

**INSTITUTIONS:
Abstract Model Theory
for Specification and Programming**

by

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INSTITUTIONS: Abstract Model Theory for Specification and Programming*

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Abstract

There is a population explosion among the logical systems used in computer science. Examples include first order logic, equational logic, Horn clause logic, higher order logic, infinitary logic, dynamic logic, intuitionistic logic, order-sorted logic, and temporal logic; moreover, there is a tendency for each theorem prover to have its own idiosyncratic logical system. We introduce the concept of *institution* to formalize the informal notion of "logical system." The major requirement is that there is a satisfaction relation between models and sentences which is consistent under change of notation. Institutions enable us to abstract from syntactic and semantic detail when working on language structure "in-the-large"; for example, we can define language features for building large structures from smaller ones, possibly involving parameters, without commitment to any particular logical system. This applies to both specification languages and programming languages. Institutions also have applications to such areas as database theory and the semantics of artificial and natural languages. A first main result of this paper says that any institution such that signatures (which define notation) can be glued together, also allows gluing together theories (which are just collections of sentences over a fixed signature). A second main result considers when theory structuring is preserved by institution morphisms. A third main result gives conditions under which it is sound to use a theorem prover for one institution on theories from another. A fourth main result shows how to extend institutions so that their theories may include, in addition to the original sentences, various kinds of constraint that are useful for defining abstract data types, including both "data" and "hierarchy" constraints. Further results show how to define institutions that allow sentences and constraints from two or more institutions. All our general results apply to such "duplex" or "multiplex" institutions.

1 Introduction

Recent work in theoretical computer science uses many different logical systems. Perhaps most popular are the many variants of first and higher order logic found in current generation theorem provers. But also popular are equational logic, as used to study abstract data types, and Horn clause logic, as used in "logic programming," e.g., Prolog. More exotic logical systems, such as temporal logic, second order polymorphic lambda calculus, dynamic logic, order-sorted logic, modal logic, continuous algebra, infinitary logic, intuitionistic higher order type theory, and intensional logic have been proposed to handle problems such as concurrency, overloading, exceptions, non-termination, program construction, and natural language. However, it seems apparent that many general results used in the applications are actually *completely independent* of what underlying logic is chosen. In particular, if we are correct that the essential purpose of a specification language is to say how to put (small and hopefully standard) theories together to make new (and possibly very large) specifications [13], then much of the syntax and semantics of specification does not depend upon the logical system in which the theories are expressed; similar considerations apply to implementing a specification, verifying correctness, and programming-in-the-large.

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Also, because of the proliferation of logics of programming and of logic-based programming languages, plus the great expense of implementing tools like theorem provers and compilers, it is useful to know when sentences in one logic can be translated into sentences in another logic in such a way that soundness is preserved. This will allow, for example, using a theorem prover for one logic on (translations of) sentences from another logic, or using a compiler for one logic-based language on (translations of) programs from another. Institutions provide a foundation for approaching these and many other problems in computer science.

One of the most essential elements of a logical system is its relationship of *satisfaction* between its *syntax* (i.e., its sentences) and its *semantics* (i.e., its models); this relationship is sometimes called a “model theory” and classically appear in the form of a Galois connection (as in Section 2.2 below). Whereas traditional model theory assumes a fixed vocabulary, institutions allow us to consider many different vocabularies at once. Informally, an institution consists of

- a collection of signatures (which are vocabularies for use in constructing sentences in a logical system) and signature morphisms, together with for each signature Σ ,
- a collection of Σ -sentences,
- a collection of Σ -models, and
- a Σ -satisfaction relation, of Σ -sentences by Σ -models,

such that when you change signatures (by a signature morphism), satisfaction of sentences by models changes consistently.

The first main result in this paper (Theorem 11) states that any institution whose declarations of notation (as given by signatures) can be glued together will also allow gluing together *theories* (which are collections of sentences) to form larger specifications, using colimits. A second main result (Proposition 23) states that any institution extends to another whose sentences may be either the old sentences, or else new “data constraints” which capture structural induction and are useful in defining abstract data types. Theorem 26 extends this to more general kinds of constraint, including so-called “hierarchy constraints,” and Theorem 24 says that colimits extend from signatures of the original institution to theories over the extended institution. Another result (Proposition 34) shows that a suitable *institution morphism* permits a theorem prover for one institution to be used on theories from another. Another main result (Theorem 36) shows that many institution morphisms preserve structuring operations on theories. This implies that constructions for programming-in-the-large are preserved by certain language translations. Again using the notion of institution morphism, “duplex” institutions permit combining sentences from one institution with constraints from another (Theorem 41); moreover, the duplex institution again inherits colimits from the signature category of the base institution (Theorem 42). Finally, “multiplex” institutions permit whatever combination of sentences and constraints one might desire, provided they are related by morphisms to the same base institution (Section 4.3). Altogether, this gives a very rich and flexible framework that can be used in program specification and logical programming, as well as many other other areas of computer science.

Institutions arose in our research on the specification language Clear [13] under the rather general name of “language” [15]. The present paper adds many new concepts, results, and examples, as well as an improved notation. Barwise’s approach to “abstract model theory” [6] (see also [5]) resembles our work in its intention to generalize basic results in model theory and in its use of elementary category theory; but it differs in being more concrete (for example, its syntactic structures are limited to the usual function, relation and logical symbols) and in focussing on classical results of logic.

Section 2 below gives the basic definitions and results for institutions and theories. Section 3 discusses constraints, while Section 4 considers the use of two or more institutions. Appendix A.1

briefly reviews universal algebra, emphasizing the results needed to show that equational logic is indeed an institution, while Appendix A.2 considers the (many-sorted) first order, first order with equality, Horn clause, and conditional equational institutions.

1.1 Methodology and Logical Systems

Systematic program design requires a careful specification of the problem to be solved. But experience in software engineering shows that there are major difficulties in producing consistent, rigorous specifications that adequately reflect users' requirements for complex systems. We suggest that these difficulties can be ameliorated by making specifications as modular as possible, so that they are built from small, understandable pieces, many of which may be used repeatedly (for example, those defining concepts like "ordering," "list," or "file"). Modern work in programming methodology supports the view that abstraction, and in particular data abstraction, is a useful way to obtain such modularity, and that parameterized (also called "generic") specifications can lead to further improvements.

One way to apply these ideas is through a specification language that supports putting together parameterized abstractions. Whereas a specification written directly in a logical system is an unstructured, and possibly unmanageably large, collection of sentences, a suitable specification language can make it much easier to write and to read specifications, especially for large systems. Specification languages that support modularity include Clear [13, 15, 16], OBJ [27, 47], Z [1, 52, 84], Act One [22], Act Two [24], ASL [77], and Extended ML [75]. We also suggest that modularity may be useful in proving theorems about specifications, for example, in proving that a given program actually satisfies its specification. Finally, exactly the same structuring mechanisms can be used to achieve the exactly same advantages in programming languages that are rigorously based upon some formal system of logic; we shall call such languages *logical programming languages*.

In order for a specification written in a given language to have a precise meaning, it is necessary for that language to have a precise semantics. (This may seem obvious, but the fact is that many specification languages lack such a semantics.) Part of that semantics will be an underlying logical system, which must have certain properties to be useful for this task. These include suitable notions of model and sentence, and a satisfaction relationship between sentences and models that is invariant under change of notation.

Signature morphisms play an important role in structuring specifications. Let us assume for concreteness of exposition that the signatures have sorts and operators, and then consider some specific structuring mechanisms. First, we may build a more complex specification by adding new sorts and operators to an existing signature; then the inclusion of the original signature into the extended signature is an "enrichment" signature morphism. Second, we may wish to use such an enrichment not just on one specification, but on a whole class of specifications. This leads to parameterized specifications. For instantiation, the parameter sorts and operators are bound to particular sorts and operators by a "binding" signature morphism. Third, a large specification may have name clashes: two subspecifications may happen to use the same sort or operator names. These can be eliminated by signature morphisms that define renamings. Enrichment, binding and renaming do not raise deep logical problems, but they are nonetheless important for modular structure. Using institutions, we can define such features without making a commitment to any particular logical system. Moreover, the task of giving a semantics for the language is also simplified. We feel that these considerations justify an attempt to deal with logical systems in a general way, free of the entanglements of any particular syntax and semantics.

A specification language is not a programming language. Thus, the denotation of an Algol text is a function, but the denotation of a specification text is a *theory*, that is, a collection of sentences *about* programs. Of course, a theory also has a denotation, which is the collection of all models that satisfy the sentences in the theory, and this should be taken as the ultimate denotation of

a specification text. Programmers construct such models using programming languages. For a (pure) logical programming language, the specification is also a program, so there is no need to verify that these two agree.

Despite this distinction, it may be useful to view a specification language like Clear [13, 15, 16] as a functional language with types. Its values are specifications, its functions are specification constructing operations, and its types denote classes of specifications. Since theories are used for specifications and theory morphisms for bindings, we have the following correspondence:

value	theory
function	theory morphism
application of a function to a value	pushout of theory morphisms
type	theory
values of type T	theories with a morphism from T

Clear was designed to work with any institution. To provide initiality for abstract data types, we can further elaborate values to be “theories with duplex constraints,” as discussed in Section 4.2. This elaboration makes sense for any institution, and the elaborated theories also form an institution.

1.2 Related Work and Applications

Although this paper is a direct descendent of the first paper on institutions [36], delays in publication have led to the situation that the literature on institutions and their applications is now large enough that it would be awkward to undertake more than the following brief survey of some representative results and papers.

We begin with some of our own work. The institution concept was introduced in [15] to help define the semantics of Clear [13]. Thus, Clear can be used to build large specifications from small, reusable theories over any logical system. Clear’s approach to modularity is called **parameterized programming** [31]. Parameterized programming is implemented in OBJ3 [47], which can actually be considered an implementation of Clear, and it has also been used in designing the logical programming languages Eqlog [42] and FOOPS [43], the latter of which is object-oriented. Parameterized programming can even be applied to a conventional imperative programming language like Ada, where it provides module interconnection capabilities far beyond those in the language itself; see LIL [32]. Each of these language designs relies on the machinery in this paper, especially Theorems 11 and 24, but instantiated with different institutions.

In [39] we developed so-called “charters” and “parchments” as easier ways to generate institutions, exploiting the fact that the syntax of a logical system forms an initial algebra. In particular, the difficulty of checking the Satisfaction Condition (in Definition 1 below) is avoided, since it is automatically satisfied. A quite different formalization of the intuitive notion of logical system, axiomatizing the category of theories as well as that of signatures, is given in [37].

Other researchers have produced much interesting work. Mayoh [66] proposed generalizing institutions so that satisfaction is no longer just Boolean-valued, and pointed out that this generalization would have applications to database systems, where the answer to a given query (considered as a sentence) for a given model might be some complex proposition, or set of values, rather than just *true* or *false*. This generalization is somewhat further developed in [39]. Mayoh [66] also suggested some applications to the semantics of programming and natural languages.

Two other specification languages based on institutions are ASL [77] and Extended ML [75], both due to Sannella and Tarlecki. Extended ML is integrated with the widely used functional programming language ML [50]. A number of topics in theoretical computer science have been abstracted to the level of institutions, including observational equivalence by Tarlecki and Sannella

[76] and data representations (usually called "implementations" in the abstract data type literature) by Beierle and Voss [7, 75]. Tarlecki has also studied free constructions in institutions [86, 87], and moreover has shown that certain model theoretic results generalise to institutions, including equivalence of the Craig Interpolation and the Robinson Consistency Theorem under certain conditions [85]. Probably much more could be done along these lines. Lowry has applied institutions to problem reformulation [63], and Mosses has used institutions in his elegant development of unified algebra [70] for the denotational semantics of programming languages. The machinery of the present paper has also influenced work that does not use institutions directly. In particular, our abstract notion of constraint (Section 3) has been studied by several authors in the context of algebraic specification, that is, for the equational institution, e.g., [23].

Several authors have felt the need for an abstraction of logical system that is based on deduction rather than satisfaction. Fiadeiro and Sernadas [25] introduce " π -institutions," based on a consequence relation like that of Tarski's deductive systems [89], and show that the main results of the present paper also hold in that setting. [25] also discusses some applications to conceptual modelling and knowledge representation, as do [26, 81], the latter also mentioning some interesting applications to database theory.

Logical frameworks [49, 4] are another formalism for defining logics at the same level of generality as institutions. Harper, Sannella and Tarlecki [51] define a "logical system" as a family of consequence relations indexed by signatures, obtaining a notion equivalent to that of Fiadeiro and Sernadas, but applied to logical frameworks. Poigné [72] studies "foundations," where the set of sentences associated with a signature becomes a category indexed by sets of variables; this gives a double indexing, over the constants in the signature and over variables. The extra complexity may or may not be worthwhile. Meseguer [68] synthesizes consequence and satisfaction, and uses it in a careful explication of the notion of logical programming language. This builds on Dana Scott's axiomatization of deduction [79], rather than on the more category theoretic approach (in which proofs are morphisms) advocated by Lambek and (Peter) Scott [57] which we suggest as a generalization of institutions for this purpose in Section 2.5.1.

1.3 Prerequisites

Although relatively little category theory is needed for most of this paper, we have not resisted the temptation to add some more arcane remarks for those who may be interested. We must assume the reader is already acquainted with the notions of category, functor and natural transformation. Occasional non-essential remarks use adjoint functors. There are several introductions to these ideas, including [17, 2, 48], and for the mathematically more sophisticated [58]. Familiarity with the initial algebra approach to abstract data types is helpful, but probably not necessary. Colimits are briefly explained in Section 2.4. A general motivation for the use of category theory in Computer Science is given in [33].

By way of notation, categories are boldface, $|C|$ denotes the class of objects of C , $f;g$ denotes the composition of morphisms f and g in diagrammatic order, 1_A denotes the identity at an object A , \cong denotes isomorphism, and C^{op} denotes the opposite category of C (see [58], II.2). The notation that we use for general algebra is developed in Section A.1, while Section A.2 develops first order logic.

1.4 Acknowledgements

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2 Institutions

An institution consists of a category of signatures such that associated with each signature are sentences, models, and a relationship of satisfaction that, in a certain sense, is invariant under change of signature. Two familiar examples of this setup are equational logic and first order logic (or model theory). In equational logic, a signature Σ declares the function symbols that are available, Σ -sentences are equations using these function symbols, and Σ -models are Σ -algebras. In first order logic, signatures in addition give relation symbols, sentences are the usual first order sentences, and models are the usual first order structures. In both cases, satisfaction is the familiar relation. The appendix discusses the many-sorted variants of these two examples in some detail.

2.1 Definition and Examples

The essence of the institution notion is that a change of signature (by a signature morphism) induces "consistent" changes in sentences and models, in a sense made precise by the "Satisfaction Condition" in Definition 1 below. This goes a step beyond Tarski's classic "semantic definition of truth" [90], and also generalizes Barwise's "Translation Axiom" [6]. The wide range of consequences, and the fact that even for equational logic, the Satisfaction Condition is not entirely trivial, suggest that this step has some substance. Moreover, it is a basic and familiar fact that the truth of a sentence (in logic) is independent of the symbols chosen to represent its functions and relations. This can be summed up in the slogan

Truth is invariant under change of notation

It is also fundamental that sentences translate in the same direction as the change of notation, whereas models translate in the opposite direction. Since reversing the direction of morphisms gives a contravariant functor, the definition below uses Cat^{op} , the opposite of the category of categories.

Definition 1 An institution I consists of

1. a category Sign , whose objects are called **signatures**,
2. a functor $\text{Sen}: \text{Sign} \rightarrow \text{Set}$, giving for each signature a set whose elements are called **sentences** over that signature,
3. a functor¹ $\text{Mod}: \text{Sign} \rightarrow \text{Cat}^{op}$ giving for each signature Σ a category whose objects are called Σ -**models**, and whose arrows are called Σ -**(model) morphisms**, and

¹Here, and at other places in this paper, some readers may have questions about set-theoretic foundations, since Cat clearly needs to include "large" categories. In fact, we stay well away from anything genuinely problematical, and nearly any foundation that has been proposed for category theory will do, in particular, the "hierarchy of universes" discussed e.g., by MacLane [58], in Section I.6.

