar Birkhoff's Completeness Theorem for Equational Calculus in *many-sorted* case.

Theorem 2.5 (Soundness and Completeness of $\tilde{D}$) : *The least cross-fully invariant congruence is the same as the the least fixpoint of the deduction map $\tilde{D}$ defined as below, given any $\tilde{\Gamma}$,*

1. *(identity)* $\frac{t \simeq_{\vec{X}} u \in \tilde{\Gamma}}{t \simeq_{\vec{X}} u \in D_{\vec{X}}(\tilde{\Gamma})}$

2. *(reflectivity)* $\frac{t \in \mathbf{T}_\Sigma(\vec{X})}{t \simeq_{\vec{X}} t \in D_{\vec{X}}(\tilde{\Gamma})}$

3. *(symmetricity)* $\frac{t \simeq_{\vec{X}} u \in \tilde{\Gamma}}{u \simeq_{\vec{X}} t \in D_{\vec{X}}(\tilde{\Gamma})}$

4. *(transitivity)* $\frac{t \simeq_{\vec{X}} u, u \simeq_{\vec{X}} v \in \tilde{\Gamma}}{t \simeq_{\vec{X}} v \in D_{\vec{X}}(\tilde{\Gamma})}$

5. *(compositionality)* $\frac{\sigma\in\Sigma_{\kappa,i};\ t_m\simeq_{\vec{X}}u_m\in\tilde{\Gamma}(|\vec{t}|=|\vec{u}|=|\kappa|)}{\sigma(\vec{t})\simeq_{\vec{X}}\sigma(\vec{u})\in D_{\vec{X}}(\tilde{\Gamma})}$

6. *(cross-substitution)* $\frac{t\simeq_{\vec{X}}u\in\tilde{\Gamma};\ \alpha:\mathbf{T}_\Sigma(\vec{X})\rightarrow\mathbf{T}_\Sigma(\vec{Y})}{\alpha(t)\simeq_{\vec{Y}}\alpha(u)\in D_{\vec{Y}}(\tilde{\Gamma})}$

The variable indices in the cross-substitution rule play a crucial role; otherwise we can derive $true \simeq false$ as strikingly demonstrated in [6].

The main idea in the proof of Theorem 2.5 runs as follows : all sound equations in the family of term algebras forms least cross-fully invariant congruence over the collection of all those term algebras.

3 Elimination of Variable Index in Equational Calculus

As we point out previously in Theorem 2.2, the index $\vec{X}$ plays two different roles. In Theorem 2.4, these different roles have been represented by different index $\vec{X}$ and $\vec{Y}$ such as $\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{Y}} u$. For simplicity, we collect all these $t \simeq_{\vec{Y}} u$ and name them the *consequence family* $\{\tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{Y}} | \vec{X}, \vec{Y} \subseteq \vec{V}\}$, i.e. $< t, u > \in \tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{Y}}$ iff $\mathbf{T}_\Sigma(\vec{X})/\bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models t \simeq_{\vec{Y}} u$. So, the index $\vec{X}$ in $\tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{Y}}$ reflects the size of term algebras, and the index $\vec{Y}$ indicates the index for equations. Thus, we can define that $\tilde{D}$ is *variable eliminatable* (or variable *index-free*) iff for all $\vec{X}, \vec{Y} \subseteq \vec{V}$, $\vec{X} \subseteq \vec{Y}$ such that (a) $\tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{Y}} \lceil_{\mathbf{T}_\Sigma(\vec{X})\times\mathbf{T}_\Sigma(\vec{X})} = \tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{X}}$ and (b) $\tilde{C}_{\vec{Y}}(\mathcal{K})_{\vec{Y}} \lceil_{\mathbf{T}_\Sigma(\vec{X})\times\mathbf{T}_\Sigma(\vec{X})} = \tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{X}}$.

On the other hand, Theorem 2.4 can be restate as follows : let $\vec{X}, \vec{Y} \subseteq \vec{V}$ and $\vec{X} \subseteq \vec{Y}$,

$$\begin{array}{llll} \tilde{C}_{\vec{Y}}(\mathcal{K})_{\vec{X}} & & & \\ & \subseteq & & \\ ?\cup & & \tilde{C}_{\vec{Y}}(\mathcal{K})_{\vec{Y}} \lceil_{\mathbf{T}_\Sigma(\vec{X})\times\mathbf{T}_\Sigma(\vec{X})} & \tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{Y}} \lceil_{\mathbf{T}_\Sigma(\vec{X})\times\mathbf{T}_\Sigma(\vec{X})} \overset{\ddot{\exists}\alpha}{\subseteq} \tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{X}} \\ & = & & \\ \tilde{D}_{\vec{X}}(\mathcal{K})_{\vec{X}} & & & \end{array}$$

where $\overset{\ddot{\exists}\alpha}{\subseteq}$ means that the containment holds (only) if there is a homomorphism α : $\mathbf{T}_\Sigma(\vec{Y}) \rightarrow \mathbf{T}_\Sigma(\vec{X})$.

