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Petri Nets as Quantales

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by

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1 Introduction

The work presented here was motivated by a desire to understand more fully the connection between Petri nets and linear logic. When linear logic was introduced, it was suggested that it would prove to be a natural logic for reasoning about concurrent systems (see [Gir86]. It is now well-known, due to independent results of Asperti [Asp87], Gunter and Gehlot [GG89] and Brown [Bro89], that evolution in Petri nets corresponds to linear proof, and in fact that the simple tensorial fragment of linear logic suffices to model Petri nets. Attempts to understand the other connectives of linear logic in terms of nets have also been made, in particular in [Bro89] and [MOM89]. Quantales were introduced by Mulvey [Mul86], and studied by Niefield and Rosenthal [NR88] and by Abramsky and Vickers [AV88]. We here make a connection with the results of Yetter [Yet] showing that quantales are models of linear logic. This allows us to interpret a large fragment of linear logic using the behaviour of Petri nets (an extension allows us to interpret the whole of linear logic, as is shown in the related, independent work of Winskel and Engberg [WE89]).

We show how a quantale can be generated by a Petri net, and how such quantales model intuitionistic linear logic. A simple construction (observed by Abramsky and Vickers [AV88]), turns a quantale into a model of classical linear logic.

2 Linear Logic

We present here the rules of linear intuitionistic logic as they are to be found in [GL87].

$$\frac{}{A \vdash A}(\text{Identity}) \qquad \frac{}{\Gamma,\ \mathrm{F}\ \vdash A}(\mathrm{F}) \qquad \frac{}{\Gamma \vdash \mathrm{T}}(\mathrm{T})$$

$$\frac{\Gamma \vdash A}{\Gamma, \mathbf{1} \vdash A}(1\mathrm{L}) \qquad \frac{}{\vdash \mathbf{1}}(1\mathrm{R})$$

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}(\text{Cut}) \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}(\text{Exchange})$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}(\otimes\mathrm{R}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}(\otimes\mathrm{L})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}(\wedge R)$$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}(\wedge L1) \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \wedge B \vdash C}(\wedge L2)$$

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Delta, \Gamma, A \multimap B \vdash C}(\multimap L) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}(\multimap R)$$

Linear logic differs from intuitionistic logic primarily in the absence of the structural rules of *weakening* and *contraction.* Weakening allows us to prove a proposition in the context of irrelevant (unused) assumptions, while contraction allows us to use a premise an arbitrary number of times. Because of this feature, linear logic has been called a "resource-conscious logic", since the premises of a sequent must appear exactly as many times as they are used. If these two rules were added to the logic, then the rules for $\otimes$ and $\wedge$ would be inter-derivable, and we would lose the distinction which linear logic makes between these two "flavours" of *and.* $A \otimes B$ is a resource consisting of exactly one resource A and one resource B: by contrast, $A \wedge B$ has the potential to be either a resource A *or* a resource B, but cannot be both.

Dropping weakening and contraction decreases the expressibility of the logic in some ways. We can, however, regain their power in a controlled way by using the "of course" operator, (a modal operator) which has the following proof rules:

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}(\text{Cont}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}(\text{Der})$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}(\text{Weak}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}(!R)$$

These rules are together equivalent to the single rule

$$\frac{}{!A \vdash \mathbf{1} \wedge A \wedge (!A \otimes !A)}(!)$$

From this rule we see that to assert $!A$, we must be able to make arbitrarily many (that is, zero or more) assertions of A.

For any choice of the logical constant $\perp$, we can derive negation in the usual way by defining

$$A^{\perp} = A \multimap \perp.$$

3 Constructing a Quantale from a Net

Two sorts of model have been given for the linear logic proof rules presented above, the categorical and the algebraic. The phase semantics of [Gir86], an alegebraic model, is an example of a more general construction, a quantale. Several people ([AV88], [Yet]) have shown that quantales give an algebraic semantics to linear intuitionistic logic just as complete Heyting algebras do for intuitionistic logic.

We start with a definition of a quantale. This differs slightly from the original definition of Mulvey [Mul86], which did not assume commutativity and required idempotency.