4. a relation $\models_{\Sigma} \subseteq |\mathbf{Mod}(\Sigma)| \times \mathit{Sen}(\Sigma)$ for each $\Sigma \in |\mathbf{Sign}|$, called Σ -satisfaction,

such that for each morphism $\phi: \Sigma \rightarrow \Sigma'$ in \mathbf{Sign} , the **Satisfaction Condition**

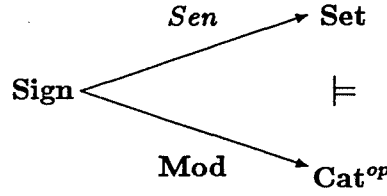
$$m' \models_{\Sigma} \mathit{Sen}(\phi)(e) \text{ iff } \mathbf{Mod}(\phi)(m') \models_{\Sigma} e$$

holds for each $m' \in |\mathbf{Mod}(\Sigma')|$ and each $e \in \mathit{Sen}(\Sigma)$. We will write $\phi(e)$ or even ϕe for $\mathit{Sen}(\phi)(e)$, and $\phi(m')$ or $\phi m'$ for $\mathbf{Mod}(\phi)(m')$; also we will drop the signature subscripts on the satisfaction relation when it is not too confusing. \square

These conventions are used in the following condensed form of the Satisfaction Condition,

$$m' \models \phi e \text{ iff } \phi m' \models e.$$

The following picture may help in visualizing these relationships:



$$\begin{array}{ccccc}
 \Sigma & & \mathbf{Mod}(\Sigma) & \models_{\Sigma'} & \mathit{Sen}(\Sigma) \\
 \sigma \downarrow & & \uparrow \mathbf{Mod}(\sigma) & & \downarrow \mathit{Sen}(\sigma) \\
 \Sigma' & & \mathbf{Mod}(\Sigma') & \models_{\Sigma} & \mathit{Sen}(\Sigma')
 \end{array}$$

The appendix shows that a number of logical systems satisfy Definition 1, including equational logic, first order logic, Horn clause logic, Horn clause logic with equality, and first order logic with equality. We note that it can be non-trivial to establish the Satisfaction Condition. For some purposes, Definition 1 can be simplified by replacing the functor $\mathbf{Mod}: \mathbf{Sign} \rightarrow \text{Cat}^{op}$ by $\mathit{Mod}: \mathbf{Sign} \rightarrow \text{Set}^{op}$, where $\mathit{Mod}(\Sigma)$ is the collection of all Σ -models; the two versions of the definition are thus related by the equation $\mathit{Mod}(\Sigma) = |\mathbf{Mod}(\Sigma)|$. Indeed, this was our original version [15]. Some reasons for changing it are: first, it is more consistent with the categorical point of view to consider morphisms of models along with models; and second, we want every liberal institution to be an institution, rather than just determine one (liberal institutions are discussed later in this paper). Section 2.5 gives a more categorical definition of institutions which replaces the rather *ad hoc* looking family of satisfaction relations by a functor into a category of “twisted relations” and also includes many variations of the institution concept, including one with $\mathit{Sen}: \mathbf{Sign} \rightarrow \text{Cat}$ which includes deduction.

2.2 Theories and Theory Morphisms

If, as suggested in the introduction, a *specification* provides a mathematical theory of the intended behavior of a program, and if a *theory* consists of all the sentences that are true of that behavior, then it will be important to define and develop the basic properties of theories over an arbitrary institution. A theory consists of a signature Σ and a “closed” collection of Σ -sentences, as in first

order logic. Thus, this notion differs from the Lawvere notion of “algebraic theory” [60], which is independent of any choice of signature, and also simplifies our “signed theories” in [13]. Usually, these theories contain an infinite number of sentences, but it is often convenient to define them by a finite subset.

Definition 2 Let I be a fixed but arbitrary institution. Then

1. A Σ -presentation is a pair $\langle \Sigma, E \rangle$, where Σ is a signature and E is a collection of Σ -sentences.
2. A Σ -model A satisfies a presentation $\langle \Sigma, E \rangle$ if it satisfies each sentence in E ; write $A \models E$ in this case.
3. Given a collection E of Σ -sentences, let E^* be the collection of all Σ -models that satisfy each sentence in E .
4. Given a collection M of Σ -models, let M^* be the collection of all Σ -sentences that are satisfied by each model in M ; we also let M^* denote $\langle \Sigma, M^* \rangle$, called the theory of M .
5. The closure of a collection E of Σ -sentences is E^{**} , denoted E° .
6. A collection E of Σ -sentences is closed iff $E = E^\circ$.
7. A Σ -theory is a presentation $\langle \Sigma, E \rangle$ such that E is closed.
8. The Σ -theory presented by a presentation $\langle \Sigma, E \rangle$ is $\langle \Sigma, E^\circ \rangle$.
9. A Σ -sentence e is semantically entailed by a collection E of Σ -sentences, written $E \models e$, iff $e \in E^\circ$.

□

Our definition of closure is based on satisfaction rather than deduction. Of course, some institutions have a natural complete set of inference rules, for example, the many-sorted equational institution; but others do not, and we prefer the added generality.

We can also consider closed collections of models; following the terminology of the equational institution, these might be called **varieties**. It is often convenient to regard E^* as the full subcategory of $\mathbf{Mod}(\Sigma)$ with objects the models that satisfy E . The closure of a collection M of models is M^{**} , denoted M° , and a full subcategory of models is called closed iff its objects are exactly all the models of some collection of sentences.

Notice that there is a forgetful functor $\mathbf{Sign}: \mathbf{Th} \rightarrow \mathbf{Sign}$ sending $\langle \Sigma, E \rangle$ to Σ , and sending ϕ as a theory morphism to ϕ as a signature morphism.

Proposition 3 The two functions denoted “ $*$ ” in Definition 2 form what is known as a Galois connection (see, e.g., [19]), in that they satisfy the following properties, for any collections E, E' of Σ -sentences and collections M, M' of Σ -models:

1. $E \subseteq E'$ implies $E' \subseteq E^*$.
2. $M \subseteq M'$ implies $M'^* \subseteq M^*$.
3. $E \subseteq E^{**}$.
4. $M \subseteq M^{**}$.

These imply the following properties:

5. $E^* = E^{***}$.

6. $M^* = M^{***}$.

7. There is a dual (i.e., inclusion reversing) isomorphism between the closed collections of sentences and the closed collections of models; this isomorphism takes unions to intersections and intersections to unions.

8. $\bigcap_n E_n^* = (\bigcup_n E_n)^*$;

9. $(\bigcap_n E_n^*)^{**} = (\bigcup_n E_n)^*$;

10. $(\bigcup_n E_n^{**})^* = \bigcap_n E_n^*$;

11. $(\bigcup_n E_n^{**})^* = (\bigcup_n E_n)^*$;

12. $(\bigcap_n E_n^{**})^* = (\bigcup_n E_n^*)^{**}$.

Although these identities are stated for collections of sentences, there are also dual identities to 8.-12. for collections of models.

Proof: The first two assertions are straightforward. We now prove the third, assuming that $E \subseteq \text{Sen}(\Sigma)$. Then

$$E^* = \{m \mid (\forall e \in E) m \models e\},$$

and so

$$\begin{aligned} E^{**} &= \{e' \mid (\forall m \in E^*) m \models e'\} \\ &= \{e' \mid (\forall m) m \in E^* \Rightarrow m \models e'\} \\ &= \{e' \mid (\forall m) [(\forall e \in E) m \models e] \Rightarrow m \models e'\}. \end{aligned}$$

But if e' is in E , and if $m \models e$ for all $e \in E$, then certainly $m \models e'$; thus the above set contains $\{e' \mid e' \in E\} = E$. The proof of the fourth assertion is similar, and the next eight assertions are familiar from lattice theory, e.g. [9]; in fact, 8.-12. follow easily from 7. \square

The following identity is useful, for example in Theorem 36 below:

Proposition 4 Given a signature morphism $\phi: \Sigma \rightarrow \Sigma'$, a Σ -presentation E , and a functor $\phi: \text{Mod}(\Sigma') \rightarrow \text{Mod}(\Sigma)$ that is surjective on objects², then $\phi(\phi E)^* = E^*$.

Proof: Let $m \in |\text{Mod}(\Sigma)|$. Then $m \in E^*$ iff $m \models E$. And $m \in \phi(\phi E)^*$ iff $[(\exists m' \in (\phi E)^*) m = \phi m']$ iff $[(\exists m') m' \models \phi E \ \& \ m = \phi m']$ iff $[(\exists m') m \models E \ \& \ m = \phi m']$ iff $[m \models E \ \& \ (\exists m') m = \phi m']$ iff $m \models E$ (since ϕ is surjective). \square

Definition 5 If T and T' are theories, say $\langle \Sigma, E \rangle$ and $\langle \Sigma', E' \rangle$, then a **theory morphism** from T to T' is a signature morphism $\phi: \Sigma \rightarrow \Sigma'$ such that $\phi(e) \in E'$ for each $e \in E$; we will write $\phi: T \rightarrow T'$. The **category of theories** has theories as objects, and theory morphisms as morphisms, with composition and identities defined as for signature morphisms; we denote it **Th**. (It is easy to see that this is a category.) The following will frequently use capital letters to denote theory morphisms, e.g., $F: T \rightarrow T'$. \square

For equational logic, there is another category with theories as objects, but with derivors, which map operators in Σ to derived operators in Σ' , i.e., to Σ' -terms, as morphisms [38]; in fact, this kind of morphism agrees with the usual morphism notion for Lawvere theories [60].

For a theory T with signature Σ , let $\text{Mod}(T)$ and also T^* denote the full subcategory of $\text{Mod}(\Sigma)$ of all Σ -models that satisfy all the sentences in T .

²It does not matter what happens to the morphisms; in effect, we are only concerned here with the simpler notion of institution in which Mod takes values in Set^{op} .

Definition 6 Given a theory morphism $\phi: T \rightarrow T'$, the forgetful functor $\phi^*: T'^* \rightarrow T^*$ sends a T' -model m' to the T -model $\phi(m')$, and sends a T' -model morphism $f: m' \rightarrow n'$ to $\phi^*(f) = \text{Mod}(\phi)(f): \phi(m') \rightarrow \phi(n')$. This functor is also denoted $\text{Mod}(\phi)$. \square

To ensure that this definition makes sense, we should check that if a given Σ' -model m' satisfies T' , then $\phi^*(m')$ satisfies T . Let e be any sentence in T . Because ϕ is a theory morphism, $\phi(e)$ is a sentence of T' and therefore $m' \models \phi(e)$. The Satisfaction Condition now gives $\phi(m') \models e$, as desired. We also need that the morphism $\phi^*(f)$ lies in T^* , but this follows because T^* is a full subcategory of $\text{Mod}(\Sigma)$, and the source and target objects of $\phi^*(f)$ lie in T^* .

2.3 The Closure and Presentation Lemmas

Given a signature morphism $\phi: \Sigma \rightarrow \Sigma'$, a collection E of Σ -sentences, and a collection M' of Σ' -models, let us write $\phi(E)$ for $\{\phi(e) \mid e \in E\}$ and $\phi(M')$ for $\{\phi(m') \mid m' \in M'\}$. Given a collection M of Σ -models, let us also write $\phi^{-1}(M)$ for $\{m' \mid \phi(m') \in M\}$. Using this notation, we can write the Satisfaction Condition more compactly as

$$\phi^{-1}(E^*) = \phi(E)^*$$

and we also have

Lemma 7 Closure. $\phi(E^*) \subseteq \phi(E)^*$.

Proof: $\phi(E^{**})^* = \phi^{-1}(E^{***}) = \phi^{-1}(E^*) = \phi(E)^*$, using the Satisfaction Condition and 5. of Proposition 3. Therefore $\phi(E^*) = \phi(E^{**}) \subseteq \phi(E^{**})^{**} = \phi(E)^{**} = \phi(E)^*$, using 3. of Proposition 3 and the just proved equation. \square

It is worth pointing out that the inclusion $\phi(E^*) \subseteq \phi(E)^*$ can be proper. This means that while $E \models e$ implies $\phi E \models \phi e$, the converse, that $\phi E \models \phi e$ implies $E \models e$, does *not* hold. The following gives an easier to check necessary and sufficient condition for a signature morphism to be a theory morphism.

Lemma 8 Presentation. Let $\phi: \Sigma \rightarrow \Sigma'$ and suppose that $\langle \Sigma, E \rangle$ and $\langle \Sigma', E' \rangle$ are presentations. Then $\phi: \langle \Sigma, E^* \rangle \rightarrow \langle \Sigma', E'^* \rangle$ is a theory morphism iff $\phi(E) \subseteq E'^*$.

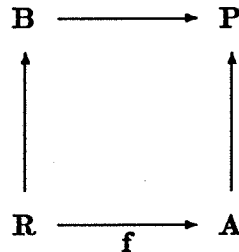
Proof: The “only if” part is trivial. For the “if” part, we have by the Closure Lemma that $\phi(E^*) \subseteq \phi(E)^*$. By hypothesis, $\phi(E) \subseteq E'^*$. Therefore, $\phi(E^*) \subseteq \phi(E)^* \subseteq E'^*$, so ϕ is a theory morphism. Conversely, if ϕ is a theory morphism, then $\phi(E^*) \subseteq E'^*$. Thus $\phi(E) \subseteq E'^*$ since $E \subseteq E^*$. \square

The Presentation Lemma tells us that to check whether ϕ is a theory morphism, we can apply ϕ to each sentence e of the source presentation E and see whether $\phi(e)$ is in the closure of E' ; there is no need to check every sentence in E^* .

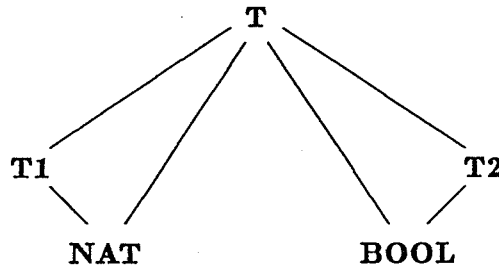
2.4 Putting Theories Together

The introduction of this paper suggests that a basic purpose of a specification language is to reduce the difficulty of describing large theories by providing mechanisms for constructing them from already available smaller theories. The specification language Clear [13, 14, 15] provided a number of such mechanisms that can be explicated using colimits in the category of theories over an arbitrary institution. These include: the *sum* of theories (in such a way as to not duplicate subtheories that may be shared; for example, NAT and BOOL might be subtheories of several theories); and the *application* of a parameterized theory (such as LIST) to an actual theory (such as NAT) to yield a result theory (such as LIST[NAT]). Clear’s parameterized theories

also involve a *formal* or *requirement* theory R which makes explicit the syntactic and semantic requirements on an actual theory in order for the result of the application to be meaningful; R will be a subtheory of the body theory B . Moreover, a *binding* f of what the actual theory provides to what the formal theory requires is needed to carry out the application, called a “fitting morphism” in Clear. The semantics of Clear [13, 14] says that the result P of the application is given by the following pushout diagram, in which P is the resulting theory:



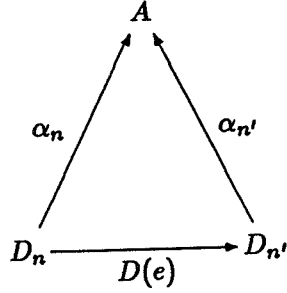
Similarly, the *sum* T of two theories $T1$ and $T2$ which share NAT and $BOOL$ as subtheories, is given by the following colimit diagram, in which all lines indicate subtheory inclusions:



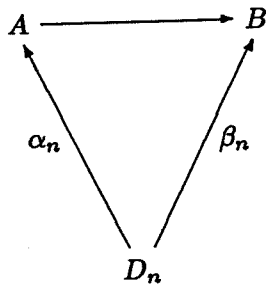
The construction of large theories as colimits of small theories connected by theory morphisms which was implicit in the first paper on Clear [13] was made explicit in [14] and [20]. For this approach to make sense, the category of theories should have finite colimits. For the equational institution, the intuitively correct syntactic pasting together of presentations exactly corresponds to colimits of theories [37]. Colimits have become a standard technique in defining the semantics of specification languages, for example in ASL [77], Extended ML [75], Act One [22], and Act Two [24], as well as Clear and OBJ. Quite similar things can be done for logical programming languages, and even for conventional imperative languages like Ada [32]. Thus, there is much motivation for studying colimits in the category of theories over an institution.

The use of colimits to explicate various ways to combine theories is not *ad hoc*. It is a general principle that a large widget can be described as the interconnection of a system of small widgets using widget-morphisms to identify the interfaces over which the interconnection is to be done; then the result of the interconnection is given by the colimit of the diagram; some motivation for this principle is given in [33] from the point of view of general systems theory. Colimits have also been used for many other things in computer science [33], such as graph grammars [21]. We now review the necessary categorical concepts.

Definition 9 A diagram D in a category C consists of a graph G together with a labelling of each node n of G by an object D_n of C , and a labelling of each edge e , say from node n to node n' in G , by a morphism $D(e)$ in C from D_n to $D_{n'}$; let us write $D: G \rightarrow C$. Then a cone α in C over the diagram D consists of an object A of C and a family of morphisms $\alpha_n: D_n \rightarrow A$, one for each node n in G , such that for each edge $e: n \rightarrow n'$ in G , the diagram



commutes in \mathbf{C} . We call D the base of the cone α , A its apex, G its shape, and we write $\alpha: D \Rightarrow A$. If α and β are cones with base D and apexes A, B (respectively), then a **morphism of cones** $\alpha \rightarrow \beta$ is a morphism $f: A \rightarrow B$ in \mathbf{C} such that for each node n in G the diagram



commutes in \mathbf{C} . Now let $\text{Cone}(D, \mathbf{C})$ denote the category of all cones over D in \mathbf{C} , with the obvious composition. Then a **colimit of D in \mathbf{C}** is an initial object in $\text{Cone}(D, \mathbf{C})$. \square

The uniqueness of initial objects up to isomorphism implies the uniqueness of colimits up to cone isomorphism. The apex of a colimit cone α is called its **colimit object**, and the morphisms α_n to the apex are called the **injections into the colimit**. The colimit object is also unique up to isomorphism.