The contaiment, not obtained from Theorem 2.4, is the one with ? beside $\cup$, i.e. $\tilde{C}_{\vec{X}}(\mathcal{K})_{\vec{X}} \subseteq \tilde{C}_{\vec{Y}}(\mathcal{K})_{\vec{X}}$. This can be easily checked. Hence, if we can eliminate $\ddot{\exists}\alpha$ on the top of $\subseteq$ from the above, we will get a variable index free $\tilde{D}$. This observation leads to the following theorem, formally,

Theorem 3.1 (variable eliminatable condition) : *Given signature Σ, the following three statements are equivalent :*

1. *for all $i, j \in Sort(\Sigma)$, $\mathbf{T}_\Sigma(X_j)_i \neq \emptyset$, where X_j is a singleton set, i.e. there is only one variable for sort j and no other variable;*

2. *for all $\vec{Y} \subseteq \vec{V}$ and $\vec{Y} \neq \emptyset$, $\mathbf{T}_\Sigma(\vec{Y})_i \neq \emptyset$ for each $i \in Sort(\Sigma)$;*

3. *the variable index in $\tilde{D}$ is eliminatable, i.e. variable index free.*

This result is very strong. It confirms that the single-sorted $\tilde{D}$ is variable index free and also justify the commonly used technique of getting variable index free $\tilde{D}$ by introducing extra constants to the signature Σ of every sort.

4 Dependent Equational Deduction

Before proceeding further, we look at the models of dependent equations first. This provides certain interesting information.

Fact 4.1 (models of dependent equations) : *Let $\tilde{\Gamma}^d$ be a collection of dependent equations (N.B. $\emptyset \mapsto \Delta_{\vec{X}}$ is an equation, $\Delta_{\vec{X}}$ for short), $Eq(\tilde{\Gamma}^d) =_{df} \{\Delta_{\vec{X}} | \Delta_{\vec{X}} \in \tilde{\Gamma}^d\}$, $pr(\tilde{\Gamma}^d) =_{df} Eq(\tilde{\Gamma}^d) \cup \{\Delta_{\vec{X}} | \Delta_{\vec{X}} \in \gamma_{\vec{X}}$ for some $\gamma_{\vec{X}} \mapsto \Delta'_{\vec{X}} \in \tilde{\Gamma}^d\}$ and $cl(\tilde{\Gamma}^d) =_{df} Eq(\tilde{\Gamma}^d) \cup \{\Delta_{\vec{X}} | \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{\Gamma}^d$ for some $\gamma_{\vec{X}}\}$,*

1. *if $Alg_{\Sigma,Eq(\tilde{\Gamma}^d)} \not\models \gamma_{\vec{X}}$, then $Alg_{\Sigma,Eq(\tilde{\Gamma}^d)\cup\{\gamma_{\vec{X}}\mapsto\Delta_{\vec{X}}\}} \supseteq Alg_{\Sigma,Eq(\tilde{\Gamma}^d)}$*
2. *if $Alg_{\Sigma,Eq(\tilde{\Gamma}^d)} \models \gamma_{\vec{X}}$, then $Alg_{\Sigma,Eq(\tilde{\Gamma}^d)\cup\{\gamma_{\vec{X}}\mapsto\Delta_{\vec{X}}\}} \subseteq Alg_{\Sigma,Eq(\tilde{\Gamma}^d)}$*
3. $$\begin{array}{ccccc} Alg_{\Sigma,pr(\tilde{\Gamma}^d)} & & & & Alg_{\Sigma,pr(\tilde{\Gamma}^d)} \\ & \subseteq & & \subseteq & \\ & & Alg_{\Sigma,Eq(\tilde{\Gamma}^d)} \ \text{and} \ Alg_{\Sigma,pr(\tilde{\Gamma}^d)} \cap Alg_{\Sigma,cl(\tilde{\Gamma}^d)} & & \\ & \subseteq & & \subseteq & \\ Alg_{\Sigma,cl(\tilde{\Gamma}^d)} & & & & Alg_{\Sigma,cl(\tilde{\Gamma}^d)} \end{array}$$
4. *if $Alg_{\Sigma,Eq(\tilde{\Gamma}^d)} \models \Delta'_{\vec{X}}$ for each $\Delta'_{\vec{X}} \in pr(\tilde{\Gamma}^d) - Eq(\tilde{\Gamma}^d)$, then $Alg_{\Sigma,\tilde{\Gamma}^d} = Alg_{\Sigma,pr(\tilde{\Gamma}^d)} \cap Alg_{\Sigma,cl(\tilde{\Gamma}^d)}$*

The first item of Fact 3.1 seems a bit surprising, since our common sense tells us that the number of models should decrease if adding new axioms (axioms can be dependent equations in this case). The item states the opposite of it.