Definition 3.1
A **quantale** *is a complete semi-lattice Q together with an associative, commutative binary operation $\otimes$ and constant* $\mathbf{1}$ *such that*

- *for all elements A of Q, $A \otimes \mathbf{1} = A$ and*
- *for any indexing set I and any $A \in Q$, $A \otimes \bigvee_{i \in I} B_i = \bigvee_{i \in I} (A \otimes B_i)$*

Following Niefield and Rosenthal [NR88], we define

Definition 3.2 *A* **closure operator** *on a quantale Q is a map $j : Q \to Q$ such that*

- $a \leq b \Rightarrow j(a) \leq j(b)$ *(j preserves order),*
- $a \leq j(a)$*, (j increasing), and*
- $j(j(a)) = j(a)$ *(j idempotent).*

Definition 3.3 *A* **quantic nucleus** *on a quantale Q is a closure operator $j : Q \to Q$ such that*

- $j(a) \otimes j(b) \leq j(a \otimes b)$.

Then we can prove (see [NR88])

Theorem 3.4 *If $j : Q \to Q$ is a quantic nucleus, then $j(Q)$ is a quantale with $a \otimes_j b = j(a \otimes b)$, and $j : Q \to Q$ is a quantale homomorphism.*

3.1 A Quantale of Markings, M

Firstly we shall construct a quantale M in which elements are just sets of markings of a given Petri net N. We consider a net N to be specified as a set *Places* of places, a set T of transitions and a flow relation $F \subseteq (Places \times T) \cup (T \times Places)$. The quantale we construct has no connection with the net N other than the fact that its elements are sets of markings of N, and does not represent the behaviour of the net in any way. Next we shall show that we can build a quantale M_{N}, which is a quotient of M and which *does* represent the behaviour of the net N.

Consider the operation of multiset union on multisets of places, which we shall denote by $+$.

Definition 3.5 *A marking of a Petri net* N *with set of places Places is a multiset, that is, a function $m : Places \to \omega + 1$* [1] *which assigns to each place the number of tokens allocated to it.*

Definition 3.6 *We define addition of markings to be multiset union, that is, for markings $m_1 : Places \to \omega + 1$ and $m_2 : Places \to \omega + 1$ their sum $(m_1 + m_2)$ is such that*

$$\forall P \in Places.((m_1 + m_2)(P) = m_1(P) + m_2(P)).$$

[1] i.e. Nat $\cup\{\omega\}$

If we consider + acting in the obvious way on sets of markings, so

$$P + Q = \{p + q \mid p \in P \text{ and } q \in Q,\}$$

then it is easy to show

Lemma 3.7 *+ is a commutative monoid on sets of markings.*

Proof: Associativity, closure and commutativity of multiset union follow immediately from the corresponding properties of integer addition.
It is clear that + on markings has as unit the constant zero marking $\mathbf{0}$ given by $\forall P \in Places.(\mathbf{0}(P) = 0)$.
On sets of markings, the unit of + is $\{\mathbf{0}\}$. □

Consider M defined as follows:

Definition 3.8

- *elements of M are sets of markings of* N,
- *the ordering on M is subset inclusion,*
- *the top element of M, T_M is the set of all possible markings of* N,
- *the bottom element of M, F_M is the empty set,*
- *the monoid operation is +, and*
- *the unit of + is $\{\mathbf{0}\}$.*

Proposition 3.9 *M is a quantale.*

Proof: Immediate. □

We shall now construct the quantale M_{N} which corresponds to the net N, with no restriction on the set of initial markings to be considered. Once the quantale is built, we have encoded all the information that was included in both the Petri net and its initial markings.
Later we shall show that restricting the set of initial markings in certain ways leads to smaller (and hence more tractable) quantales. The flow relation F of a Petri net induces a multirelation $\rightarrow$ on multisets of elements of *Places*, which we shall call the derivability relation for the net N. We obtain $\rightarrow$ from F by

Definition 3.10 *Let m and m' be markings of a net* N. *Then*

$$m \rightarrow m' \text{ iff } for\ i \in \{1, \cdots n\}\ \exists t_i \in T\ \exists\ markings\ m_i.(m\ F\ t_1\ \&\ t_1\ F\ m_1\ \& \cdots \&\ t_n\ F\ m_n = m')$$

In what follows we shall use the derivability relation rather than the flow relation.

We define a pre-order on the markings of a given net N as follows:

Definition 3.11 *Given markings m_1 and m_2 of a net* N*, we say $m_1 \leq_m m_2$ exactly when the marking m_1 is a reachable marking of the net* N *marked with m_2,* i.e.

$$m_1 \leq m_2 \text{ iff } m_2 \rightarrow m_1.$$

(The subscript m indicates that this is an order on *markings*).