Definition 10 A category \mathbf{C} is **finitely cocomplete** iff it has colimits of all finite diagrams, and is **cocomplete** iff it has colimits of all diagrams (whose base graphs are sets). A functor $F: \mathbf{C} \rightarrow \mathbf{C}'$ **reflects colimits** iff whenever D is a diagram in \mathbf{C} such that the diagram $D; F$ in \mathbf{C}' has a colimit cone $\alpha': D \Rightarrow A'$ in \mathbf{C}' , then there is also a colimit cone $\alpha: D \Rightarrow A$ in \mathbf{C} such that $\alpha' = \alpha; F$, i.e., such that $\alpha'_n = F(\alpha_n)$ for all nodes n in the base of D . \square

Since colimits in the category of theories over an institution are to be used for putting together smaller specifications to form larger ones, it would be very nice to have a powerful, general criterion for when such colimits of specifications actually exist. Perhaps surprisingly, it suffices for the category of signatures to have colimits; the following more general result is the first really non-trivial result in this paper.

Theorem 11 The forgetful functor $\text{Sign}: \mathbf{Th} \rightarrow \mathbf{Sign}$ reflects colimits.

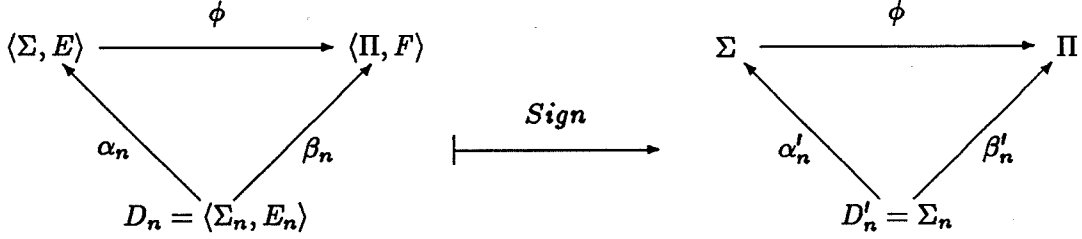
Proof: Suppose that $D: G \rightarrow \mathbf{Th}$ is a diagram in \mathbf{Th} , say with $D_n = \langle \Sigma_n, E_n \rangle$ for $n \in |G|$. Let $D' = D; \text{Sign}: G \rightarrow \mathbf{Sign}$ be the corresponding diagram in \mathbf{Sign} , in which $D'_n = \Sigma_n$. Now let $\alpha': D' \Rightarrow \Sigma$ be a colimit cone for D' . We have to find a colimit cone α for D such that $\text{Sign}(\alpha) = \alpha'$. We define $D = \langle \Sigma, E \rangle$ where

$$E = \left(\bigcup_{n \in |G|} \alpha_n(E_n) \right)^*$$

and we define $\alpha_n = \alpha'_n$ for all $n \in |G|$. Then each α_n is a theory morphism and we claim that $\alpha = \langle \alpha_n: \langle \Sigma_n, E_n \rangle \rightarrow \langle \Sigma, E \rangle \mid n \in |G| \rangle$ is a colimit cone over D in \mathbf{Th} . For suppose that $\beta'' = \langle \beta_n: \langle \Sigma_n, E_n \rangle \rightarrow \langle \Pi, F \rangle \mid n \in |G| \rangle$ is another cone over D in \mathbf{Th} . Then applying *Sign* to everything, we get a unique $\phi: \Sigma \rightarrow \Pi$ such that $\alpha_n; \phi = \beta_n$ for each $n \in |G|$. Thus, there is at most one $\psi: \langle \Sigma, E \rangle \rightarrow \langle \Pi, F \rangle$ such that $\alpha_n; \psi = \beta_n$ for all $n \in |G|$, namely ϕ . Therefore all we need to show is that ϕ is a theory morphism. Because $\beta_n: \langle \Sigma_n, E_n \rangle \rightarrow \langle \Pi, F \rangle$ is a theory morphism, we have $\beta_n(E_n) \subseteq F$. Therefore $\bigcup_{n \in |G|} \beta_n(E_n) \subseteq F$, and so

$$\phi(E) = \phi\left(\left(\bigcup_n \alpha_n(E_n)\right)^*\right) \subseteq \left(\bigcup_n \alpha_n; \phi(E_n)\right)^* = \left(\bigcup_n \beta_n(E_n)\right)^* \subseteq F$$

where the first inclusion follows from the Closure Lemma. (See the diagram below.) \square



It now follows, for example, that the category \mathbf{Th} of theories in an institution is [finitely] cocomplete if its category \mathbf{Sign} of signatures is [finitely] cocomplete. (It is easy to see that the converse is also true.) Since the category of equational signatures is finitely cocomplete ([35, 38] give a simple proof using comma categories), we conclude that the category of signed equational theories is cocomplete. Using similar techniques, we can show that the category of first order signatures (as in the Appendix) is cocomplete, and thus without effort conclude that the category of first order theories is cocomplete (this might even be a new result, especially since our notion of morphism is not quite the usual one in logic).

2.5 A More Categorical Formulation

It is somewhat aesthetically displeasing to define institutions using two functors and a family of relations; we would prefer something more categorical than a family of relations. This can be done, and moreover it will yield a notion of institution morphism as a byproduct, but some preparation is required. (The reader not already familiar with vertical and horizontal composition of natural transformations who wants to read this subsection might first consult [58], Section II.5.)

Let \mathbf{Rel} denote the category with sets A, B, C, \dots as objects; with relations $R: A \rightarrow B$ as morphisms, i.e., triples $\langle A, R, B \rangle$ where $R \subseteq A \times B$; with the “diagonal” relation $\{\langle a, a \rangle \mid a \in A\}$ as the identity on a set A ; and with composition as usual for relations, but keeping track of their sources and targets. Given a relation $R: A \rightarrow B$, let $R^\sim: B \rightarrow A$ denote its converse. We now define the category \mathbf{Trel} of “twisted relations”: its objects are relations $R: A \rightarrow B$, and its morphisms from $(R: A \rightarrow B)$ to $(R': A' \rightarrow B')$ are pairs of functions $\langle f: A' \rightarrow A, g: B \rightarrow B' \rangle$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ f \uparrow & & \uparrow g \\ A' & \xrightarrow{R'} & B' \end{array}$$

commutes in **Rel**, i.e., such that for all a' in A' and b in B , we have³ $f(a')Rb$ iff $a'R'g(b)$. The identity morphism on a relation $R: A \rightarrow B$ is the pair $\langle 1_A, 1_B \rangle$, and the composition $\langle f', g' \rangle; \langle f, g \rangle$ is $\langle f'; f, g; g' \rangle$ provided the target sets of $\langle f', g' \rangle$ equal the source sets of $\langle f, g \rangle$.

The notation for a proper categorical definition of the institution concept will be simpler if we first define two functors on **Trel**. First *Left*: **Trel** \rightarrow **Set**^{op} is defined by $Left(R: A \rightarrow B) = A$ and $Left(\langle f, g \rangle) = f$. Second, *Right*: **Trel** \rightarrow **Set** is defined by $Right(R: A \rightarrow B) = B$ and $Right(\langle f, g \rangle) = g$. Notice that $Left(f; f') = Left(f'); Left(f)$, whereas $Right(g; g') = Right(g); Right(g')$.

Definition 12 An institution is a functor $I: \mathbf{Sign} \rightarrow \mathbf{Trel}$ to twisted relations; its source **Sign** is its category of signatures; the functor composition $I; Left: \mathbf{Sign} \rightarrow \mathbf{Set}^{op}$ is the “model functor” of I , denoted Mod ; the composition $I; Right: \mathbf{Sign} \rightarrow \mathbf{Set}$ is the “sentence functor” of I , denoted Sen ; and the satisfaction relation \models_Σ is $I(\Sigma)$ for $\Sigma \in |\mathbf{Sign}|$. \square

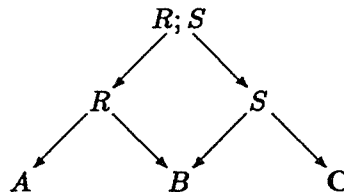
This definition gives the version of institution in which each $Mod(\Sigma)$ is a set, rather than the more advanced version with a category $Mod(\Sigma)$ of models. To get the category version, we use instead of **Trel** a category whose objects are triples $\langle \mathbf{A}, R, B \rangle$, where \mathbf{A} is a category, B is a set, and $R \subseteq |\mathbf{A}| \times B$ is a relation; morphisms are then pairs $\langle F, g \rangle$ where $F: \mathbf{A}' \rightarrow \mathbf{A}$ is a functor and $g: B \rightarrow B'$ is a function, such that the above diagram commutes with $f = |F|$. Definition 12 with this category gives the desired concept. The following is now very natural:

Definition 13 An institution morphism $I \rightarrow I'$ consists of a functor $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$ and a natural transformation $\eta: \Phi; I' \Rightarrow I$. Given $\langle \Phi, \eta \rangle: I \rightarrow I'$ and $\langle \Phi', \eta' \rangle: I' \rightarrow I''$, their composition is defined to be $\langle \Phi; \Phi', (\Phi \circ \eta); \eta' \rangle: I'' \rightarrow I$, where \circ denotes the (vertical) composition of a natural transformation with a functor. Let **INS** denote the category of institutions. \square

There are reasons for believing that it might have been better to consider the above as defining a morphism from I' to I , rather than from I to I' , even though in a certain sense the choice is arbitrary; however, we have chosen to maintain the above definition in this paper for essentially historical reasons. Section 4.1 gives a more concrete version of this definition. That **INS** really is a category, and moreover has whatever limits and colimits **Trel** has, follows from some “abstract nonsense” and “general systems theory”⁴ as described in the following subsection.

2.5.1 An Even More Categorical Formulation

In order to consider variations of the notion of institution, and also to study properties of the category of institutions, it is helpful to take a very general approach. Readers who are not especially fond of categorical “abstract nonsense” may wish to skip this subsection; also, it is somewhat sketchy. Given a category **C** with pullbacks, we first define the category **Rel(C)** of relations in **C** as follows: its objects are those of **C**; its morphisms from A to B are pairs $\langle p1: R \rightarrow A, p2: R \rightarrow B \rangle$ of morphisms in **C** with a common source; its identities have both $p1$ and $p2$ identity morphisms; and its composition is obtained by pullback, as shown in the diagram below:



³The reader may recognize the ghost of the Satisfaction Condition here.

⁴An earlier draft of this definition had the natural transformation going the other way; comparison with the general systems theory in [29] enabled us to formulate institution morphisms correctly.

Lemma 14 If \mathbf{C} has pullbacks, then $\mathbf{Rel}(\mathbf{C})$ is a category. \square

We next recall Lawvere's general comma category construction. If we are given functors $F: \mathbf{A} \rightarrow \mathbf{C}$ and $G: \mathbf{B} \rightarrow \mathbf{C}$, then the category (F/G) is defined as follows: its objects are triples $\langle A, c: F(A) \rightarrow G(B), B \rangle$, where A is an object of \mathbf{A} , B is an object of \mathbf{B} , and c is a morphism of \mathbf{C} ; its morphisms from $\langle A, c: F(A) \rightarrow G(B), B \rangle$ to $\langle A', c': F(A') \rightarrow G(B'), B' \rangle$ are pairs $\langle a, b \rangle$ with $a: A \rightarrow A'$ in \mathbf{A} and $b: B \rightarrow B'$ in \mathbf{B} ; and the composition $\langle a, b \rangle; \langle a', b' \rangle$ in (F/G) is just $\langle a; a', b; b' \rangle$. It is known⁵ that this gives a category, and that whatever limits or colimits are possessed by \mathbf{A} and \mathbf{B} and preserved by F and G , are also possessed by (F/G) .

Next we give some ways of constructing functors into relation categories. Given $F: \mathbf{A} \rightarrow \mathbf{C}$, define $F\uparrow: \mathbf{A} \rightarrow \mathbf{Rel}(\mathbf{C})$ as follows: an object A in \mathbf{A} goes to the object $F(A)$ in $\mathbf{Rel}(\mathbf{C})$; and a morphism $a: A \rightarrow A'$ in \mathbf{A} goes to a morphism $\langle 1_{F(A)}, F(a) \rangle$ from $F(A)$ to $F(A')$ in $\mathbf{Rel}(\mathbf{C})$. (It is easy to see that this is a functor.) Similarly, given $F: \mathbf{A} \rightarrow \mathbf{C}$, define $F\downarrow: \mathbf{A}^{op} \rightarrow \mathbf{Rel}(\mathbf{C})$ by: an object A in \mathbf{A} goes to the object $F(A)$ in $\mathbf{Rel}(\mathbf{C})$; and a morphism $a: A \rightarrow A'$ in \mathbf{A}^{op} goes to a morphism $\langle 1_{F(A')}, f(a) \rangle$ from $F(A')$ to $F(A)$ in $\mathbf{Rel}(\mathbf{C})$. (It is also easy to see that this is a functor.) This last construction will let us "twist" relations.

Now let's exercise this machinery. Combining the relation, comma category, and the \uparrow and \downarrow constructions will give us the categories of twisted relations that we need; and taking functors into these will give us the corresponding categories of institutions. Let's get the relation categories first. The (original) definition of institution with Mod set-valued corresponds to the twisted relation category $(1_{\mathbf{Set}}\downarrow / 1_{\mathbf{Set}}\uparrow)$ where $1_{\mathbf{A}}$ denotes the identity functor on the category \mathbf{A} . Similarly, Definition 1 corresponds to the category $(1_{\mathbf{Set}}\downarrow / U\uparrow)$, where $U: \mathbf{Cat} \rightarrow \mathbf{Set}$ is the forgetful functor that takes each category to its underlying set, and each functor to its underlying function on objects. Both enjoy the "twist" given by \downarrow , but the first gives only sets of models, while the second gives categories of models; in each case, the source category of the first functor gives the structure for models.

Another interesting variant is given by the category $(U\downarrow / U\uparrow)$, which allows morphisms between sentences as well as between models; one might want to think of a morphism from one sentence to another as a "proof" that the second follows from the first; see [39] and Section 5 for further discussion of this point.

An institution is a functor into a relation category; in particular, an institution in the sense of Definition 1 is a functor $I: \mathbf{Sign} \rightarrow (1_{\mathbf{Set}}\downarrow / U\uparrow)$. This leads to categories whose objects are functors; but since their source categories (which are their categories of signatures) may vary, their morphisms will not be just natural transformations. In fact, these morphisms are pairs of the form $\langle \Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}', \eta: \Phi; I' \Rightarrow I \rangle$, and are a special case of a general construction for the category of diagrams (with varying shape) over a category: Given a category \mathbf{T} , the objects of $\mathbf{Dgm}(\mathbf{T})$ are functors $I: \mathbf{S} \rightarrow \mathbf{T}$, its morphisms $I \rightarrow I'$ are pairs $\langle \Phi: \mathbf{S} \rightarrow \mathbf{S}', \eta: \Phi; I' \Rightarrow I \rangle$, and its composition $\langle \Phi: \mathbf{S} \rightarrow \mathbf{S}', \eta: \Phi; I' \Rightarrow I \rangle; \langle \Phi': \mathbf{S}' \rightarrow \mathbf{S}'', \eta': \Phi'; I'' \Rightarrow I' \rangle$ is $\langle \Phi; \Phi', \Phi \circ (\eta'; \eta) \rangle$, where in the last expression \circ denotes the "horizontal" and $;$ denotes the "vertical" composition of natural transformations. That $\mathbf{Dgm}(\mathbf{T})$ is a category is shown, for example, in [29] in the context of general systems.

This abstract view is also useful for getting other variants of the institution notion; for example, to get partial satisfaction we might let \mathbf{C} be the category \mathbf{Pfn} of partial functions; it seems worth exploring this further. This view of institutions also allows us to get completeness results for the category of institutions using general results about relation, diagram, and comma categories; see also [88]. [39] discusses an even more abstract formulation based on "wedges."

⁵Actually, somewhat stronger results are known; see [38, 88].

3 Constraints

To avoid over-specifying problems, we sometimes want to use *loose* specifications, that is, specifications for which non-isomorphic models are acceptable. On the other hand, most problems require some fixed data types, such as the natural numbers or truth values. In such cases, the subtheories that correspond to these data types must be given *standard* interpretations. Finally, sometimes we want to consider *parameterized* standard data types, such as $\text{SET}[\mathbf{X}]$ and $\text{LIST}[\mathbf{X}]$, for which sets and lists are to be given standard interpretations, once an interpretation is given for \mathbf{X} , which is loose with respect to some requirement theory.

We have already considered the category $T^* = \text{Mod}(T)$ of *all* interpretations of a theory T in a fixed institution; now we consider how to impose constraints on these interpretations. One kind of constraint requires that some parts of T have a “standard” interpretation relative to other parts; these are the “data constraints” of [15], which generalize and relativize the “initial algebra” approach to abstract data types of ADJ [45]. Data constraints make sense for any institution, and are much more expressive even for the equational institution.