For dependent equations, we can easily obtain an extension of Theorem 2.3, formally

Theorem 4.2 (a similar Birkhoff's theorem) : *The following three statements are equivalent,*

1. *$\mathcal{K} \models \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$;*
2. *if $< t_m, u_m > \in \bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A})$ for all m (or simply $\gamma_{\vec{X}} \subseteq \bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A})$), then $< t, u > \in \bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A})$ (or simply $\Delta_{\vec{X}} \in \bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A})$;*
3. *$\mathbf{T}_\Sigma(\vec{X}) / \bigcap_{\mathbf{A}\in\mathcal{K}} Ker_{\vec{X}}(\mathbf{A}) \models \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$.*

An indexed map $\tilde{D}^d_{\simeq}$ from the indexed product $\tilde{\mathcal{P}}(\tilde{\mathcal{P}}(\tilde{V} \times \mathbf{T}_\Sigma(\vec{V}) \times \mathbf{T}_\Sigma(\vec{V})) \times (\tilde{V} \times \mathbf{T}_\Sigma(\vec{V}) \times \mathbf{T}_\Sigma(\vec{V})))$ to itself is said to be a deduction map of dependent equations iff it is defined as follows : let $\tilde{\Gamma}^d$ be a family of collections of dependent equations. Then,

1. (identity) $\frac{\gamma_{\vec{X}}\mapsto\Delta_{\vec{X}}\in\tilde{\Gamma}^d}{\gamma_{\vec{X}}\mapsto\Delta_{\vec{X}}\in\tilde{D}^d_{\simeq}(\tilde{\Gamma}^d)}$
2. (reflectivity) $\frac{t\in\mathbf{T}_\Sigma(\vec{X})}{\emptyset\mapsto t\simeq_{\vec{X}} t\in\tilde{D}^d_{\simeq}(\tilde{\Gamma}^d)}$
3. (symmetricity) $\frac{t,t'\in\mathbf{T}_\Sigma(\vec{X})}{t\simeq_{\vec{X}} t'\mapsto t'\simeq_{\vec{X}} t\in\tilde{D}^d_{\simeq}(\tilde{\Gamma}^d)}$
4. (transitivity) $\frac{t,t',t''\in\mathbf{T}_\Sigma(\vec{X})}{\{t\simeq_{\vec{X}} t', t'\simeq_{\vec{X}} t''\}\mapsto t\simeq_{\vec{X}} t''\in\tilde{D}^d_{\simeq}(\tilde{\Gamma}^d)}$

5. (compositionality) $\frac{\vec{t}_m, \vec{t'}_m \in \mathbf{T}_\Sigma(\vec{X})}{\{\vec{t}_m \simeq_{\vec{X}} \vec{t'}_m \mid 1 \leq m \leq |\vec{t}| = |\vec{t'}|\} \mapsto \sigma(\vec{t}) \simeq_{\vec{X}} \sigma(t') \in \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)}$

6. (substitution) $\frac{\emptyset \mapsto \Delta_{\vec{X}} \in \tilde{\Gamma}^d_{\vec{X}}; \alpha : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{Y})}{\emptyset \mapsto \alpha(\Delta_{\vec{X}}) \in \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)_{\vec{Y}}}$, where $\alpha(t \simeq_{\vec{X}} t') =_{df} \alpha(t) \simeq_{\vec{Y}} \alpha(t')$ and $\alpha(\gamma_{\vec{X}})$ is its natural extension to a collection of equations (note : $\alpha(\emptyset) = \emptyset$).

7. (axiom introduction) $\frac{\emptyset \mapsto \Delta_{\vec{X}} \in \tilde{\Gamma}^d; \gamma_{\vec{X}} \subseteq \mathbf{T}_\Sigma(\vec{X}) \times \mathbf{T}_\Sigma(\vec{X})}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)}$

8. (modus ponens) $\frac{\gamma'_{\vec{X}} \mapsto \Delta''_{\vec{X}} \in \tilde{\Gamma}^d; \{\gamma_{\vec{X}} \mapsto \Delta'_{\vec{X}} \in \tilde{\Gamma}^d \mid \Delta' \in \gamma'_{\vec{X}}\}}{\gamma_{\vec{X}} \mapsto \Delta''_{\vec{X}} \in \tilde{D}^{d}_{-}(\Gamma^d)}$

The deduction closure $\tilde{\Gamma}^{d}_{-}$ is the least fixed point of $\tilde{D}^{d}_{-}$ containing $\tilde{\Gamma}^d$, written as $\sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)$ or simply $\tilde{D}^{d}_{-}(\tilde{\Gamma}^d)$.