As a result of the linear behaviour of markings of Petri nets, it is easy to show

Lemma 3.12 *If m_1, m_1', m_2 and m_2' are markings of a net such that $m_1 \leq_m m_2$ and $m_1' \leq_m m_2'$ then $(m_1 + m_1') \leq_m (m_2 + m_2')$.* □

We extend the ordering $\leq_m$ to sets of markings as follows:

Definition 3.13 *Given two sets, A and B, whose elements are markings of a given net* N*, we say $B \leq_S A$ exactly when every marking in B is reachable from some marking in A. That is,*

$$B \leq_S A \text{ iff } \forall b \in B \;\; \exists a \in A.(b \leq_m a).$$

Definition 3.14 *We define forwards closure under evolution, $\downarrow^m$ as a map between sets of markings by*

$$\downarrow^m \{A\} = \{m \mid \exists a \in A.(a \leq_m m)\}$$

Thus $\downarrow^m \{A\}$ is the downwards closure of the set A with respect to the ordering $\leq_m$.

Remark 3.15

- *For any sets of markings A and B, $A \subseteq B \Rightarrow A \leq_S B$.*
- *On sets which are downwards closed under $\leq_m$, $\leq_S$ coincides with $\subseteq$, the inclusion ordering.*

Proposition 3.16 $\downarrow^m$: *$M \rightarrow M$ is a quantic nucleus.*

Proof: $\downarrow^m$ is a closure operator:

for all sets of markings A and B,

- $\downarrow^m A$ is clearly increasing,
- $\downarrow^m$ preserves order: $A \leq_S B \Rightarrow A \leq_S \downarrow^m B$ (as $\downarrow^m$ increasing)
  $\Rightarrow \downarrow^m A \leq_S \downarrow^m B$ (since $\downarrow^m B$ is downwards closed)

and

- $\downarrow^m$ is clearly idempotent.

It remains to show that $\downarrow^m (A) + \downarrow^m (B) \leq \downarrow^m (A+B)$.

This follows essentially from Lemma 3.12, since

$$\begin{aligned}\downarrow^m (A) + \downarrow^m (B) &= \{m \mid \exists a \in A.(m \leq_m a)\} + \{m \mid \exists b \in B.(m \leq_m b)\} \\ &\subseteq \{p \mid \exists m \in (A+B).(p \leq_m m)\} \\ &= \downarrow^m (A+B)\end{aligned}$$

Using Remark 3.15, we have $\downarrow^m (A) + \downarrow^m (B) \leq \downarrow^m (A+B)$.

□

Applying Theorem 3.4, we see that the image of M under $\downarrow^m$ is a quantale, in which

- elements are sets of markings of the net N closed under evolution,
- the ordering is subset inclusion, $\subseteq$,
- the top element $\mathrm{T}_{\downarrow^m(M)}$ is the set of all possible markings of N,
- the bottom element $\mathrm{F}_{\downarrow^m(M)}$ is the empty set,
- the monoid operation $\otimes$ is given by $A \otimes B = \downarrow^m (A + B)$,
- the unit of $\otimes$ is $\downarrow^m \{\mathbf{0}\}$.

Note that commutativity of $\otimes$ follows from commutativity of $+$.

We shall denote the quantale $\downarrow^m (M)$ by M_{N}. The quantale M_N has as objects those sets of markings of the net N which are downwards closed under the order $\leq_m$ (in other words, those sets which are forwards closed under evolution of the net). This is why M_{N} describes the behaviour of the net N, while M does not. Remark 3.15 shows that we have chosen our objects in such a way that the ordering $\leq_S$, which represents the behaviour of the net, is reduced to simple subset inclusion, $\subseteq$.
In what follows, where it is possible without causing confusion, we shall use M_N to refer both to the quantale representing the net N, and to its underlying set.

In M_N, we have encoded all the structure of the Petri net N algebraically. The behaviour of the net can now be examined without reference to specific transitions, and certain aspects of its structure (in the sense of patterns of behaviour) may become more apparent when it is viewed in this way. An example of such a pattern is given in Section 4.

3.2 Backwards closed sets of markings

Consider the map $\uparrow^m$ between sets of markings defined by

$$\uparrow^m A = \{m \mid \exists a \in A.(m \to a)\}$$

It is easy to see that $\uparrow^m$ is also a quantic nucleus. The quantale $\uparrow^m (M)$ is the quantale discussed in [WE89], whose objects are sets closed under backward evolution of the net, and indicate what resources are needed for the net to evolve to a given marking.