To require that some subtheories T_1, \dots, T_n of a theory T are initially interpreted, we could try to use the subtheories themselves as constraints. In this case, a model M would satisfy T with constraint (T_1, \dots, T_n) iff M satisfies T and M restricted to each T_i is initial. More precisely now, given $F_i: T_i \rightarrow T$ for $i = 1, \dots, n$, we can define a satisfaction relation \models between models and theories with constraints by

$$M \models \langle T, (T_1, \dots, T_n) \rangle \text{ iff } M \models T \text{ and } \text{Mod}(F_i)M \text{ is initial in } \text{Mod}(T_i) \text{ for } i = 1, \dots, n.$$

Unfortunately, things are not quite so simple. To deal with parameterized theories, we need not just initial models (e.g., for the natural numbers), but also free extensions of models (e.g., to form sets from elements). Also, it would be very convenient if given an institution I , we could construct a new institution $\mathcal{C}(I)$ such that its sentences are either I -sentences or else constraints, and its models are I -models, with an appropriate satisfaction. To make this work, we will need to provide translations of constraints under signature morphisms. Then, if I -signatures have colimits, it will follow by Theorem 11 that $\mathcal{C}(I)$ -theories also have colimits, and thus can be glued together.

Actually, our general results will apply to many different notions of constraint. In particular, Section 3.3 will show that so-called “generating constraints” are a special case, and Section 4.2 will generalize to “duplex constraints” that allow constraints in an institution different from the one in which models are taken; more generally still, Section 4.3 considers multiplex institutions.

3.1 Free Interpretations

Suppose that we want to define the natural numbers in the equational institution. To this end, consider a theory \mathbf{N} with one sort Nat , and with a signature Σ containing one constant 0 and one unary operator inc ; there are no equations in the presentation of this theory. Now this theory has many algebras, including some where $\text{inc}(0) = 0$. But the natural numbers, no matter how represented, give an *initial algebra* in the category Alg_Σ of all algebras for this theory, in the sense that there is exactly one Σ -homomorphism from it to any other Σ -algebra. It is easy to prove that any two initial Σ -algebras are Σ -isomorphic; this means that the property “being initial” determines the natural numbers uniquely up to isomorphism, that is, different representations for the natural numbers give different (but isomorphic) initial algebras. This characterization of the natural numbers is due to Lawvere [61], and a proof that it is equivalent to Peano’s axioms can be found in [59], pages 67-70.

The initial algebra approach to abstract data types [45] takes this “Lawvere-Peano” characterization of the natural numbers as paradigmatic for defining other abstract data types; the method has been used to specify sets, lists, stacks, and many many other data types, as well as

database systems and programming languages, among many other things. The essential ideas here are that concrete data types are algebras, and that “abstract” in “abstract data type” means *exactly* the same thing as “abstract” in “abstract algebra,” namely, uniquely defined up to isomorphism. A number of less abstract equivalents to initiality for the equational institution, including generalized Peano axioms, are given in [69].

Let us now consider the case of the parameterized abstract data type of sets of elements of a sort S . We add a new sort Set , and operators⁶

$$\begin{aligned} \emptyset &: \rightarrow \text{Set} \\ \{-\} &: S \rightarrow \text{Set} \\ _ \cup _ &: \text{Set}, \text{Set} \rightarrow \text{Set} \end{aligned}$$

subject to the following equations, where S, S' and S'' are variables of sort Set ,

$$\begin{aligned} \emptyset \cup S &= S \\ S \cup (S' \cup S'') &= (S \cup S') \cup S'' \\ S \cup S' &= S' \cup S \\ S \cup S &= S \end{aligned}$$

Although we want these operators to be interpreted “initially” in some sense, we do *not* want the initial algebra of the theory having sorts S and Set and the operators above. Indeed, the initial algebra of this theory has the *empty* carrier for the sort S (since there are no operators to generate elements of S) and has only the element \emptyset of sort Set . Rather, we want to permit *any* interpretation for the parameter sort S , and then require that the new sort Set and its new operators are interpreted freely *relative* to the given interpretation of S .

Let us make this precise. Suppose that $F: T \rightarrow T'$ is a theory morphism. Then there is a forgetful functor from the category of T' -models to the category of T -models, $F^*: T'^* \rightarrow T^*$ as in Definition 6. In the equational case $T^* = \text{Alg}(T)$, and a very general result of Lawvere [60] says that for every T -model A there is a T' -model A^\ddagger and a T -homomorphism $\eta_A: A \rightarrow F^*(A^\ddagger)$, called the **universal morphism** and characterized by the following “universal property”: given any T' -model B and any T -morphism $f: A \rightarrow F^*(B)$, there is a unique T' -morphism $f^\#: F^*(A^\ddagger) \rightarrow B$ such that the following diagram commutes in $\text{Mod}(T)$:

$$\begin{array}{ccc} & & F^*(B) \\ & \nearrow f & \uparrow F^*(f^\#) \\ A & \xrightarrow{\eta_A} & F^*(A^\ddagger) \end{array}$$

Such an object A^\ddagger in $\text{Mod}(T)$ will be called a **free extension** of A along F ; the universal property determines A^\ddagger uniquely up to isomorphism in $\text{Mod}(T')$. If each A in $\text{Mod}(T)$ has a free extension along F , then there is unique way to define a functor $F^\ddagger: \text{Mod}(T) \rightarrow \text{Mod}(T')$ such that $F^\ddagger(A) = A^\ddagger$ [58]. $F^\ddagger: T^* \rightarrow T'^*$ is called the **free functor** determined by F . (In a more technical language, F^\ddagger is a left adjoint to F^* , and η is a natural transformation, the unit of this adjunction.) This discussion of the equational institution motivates the following:

Definition 15 An institution is **liberal** iff for every theory morphism $F: T \rightarrow T'$ and every T -model A , there is a T' -model A^\ddagger , called the **free extension** of A along F , such that there is a **universal morphism** $\eta_A: A \rightarrow F^*(A^\ddagger)$ with the property that for each T -model B and each T -morphism $f: A \rightarrow F^*(B)$, there is a unique T' -morphism $f^\#: A^\ddagger \rightarrow B$ such that the above diagram commutes (in the category $\text{Mod}(T)$). \square

⁶We use “mixfix” (also called “distfix”) declarations in this signature, in the style of OBJ: each underbar is a placeholder for an element of the corresponding sort from the list that follows the colon.

As in the equational case, giving a free extension A^\sharp of A along F for each T -model A determines a unique functor F^\sharp having the value A^\sharp on the object A ; this functor F^\sharp is called a **free (extension) functor**, just as F^* is called a **forgetful functor**. (F^\sharp is left adjoint to F^* , and is unique up to natural isomorphism if it exists. Thus an institution is liberal iff the forgetful functors induced by its theory morphisms always have left adjoints.) The equational institution is liberal, as are the institutions of Horn clause logic with equality, and of conditional equations; however the first order logic institution is not liberal (the latter three institutions are defined in the Appendix). Notice that even in a non-liberal institution, there may be many models which have a free extension along a given theory morphism, and there may even be many theory morphisms that have a free extension functor. Hence, in the following we may use the notation A^\sharp for a free extension of A along F , or even the notation $F^\sharp(A)$, in cases where there is no functor F^\sharp for the given theory morphism F .

Returning to our set example, consider the theory morphism Set that is the inclusion of the trivial theory, **TRIV** having just the sort S , into the theory of sets of S , let's call it **SET**, obtained by adding the sort Set and the operators and equations given above. Then Set^* takes a **SET**-algebra and forgets the new sort Set and the three new operators, giving an algebra that has just the carrier of S -sorted elements, and $Set^\sharp(A)$ is the algebra of all finite subsets of A . The free functor Set^\sharp takes a **TRIV**-algebra A (i.e., a set) and extends it freely to a **SET**-algebra, the new operators giving distinct results except where the equations of **SET** force equality.

Given a **SET**-algebra B , there is a natural way to check whether or not its Set sort and operators are free over its parameter sort S : in the above diagram, let $A = Set^*(B)$ and let $f = 1_{Set^*(B)}$; then $f^\sharp: (Set^*(B))^\sharp \rightarrow B$ should be an isomorphism. In general, the morphism $(1_{F^*(B)})^\sharp: (F^*(B))^\sharp \rightarrow B$ is called the **counit** (of the adjunction, if there is one) and is denoted ϵ_B . In a liberal institution, it has the following **couniversal** property, dual to that of the unit: given any B in $\text{Mod}(T')$ and $f: F^\sharp(B') \rightarrow B$ in $\text{Mod}(T)$, there is a unique $u: B' \rightarrow F^*(B)$ such that $F^\sharp(u); \epsilon_B = f$, i.e., such that the diagram

$$\begin{array}{ccc}
 F^\sharp(B') & & \\
 \downarrow F^\sharp(u) & \searrow f & \\
 F^\sharp(F^*(B)) & \xrightarrow{\epsilon_B} & B
 \end{array}$$

commutes in $\text{Mod}(T)$. This motivates the following.

Definition 16 Let $F: T \rightarrow T'$ be a theory morphism. Then a T' -model B is F -free iff $F(B)$ has a free extension along F such that $\epsilon_B = (1_{F^*(B)})^\sharp$ is an isomorphism. Let us call a functor $G: \text{Mod}(T) \rightarrow \text{Mod}(T')$ **extensive** iff $G(A)$ is F -free for every T -model A . \square

Of course, we will mostly be interested in whether or not a free extension functor is extensive. The notion of F -free for the equational case is due to Thatcher, Wagner and Wright [91]; here, we generalize to arbitrary institutions, and do not assume that there is necessarily a functor F^\sharp . Originally, we defined B to be F -free if B and $F^\sharp(F^*(B))$ were isomorphic [15]. However, there are examples⁷ where these two objects are isomorphic, but not naturally so by the morphism ϵ . There is also a concept that is dual to F -free in a certain sense:

⁷Our thanks to Eric Wagner for this comment.

Definition 17 Let $F: T \rightarrow T'$ be a theory morphism. Then a T -model A is F -protected iff it has a free extension along F such that the universal morphism $\eta_A: A \rightarrow F^*(A^\sharp)$ is an isomorphism. Moreover, the functor F^\sharp is persistent iff every T -model A is F -protected. \square

This definition generalizes the concept of persistence in [91] to any liberal institution. Let us define a theory morphism $F: T \rightarrow T'$ to be conservative iff $e \notin T$ implies $F(e) \notin T'$. A syntactic characterization of persistence for the equational institution is given by Goguen and Meseguer in [40]⁸: when F is injective on sorts, F^\sharp is persistent iff it is conservative and whenever t' is a Σ' -term with its variables and its sort in $F(T)$, then there is an equation $(\forall X) t' = F(t)$ in T' with t a Σ -term (where Σ is the signature of T and Σ' is the signature of T' .) Unfortunately, this result uses some rather subtle constructions in the equational institution which it is not obvious how to generalize to other institutions. The following simple result relates the concepts of Definitions 16 and 17.

Proposition 18 Let $F: T \rightarrow T'$ be a theory morphism in a liberal institution.

1. If a T -model A is F -protected then $F^\sharp(A)$ is F -free.
2. If a T' -model A' is F -free then $F^*(A')$ is F -protected.
3. If F^\sharp is persistent then F^\sharp is extensive.

Proof: For the first assertion, we use the unit isomorphism $\eta_A: A \rightarrow F^*(F^\sharp(A))$ and the fundamental equation for the adjunction (see [58], Theorem IV.2, p.81), $F^\sharp(\eta_A); \epsilon_{F^*(A)} = 1_{F^\sharp(A)}$, to see that

$$\epsilon_{F^*(A)}: F^\sharp(F^*(F^\sharp(A))) \rightarrow F^\sharp(A)$$

is also an isomorphism. The second assertion is proved in the same way. The third assertion, that F^\sharp persistent implies $F^\sharp(A)$ is F -free for every A , follows directly from the first. \square

The following remarks may be of some interest to those familiar with adjoint functors: Let $F: T \rightarrow T'$ be a theory morphism in a liberal institution. Then F^\sharp is persistent iff the unit $\eta: 1_{\mathbf{Mod}(T)} \Rightarrow F^\sharp; F^*$ of the adjunction between F^\sharp and F^* is an isomorphism. Moreover, in this case F^\sharp can be chosen so that this isomorphism is actually an equality, i.e., such that F^\sharp is what [91] call **strongly persistent**.

Results of Mahr and Makowsky [65] show that there is a sense in which the most general sublanguage of infinitary first order logic admitting initial models is Horn clause logic with infinitary conditions; further, the most general finitary sublanguage uses finitary Horn clauses; the most general equational sublanguage uses (infinitary) conditional equations; and the most general finitary equational sublanguage consists of finitary conditional equations. These results are in the style of abstract model theory, a framework much closer to first order logic than to institutions; also, the results concern the existence of initial models rather than left adjoints to forgetful functors. Tarlecki has extended this work of Mahr and Makowsky to characterize liberal sublanguages [87], and has also generalized to what he calls “abstract algebraic institutions” [86].

3.2 Constraining Theories

It is very convenient in program specification to distinguish between sets of sentences which are to be interpreted “loosely” (i.e., any model satisfying the theory will do) and those which require a standard interpretation, say an initial model, or more generally, a free extension. For example, in a parameterized specification, we want to permit any interpretation of the parameter (say, any

⁸This result is actually slightly more general, since it allows theory morphisms that map operators to terms.

partially ordered set) but we also want to constrain the interpretation of the specification which enriches this parameter to be free over it (say, strings of elements of the set, instead of just some arbitrary monoid over them).

In a specification language, we may want to let theories include not just the sentences provided by some institution, but also sentences that constrain certain subtheories to be interpreted freely relative to others, in the sense of Section 3.1. We call such sentences **constraints** and we call theories that can include them **constraining theories**. For example, in a first order theory we might introduce a sentence which constrains interpretations of a theory of the natural numbers to be standard, i.e., to actually be the natural numbers. Our approach will provide a general logic-independent way of moving to a more powerful language that can impose such constraints. We will show that theories with such constraints in them can be treated just like ordinary theories. Moreover, we will show that an institution whose signatures can be glued together (with colimits) yields another institution whose theories may have constraints, such that these theories can also be glued together (with colimits).

Returning to the example of the previous subsection, let us further enrich **SET** with some new sorts and operators to get a theory **SETCH** which adds an operator

$$\text{choose: Set} \rightarrow \mathbf{S}$$

that chooses an arbitrary element of any non-empty set, as described by the conditional equation

$$\text{choose}(S) \in S \text{ if } S \neq \emptyset$$

where S is a variable of sort **Set**. This is a good example of a loose specification. Now let $Etc: \mathbf{SET} \rightarrow \mathbf{SETCH}$ be the theory inclusion, with $Etc: \text{Sign}(\mathbf{SET}) \rightarrow \text{Sign}(\mathbf{SETCH})$ the corresponding signature morphism. Then a **SETCH**-algebra A interprets sets as intended iff $Etc^*(A)$ satisfies **SET** and is Etc -free. This motivates

Definition 19 Let Σ be a signature. Then a Σ -constraint is a pair

$$\langle F: T'' \rightarrow T', \theta: \text{Sign}(T') \rightarrow \Sigma \rangle$$

consisting of a theory morphism and a signature morphism. (We may call a Σ -constraint a Σ -**data constraint** if it is used to define a data type.) A Σ -model A satisfies the Σ -constraint $c = \langle F: T'' \rightarrow T', \theta: \text{Sign}(T') \rightarrow \Sigma \rangle$ iff θA satisfies T' and is F -free, which means that $F^*(\theta A)$ has a free extension along F such that $(1_{F^*(\theta A)})^\# = \epsilon_{\theta A}: F^\$(F^*(\theta A)) \rightarrow \theta(A)$ is an isomorphism; in this case we write $A \models_\Sigma c$. \square

Notice that T'' may be the empty theory. The following picture of the situation in this definition may help:

$$\begin{array}{ccc} T'' & \xrightarrow{F} & T' & \text{Sign}(T') & \xrightarrow{\theta} & \Sigma \\ & & & & & \\ T''^* & \xleftarrow{F^*} & T'^* & \text{Mod}(\Sigma') & \xleftarrow{\theta^*} & \text{Mod}(\Sigma) \\ & \xrightarrow{F^\$} & & & & \end{array}$$

In our set example, $F: T'' \rightarrow T'$ is the inclusion $\text{Set}: \mathbf{TRIV} \rightarrow \mathbf{SET}$ and $\theta: \text{Sign}(T') \rightarrow \Sigma$ is the signature morphism that underlies the theory inclusion $Etc: \mathbf{SET} \rightarrow \mathbf{SETCH}$. For any Σ -algebra A , it makes sense to ask whether θA satisfies T' and is F -free, as in Definition 19. It is not necessary that the institution involved is liberal.