For the soundness and completeness of $\tilde{D}^{d}_{-}$ with respect to equations, we have that the soundness and completeness of $\tilde{D}^{d}_{-}$ with respect to Σ-equations are $\phi(\tilde{D}^{d}_{-}(\tilde{\Gamma}^d))\lceil_{\simeq} = \bigcap_{\mathbf{A} \in Alg_{\Sigma, \tilde{\Gamma}^d}} \tilde{\mathcal{K}}(\mathbf{A})$, where $\phi(\tilde{D}^{d}_{-}(\tilde{\Gamma}^d)_{\vec{X}})\lceil_{\simeq} = \{< t, t' > \mid \emptyset \mapsto t \simeq_{\vec{X}} t' \in \tilde{D}^{d}_{-}(\tilde{\Gamma})\}$. This can be achieved by the same reasoning as we did to $\tilde{D}$, with the fact that the rules from 1 to 6 and 8 of $\tilde{D}^{d}_{-}$ include all rules of $\tilde{D}$.

The *soundness and completeness of* $\tilde{D}^{d}_{-}$ for all possible dependent Σ-equations are whether we have that $\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)$ iff $\gamma_{\vec{X}} \subseteq \sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)$ implies $\Delta_{\vec{X}} \in \sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)$ (i.e. $\emptyset \mapsto \Delta_{\vec{X}} \in \sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)$ when $\emptyset \mapsto \Delta'_{\vec{X}} \in \sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)$ for every $\Delta'_{\vec{X}} \in \gamma_{\vec{X}}$). The proof for the soundness direction is easy. But for the completeness direction, we have a counter-example, see Example 4.3 below. Therefore, $\tilde{D}^{d}_{-}$ is *incomplete* for the dependent equations.

Example 4.3 (counter-example for completeness of $\tilde{D}^{d}_{-}$) : *A counter-example for completeness of $\tilde{D}^{d}_{-}$ is provided as follows : let i be a sort in I, $\Sigma_{i,i} = \{c_*\}$ and other $\Sigma_{\kappa,j} = \emptyset$, $\mathbf{A}$ be an algebra $< A, \mathcal{A} >$ where $A_i = \{\bullet, *\}$ and $\mathcal{A}_{c_*}(a) = *$ for every $a \in A_i$, $\tilde{\Gamma}^d = \{c_*(x) \simeq_{\{x,y\}} c_*(y), c_*(x) \simeq_{\{x,y\}} c_*(y) \mapsto c_*(c_*(x)) \simeq_{\{x,y\}} c_*(y)\}$. The remaining is to show $x \simeq_{\{x,y\}} y \mapsto c_*(x) \simeq_{\{x,y\}} y$ is not derivable by $\tilde{D}^{d}_{-}$. I give a hint and leave it as an exercise. (Hint : to show this, you can define $n_*(t)$ as the number of c_* occurring in t and show that every derivable $\frac{\gamma_{\vec{X}}}{t \simeq_{\vec{X}} t'}$ has the property of $t = t'$ or $n_*(t) + n_*(t') > 1$.)*

The substitution rule (6 in $\tilde{D}^{d}_{-}$) can not be generalized to $\frac{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{\Gamma}^d_{\vec{X}}; \alpha : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{Y})}{\alpha(\gamma_{\vec{X}}) \mapsto \alpha(\Delta_{\vec{X}}) \in \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)_{\vec{Y}}}$. Why? (Hint : because the generalized rule is not sound, see Example 4.3 and include $x \simeq_{\{x,x',y,y'\}} x' \mapsto y \simeq_{\{x,x',y,y'\}} y'$ into the $\tilde{\Gamma}^d$ and consider $c_*(x) \simeq_{\{x,x'y,y'\}} c_*(x') \mapsto y \simeq_{\{x,x',y,y'\}} y'$, which is a result of a generalized substitution; on the other hind, this provide a counter-example against the soundness of Selman's equational implication calculus in [14]).