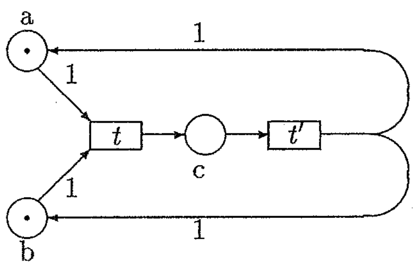
4 Traps

We now explore a simple instance of structure in the quantale which expresses certain properties of the corresponding net rather neatly.

Consider the atoms of the lattice $\langle M, \leq \rangle$: that is, those elements A of the lattice for which $X \leq A \Rightarrow (X = \perp \text{ or } X = A)$.
The atoms of form $\{A,\ \phi\}$ correspond exactly to traps in the net, that is, to markings (or possibly, sets of markings) from which the net can never escape to a different marking. This

is because the lattice atoms are only greater than $\perp$ and so have no potential for any other evolutions.

The lattice atoms with more than one non-trivial element correspond to cyclic states in which the behaviour of the net has stabilised to the point where it cycles repreatedly through the finite number of states contained in the atom. Consider, for example, the net with two transitions shown below:



This net oscillates between the two markings C and $A \otimes B$, and never halts. Therefore the set $\{A \otimes B,\ C,\ \phi\}$ is an atom in the quantale corresponding to this net. For convenience, we shall say that when a net has reached such a stable cyclic state, its evolution has terminated in state p. Here, $p = \{A \otimes B,\ C,\ \phi\}$.

There are some interesting properties of sets of atoms which enable us to make statements about the possible behaviour of a net:

1. If a is atomic then for any marking m,

$$\downarrow^m \{m\} \wedge a = \begin{cases} \{\mathrm{a}\} & \text{iff } m \text{ can terminate in marking } a \\ \phi & \text{otherwise} \end{cases}$$

2.

$$\bigwedge\{q \wedge a \mid q \in \downarrow^m \{m\} \text{ and } a \text{ atomic}\} = \begin{cases} p & \text{if } p \text{ is the only state in which } m \text{ can terminate} \\ \phi & \text{if } m \text{ can terminate in more than one state} \end{cases}$$

3.

$$\bigvee\{q \wedge a \mid q \in \downarrow^m \{m\} \text{ and } a \text{ atomic}\} = \{p \mid p \text{ atomic, and } m \text{ can terminate in } p\} \cup \{\phi\}$$

5 Interpreting Linear Logic in a Net-Quantale

In the quantale corresponding to a net, we can interpret the linear logic connectives $\oplus$, $\wedge$, $\otimes$, $\multimap$, $(-)^{\perp}$ and !, and also the constants T, $\perp$, **1** and F.

The connective $\otimes$ is interpreted by the monoid operation on the quantale, and the constant **1** by its unit. By analogy with Heyting algebras, we interpret

- linear entailment by the ordering $\subseteq$ on the net-quantale,
- T and F by the top and bottom elements of the lattice, repectively, and
- implication ($\multimap$) by defining

  $$A \multimap B \triangleq \bigvee\{C \mid C \otimes A \subseteq B\}$$

  This gives us the usual adjunction

  $$C \otimes A \subseteq B \text{ iff } C \subseteq A \multimap B$$

  which we expect, since linear $\otimes$ is here playing the role of the intuitionistic *and.*

We shall assume that places interpret the atomic propositions of the linear calculus. Our interpretation is then parametric in the interpretation of these atomic propositions. We shall write m_A for the marking which consists of a single token at the place A. The interpretation of linear logic in M_{N} is as follows:

- $[\![A]\!] = \{m \mid \exists a \in A.(a \rightarrow m)\}$ for an atomic proposition A
- $[\![\mathrm{T}]\!] = \{m \mid m$ a possible marking of the net N$\} = \mathrm{T}_{M_N}$
- $[\![\mathrm{F}]\!] = \phi = \mathrm{F}_{M_N}$
- $[\![\mathbf{1}]\!] = \downarrow^m \{\mathbf{0}\}$
- $[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!] = \downarrow^m \{a + b \mid a \in [\![A]\!]$ and $b \in [\![B]\!]\}$
- $[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]$
- $[\![A \oplus B]\!] = [\![A]\!] \cup [\![B]\!]$
- $[\![A \multimap B]\!] = \bigcup\{C \mid [\![C \otimes A]\!] \subseteq [\![B]\!]\}$

With a suitable interpretation of the linear logic constant $\perp$, we can now interpret $A^{\perp}$ in the quantale in the usual way by putting $[\![A^{\perp}]\!] = [\![A \multimap \perp]\!] = \bigvee\{C \mid [\![C \otimes A]\!] \subseteq [\![\perp]\!]\}$.