Our work on constraints dates from the Spring of 1979, and was influenced by a lecture of Reichel in Poland in 1978 and by the use of functors to handle parametric data types by Thatcher, Wagner and Wright in [91]. Historically the first work in this area seems to be the little known 1971 paper [56] of Kaphengst and Riechel, which apparently considered the case of a single chain of theory inclusions. This was later generalized to “initially restricting algebraic theories” by Reichel [73]. There are three main differences between our “data constraints” and the “initial restrictions” of [73]: first, an initial restriction on an algebraic theory is a pair of subtheories, whereas we use a pair of theories with an arbitrary theory morphism between them, plus a signature morphism from the target theory. It seems very natural to use subtheories, but this does not give rise to an institution⁹; also, the added generality (of arbitrary theory morphisms) seems to permit some interesting additional examples. The second difference is simply that we are doing our work over an arbitrary institution. The third difference lies in the manner of adding constraints: whereas [73] defines a “canon” to be an algebraic theory together with some initial restrictions on it, we will define a new institution whose sentences on a signature Σ include both the old Σ -sentences and also Σ -constraints. This route has the technical advantage that both kinds of new sentence refer to the same signature. A slightly more complex kind of constraint is used by Ehrig, Wagner and Thatcher in [23]; in addition to $\theta: \text{Sign}(T') \rightarrow \Sigma$, they also have $\theta_0: \Sigma_0 \rightarrow \Sigma$, in order to deal with “derive.” To make this work, the model functor Mod must preserve pushouts.

We now show that constraints behave like sentences even though they have a very different structure. Like sentences, they impose restrictions on the allowable models. Moreover, a signature morphism from Σ to Σ' translates Σ -constraints to Σ' -constraints, just as it translates from Σ -sentences to Σ' -sentences.

Definition 20 Let $\phi: \Sigma \rightarrow \Sigma'$ be a signature morphism and let $c = \langle F, \theta \rangle$ be a Σ -constraint. Then the translation of c by θ is the Σ' -constraint $\langle F, \theta; \phi \rangle$; we write this as $\phi(c)$. \square

It is the need for the translation of a constraint to be a constraint that leads to constraints in which θ is not an inclusion. We now prove the Satisfaction Condition for constraints.

Lemma 21 Constraint Satisfaction. If $\phi: \Sigma \rightarrow \Sigma'$ is a signature morphism, if c is a Σ -constraint, and if B is a Σ' -model then

$$B \models \phi(c) \text{ iff } \phi(B) \models c.$$

Proof: Suppose that $c = \langle F, \theta \rangle$ with $F: T'' \rightarrow T'$ and $\theta: \text{Sign}(T') \rightarrow \Sigma$. Then $B \models \phi(c)$ means that $B \models \langle F, \theta; \phi \rangle$, which means that $(\theta; \phi)(B)$ satisfies T' and is F -free. On the other hand, $\phi(B) \models c$ means that $\phi(\theta(B))$ satisfies T' and is F -free. To show that these are the same, we may show that $(\theta; \phi)(B) = \phi(\theta(B))$, which is just the functoriality of Mod . \square

Given an institution I , we can construct another institution having as its sentences both the sentences of I and also constraints. This construction is of particular interest for liberal institutions, where we know that there is always a counit morphism with the couniversal property.

Definition 22 Given an arbitrary institution I , construct the institution $\mathcal{C}(I)$ whose theories contain both constraints and I -sentences as follows: the category of signatures of $\mathcal{C}(I)$ is the category Sign of signatures of I ; if Σ is an I -signature, then $\text{Sen}_{\mathcal{C}(I)}(\Sigma)$ is the (disjoint) union of the collection $\text{Sen}_I(\Sigma)$ of all Σ -sentences from I with the collection of all Σ -constraints¹⁰; also $\text{Mod}(\Sigma)$ is the same for $\mathcal{C}(I)$ as for I ; we use the constraint translation of Definition 20 to define $\text{Sen}(\phi)$ on constraints; finally, satisfaction for $\mathcal{C}(I)$ is as in I for Σ -sentences from I , and is as in Definition 19 for Σ -constraints. \square

⁹Andrzej Tarlecki has shown us a counterexample.

¹⁰There are some foundational questions about the size of the closure of a constraint theory that we will ignore here; they can be solved for example by limiting the size of the category of signatures used in the original institution.

Proposition 23 If I is an institution, then $C(I)$ is an institution.

Proof: Clearly the first condition of Definition 1 is satisfied, since the signature category of $C(I)$ is that of I . For the second condition we must show the functorality of $Sen_{C(I)}$. Let $\phi: \Sigma \rightarrow \Sigma'$ in **Sign**; then $Sen_{C(I)}(\phi): Sen_{C(I)}(\Sigma) \rightarrow Sen_{C(I)}(\Sigma')$ is the disjoint union of the map $Sen_I(\theta)$ defined on $Sen_I(\Sigma)$ and the map sending Σ -constraints to their translations by ϕ . Since Sen_I is already functorial, it suffices to prove functorality of constraint translation. But this is obvious.

There is nothing to check for the third condition. This leaves the Satisfaction Condition: we already know that it is satisfied for $C(I)$ -sentences that are I -sentences; and Lemma 21 shows that it is satisfied for constraints. \square

This means that all the concepts and results of Section 2 can be applied to theories that include constraints as well as sentences, that is, to $C(I)$ -theories¹¹. We call such theories **constraining theories**. Thus, we get notions of presentation and closure, as well as theory. In particular, the Presentation and Closure Lemmas hold, and we also get the following important result, which enables us to glue together constraining theories:

Theorem 24 Given an institution with a [finitely] cocomplete category of signatures, then its category of constraining theories is also [finitely] cocomplete.

Proof: Immediate from Theorems 11 and 23. \square

Let us consider what this means for the equational institution. While rules of deduction for inferring that an equation is in the closure of a set of equations are familiar, we have no such general rules for constraints. However, it should be possible to obtain such rules (although they will not be first order, and in general may not be complete) because a constraint corresponds to an induction principle plus some inequalities [16, 69]; in particular, the constraint that sets are to be interpreted freely gives us all the consequences of the induction principle for sets. In more detail, this constraint for sets demands that all elements of sort **Set** are generated by the operators \emptyset , $\{-\}$ and $-\cup-$, which can be expressed as a principle of structural induction over these generators (notice that structural induction is not first order). The constraint also demands that two elements of sort **Set** are *unequal* unless they can be proved equal using the given equations. One way to express this distinctness is to add a new boolean-valued binary operator on the new sort, say \equiv , with some new equations such that $t \equiv t' = false$ iff $t \neq t'$, and also such that $true = false$ does not hold [69]. Or we might consider an institution with disequations.

Let now us consider an easier example, involving equational theories with the following three (unconstrained) presentations:

1. **E** – the empty theory: no sorts, no operators, no equations.
2. **N** – the theory with one sort **Nat**, one constant 0 , one unary operator *inc*, and no equations.
3. **NP** – the theory with one sort **Nat**, two constants 0 and 1 , one unary operator *inc*, one binary infix operator $+$, and equations $0 + n = n$, $inc(m) + n = inc(m + n)$, $1 = inc(0)$.

Let Σ^N and Σ^{NP} be the signatures of **N** and **NP** respectively, and let $F^N: \mathbf{E} \rightarrow \mathbf{N}$ and $F^{NP}: \mathbf{N} \rightarrow \mathbf{NP}$ denote the inclusion morphisms. Now **NP** as it stands has many different interpretations, for example the integers modulo 10, or the truth values with $0 = false$, $1 = true$, $inc(false) = false$, $inc(true) = true$, $false + n = false$, $true + n = true$. In the latter model, $+$ is not commutative. In order to get the *standard* model suggested by the notation, we need to impose the constraint $\langle F^N, Sign(F^{NP}) \rangle$ on the theory **NP**. Then the only model (up to isomorphism) is the natural numbers with the usual addition. Note that the equation $m + n = n + m$ is satisfied

¹¹Note that there is no reason to suppose that $C(I)$ is liberal if I is.

by this model and therefore appears in the equational closure of the presentation; it is a property of $+$ provable by induction. There are also extra constraints in the closure, for example $\langle F', \phi \rangle$, where F' is the inclusion of the empty theory \mathbf{E} in the theory with $0, 1$ and $+$ with identity, associativity and commutativity equations, and ϕ is its inclusion into \mathbf{NP} . This constraint is satisfied in all models that satisfy the constraint $\langle F^N, \text{Sign}(F^{NP}) \rangle$. In this sense, the constraint on $0, 1$ and $+$ gives a derived induction principle. Further examples can be found in [16].

In general, the closure of a constraining presentation to a constraining theory adds some new equations derivable by induction principles corresponding to the constraints; and it also adds some new constraints corresponding to derived induction principles. The new equations are important in giving a precise semantics for programming methodology. For example, we may want to supply \mathbf{NP} as an actual parameter theory to a parameterized theory whose requirement demands a commutative binary operator. The new constraints seem less essential. For the Horn clause institution, constraints enable us to define predicates by induction.

What may be a promising approach to the proof-theoretic aspect of constraining theories is discussed by Clark [18], Reiter [74], McCarthy [67], and others who have been concerned with ways to get new sentences that are satisfied under a "closed world" assumption (the common sense assumption that the information actually given about a predicate is all the relevant information about that predicate; McCarthy identifies this with Occam's famous razor). We, of course, identify that "closed world" with the initial model of the given theory. Clark's scheme, called "predicate completion," is simply to infer the converse of any Horn clause. This is sound for ground queries in the institutions of conditional equations and first order Horn clauses (in the sense that all the answers thus obtained are true of the initial model); but it is not complete [82]. McCarthy calls his scheme "circumscription" and is interested in its application in the context of full first order logic [67]; he has shown that it is sound when minimal models exist, but it can be unsound when such models do not exist.

It is worth considering what happens if we add extra silly equations to \mathbf{NP} constrained by $\langle F^N, F^{NP} \rangle$. For example, $1 + n = n$ contradicts the constraint. In fact, if we add it, we simply get a constraining theory which is *inconsistent*, in the sense that it has no models.

There is a rather elegant construction that gives a somewhat different notion of constraining theory than that given by $\mathcal{C}(I)$. Let I be an institution. Then the signature category of $\mathcal{C}'(I)$ is the category of theories of I . If T is a theory of I , then a T -model for $\mathcal{C}'(I)$ is just a $\text{Sign}(T)$ -model in I that satisfies T [thus $\text{Mod}_{\mathcal{C}'(I)}(T) = \text{Mod}_I(T)$], and a T -sentence for $\mathcal{C}'(I)$ is a pair $\langle F: T'' \rightarrow T', \theta: T' \rightarrow T \rangle$ of morphisms of theories in I . Finally, if A is a T -model and $\langle F, \theta \rangle$ is a T -sentence, then $A \models \langle F, \theta \rangle$ iff θA is F -free. This use of pairs of theory morphisms for constraints is perhaps more elegant than Definition 19; but theories in $\mathcal{C}'(I)$ do not contain the consequences in I of these constraints – they contain only other (given and derived) constraints. As noted above, some important applications require the new I -sentences that follow from the added constraints.

3.3 Other Kinds of Constraint

Several variations of the data constraint notion have been proposed for the equational institution, and it seems worthwhile to generalize these variations to the level of institutions. In fact, very little of Section 3.2 depends on " F -freeness" in the definition of constraint satisfaction. This suggests weakening that notion. Recall that a Σ -model A satisfies a data constraint $c = \langle F: T \rightarrow T', \theta: \text{Sign}(T') \rightarrow \Sigma \rangle$ iff θA satisfies T' and is F -free, which means that $F^*(\theta A)$ has a free extension along F , and $(F_{F^*(\theta A)})^\# = \epsilon_{\theta A}: F^\$(F^*(\theta A)) \rightarrow \theta A$ is an isomorphism. For the equational institution, the most obvious ways of weakening the F -free concept are to require that $\epsilon_{\theta A}$ is only injective or only surjective, rather than bijective as for F -free. For $\epsilon_{\theta A}$ to be injective generalizes the "no confusion" condition of [17]; it means that no elements of θA are identified by the natural mapping from $(F^*(\theta A))^\$$. Some work of Poigné uses of this notion under a name like

“protecting.” For $\epsilon_{\theta A}$ to be surjective corresponds to what is called a “generating constraint” in [23]; it means that all elements of θA are generated by elements of $F^*(\theta A)$; this condition generalizes the “no junk” condition of [17]; it is also related to various concepts of “hierarchy constraint” found in the literature, e.g. [92] and [78]. Of course, injectivity and surjectivity of $\epsilon_{\theta A}$ together imply F -freeness. Now the general notion:

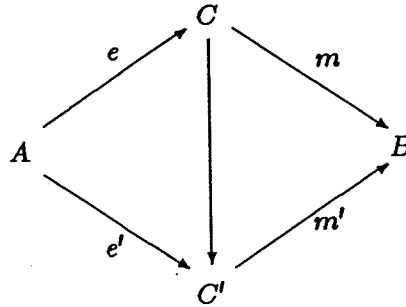
Definition 25 Let I be an institution, let \mathcal{A} denote a class of model morphisms for each signature Σ of I , let $c = \langle F: T \rightarrow T', \text{Sign}(T') \rightarrow \Sigma \rangle$ be a constraint from I , and let A be a Σ -model from I . Then A \mathcal{A} -satisfies c iff θA satisfies T' and has a free extension along F such that $(1_{F^*(\theta A)})^\# = \epsilon_{\theta A}$ lies in \mathcal{A} . (This could be further generalized to allow a different \mathcal{A} for each Σ -theory T .) \square

Proposition 26 Modifying the construction of $\mathcal{C}(I)$ to use \mathcal{A} -satisfaction gives an institution, and theories in this institution will be as cocomplete as I is. \square

The proofs are similar to those Proposition 23 and Theorem 24. In particular, this result implies that we can glue together theories with generating constraints using the usual colimit constructions, and we can do the specification language constructions of Clear [15]. To study generating constraints a little more closely, we assume that the morphisms in the categories of models of an institution have factorizations, in the following sense:

Definition 27 An image factorization situation for a category \mathbf{C} consists of

- (1) a class \mathcal{M} of monics and a class \mathcal{E} of epics in \mathbf{C} such that
- (2) both \mathcal{E} and \mathcal{M} are closed under composition,
- (3) all isomorphisms are in both \mathcal{M} and \mathcal{E} , and
- (4) every morphism f can be factored as $e; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ “uniquely up to isomorphism” in the sense that if $e'; m'$ is another factorization of f with $e' \in \mathcal{E}$ and $m' \in \mathcal{M}$, then there is a unique isomorphism from the center object C of $e; m$ (i.e., the target of e and the source of m) to the center object C' of $e'; m'$ such that the following diagram commutes:

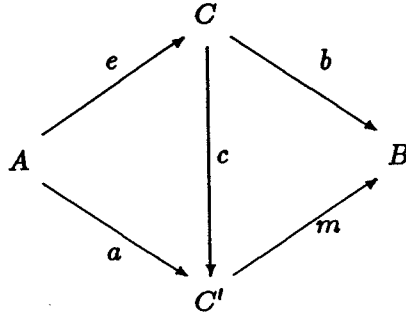


\square

This concept, which seems originally due to Isbell, is actually a bit stronger than our application demands, but it has been well-studied, and has many pleasant properties; for example, it was used in [37] for a general study of some institution-like concepts. Herrlich and Strecker [53] give more detail on image factorization situations, including proofs of the following:

Proposition 28 Let \mathcal{E}, \mathcal{M} be an image factorization for a category \mathbf{C} . Then

1. (Diagonal Fill-in Property.) If $c; b = a; m$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ and if C and C' are the center objects of the two factorizations, then there is a unique morphism $c: C \rightarrow C'$ such that $e; c = a$ and $c; m = b$.



2. If $f \in \mathcal{E}$ and $f \in \mathcal{M}$ then f is an isomorphism.

3. If $a; b \in \mathcal{E}$ then $b \in \mathcal{E}$, and if $a; b \in \mathcal{M}$ then $a \in \mathcal{M}$.

□

Definition 29 An institution I has **image factorization** iff it has an image factorization situation for each of its categories of Σ -models. Let I be such an institution, let $F: T \rightarrow T'$ be a theory morphism and let A be a T' -model. Then A is F -generated iff there is a morphism $e: F^\sharp(F^*(A)) \rightarrow A$ in \mathcal{E} with the couniversal property. □

The following is a direct consequence of the definitions.

Fact 30 Let I be an institution with image factorization, let $F: T \rightarrow T'$ be a theory morphism, and let A satisfy T' . Then A is F -generated iff it satisfies the \mathcal{E} -constraint $\langle F, 1_{\text{Sign}(T')} \rangle$. □

We next show that F -generation in this sense is (essentially) equivalent to a condition called F -generation by Ehrig, Wagner and Thatcher in [23], but which we call F -prime¹² (our F -prime concept differs from their F -generation concept only in that we replace equality by isomorphism, and of course in that their concept is restricted to the equational institution). The following result greatly increases our confidence that the concept in question has been correctly captured in the institutional framework:

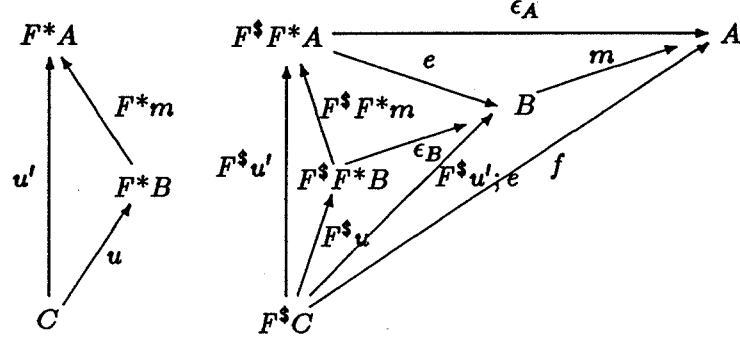
Proposition 31 Let I be an institution with image factorizations, let $F: T \rightarrow T'$ be a theory morphism, and let A and B be T' -models. Call A F -prime iff whenever $m: B \rightarrow A$ is in \mathcal{M} and $F^*m: F^*B \rightarrow F^*A$ is an isomorphism, then m is an isomorphism. Then A is F -prime iff A is F -generated.