The deduction map $\tilde{D}^d$ is defined as : given any $\tilde{\Gamma}^d$,

1. $\frac{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d)}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^d(\tilde{\Gamma}^d)}$

2. $\frac{\gamma_{\vec{X}} \not\subseteq \sqcup \tilde{D}^{d}_{-}(\tilde{\Gamma}^d); \Delta_{\vec{X}} : \mathbf{T}_\Sigma(\vec{X}) \times \mathbf{T}_\Sigma(\vec{X})}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^d(\tilde{\Gamma}^d)}$

where $\sqcup \tilde{D}^d(\tilde{\Gamma}^d)$ means the least fixed point of $\tilde{D}^d$ contained $\tilde{\Gamma}^d$ with the usual inclusion order. Sometimes we will omit $\sqcup$ for simplicity.

Since $\tilde{D}^d_-$ is complete for equations and $\tilde{D}^d$ does not increase the derivable equations of $\tilde{D}^d_-$, we have that $\tilde{D}^d$ is sound and complete with respect to equations. For soundness and completeness of $\tilde{D}^d$, we have a theorem below.

Theorem 4.4 (soundness and completeness of $\tilde{D}^d$) : *$\tilde{D}^d$ is a sound and complete deduction system of dependent Σ-equations for any given $\tilde{\Gamma}^d$, i.e. $\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^d(\tilde{\Gamma}^d)$ iff $\gamma_{\vec{X}} \subseteq \tilde{D}^d(\tilde{\Gamma}^d)$ implies $\Delta_{\vec{X}} \in \tilde{D}^d(\tilde{\Gamma}^d)$ (i.e. $\emptyset \mapsto \Delta_{\vec{X}} \in \tilde{D}^d(\tilde{\Gamma}^d)$ when $\emptyset \mapsto \Delta'_{\vec{X}} \in \tilde{D}^d(\tilde{\Gamma}^d)$ for every $\Delta'_{\vec{X}} \in \gamma_{\vec{X}}$).*

Since $\tilde{D}^d$ is not monotonic (due to 2 of $\tilde{D}^d$), there is an interesting question left below,

Open Problem 4.5 (monotonic $\tilde{D}^d$?) : *Whether can we combine the two definitions of $\tilde{D}^d_-$ and $\tilde{D}^d$ together into one (for $\tilde{D}^d$) such that the new $\tilde{D}^d$ is monotonic?*

Since a dependent equation $\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$ is actually an equation $\Delta_{\vec{X}}$ if $\gamma_{\vec{X}} = \emptyset$, the condition for variable index-free deduction system $\tilde{D}^d$ is still the same as it previously expressed in section 3.

5 Quasi-dependent Equational Deduction

First of all, we will try to find out the relationship between dependent equations and quasi-dependent equations. For this purpose, we define an obvious translation between dependent equations and quasi-dependent equations. Formally, We define a natural translation q-d from quasi-dependent equations to dependent equations as $q\text{-}d[[\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}]] =_{df} \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$. It is easy to verify that q-d is semantically sound since $\mathbf{A} \models \gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$ implies $\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$. Then, we extend it naturally to translate a collection of quasi-dependent equations to a collection of dependent equations. The models between dependent equations and quasi-dependent equations under the translation have the following property : given a $\tilde{\Gamma}^q$ and let $\tilde{\Gamma}^d$ be $q\text{-}d[[\tilde{\Gamma}^q]]$, since $\mathbf{A} \models \gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$ implies $\mathbf{A} \models \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$, we would have $Alg_{\Sigma,\tilde{\Gamma}^q} \subseteq Alg_{\Sigma,\tilde{\Gamma}^d}$. Hence, $Alg_{\Sigma,\tilde{\Gamma}^d} \models t \simeq_{\vec{X}} t'$ implies $Alg_{\Sigma,\tilde{\Gamma}^q} \models t \simeq_{\vec{X}} t'$. Another way to understand this property is that for a (non-empty) $\mathcal{K}$, $\mathcal{K} \models \gamma_{\vec{x}} \hookrightarrow \Delta_{\vec{x}}$ implies $\mathcal{K} \models \gamma_{\vec{X}} \mapsto \Delta_{\vec{X}}$. Such an observation leads us to understand that $\tilde{D}^d(\Gamma^d)\lceil_{eq} \subseteq \tilde{D}^q(\Gamma^q)\lceil_{eq}$ (i.e. whenever an equation can be deduced by $\tilde{D}^d$, it can also deduced by $\tilde{D}^q$, if there is a sound and complete $\tilde{D}^q$). With a surprise, I discover that Birkhoff method (i.e. using fully invariant congruences and their quotient algebras to capture a completeness) is not powerful enough to reach a sound and complete deduction system $\tilde{D}^q$ for quasi-dependent equations. However, we are still able to reach a sound and complete $\tilde{D}^q_-$ only with respect to equations (valid equations are the central concern in many cases anyway). Actually, the obtained $\tilde{D}^q_+$ is complete but not sound, as we will see the result.