We define semantic entailment in the quantale by

$$A_1 \otimes \cdots \otimes A_n \models A \text{ iff } [\![A_1]\!] \otimes \cdots \otimes [\![A_n]\!] \subseteq [\![A]\!]$$

The idea behind this interpretation is that a marking m is denoted by its consequences, or in other words, by the set of all things we could gain, if we knew that we had m. Thus anything gained by having an element of $[\![A \otimes B]\!]$ must be a possible gain when we have some $a \in [\![A]\!]$ and some $b \in [\![B]\!]$ *at the same time.*

Also, any consequence of having some resource which came from a non-deterministic choice between A and B must be either a consequence of having some element of A or of having some element of B. Accordingly, we interpret $A \oplus B$ as the union of consequences of A and consequences of B.

Similarly, whenever we have a consequence x of $A \wedge B$, we can make a determined choice of A and we know that x will be a consequence of A. Similarly, we know that if we choose B, x must be a consequence of our choice. We must therefore insist that x be a consequence of both A and B, and so we interpret $A \wedge B$ by the intersection of consequences of A and B.

Our interpretation of $A \multimap B$ expresses the property of implication that no consequence of $A \multimap B$ can give any more gain when taken in conjunction ($\otimes$) with some consequence a of A than could be gained from an appropriate b in $[\![B]\!]$.

The interpretations of **1**, T and F follow simply from their required behaviour as constants of the logic. We can gain nothing more from the set of all possible markings than what was already possible, and this explains the choice of interpretation for T. Also, if we have an element of $[\![\mathrm{F}]\!]$, we should be able to deduce even impossible markings - as there can be no such element, F is interpreted by the empty set. $[\![\mathbf{1}]\!]$ is just the set of all things that can be gained from nothing, as we would expect.

In [WE89], Winskel and Engberg show that the interpretation of "of course" A should be

$$[\![!A]\!] = \bigcup \{C \in M_{\mathrm{N}} \mid C \text{ is a postfixed point of } f_A\},$$

where $f_A : M_{\mathrm{N}} \to M_{\mathrm{N}}$ is the function given by

$$x \mapsto \mathbf{1} \wedge [\![A]\!] \wedge (x \otimes x).$$

This follows a suggestion of Girard in [GL87].

We take $\models A$ to abbreviate $\mathbf{1} \models A$. It is easy to show

Proposition 5.1

1. $\models A$ *iff* $\mathbf{0} \in [\![\mathrm{A}]\!]$
2. $A \models B$ *iff* $[\![\mathrm{A}]\!] \subseteq [\![\mathrm{B}]\!]$ *iff* $\models A \multimap B$
3. $\models m \multimap m'$ *iff* N *can evolve from marking* m' *to marking* m.

Proposition 5.2 *The quantale* M_{N} *with the above interpretation is sound with respect to the rules of linear logic given in Section 2, i.e.*

$$\Gamma \vdash A \Rightarrow \Gamma \models A.$$

Proof: By case analysis. □

We can now make assertions about the behaviour of the net N whose behaviour has been encoded in the quantale. For instance,

- $\models m$ asserts that marking m can evolve to the empty marking, $\mathbf{0}$,
- $\models (A \otimes B) \multimap C$ asserts that from a token a place C, the net N can evolve to the marking which consists of a token at place A and a token at place B, and
- $\models (m_1 \wedge m_2) \multimap m$ asserts that the marking m can evolve to marking m_1 and also to marking m_2.

6 Linear Negation

We now suggest one possible choice for the interpretation of the logical constant $\perp$, and show how it can be used to make furtherassertions about the behaviour of a net. In general, we can use negation to assert things which a net *cannot* do, rather than things which it can do.

let M_F be an arbitrary set of markings. Now put

$$\perp = \{m \mid \forall m' \in M_F.(m \not\rightarrow m')\}.$$

Then (writing $m \not\rightarrow M_F$) for $\forall m' \in M_F.(m \not\rightarrow m')$), we have

$$[\![A^\perp]\!] = \{m \mid \forall a \in [\![A]\!] .((m + a) \not\rightarrow M_F)\}$$

In particular, $\models A^\perp$ iff $(\forall a \in [\![A]\!] .(a \not\rightarrow M_F))$
iff $[\![A]\!] \cap M_F = \phi$.

So if $\models A^\perp$ then there is no marking in A from which the net N can evolve to any marking in M_F.