Proof: Assume that A is F -generated, that $F^*m: F^*B \rightarrow F^*A$ is an isomorphism, and that $m: B \rightarrow A$ is in \mathcal{M} ; then we must show that m is an isomorphism. We know that $\epsilon_A: F^\sharp(F^*A) \rightarrow A$ lies in \mathcal{E} . Then the desired conclusion follows from the facts that the m lies in both \mathcal{M} and \mathcal{E} , using (2) and (3) of Proposition 28, and that the left upward arrow in the diagram below is an isomorphism.

$$\begin{array}{ccc}
 & \epsilon_A & \\
 F^\sharp(F^*A) & \longrightarrow & A \\
 \uparrow & & \uparrow m \\
 F^\sharp(F^*B) & \xrightarrow{\epsilon_B} & B
 \end{array}$$

¹²Our terminology is from [86]; intuitively, A is F -prime if it cannot have a proper (i.e., non-isomorphic) \mathcal{M} -subobject $m: B \rightarrow A$ unless m is already proper when viewed through F^* .

For the converse, we must show that if ϵ_A is not in \mathcal{M} then there is some $m: B \rightarrow A$ in \mathcal{M} not an isomorphism such that F^*m is an isomorphism. The proof is a chase of the following diagram:



where $e; m$ is a factorization of ϵ_A with center object B , where u' is defined by the universal property of ϵ_A using the morphism f , and then u is defined by the universal property of ϵ_B using the morphism $F^*u'; e$. Then m is not an isomorphism since ϵ_A is not in \mathcal{E} , and all triangles commute, using various properties and definitions.

In order to show that $F^*B \cong F^*A$, we show that $\epsilon_B; m$ satisfies the universal property for ϵ_A , i.e., given C and $f: F^*C \rightarrow A$, there is a unique $u: B \rightarrow F^*B$ such that $F^*u; \epsilon_B; m = f$. Existence follows from the above diagram. For the uniqueness, assume $v: C \rightarrow F^*B$ such that $F^*v; \epsilon_B; m = f$. Then $F^*v; \epsilon_B = F^*u; \epsilon_B (= F^*u'; e)$, so $v = u$, by the universal property of ϵ_B . This implies that there is a unique isomorphism $h: F^*B \rightarrow F^*A$ such that $u' = u; h$. But F^*m is such a morphism. \square

(We thank Andrzej Tarlecki for pointing out an error in an earlier version of this proof; the reader may also wish to see [86] for a proof of a similar result for the special case of “algebraic” institutions.)

4 Institution Morphisms — Using More than One Institution

After the work of Section 3.2, we know how to express constraints in any institution; in particular, we can use constraints in the equational institution to specify parameterized abstract data types. We can also give loose specifications in any institution. Constraints seem most natural for liberal institutions, since we are guaranteed that satisfaction is possible. But liberality is a rather significant restriction, since non-liberal institutions can often be more expressive than liberal institutions; for example, if one adds negation to the equational institution, it ceases to be liberal; also, first order logic is not liberal. Thus, the ambitious specifier might want both the rich expressive power of first order logic and also the data structure definition power of the equational institution. This section shows he can eat his cake and have it too. The basic idea is to precisely describe a relationship between two institutions in the form of an institution morphism, and then permit constraints that use theories from the second institution as an additional kind of sentence in this “duplex” institution; we can even have more than two institutions. There are also other uses for institution morphisms; in particular, we will use this concept in considering when a theorem prover for one institution is sound for theories from another institution.

4.1 Institution Morphisms

Let us consider the relationship between the institution of first order logic with equality, $\mathcal{FOL}\mathcal{EQ}$, and the equational institution, \mathcal{EQ} . First of all, any first order signature can be reduced to an equational signature just by forgetting all its predicate symbols. Secondly, any equation can

be regarded as a first order sentence just by regarding the equal sign in the equation as the distinguished binary predicate symbol that is interpreted as actual identity in models (the equal sign in the equations of the equational institution is *not* a predicate symbol, but just a punctuation symbol that separates the left and right sides). Thirdly, any first order model can be viewed as an algebra just by forgetting all its predicates. These three functors are the substance of an institution morphism $\mathcal{FOEQ} \rightarrow \mathcal{EQ}$.

Definition 32 (More concrete version of Definition 13) Let I and I' be institutions. Then an institution morphism $\Phi: I \rightarrow I'$ consists of

1. a functor $\Phi: \text{Sign} \rightarrow \text{Sign}'$,
2. a natural transformation $\alpha: \text{Sen}' \Rightarrow \text{Sen}$, and
3. a natural transformation $\beta_\Sigma: \text{Mod}(\Sigma) \rightarrow \text{Mod}'(\Phi(\Sigma))$

such that the following Satisfaction Condition holds

$$A \models_\Sigma \alpha_\Sigma(e') \text{ iff } \beta_\Sigma(A) \models'_{\Phi(\Sigma)} e'$$

for any Σ -model A from I and any $\Phi(\Sigma)$ -sentence e' from I' . \square

We leave it as an exercise for the interested reader to prove that this definition agrees with Definition 13. In fact, α is η ; *Right* and β is η ; *Left*. The reader who prefers her definitions as concrete as possible may wish to see what “naturalness” means in conditions 2. and 3. above: the following two diagrams spell this out, for $\phi: \Sigma \rightarrow \Sigma'$ a signature morphism in I :

$$\begin{array}{ccc} \text{Sen}'(\Phi(\Sigma)) & \xrightarrow{\alpha_\Sigma} & \text{Sen}(\Sigma) \\ \text{Sen}'(\Phi(\phi)) \downarrow & & \downarrow \text{Sen}(\phi) \\ \text{Sen}'(\Phi(\Sigma')) & \xrightarrow{\alpha_{\Sigma'}} & \text{Sen}'(\Sigma') \end{array}$$

$$\begin{array}{ccc} \text{Mod}(\Sigma) & \xrightarrow{\beta_\Sigma} & \text{Mod}'(\Phi(\Sigma)) \\ \text{Mod}(\phi) \downarrow & & \downarrow \text{Mod}'(\Phi(\phi)) \\ \text{Mod}(\Sigma') & \xrightarrow{\beta_{\Sigma'}} & \text{Mod}'(\Phi(\Sigma')) \end{array}$$

The reader may also verify that $\Phi: \mathcal{FOEQ} \rightarrow \mathcal{EQ}$ as sketched in the first paragraph of this subsection really is an institution morphism; this amounts to little more than verifying the two diagrams above and the Satisfaction Condition.

Just as there is an elegant equational way to state the Satisfaction Condition for institutions (see the beginning of Section 2.3), there is also one for institution morphisms. From the basic Satisfaction Condition

$$m \models_\Sigma \alpha_\Sigma(e') \text{ iff } \beta_\Sigma(m) \models' e'$$

for m an Σ -model from I and e' a $\Phi(\Sigma)$ -sentence from I' , we get

$$M \models_\Sigma \alpha_\Sigma(e') \text{ iff } \beta_\Sigma(M) \models' e'$$

for M a collection of Σ -models, and therefore

$$\{e' \mid \beta_\Sigma(M) \models' e'\} = \{e' \mid M \models_\Sigma \alpha_\Sigma(e')\}$$

so that

$$(\beta_{\Sigma}M)^* = \{e' \mid \alpha_{\Sigma}(e') \in M^*\},$$

which we can rewrite more elegantly as

$$(\beta M)^* = \alpha^{-1}(M^*).$$

Similarly, we can show that

$$(\alpha E)^* = \beta^{-1}(E^*).$$

We now turn to the question of when it is sound to use a theorem prover for one institution on (translations of) sentences from another. The basic concept is given by the following, which is a kind of surjectivity condition:

Definition 33 An institution morphism $\Phi: I \rightarrow I'$ is **sound** iff for every signature Σ' and every Σ' -model A' from I' , there are a signature Σ and a Σ -model A from I such that $A' = \beta_{\Sigma}(A)$. \square

In particular, the institution morphism $\mathcal{FOELQ} \rightarrow \mathcal{ELQ}$ discussed above is sound.

Proposition 34 If $\Phi: I \rightarrow I'$ is a sound institution morphism and if E' is a collection of Σ' -sentences from I' , then a Σ' -sentence e' is in E'^* iff $\alpha_{\Sigma}(e')$ is in $\alpha_{\Sigma}(E')^*$, where Σ is a signature from I such that $\Sigma' = \Phi(\Sigma)$.

Proof: The two conditions to be shown equivalent are:

1. for all m' , if $m' \models E'$ then $m \models e'$.
2. for all A , if $A \models \alpha E'$ then $A \models \alpha e'$.

Let us assume the second condition and prove the first from it. Thus we assume that $m' \models E'$. Now using soundness of Φ , we get a model m from I such that $m' = \beta_{\Sigma}(m)$. Because Φ is an institution morphism, we have that $m \models \alpha_{\Sigma}(E)$. Then condition 2. gives us that $m \models \alpha_{\Sigma}(e')$. Therefore $m' \models e'$, and we are done.

For the converse, assume that $A \models \alpha_{\Sigma}(E')$ and let $m' = \beta_{\Sigma}(m)$. Then $m' \models E'$ because Φ is a morphism. Now condition 1. gives us that $m' \models e'$, and again using that Φ is a morphism, we conclude that $m \models \alpha(e')$. \square

It now follows using the institution morphism from \mathcal{FOELQ} to \mathcal{ELQ} , that a theorem prover for first order logic with equality can be used for equational theories.

We can rewrite the conclusion of the above proposition more compactly as

$$E'^{**} = \alpha_{\Sigma}^{-1}(\alpha_{\Sigma}(E'))^{**}$$

and from this we can derive

Corollary 35 If $\Phi: I \rightarrow I'$ is a sound institution morphism and if E is a collection of Σ -sentences in I , then

$$\alpha^{-1}(E^{**}) = (\alpha^{-1}E)^{**}.$$

Proof: This follows by substituting $\alpha^{-1}E$ for E in Proposition 34 and simplifying. \square

Notice that an institution morphism $\Phi: I \rightarrow I'$ induces a functor $\Phi: \mathbf{Th}_I \rightarrow \mathbf{Th}_{I'}$ on the corresponding categories of theories by sending a Σ -theory T to the $\Phi(\Sigma)$ -theory $\beta_{\Sigma}(T^*)^*$; assuming that $T = E^{**}$, we know by the above corollary that $\Phi(T) = \alpha_{\Sigma}(E^{**}) = (\alpha_{\Sigma}^{-1}E)^{**}$. We also have the following useful result about this translation of theories:

Theorem 36 Given a sound institution morphism $\Phi: I \rightarrow I'$ such that $\Phi: \mathbf{Sign} \rightarrow \mathbf{Sign}'$ is [finitely] cocontinuous and such that both $\mathbf{Mod}: \mathbf{Sign} \rightarrow \mathbf{Cat}^{op}$ and $\mathbf{Mod}': \mathbf{Mod}' \rightarrow \mathbf{Cat}^{op}$ preserve colimits, then $\Phi: \mathbf{Th}_I \rightarrow \mathbf{Th}_{I'}$ is also [finitely] cocontinuous.

Proof: Let $D: G \rightarrow \mathbf{Th}_I$ be a diagram with $D_n = \langle \Sigma_n, E_n \rangle$, and let $\gamma: D \Rightarrow T$ be its colimit in \mathbf{Th}_I where $T = \langle \Sigma, E \rangle$. Then by the construction of Theorem 11, Σ is the colimit of the diagram $D; \mathbf{Sign}$ in \mathbf{Sign} , and

$$E = \left(\bigcup_{n \in |G|} \gamma_n(E_n) \right)^*.$$

Now let $D': G \rightarrow \mathbf{Th}_{I'}$ be the diagram $D; \Phi$. Then by assumption, $\gamma' = \Phi(\gamma): D; \Phi \Rightarrow \Sigma'$ is a colimit cone in \mathbf{Sign}' , where $\Sigma' = \Phi(\Sigma)$. Because the forgetful functor \mathbf{Sign}' also reflects colimits, we know that $\gamma': D' \Rightarrow T'$ is also a colimit cone, where $T' = \langle \Sigma', E' \rangle$ with

$$E' = \left(\bigcup_{n \in |G|} \gamma'_n(E'_n) \right)^*$$

where

$$E'_n = \Phi(E_n) = \beta_{\Sigma_n}(E_n^*)^*.$$

What we wish to show is that $\langle \Sigma', E' \rangle = \Phi(\langle \Sigma, E \rangle)$, which reduces to showing that $E' = \Phi(E)$. So, let us calculate:

$$\begin{aligned} E' &= \left(\bigcup_n \gamma'_n E'_n \right)^{**} && \text{by definition} \\ &= \left\{ \bigcup_n \gamma'_n [\beta_{\Sigma_n} E_n^*]^* \right\}^{**} && \text{by definition} \\ &= \left\{ \bigcup_n \gamma'_n [\beta_{\Sigma_n} (\gamma_n (\gamma_n E_n)^*)]^* \right\}^{**} && \text{by Proposition 4} \\ &= \left\{ \bigcup_n \gamma'_n [\gamma'_n (\beta_{\Sigma} (\gamma_n E_n)^*)]^* \right\}^{**} && \text{by naturality of } \beta \\ &= \left\{ \bigcup_n [\beta_{\Sigma} (\gamma_n E_n)^*]^* \right\}^{**} && \text{by Proposition 4 again} \\ &= \left\{ \bigcup_n \alpha_{\Sigma}^{-1} (\gamma_n E_n)^{**} \right\}^{**} && \text{by the Satisfaction Condition} \\ &= \left\{ \bigcup_n [\alpha_{\Sigma}^{-1} \gamma_n E_n]^{**} \right\}^{**} && \text{by Corollary 35} \\ &= \left\{ \bigcup_n \alpha_{\Sigma}^{-1} \gamma_n E_n \right\}^{**} && \text{by 11. of Proposition 3} \end{aligned}$$

In the applications of Proposition 4 above, we need to know that γ and γ' each induce surjective functions on models. This follows because each γ and γ' are colimit injections, and so therefore also are $\mathbf{Mod}(\gamma)$ and $\mathbf{Mod}(\gamma')$, by hypothesis. This then implies that they are surjective (on models).

For the right hand side, we note that

$$\begin{aligned} \Phi(E) &= \left\{ \beta_{\Sigma} [\bigcup_n \gamma_n E_n]^{***} \right\}^* && \text{by definitions} \\ &= \left\{ \beta_{\Sigma} [\bigcap_n (\gamma_n E_n)^*]^{**} \right\}^* && \text{by 8. of Proposition 3} \\ &= \alpha_{\Sigma}^{-1} [\bigcap_n (\gamma_n E_n)^*]^* && \text{by the Satisfaction Condition} \\ &= \alpha_{\Sigma}^{-1} [\bigcup_n \gamma_n E_n]^{**} && \text{by 8. of Proposition 3} \\ &= \left\{ \alpha_{\Sigma}^{-1} \bigcup_n \gamma_n E_n \right\}^{**} && \text{by Corollary 35} \\ &= \left\{ \bigcup_n \alpha_{\Sigma}^{-1} \gamma_n E_n \right\}^{**} && \text{since inverse image commutes with union} \end{aligned}$$

□

Actually, this proof also supports a stronger conclusion: Φ preserves on theories whatever colimits it preserves on signatures. Theorem 36 implies that if a large theory in the source institution is expressed as a colimit of smaller theories, then the corresponding theory in the target institution can also be so expressed. This can help with the practicalities of using a theorem prover for one institution on sentences from another, by modularizing the proofs. Some experiments using OBJ3 [47] as a theorem prover have exploited this viewpoint [34]. It would also be interesting to see if the hypotheses on this theorem can be weakened. For example, the assumption that \mathbf{Mod} preserves colimits seems stronger than is needed to get that $\mathbf{Mod}(\gamma)$ is surjective on objects; however, it is worth noting that this assumption holds in most institutions that have been studied.

4.2 Duplex Institutions

This subsection gives a construction for an institution whose theories can contain both sentences from an institution I and constraints from another institution I' . For example, I might be first order logic and I' equational logic. What is needed to make the construction work is an institution morphism $\Phi: I \rightarrow I'$ expressing the relationship between the two institutions. This idea was introduced informally and applied to some examples in [16]. Note that all the results of this section generalize to the notion of \mathcal{A} -satisfaction given in Definition 25, although they are stated for the case of data constraints, i.e., for the case where \mathcal{A} is the class of isomorphisms.