On contrast of equational case, i.e. Theorem 2.3, we do not have the three equivalent statements for quasi-dependent equations rather we only have two equivalent statements corresponding to it. Formally,

Theorem 5.1 (a similar Birkhoff's theorem) : *Let $\mathcal{K}$ be a class of Σ-algebras. Then, following two statements are equivalent :*

1. $\mathbf{T}_\Sigma(\vec{X})/Ker_{\vec{X}}(\mathbf{A}) \models \gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$.

2. *for every $\breve{\alpha} : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{X})$, if $\breve{\alpha}(\gamma_{\vec{X}}) \subseteq Ker_{\vec{X}}(\mathbf{A})$, then $\breve{\alpha}(\Delta_{\vec{X}}) \in Ker_{\vec{X}}(\mathbf{A})$.*

From above, we understand that quasi-dependent deduction have a very close relation with dependent deduction. This close relation has been explicitly expressed, say in 2 of the above. Such a relation enables us to define a $\tilde{D}^q_+$ which is an extension of $\tilde{D}^d$.

Although we do not have three equivalence, we do have two equivalent statements of the above and plus an (extra) implication below.

Theorem 5.2 (quasi-dependent equation character) : $\mathbf{A} \models \gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$ *implies that for every* $\breve{\alpha} : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{X})$, *if* $\breve{\alpha}(\gamma_{\vec{X}}) \subseteq Ker_{\vec{X}}(\mathbf{A})$, *then* $\breve{\alpha}(\Delta_{\vec{X}}) \in Ker_{\vec{X}}(\mathbf{A})$.

The deduction map $\tilde{D}^q_-$ is defined as follows : let $\tilde{\Gamma}^q$ be a family of collections $\tilde{\Gamma}^q_{\vec{X}}$ of Σ-equations, dependent Σ-equations and quasi-dependent Σ-equations,

1. (identity) $\frac{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{\Gamma}^q}{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$ *and* $\frac{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{\Gamma}^q}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$

2. (reflectivity) $\frac{t \in \mathbf{T}_\Sigma(\vec{X})}{\emptyset \hookrightarrow t \simeq_{\vec{X}} t \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$

3. (symmetricity) $\frac{t, t' \in \mathbf{T}_\Sigma(\vec{X})}{t \simeq_{\vec{X}} t' \hookrightarrow t' \simeq_{\vec{X}} t \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$

4. (transitivity) $\frac{t, t', t'' \in \mathbf{T}_\Sigma(\vec{X})}{\{t \simeq_{\vec{X}} t', t' \simeq_{\vec{X}} t''\} \hookrightarrow t \simeq_{\vec{X}} t'' \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$

5. (compositionality) $\frac{\vec{t}_m, \vec{t'}_m \in \mathbf{T}_\Sigma(\vec{X})}{\{\vec{t}_m \simeq_{\vec{X}} \vec{t'}_m \mid 1 \leq m \leq |\vec{t}| = |\vec{t'}|\} \hookrightarrow \sigma(\vec{t}) \simeq_{\vec{X}} \sigma(\vec{t'}) \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$

6. (substitution) $\frac{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{\Gamma}^q ; \alpha : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{Y})}{\alpha(\gamma_{\vec{X}}) \hookrightarrow \alpha(\Delta_{\vec{X}}) \in \tilde{D}^q_-(\tilde{\Gamma}^q)_{\vec{Y}}}$ and
$\frac{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{\Gamma}^q ; \{\emptyset \mapsto \Delta'_{\vec{X}} \in \tilde{\Gamma}^q \mid \Delta'_{\vec{X}} \in \gamma_{\vec{X}}\} ; \alpha : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{Y})}{\alpha(\gamma_{\vec{X}}) \mapsto \alpha(\Delta_{\vec{X}}) \in \tilde{D}^q_-(\tilde{\Gamma}^q)_{\vec{Y}}}$

7. (axiom introduction) $\frac{\emptyset \hookrightarrow \Delta_{\vec{X}} \in \tilde{\Gamma}^q ; \gamma_{\vec{X}} \subseteq \mathbf{T}_\Sigma(\vec{X}) \times \mathbf{T}_\Sigma(\vec{X})}{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$,