This enables us to make negative assertions about a net's behaviour, and hence to specify safety properties of a net. It is also possible to assert that there is no marking reachable from A in which a particular multiset is marked. For instance, if we put

$$M_F = \{B^n \mid n \textit{ an integer}\}$$

then $\models A^\perp$ asserts that there is no marking reachable from A in which place B is marked in any multiplicity.

7 Equivalences on Nets

We have seen that every net generates a quantale. We would like to know under what circumstances two nets generate the same quantale.

Two nets N_0 and N_1 can only generate the same quantale if they have the same set of places. We shall say that N_0 and N_1 with the same set of places are **equivalent** whenever they make exactly the same set of propositions valid, in the sense of Section 5. We shall write $\models_N A$ to mean that A is valid in the quantale generated by N, in the sense of Section 5.

Definition 7.1 *If* N_0 *and* N_1 *have the same set of places, then*

$$N_0 \sim N_1 \text{ iff } (\forall \textit{ propositions } A,\ (\models_{N_0} A) \iff (\models_{N_1} A)).$$

It is easy to see that $\models_N m \multimap m'$ *iff* the net N can evolve from a marking m' to a marking m. It follows that if $N_0 \sim N_1$, then the nets N_0 and N_1 have the same derivability relation. Since the derivability relation $\rightarrow$ of a net is what determines its net-quantale, if $N_0 \sim N_1$ then N_0 and N_1 generate the same quantale.

Naturally, since the derivability relation $\rightarrow$ contains the identity relation and is transitive, any alteration to a net N which does not affect its derivability relation does not change its equivalence class with respect to $\sim$. In particular, we can augment a net N without changing its equivalence class if we add identity transitions (that is, transitions which leave some multiset of tokens unchanged), or compositions of transitions (that is, transitions which transform some multiset of tokens in the same way as some sequence of events). We shall call such composite transitions "short-cuts", and define them as follows:

Definition 7.2 *A transition s is a short cut* iff *whenever the net* N *can evolve from marking m to marking m' by a firing of s, there exists some sequence of firings $s_0; s_1; \cdots; s_n$ which is disjoint from s, and which also takes* N *from the marking m to the marking m'.*

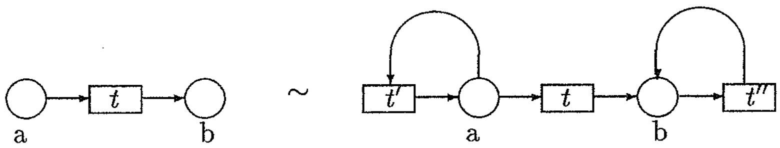
Thus the equivalence $\sim$ identifies with any net N all augmentations of N by identity or short-cut transitions. From a computational point of view, we ignore identity transitions because we are not interested in specifying actions which do not alter the net's state or environment in any way.The only way in which we could distinguish the net with short cuts from that without would be where we had some measure on the number of transitions (or possibly events) by which a given marking is to be reached. This equivalence only means that in this framework we cannot address such issues of computational complexity.

Another feature of this equivalence is that we cannot tell from the quantale the order in which the events of a cycle occur.

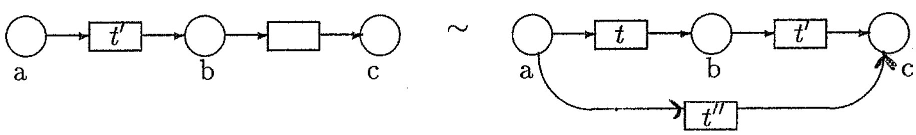
Notice also, that since the quantale is determined by the reachability relation, it contains no information about irrelevant firings (that is, firings which never become enabled).

Examples of equivalent nets

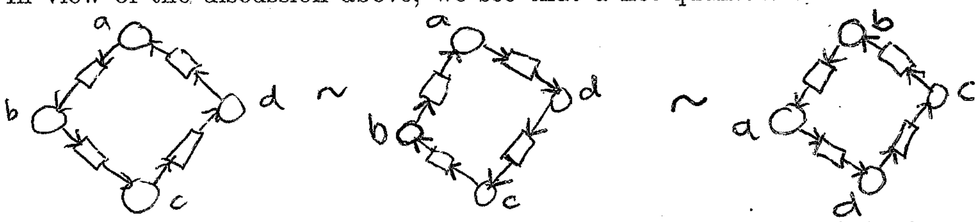
(1) Augmenting with identity transitions:



(2) Augmenting with short-cuts:



(3) Cycles: In view of the discussion above, we see that a net-quantale determines its corresponding net up to the equivalence defined above. In particular, we have the following result:



Proposition 7.3 *Any acyclic, irrelevance-free net without short-cuts is determined uniquely by the corresponding net-quantale.*

8 Restrictions on the top element of a net-quantale

The top element of the quantale M_N constructed above is very large. In particular, if $||Places|| = \alpha$ and no more than β tokens are allowed to occupy a place at any one time, then the class T of possible markings of the net is of cardinality β^α. T may be *finite* (α, β both finite), *countable* (α finite, β countable) or *uncountable* (α countable).