Definition 37 Let $\Phi: I \rightarrow I'$ be an institution morphism and let Σ be a signature from I . Then a Σ -duplex constraint is a pair

$$c = \langle F: T'' \rightarrow T', \theta: \text{Sign}(T') \rightarrow \Phi(\Sigma) \rangle,$$

where F is a theory morphism from I' and θ is signature morphism from I . Furthermore, a Σ -model A from I satisfies the duplex constraint c iff $\theta(\beta_\Sigma(A))$ satisfies T' and is F -free; as usual, we write $A \models_\Sigma c$. \square

The following picture of the general situation in this definition may help.

$$\begin{array}{ccccc} T'' & \xrightarrow{F} & T' & & \text{Sign}(T') \xrightarrow{\theta} \Phi(\Sigma) \\ & & & & \\ T''^* & \xleftarrow{F^*} & T'^* & \xleftarrow{\theta^*} & \text{Mod}'(\text{Sign}(T')) \xleftarrow{\beta_\Sigma} \text{Mod}(\Phi(\Sigma)) \xleftarrow{\beta_\Sigma} \text{Mod}(\Sigma) \\ & & & & \end{array}$$

It is worth remarking that in practice T'' and T' could be just presentations, as long as F is a theory morphism in I' .

Definition 38 Let $\Phi: I \rightarrow I'$ be an institution morphism, let Σ be a signature from I , let $c = \langle F, \theta \rangle$ be a Σ -duplex constraint, and let $\phi: \Sigma \rightarrow \Sigma'$ be a signature morphism from I . Then the translation of c by ϕ is the Σ' -duplex constraint $\phi c = \langle F, \theta; \Phi(\phi) \rangle$. \square

Lemma 39 Satisfaction. Let $\Phi: I \rightarrow I'$ be an institution morphism, let Σ be a signature from I , let $c = \langle F, \theta \rangle$ be a Σ -duplex constraint, let $\phi: \Sigma \rightarrow \Sigma'$ be a signature morphism from I , and let B be a Σ -model from I . Then

$$B \models_{\Sigma'} \phi c \text{ iff } \phi B \models_\Sigma c.$$

Proof: Suppose that $c = \langle F, \theta \rangle$ with $F: T'' \rightarrow T'$ and $\theta: \text{Sign}(T') \rightarrow \Phi(\Sigma)$. Then $B \models \phi c$ means that $B \models \langle F, \theta; \Phi(\phi) \rangle$, which means that $\theta(\Phi(\phi)(\beta_{\Sigma'}(B)))$ satisfies T' and is F -free. On the other hand, $\phi B \models c$ means that $\theta(\beta_\Sigma(\phi B))$ satisfies T and is F -free. To show these are the same, we need only know that $\Phi(\phi)(\beta_{\Sigma'}(B)) = \beta_\Sigma(\phi B)$, which is just the naturality of β (the second diagram after Definition 32). \square

Definition 40 Let $\Phi: I \rightarrow I'$ be an institution morphism and let Σ be a signature from I . Then the duplex institution over Φ , denoted $\mathcal{D}(\Phi)$ has: its signatures those from I ; its Σ -sentences the Σ -sentences from I plus the Σ -duplex constraints; its Σ -models the Σ -models from I ; and satisfaction is as defined separately for the sentences from I and the duplex constraints. \square

For example, if we let $\Phi: \mathcal{FOEQ} \rightarrow \mathcal{EQ}$ as above, then we can write loose specifications in full first order logic, and at the same time we can impose constraints for defining data structures in the liberal institution of equational logic. This actually gives quite a powerful framework for specification, as shown by some examples written in Clear [16].

Theorem 41 If I is an institution and $\Phi: I \rightarrow I'$ is an institution morphism, then $\mathcal{D}(\Phi)$ is an institution. \square

The proof is similar to that of Proposition 23 and is omitted here. By Theorem 11, this result implies

Theorem 42 If the category of signatures of I is [finitely] cocomplete, then so is the category $\mathbf{Th}_{\mathcal{D}(\Phi)}$ of theories of the duplex institution $\mathcal{D}(\Phi)$. \square

Thus, we can do parameterized specification with constraining theories that use both sentences from I and duplex constraints constructed with I' , since these are the theories over the duplex institution $\mathcal{D}(\Phi)$. Of course, we can also use the more general notion of \mathcal{A} -satisfaction of constraints.

There is another much simpler way to use an institution morphism $\Phi: I \rightarrow I'$ to construct a new institution in which one simply permits sentences from either I or I' . For example, it may be convenient to use already existing equational theories when constructing new first order theories.

Definition 43 Let $\Phi: I \rightarrow I'$ be an institution morphism. Then $\mathcal{T}(\Phi)$ is the institution with: its signatures Σ those from I ; its Σ -sentences either Σ -sentences from I , or else pairs of the form $c = \langle T', \theta: \text{Sign}(T') \rightarrow \Phi(\Sigma) \rangle$, where T' is a theory from I' and θ is a signature morphism; the Σ -models of $\mathcal{T}(\Phi)$ are those of I ; and c is satisfied by a Σ -model A iff $\theta(\beta_{\Sigma}(A))$ satisfies T' . \square

This construction differs from that of Definition 40 primarily in the notion of satisfaction involved: in $\mathcal{D}(\Phi)$, satisfaction of a pair $\langle F: T'' \rightarrow T', \theta: \text{Sign}(T') \rightarrow \Phi(\Sigma) \rangle$ is constraint satisfaction, involving F -freeness, whereas in $\mathcal{T}(\Phi)$, satisfaction of $\langle T', \theta: \text{Sign}(T') \rightarrow \Phi(\Sigma) \rangle$ is simple satisfaction in I' .

Proposition 44 $\mathcal{T}(\Phi)$ is an institution. \square

4.3 Multiplex Institutions

Actually, we can use many different institutions all at once, some for various kinds of constraints, and some for additional expressive power; all that is needed is a morphism to each from some fixed base institution. Let I be an institution and let $\Phi_i: I \rightarrow I_i$ be institution morphisms for $1 \leq i \leq m+n$. Now define $\mathcal{P}(\Phi_1, \dots, \Phi_n; \Phi_{n+1}, \dots, \Phi_{n+m})$ to be the institution with: signatures those from I ; sentences either those from I , or else constraints $\langle F: T'' \rightarrow T', \theta: \text{Sign}(T') \rightarrow \Phi_i(\Sigma) \rangle$ with θ, T'', T' from I_i for some $1 \leq i \leq n$ or else pairs $\langle T, \theta: \text{Sign}(T) \rightarrow \Phi_i(\Sigma) \rangle$ with θ, T from I_i for some $n+1 \leq i \leq n+m$; with its models those of I ; and with satisfaction as usual for the I -sentences and for the constraints, and as in Definition 43 for the others. Notice that the institutions I_i for $i = 1, \dots, n$ can each use a different collection \mathcal{A}_i of morphisms to define constraint satisfaction. One can even use the same institution with different classes \mathcal{A}_i . For example, this means that one can glue together (with colimits) first order theories that use combinations of generating constraints, hierarchy constraints, and data constraints in the equational institution, as well as loose first order axioms.

For example, consider the diagram in the category of institutions, with $\Phi_1: \mathcal{FOEQ} \rightarrow \mathcal{EQ}$ and $\Phi_2: \mathcal{HCL} \rightarrow \mathcal{EQ}$, where \mathcal{FOEQ} is first order logic with equality and \mathcal{HCL} is Horn clause

logic. Then we can define a data type, such as **STRING-OF-NAT**, using initiality in $\mathcal{E}\mathcal{Q}$, and inductively define a predicate, such as *even* in \mathcal{HCL} , e.g., by

$$\begin{aligned} & \text{even}(0) \\ & \text{even}(n) \Rightarrow \text{even}(n + 2) \end{aligned}$$

These both use data constraints, and the institutions are liberal (for \mathcal{HCL} , “data” is a misnomer, since we are inductively defining predicates, not data). We can then define extra functions and predicates (“loosely” interpreted) in $\mathcal{FOE}\mathcal{Q}$, using both universal and existential quantifiers as convenient.

5 Summary and Future Research

We have formalized the intuitive notion of “logical system” or “abstract model theory” with the institution concept, and have shown that institutions whose signatures have finite colimits are useful in programming methodology. Many properties have been proved that relate classes of models and of sentences. We have also shown how to define data types by several kinds of “constraint,” which are abstract induction principles, including data and generating constraints. Liberal institutions, where the forgetful functors induced by theory morphisms have corresponding free functors, are particularly suitable for instantiating constraints. We have also introduced the notion of an institution morphism and shown how to use it in determining (for example) when a theorem prover for one institution can be soundly used on (translations of) theories from another institution. Institution morphisms were also used in defining duplex and multiplex institutions, which formalize the simultaneous use of more than one institution; this permits combining the expressive capabilities of several institutions. We showed that many institution morphisms preserve finite colimits of theories, which means that they preserve theory structuring mechanisms. These mechanisms support generic modules for logical programming languages such as (pure) Prolog, OBJ and Eqlg, as well as specification languages like Clear, and program development systems like LIL.

This paper has dealt with sets of sentences, using satisfaction to define the closure of a set of sentences, but has said little about proofs. In fact, it is attractive to include proofs in the notion of institution, by letting an inference step be an arrow from a tuple of sentences to another sentence. Such arrows can be composed, yielding a category, as in Lawvere theories. Then $\text{Sen}(\Sigma)$ becomes a category, and Sen takes a signature morphism covariantly to a functor ([39] gives the definition and some results). Proof theory has recently taken on new interest in logic programming and type theory (in the sense of Martin-Löf) because of doctrines like “computation is deduction” and “constructive proofs yield programs.” In such a setting, complexity of proof translates into complexity of computation. This provides further motivation for studying institutions with proofs.

Some problems mentioned but not discussed in detail in this paper include the following:

1. It can be tedious to show that something really is an institution, particularly to check the Satisfaction Condition. This has led us to develop a method for constructing institutions from simpler structures called “charters,” in a way that guarantees the Satisfaction Condition [39]. Charters are more concrete than institutions, since they specify the syntax of sentences. Parchments go even further in this direction [39].
2. An open problem mentioned in a preliminary version of this paper [36], to characterize when institutions are liberal and relate this to work of Mahr and Markowsky [64], has been solved for “strongly liberal” institutions (where $F^{\$}A$ is F -generated) by Tarlecki [86].
3. More thorough explorations of the properties of the category of institutions, including some completeness results appear in [39] and [85]; see also [88].

We conclude this section by mentioning two broad research programs that arise naturally from this paper. One is to extend the range of constructions treated institutionally; for example, Sannella and Tarlecki [76] discuss “observational equivalence,” i.e., abstract machines, in an arbitrary institution. We recommend doing as much computer science as possible in as general an institution as possible. The second program is to carry out as much abstract model theory as possible in the more general framework of institutions, as begun by Tarlecki [85].

A Examples of Institutions

This appendix first presents the details needed to establish equational logic (i.e., many-sorted general algebra) as an institution. Then, using this material, the second subsection develops first order logic and some related institutions.

A.1 General Algebra

This section is quick review of many-sorted general algebra, which provides a first example of the institution concept, and also aids in working out the details of several other institutions. If you know all about general algebra, you can skip to Section A.2. We use notation of [30] (see also [45]) based on “indexed sets,” in contrast to the more complex notations of [54], [8] and [10]. If I is a set (of “indices”), then an I -indexed set A (also called a family of sets indexed by I) is just an assignment of a set A_i to each index i in I . If A and B are I -indexed sets, then a mapping, map, or morphism of I -indexed sets, $f: A \rightarrow B$, is just an I -indexed family of functions, $f_i: A_i \rightarrow B_i$ for $i \in I$. There is an obvious composition of I -indexed mappings, $(f;g)_i = f_i;g_i$ (note that the composition $f;g$ of functions f, g is defined by $(f;g)(x) = g(f(x))$). This gives a category Set_I of I -indexed sets. We may use the notations $A = \langle A_i \mid i \in I \rangle$ for an I -indexed set with components A_i and $f = \langle f_i \mid i \in I \rangle$ for an I -indexed mapping $A \rightarrow B$ of I -indexed sets where $f_i: A_i \rightarrow B_i$. All the basic concepts of set theory extend component-wise to I -indexed sets. Thus, $A \subseteq B$ means that $A_i \subseteq B_i$ for each i in I , $A \cap B = \langle A_i \cap B_i \mid i \in I \rangle$, etc. This can be very helpful in simplifying notation.

A.1.1 Equational Signatures

Intuitively speaking, an equational signature declares some sort symbols (to serve as names for the various kinds of data) and some operator symbols (to serve as names for functions on data), each with a declaration that gives a list of input sorts and one output sort. Then a morphism between signatures should map sorts to sorts and operators to operators, preserving their input and output sorts.

Definition 45 An equational signature is a pair $\langle S, \Sigma \rangle$, where S is a set (of sort names) and Σ is a family of sets (of operator names), indexed by $S^* \times S$; we will often write just Σ instead of $\langle S, \Sigma \rangle$. σ in $\Sigma_{u,s}$ is said to have arity u , sort s , and rank u, s ; we may write $\sigma: u \rightarrow s$ to indicate this. \square

Definition 46 An equational signature morphism ϕ from a signature $\langle S, \Sigma \rangle$ to a signature $\langle S', \Sigma' \rangle$ is a pair $\langle f, g \rangle$ consisting of a map $f: S \rightarrow S'$ of sorts and an $S \times S^*$ -indexed family of maps $g_{u,s}: \Sigma_{u,s} \rightarrow \Sigma'_{f^*(u),f(s)}$ of operator symbols, where $f^*: S^* \rightarrow S'^*$ is the extension of f to strings¹³. We will sometimes write $\phi(s)$ for $f(s)$, $\phi(u)$ for $f^*(u)$, and $\phi(\sigma)$ or even $\phi\sigma$ for $g_{u,s}(\sigma)$ when $\sigma \in \Sigma_{u,s}$. \square

¹³This extension is defined by: $f^*(\lambda) = \lambda$, where λ denotes the empty string and $f^*(ws) = f^*(w)f(s)$, for w in S^* and s in S .

In the language of programming methodology, a signature declares the interface for a module, package, capsule, object, abstract machine, abstract data type, ... (unfortunately, there is much variation of terminology in this area). Signature morphisms are useful for expressing the *binding* of an actual parameter to the formal parameter of a parameterized software module.

Definition 47 The category of equational signatures, denoted **Sig**, has equational signatures as its objects, and has equational signature morphisms as its morphisms. The identity morphism on $\langle S, \Sigma \rangle$ is the corresponding pair of identity maps, and the composition of morphisms is the composition of their corresponding components as maps. (This clearly forms a category.) \square

For many purposes, such as showing that the usual term algebra construction yields an initial algebra, it is important to make some additional assumption about signatures, such as that whenever $\sigma: w \rightarrow s$ and $\sigma: w' \rightarrow s'$ with $\text{length}(w) = \text{length}(w')$, then $w = w'$ and $s = s'$. We shall assume that all subsequent signatures satisfy this assumption. Order-sorted algebra allows relaxing this assumption [44]. (For practical purposes, it suffices to assume that expressions are somehow disambiguated by a parser, possibly with user assistance.)

A.1.2 Algebras

Given a signature Σ , a Σ -algebra interprets each sort symbol as a set and each operator symbol as a function. Algebras, in the intuition of programming, correspond to concrete data types, i.e., to data representations in the sense of [55].

Definition 48 Let $\langle S, \Sigma \rangle$ be a signature. Then a Σ -algebra A is an S -indexed family of sets $|A| = \langle A_s \mid s \in S \rangle$ called the carriers of A , together with an $S^* \times S$ -indexed family α of maps $\alpha_{u,s}: \Sigma_{u,s} \rightarrow [A_u \rightarrow A_s]$ for u in S^* and s in S , where $A_{s_1 \dots s_n} = A_{s_1} \times \dots \times A_{s_n}$ and $[A \rightarrow B]$ denotes the set of all functions from A to B . (We may sometimes write A for $|A|$ and $|A|_s$ for A_s .) For $u = s_1 \dots s_n$, for σ in $\Sigma_{u,s}$ and for $\langle a_1, \dots, a_n \rangle$ in A_u we will write $\sigma(a_1, \dots, a_n)$ for $\alpha_{u,s}(\sigma)(a_1, \dots, a_n)$ if there is no ambiguity. \square

Definition 49 Given a signature Σ , a Σ -homomorphism from a Σ -algebra $\langle A, \alpha \rangle$ to another $\langle A', \alpha' \rangle$ is an S -indexed map $f: A \rightarrow A'$ such that for all σ in $\Sigma_{u,s}$ and all $a = \langle a_1, \dots, a_n \rangle$ in A_u the homomorphism condition

$$f_s(\alpha(\sigma)(a_1, \dots, a_n)) = \alpha'(\sigma)(f_{s_1}(a_1), \dots, f_{s_n}(a_n))$$

holds, i.e., such that the following diagram commutes:

$$\begin{array}{ccc} A_u & \xrightarrow{f_u} & A'_u \\ \alpha(\sigma) \downarrow & & \downarrow \alpha'(\sigma) \\ A_s & \xrightarrow{f_s} & A'_s \end{array}$$

\square

Definition 50 The category Alg_Σ of Σ -algebras has Σ -algebras as objects and Σ -homomorphism as morphisms; composition and identity in Alg_Σ are composition and identity as maps. (This clearly forms a category). \square

Alg extends to a functor on the category **Sig** of signatures, associating with each signature Σ the category $\text{Alg}(\Sigma)$ of all Σ -algebras, and also defining the effect of signature morphisms on algebras:

Definition 51 The functor $\mathbf{Alg}: \mathbf{Sig} \rightarrow \mathbf{Cat}^{op}$ sends each signature Σ to the category $\mathbf{Alg}\Sigma$ of all Σ -algebras, and sends each signature morphism $\phi = \langle f: S \rightarrow S', g: \Sigma \rightarrow \Sigma' \rangle$ to the functor $\mathbf{Alg}(\phi): \mathbf{Alg}\Sigma' \rightarrow \mathbf{Alg}\Sigma$ that

1. sends a Σ -algebra $\langle A', \alpha' \rangle$ to the Σ -algebra $\langle A, \alpha \rangle$ with $A_s = A'_f(s)$ and $\alpha = g; \alpha'$, and
2. sends a Σ' -homomorphism $h': A' \rightarrow B'$ to the Σ -homomorphism

$$\mathbf{Alg}(\phi)(h') = h: \mathbf{Alg}(\phi)(A') \rightarrow \mathbf{Alg}(\phi)(B') \text{ defined by } h_s = h'_{f(s)}.$$

It is often convenient to write $\phi(A')$ or $\phi A'$ for $\mathbf{Alg}(\phi)(A')$ and to write $\phi(h')$ for $\mathbf{Alg}(\phi)(h')$. \square

If S is the sort set of Σ , then there is a forgetful functor $U: \mathbf{Alg}\Sigma \rightarrow \mathbf{Set}_S$ which sends each algebra to its S -indexed family of carriers, and sends each Σ -homomorphism to its underlying S -indexed map. (Functoriality follows from the fact that $U = \mathbf{Alg}(\phi)$, where ϕ is the signature inclusion $\langle S, \emptyset \rangle \rightarrow \Sigma$.)