8. (modus ponens) $\frac{\gamma'_{\vec{X}} \hookrightarrow \Delta''_{\vec{X}} \in \tilde{\Gamma}^q ; \{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{\Gamma}^q \mid \Delta' \in \gamma'_{\vec{X}}\}}{\gamma_{\vec{X}} \hookrightarrow \Delta''_{\vec{X}} \in \tilde{D}^q_-(\Gamma^d)}$ and
$\frac{\gamma'_{\vec{X}} \mapsto \Delta''_{\vec{X}} \in \tilde{\Gamma}^q ; \{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{\Gamma}^q \mid \Delta' \in \gamma'_{\vec{X}}\}}{\gamma_{\vec{X}} \mapsto \Delta''_{\vec{X}} \in \tilde{D}^q_-(\Gamma^d)}$

9. (replacement of quasi-dependent equations by dependent equations) $\frac{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{\Gamma}^q}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^q_-(\tilde{\Gamma}^q)}$

For the soundness and completeness of $\tilde{D}^q_-$ with respect to equations we know that the rules from 1 to 6, 8 and 9 of the above implies all rules of $\tilde{D}$. So, the soundness and completeness of $\tilde{D}^q_-$ with respect to Σ-equations can be obtained by a same reasoning as the one for $\tilde{D}$.

Analogous to $\tilde{D}^d$, we can define a $\tilde{D}^q_d$. Formally, A deduction map $\tilde{D}^q_d$ is defined as : given any $\tilde{\Gamma}^q$,

1. $\frac{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \sqcup \tilde{D}^q_-(\tilde{\Gamma}^q)}{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{D}^q_d(\tilde{\Gamma}^q)}$, *and* $\frac{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \sqcup \tilde{D}^q_-(\tilde{\Gamma}^q)}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^q_d(\tilde{\Gamma}^q)}$

2. $\frac{\gamma_{\vec{X}} \not\subseteq \sqcup \tilde{D}^q_-(\tilde{\Gamma}^q) ; \Delta_{\vec{X}} \in \mathbf{T}_\Sigma(\vec{X}) \times \mathbf{T}_\Sigma(\vec{X})}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^q_d(\tilde{\Gamma}^q)}$

For the soundness and completeness of $\tilde{D}^q_d$ with respect to dependent equations, with the same reasoning as the one for $\tilde{D}^d$, we come to that $\tilde{D}^q_d$ is sound and complete with respect to dependent equations. At the meantime, this gives a positive answer to whether $\tilde{D}^q$ (if exists) and $\tilde{D}^d$ are equivalent under the translation q-d if we only consider equations.

Further extension of $\tilde{D}^q_d$ is $\tilde{D}^q_+$. Formally, a deduction map $\tilde{D}^q_+$ is defined as : given any $\tilde{\Gamma}^q$,

1. $\frac{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{D}^q_d(\tilde{\Gamma}^q)}{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{D}^q_+(\tilde{\Gamma}^q)}$, *and* $\frac{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^q_d(\tilde{\Gamma}^q)}{\gamma_{\vec{X}} \mapsto \Delta_{\vec{X}} \in \tilde{D}^q_+(\tilde{\Gamma}^q)}$

2. $\frac{\{\alpha(\gamma_{\vec{X}}) \mapsto \alpha(\Delta_{\vec{X}}) \in \tilde{D}^q_d(\tilde{\Gamma}^q) \mid \alpha : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{Y})\}}{\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}} \in \tilde{D}^q_+(\tilde{\Gamma}^q)}$

With regard to the soundness and completeness of $\tilde{D}^q_+$, we have that $\tilde{D}^q_+$ is sound and complete with respect to dependent equations, since $\tilde{D}^q_d$ is sound and complete with respect to dependent equations and $\tilde{D}^q_+$ does not add more dependent equations. But, in general, we do not even have a soundness, see Example 5.3 below.

Example 5.3 (counter-example against soundness of $\tilde{D}^q_+$) : *A counter-example of that for every $\breve{\alpha} : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{X})$, if $\breve{\alpha}(\gamma_{\vec{X}}) \subseteq Ker_{\vec{X}}(\mathbf{A})$, then $\breve{\alpha}(\Delta_{\vec{X}}) \in Ker_{\vec{X}}(\mathbf{A})$ implies $\mathbf{A} \models \gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$. It also is a counter-example against the soundness of Kaplan's condition equational calculus in [11]. Let $\tilde{\Gamma}^q = \{c_*(x) \simeq_{\{x,y\}} c_*(y), c_*(x) \simeq_{\{x,y\}} c_*(y) \hookrightarrow c_*(c_*(x)) \simeq_{\{x,y\}} c_*(y)\}$, and $\mathbf{A}$ be the algebra in Example 4.3. Thus, $\mathbf{A} \not\models x \simeq_{\{x,y\}} y \hookrightarrow c_*(x) \simeq_{\{x,y\}} y$ but for every $\breve{\alpha} : \mathbf{T}_\Sigma(\vec{X}) \to \mathbf{T}_\Sigma(\vec{X})$, if $\breve{\alpha}(x \simeq_{\{x,y\}} y) \subseteq Ker_{\vec{X}}(\mathbf{A})$, then $\breve{\alpha}(c_*(x) \simeq_{\{x,y\}} y) \in Ker_{\vec{X}}(\mathbf{A})$.*

Below we show an example, which demonstrates the necessity of $\tilde{D}^q_+$.