There are ways in which the top element of the quantale can be made smaller, however. In what follows, we construct a smaller quantale $M_{(\mathrm{N},P)}$, which corresponds to a net's behaviour on a subset P of possible markings, where elements of P are called "permitted markings". There are various ways in which the notion of permitted marking may be chosen.

Some possibilities to consider include markings which have:

- no more than n tokens on any one place at once,
- no more than n tokens on a particular place at once,
- no more than n_a tokens on place A, n_b on place B, and so on,
- no more than n tokens shared between some specified set of places,
- no more than n tokens altogether,
- only even numbers of tokens on any place,
- no fewer than n tokens on any place,
- or no fewer than n tokens in the marking altogether.

It turns out that all except the last three of these notions of permitted marking are suitable for constructing quantales.

Notation 8.1 *We write* $p \leq_1 m$ iff *there is some set of transitions which can occur simultaneously, taking net* N *with marking* m *to marking* p *in one step.*

Definition 8.2 *We say there exists a* permitted derivation *of* m_1 *from* m_2 iff *for* $i \in \{1, ...n\}$, $\exists p_i.(m_1 = p_1 \leq_1 p_2 \cdots \leq_1 p_n = m_2)$, *where each* p_i *is permitted.*

Definition 8.3 *We now say* $m_1 \leq_{m_p} m_2$ iff $\exists$ *a permitted derivation of* m_1 *from* m_2.
$\leq_{S_p}$ *is the obvious extension of* $\leq_m$*, as before.*

We next consider under what circumstances it is possible to construct the desired quantale $M_{(N,P)}$, where we restrict our consideration to permitted markings. It turns out that a straightforward condition on the notion of permitted markings allows the construction to go through as before. We insist that our notion of permitted marking is such that

$$(a + b) \textit{ permitted } \Rightarrow a \textit{ permitted} \text{ and } b \textit{ permitted}$$

We shall say that any set of multisets which has this property is *closed under subtraction.* Notice that, provided the set of permitted markings is non-empty, the marking $\mathbf{0}$, which generates the unit of $\otimes$, will always be permitted, since whenever m is permitted, $m+\mathbf{0}$ must be permitted.

As is easy to see, most of the notions of permitted marking suggested at the start of this section are closed under subtraction. In particular, this is the case for nets where markings are restricted to a maximum number of tokens on a place at any time, and so to the widely used *safe* nets (see [Win87]), whose markings have no more than one token on a place at any time.

We now construct $M_{(\mathrm{N},P)}$, and show that it is a quantale.

Definition 8.4 *In the same way as in Section 3,*

- $M_{(\mathrm{N},P)}$ *has as objects those sets of permitted markings which are downward closed under* $\leq_{m_p}$*, these objects being ordered by* $\subseteq$.
- $A \wedge B = A \cap B$ *and* $A \vee B = A \cup B$ *for all* $A, B \in M_{(\mathrm{N},P)}$.
- $\mathrm{T}_{M_{(\mathrm{N},P)}}$ *is the set of all permitted markings and their permitted derivatives, while* $\mathrm{F}_{M_{(\mathrm{N},P)}}$ *is just the empty set.*

It is easy to see that

Proposition 8.5

- *On* $M_{(\mathrm{N},P)}$, $\leq_S$ *coincides with inclusion,* $\subseteq$*, and*
- $M_{(\mathrm{N},P)}$ *is a complete lattice under* $\subseteq$. □

We now define the appropriate notion of $\otimes$ restricted to $M_{(\mathrm{N},P)}$ by

Definition 8.6

$$A \otimes_p B = \downarrow^{m_p} \{a + b \mid a \in A \text{ and } b \in B \text{ and } (a+b) \textit{ permitted}\} \qquad (*)$$

Lemma 8.7 $\otimes_p$ *is a commutative monoid operation on* $M_{(\mathrm{N},P)}$.