Example 5.4 (necessity of $\tilde{D}^q_+$) : *show that $c_*(x) \simeq_{\{x,y\}} y \hookrightarrow c_*(c_*(x)) \simeq_{\{x,y\}} y$ is not deductible without 2 of the definition of $\tilde{D}^q_+$ (hint : all deductible $\gamma_{\vec{X}} \hookrightarrow t \simeq_{\vec{X}} t'$ without 2 of the definition of $\tilde{D}^q_+$ has the property that $t = t'$ or $0 < min\{n_*(t), n_*(t')\}$ in Example 5.3.*

In reality, what 2 of $\tilde{D}^q_+$ does is to collect all those dependent equations which are closed under substitutions and make them become quasi-dependent equations. But such a collecting gets more quasi-dependent equations than necessary (see the previous counter-example against soundness of $\tilde{D}^q_+$).

In another words, $\tilde{D}^q_+$ is actually complete but not sound for quasi-dependent equations. We state it as a theorem below.

Theorem 5.5 : *Given any $\tilde{\Gamma}^q$,*

- *$\tilde{D}^q_+$ is a sound and complete deduction system with respect to equations.*
- *$\tilde{D}^q_+$ is a sound and complete deduction system with respect to dependent equations.*
- *$\tilde{D}^q_+$ is complete but not sound deduction system for quasi-dependent equations, although it is a sound and complete deduction system of $\mathbf{T}_\Sigma(\vec{V})/\tilde{K}(Alg_{\Sigma,\tilde{\Gamma}^q})$ for quasi-dependent equations.*

Since a quasi-dependent equation $\gamma_{\vec{X}} \hookrightarrow \Delta_{\vec{X}}$ is an equation $\Delta_{\vec{X}}$ if $\gamma_{\vec{X}}$ is empty, the condition for variable index free $\tilde{D}^q_+$ should be the same as it states in Theorem 3.1.

6 Conclusion

After all, we have presented sound and complete deduction systems $\tilde{D}$, $\tilde{D}^d_-$ and $\tilde{D}^q_-$. for equations, dependent equations and quasi-dependent equations only with respect to equations. These results are good enough for us to have natural deduction systems (or calculi) corresponding to them. Since a deduction system to be a calculus (only) if the corresponding deduction map is monotonic, we point out the following facts :

- $\sqcup\tilde{D}(\tilde{\Gamma}) = \bigcup_{n\in Nat} \tilde{D}^n(\tilde{\Gamma})$ for every $\tilde{\Gamma}$,
- $\sqcup\tilde{D}^d_-(\tilde{\Gamma}^d) = \bigcup_{n\in Nat} \tilde{D}^{d^n}_-(\tilde{\Gamma}^d)$ for every $\tilde{\Gamma}^d$ and
- $\sqcup\tilde{D}^q_-(\tilde{\Gamma}^q) = \bigcup_{n\in Nat} \tilde{D}^{q^n}_-(\tilde{\Gamma}^q)$ for every $\tilde{\Gamma}^q$.

This also explain why we sometime interchange the terms of the deduction systems and their calculi.

Mis-claims in [11, 14] have been clarified. Counter-examples against their claims have been presented.

About $\tilde{D}^q$, I discover that Birkhoff's approach is not powerful enough to totally capture it. Therefore, to conclude this paper, I name the existence of $\tilde{D}^q$ as an open problem and raise some closely-related questions below.

Open Problem 6.1 : *Whether is there a condition, under which $\tilde{D}^q_+$ totally captures valid quasi-dependent equations?*

Open Problem 6.2 : *Is there a sound and complete deduction system $\tilde{D}^q$ for quasi-dependent equations?*

The last problem is related to the derivability of the valid quasi-dependent equations.

Open Problem 6.3 : *Is there a sound and complete* **monotonic** *deduction system $\tilde{D}^q$ for quasi-dependent equations?*

7 Acknowledgement

I would like to thank Joseph Goguen for his comment on the early draft of this paper. He also point out that the first statement in Theorem 5.5 coincides with Proposition 2 in [7]. Therefore, Theorem 5.5 can be regarded as an improvement of the result of the proposition.

8 References

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