Proof:

- Closure is easy to show.
- For associativity, notice that

$$\begin{aligned} A\otimes_p(B\otimes_p C) &= \{p \mid \exists a \in A\, \exists r \in \{r \mid \exists b \in B\, \exists c \in C.((b+c) \geq_{m_p} r)\} \text{ s.t. } ((a+r) \geq_{m_p} p)\} \\ &= \{p \mid \exists a \in A\, \exists b \in B\, \exists c \in C.\ (((a+b+c) \geq_{m_p} p) \text{ and } (b+c) \textit{ permitted})\} \end{aligned}$$

Similarly,

$$(A \otimes_p B) \otimes_p C = \{q \mid \exists a \in A\, \exists b \in B\, \exists c \in C.\ (((a+b+c) \geq_{m_p} p) \text{ and } (a+b) \textit{ permitted})\ \}$$

For equality of $A \otimes_p (B \otimes_p C)$ and $(A \otimes_p B) \otimes_p C$, we need

$$(a+b+c) \textit{ permitted} \text{ and } (a+b) \textit{ permitted} \iff (a+b+c) \textit{ permitted} \text{ and } (b+c) \textit{ permitted}$$

For this, it suffices if our notion of permitted marking is such that the set of all permitted markings is closed under subtraction.
This is precisely why the condition on permitted markings was introduced above.

- Commutativity follows as before from the commutativity of multiset addition.
- As we noted above, **0** is permitted (except in the trivial case), and so the unit of $\otimes_p$, as before, is $\downarrow^{m_p} \{\mathbf{0}\}$ □

It only remains to prove distributivity:

Lemma 8.8

$$A \otimes_p \bigvee B_i = \bigvee (A \otimes_p B_i)$$

Proof:

$$\begin{aligned}\text{Now } A \otimes_p \bigvee B_i &= \downarrow^{m_p} \{a + c \mid a \in A \text{ and } c \in \bigcup B_i \text{ and } (a+c)\ \textit{permitted}\ \} \\ &= \downarrow^{m_p} \{a + b \mid a \in A \text{ and } (b \in B_i \text{ some } i) \text{ and } (a+b)\ \textit{permitted}\ \} \\ &= \{p \mid \exists a \in A\ \exists i \in I\ \exists b \in B_i.((a+b)\ \textit{permitted} \text{ and } (p \leq_{m_p} (a+b)))\}\end{aligned}$$

$$\begin{aligned}\text{Also, } \bigvee (A \otimes_p B_i) &= \bigcup (A \otimes_p B_i) \\ &= \bigcup (\downarrow^{m_p} \{a + b \mid a \in A \text{ and } b \in B_i \text{ and } (a+b)\ \textit{permitted}\ \} \\ &= \bigcup \{p \mid \exists a \in A\ \exists b \in B_i.\ ((a+b)\ \textit{permitted} \text{ and } (p \leq_{m_p} (a+b)))\} \\ &= \{p \mid \exists a \in A\ \exists i \in I\ \exists b \in B_i.\ ((a+b)\ \textit{permitted} \text{ and } (p \leq_{m_p} (a+b)))\}\end{aligned}$$

□

We have now shown that $M_{(\mathrm{N},P)}$ is the net-quantale corresponding to the behaviour of net N under permitted initial markings, where

- Elements of $M_{(\mathrm{N},P)}$ are subsets of P, that is, sets of permitted markings of N which are downwards closed under $\leq_{m_p}$, where $m \leq_{m_p} p$ *iff* there is a permitted derivation of m from p in N,
- The ordering on $M_{(\mathrm{N},P)}$ is $\subseteq$,
- $\mathrm{T}_{M_{(\mathrm{N},P)}}$ is P, the set of all permitted markings of N,
- $\mathrm{F}_{M_{(\mathrm{N},P)}}$ is the empty set,
- The monoid operation $\otimes_p$ on $M_{(\mathrm{N},P)}$ is given by

$$A \otimes_p B = \downarrow^{m_p} \{a + b \mid a \in A \text{ and } b \in B \text{ and } (a+b)\ \textit{permitted}\}, \text{ and}$$

- The unit of $\otimes_p$ is $\mathbf{1} = \downarrow^{m_p} \{\mathbf{0}\}$.

Example

Every irrelevance-free safe net is represented by a quantale where the set P of permitted markings is just those markings in which no place is marked more than once.

It is easy to see that this finite quantale is sound with respect to the linear logic proof rules given in Section 2.

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