For each S -indexed set X , there is a free algebra (also called a “term” or “word” algebra), denoted $T_\Sigma(X)$, with $|T_\Sigma(X)|_s$ consisting of all the Σ -terms of sort s using “variable” symbols from X ; i.e., $(T_\Sigma(X))_s$ contains all the s -sorted terms with variables from X that can be constructed using operator symbols from Σ ; moreover, the S -indexed set $T_\Sigma(X)$ forms a Σ -algebra in a natural way. In order to make this more precise, we first consider special case, defining $(T_\Sigma)_s$ to be the least set of strings of symbols such that

1. $\Sigma_{\lambda,s} \subseteq T_{\Sigma,s}$ and
2. $\sigma \in \Sigma_{s_1 \dots s_n, s}$ and $t_i \in T_{\Sigma, s_i}$ imply that the string $\sigma(t_1, \dots, t_n)$ is in $T_{\Sigma, s}$.

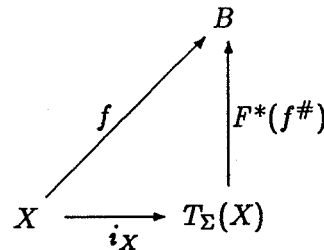
Then the Σ -structure of T_Σ is given by α defined by:

1. for $\alpha \in \Sigma_{\lambda,s}$ let $\alpha(\sigma)$ be the string σ of length 1 in $T_{\Sigma, s}$; and
2. for $\sigma \in \Sigma_{s_1 \dots s_n, s}$ and $t_i \in T_{\Sigma, s_i}$ let $\alpha(\sigma)(t_1, \dots, t_n)$ be the string $\sigma(t_1, \dots, t_n)$ in $T_{\Sigma, s}$.

Next, define $\Sigma(X)$ to be the S -sorted signature with $(\Sigma(X))_{\lambda,s} = \Sigma_{\lambda,s} \cup X_s$ and $(\Sigma(X))_{u,s} = \Sigma_{u,s}$ if $u \neq \lambda$. Then $T_\Sigma(X)$ is just $T_{\Sigma(X)}$ regarded as a Σ -algebra rather than as a $\Sigma(X)$ -algebra ¹⁴. The freeness of $T_\Sigma(X)$ is expressed by

Theorem 52 Let $i_X: X \rightarrow U(T_\Sigma(X))$ denote the inclusion. Then the following “universal” property holds: for any Σ -algebra B , every (S -indexed) map $f: X \rightarrow B$, called an assignment, extends uniquely to a Σ -homomorphism $f^\#: T_\Sigma(X) \rightarrow B$ such that $i_X; U(f^\#) = f$. \square

A proof may be found, for example, in [17]. We may omit the U 's in such equations, as in the following traditional diagram of S -indexed sets and mappings for the above equation.



In particular, taking $X = \emptyset$, we see that there is a unique Σ -homomorphism from T_Σ to any other Σ -algebra; thus, T_Σ is the initial Σ -algebra.

¹⁴I.e., $\phi(T_{\Sigma(X)})$ where $\phi: \Sigma \rightarrow \Sigma(X)$ is the signature inclusion.

A.1.3 Equations and Satisfaction

We now define equations, and the satisfaction of an equation by an algebra. Let us fix, for the rest this subsection, an infinite set \mathcal{X} of “variable symbols”. Then a **sort assignment** is a partial function $X: \mathcal{X} \rightarrow S$ where S is a set of sorts; we also let X denote the S -indexed set defined by¹⁵ $X_s = \{x \in \mathcal{X} \mid X(x) = s\}$. This is used in the following:

Definition 53 A Σ -equation e is a triple $\langle X, t1, t2 \rangle$, where X is a sort assignment $\mathcal{X} \rightarrow S$ with S the set of sorts of Σ , where $t1$ and $t2$ in $|T_\Sigma(X)|_s$ are terms over (the S -sorted set) X having the same sort $s \in S$. We may write such an equation in the form $(\forall X) t1 = t2$. \square

The necessity for variable declarations in equations has been shown by Goguen and Meseguer in [41]: without them, the expected rules of deduction for many-sorted equational logic are unsound¹⁶.

Definition 54 A Σ -algebra A satisfies a Σ -equation $(\forall X) t1 = t2$ iff $a^\#(t1) = a^\#(t2)$ for every assignment $a: X \rightarrow |A|$. In this case we write $A \models e$. \square

We now define another functor, Eqn , on the category of signatures. In order to do so, we first define for each signature morphism $\phi: \Sigma \rightarrow \Sigma'$ a function ϕ^\sim from Σ -terms to Σ' -terms.

Definition 55 Let $\phi: \Sigma \rightarrow \Sigma'$ be a signature morphism $\langle f: S \rightarrow S', g \rangle$, let $X: \mathcal{X} \rightarrow S$ be a sort assignment, and let X' be the sort assignment $X; f$. The following will define an S -indexed map $\phi^\sim: |T_\Sigma(X)| \rightarrow |\phi(T_{\Sigma'}(X'))|$: First, note that¹⁷ $X \subseteq |\phi(T_{\Sigma'}(X'))|$ since if $x \in X_s$ then $x \in X'_{f(s)}$ and $X'_{f(s)} \subseteq |T_{\Sigma'}(X')|_{f(s)} = |\phi(T_{\Sigma'}(X'))|_s$; let $j: X \rightarrow |\phi(T_{\Sigma'}(X'))|$ denote this inclusion. Then j has a unique extension as a Σ -homomorphism $j^\#: T_\Sigma(X) \rightarrow \phi(T_{\Sigma'}(X'))$ by Theorem 52, and we simply define $\phi^\sim = |j^\#|$. \square

Definition 56 The functor $Eqn: \text{Sig} \rightarrow \text{Set}$ takes each signature Σ to the set $Eqn(\Sigma)$ of all Σ -equations, and takes each $\phi = \langle f, g \rangle: \Sigma \rightarrow \Sigma'$ to the function $Eqn(\phi): Eqn(\Sigma) \rightarrow Eqn(\Sigma')$ defined by

$$Eqn(\phi)(\langle X, t1, t2 \rangle) = \langle X; f, \phi^\sim(t1), \phi^\sim(t2) \rangle.$$

It is often convenient to write $\phi(e)$ or ϕe instead of $Eqn(\phi)(e)$. \square

Proposition 57 Satisfaction Condition. If $\phi: \Sigma \rightarrow \Sigma'$, if e is an Σ -equation, and if A' is a Σ' -algebra, then

$$A' \models \phi(e) \text{ iff } \phi(A') \models e.$$

\square

We omit the proof of this result. An elegant proof using the machinery of charters and parchments may be found in [39], and a direct, but rather lengthy, proof by calculation may be found in [36]. Summarizing, we have

Example 58 Many-Sorted Equational Logic is an institution, with Sign the category Sig of equational signatures (see Definition 47), with Mod the functor Alg of Definition 51, with Sen the functor Eqn of Definition 56, and with equational satisfaction as in Definition 54. The Satisfaction Condition holds by Proposition 57. We denote this institution \mathcal{EQ} . \square

¹⁵We thank Andrzej Tarlecki for pointing out a problem with our approach to variables in [36].

¹⁶Some special assumptions about the form of signatures, such as that each sort has at least one constant, will also ensure soundness [41].

¹⁷Of course this means that $X_s \subseteq |\phi(T_{\Sigma'}(X'))|_s$ for each $s \in S$.

A.2 First Order Logic and Related Institutions

We follow a path like that in Section A.1, but now show that a more complicated logical system is an institution. The previous work on equational logic will greatly aid with this task.

Definition 59 A first order signature Ω is a triple $\langle S, \Sigma, \Pi \rangle$, where

1. S is a set (of sorts),
2. Σ is an $S^* \times S$ -indexed family of sets (of operator or function symbols), and
3. Π is an S^* -indexed family of sets (of predicate or relation symbols).

A morphism of first order signatures, from Ω to Ω' , is a triple $\langle \phi_1, \phi_2, \phi_3 \rangle$, where

1. $\phi_1: S \rightarrow S'$ is a function,
2. $\phi_2: \Sigma \rightarrow \Sigma'$ is an $S^* \times S$ -indexed family of functions $(\phi_2)_{u,s}: \Sigma_{u,s} \rightarrow \Sigma'_{\phi_1^*(u), \phi_1(s)}$ and
3. $\phi_3: \Pi \rightarrow \Pi'$ is an S^* -indexed family of functions $(\phi_3)_u: \Pi_u \rightarrow \Pi'_{\phi_1^*(u)}$.

Let **FoSig** denote the category with first order signatures as its objects, with first order signature morphisms as its morphisms, and with the obvious identities and composition. \square

Definition 60 For Ω a first order signature, an Ω -model (or Ω -structure) A consists of

1. an S -indexed family $|A|$ of non-empty sets $\langle A_s \mid s \in S \rangle$, where A_s is called the carrier of sort s , and
2. an $S^* \times S$ -indexed family α of functions $\alpha_{u,s}: \Sigma_{u,s} \rightarrow [A_u \rightarrow A_s]$ assigning a function to each function symbol, an S^* -indexed family β of functions $\beta_u: \Pi_u \rightarrow Pow(A_u)$ assigning a relation to each predicate symbol, where $Pow(A)$ denotes the set of all subsets of A .

For $\pi \in \Pi_u$ with $u = s_1 \dots s_n$ and $a_i \in A_{s_i}$ for $i = 1, \dots, n$, we say that “ $\pi(a_1, \dots, a_n)$ holds” (in A) iff $(a_1, \dots, a_n) \in \beta(\pi)$; as usual, this may be abbreviated “ $\pi(a_1, \dots, a_n)$.”

Next, we define a first order Ω -homomorphism $f: A \rightarrow A'$ of Ω -models to be an S -indexed family of functions $f_s: A_s \rightarrow A'_s$ such that the homomorphism condition holds for Σ (as in Definition 5) and such that for $\pi \in \Pi_u$ with $u = s_1 \dots s_n$, and with a_i in A_{s_i} for $i = 1, \dots, n$,

$$\beta(\pi)(a_1, \dots, a_n) \text{ implies } \beta'(\pi)(f_{s_1}(a_1), \dots, f_{s_n}(a_n)).$$

(Some readers might think that “implies” above should be “iff”; however, such a notion of homomorphism would be too strong, eliminating many maps that are necessary for the term model to be initial in the Horn clause institution; see [42].)

Let **FoMod** denote the category with first order models as objects and with first order morphisms under the obvious composition. **FoMod** extends to a functor $\text{FoSig} \rightarrow \text{Cat}^{op}$ as follows: Given a first order signature morphism $\phi: \Omega \rightarrow \Omega'$, define the functor

$$\text{FoMod}(\phi): \text{FoMod}(\Omega') \rightarrow \text{FoMod}(\Omega)$$

to send: first of all, A' in $|\text{FoMod}(\Omega')|$ to $A = \phi A'$ defined by

1. $A_s = A'_{s'}$, for $s \in S$ with $s' = \phi_1(s)$,
2. $\alpha_{u,s}(\sigma) = \alpha'_{u',s'}((\phi_2)_{u,s}(\sigma))$ for $u \in S^*$, $s \in S$ and $\sigma \in \Sigma_{u,s}$ where $u' = \phi_1^*(u)$ and $s' = \phi_1(s)$,

3. $\beta_u(\pi) = \beta'_{u'}((\phi_3)_u(\pi))$ for $u \in S^*$ and $\pi \in \Pi_u$ with u' as above;

and secondly, to send $f': A' \rightarrow B'$ in $\text{FoMod}(\Omega')$ to $f = \phi f': A \rightarrow B$ in $\text{FoMod}(\Omega)$, where $A = \phi A'$ and $B = \phi B'$, defined by $f_s = f'_{s'}$, where $s' = \phi_1(s)$. This construction extends that of Definition 7, and it is easy to see that it does indeed give a functor. \square

The next step is to define the sentences over a first order signature Ω . We do this in the usual way, by first defining terms and formulas. However, we have to be somewhat careful about variables. To this end, let \mathcal{X} be a fixed infinite set of variable symbols, and let $X: \mathcal{X} \rightarrow S$ be a partial function i.e., sort assignment; as before, X also denotes the S -indexed set with $X_s = \{x \in \mathcal{X} \mid X(x) = s\}$. Now define the S -indexed family $TERM_X(\Omega)$ of (Ω, X) -terms to be the carriers of $T_\Sigma(X)$, the free Σ -algebra with generators X , and define the (S -indexed) function $Free$ on $TERM_X(\Omega)$ inductively by

1. $Free_s(x) = \{x\}$ for $x \in X_s$,
2. $Free_s(\sigma(t_1, \dots, t_n)) = \bigcup_{i=1}^n Free_{s_i}(t_i)$.

Now define $TERM(\Omega)$ to be the (disjoint) union of all the $TERM_X(\Omega)$; this means that we assume (as in the equational case) that each Ω -term comes with an explicit indication of what its variables are.

Definition 61 A (well-formed) (Ω, X) -formula is an element of the carrier of the (one-sorted) free algebra $WFF_X(\Omega)$ having the atomic (Ω, X) -formulae

$$\{\pi(t_1, \dots, t_n) \mid \pi \in \Pi_u \text{ with } u = s_1 \dots s_n \text{ and } t_i \in TERM_X(\Omega)_{s_i}\}$$

as generators, and having the following as its (one-sorted) signature:

1. a constant *true*,
2. a unary prefix operator \neg ,
3. a binary infix operator $\&$, and
4. a unary prefix operator $(\forall x)$ for each x in X .

Let $WFF(\Omega)$ be the union of all $WFF_X(\Omega)$.

The functions Var and $Free$, giving the sets of variables and of free variables that are used in Ω -formulae, can be defined inductively over the above logical connectives by

1. $Var(true) = Free(true) = \emptyset$,
2. $Var(\pi(t_1, \dots, t_n)) = Free(\pi(t_1, \dots, t_n)) = \bigcup_{i=1}^n Free(t_i)$,
3. $Var(\neg P) = Var(P)$, and $Free(\neg P) = Free(P)$,
4. $Var(P \& Q) = Var(P) \cup Var(Q)$, and $Free(P \& Q) = Free(P) \cup Free(Q)$, and
5. $Var((\forall x)P) = Var(P) \cup \{x\}$, and $Free((\forall x)P) = Free(P) - \{x\}$.

We now define an Ω -sentence to be a closed Ω -formula, that is, an Ω -formula P with $Free(P) = \emptyset$, and let $FoSen(\Omega)$ denote the set of all Ω -sentences; it is the union of the sets $FoSen_X(\Omega)$ of closed (Ω, X) -formulae. For convenience, we can now define the remaining logical connectives, *false*, \vee , \Rightarrow , \Leftrightarrow , and $(\exists x)$ in terms of the basic ones given above in the usual way. \square

These definitions of $TERM_X(\Omega)$ and $WFF_X(\Omega)$ follow the “initial algebra semantics” advocated by [30] and [46] in that they rely only on the freeness of the algebras involved. We exploit this to define the effect of $FoSen$ on first order signature morphisms, so that it becomes a functor $FoSigSet$. Given $\phi: \Omega \rightarrow \Omega'$, we will define $FoSen_X(\phi): FoSen_X(\Omega) \rightarrow FoSen_X(\Omega')$ using initiality. Since $(\phi_1, \phi_2): \Sigma \rightarrow \Sigma'$ is a signature morphism, there is an induced morphism $\psi: T_\Sigma(X) \rightarrow T_{\Sigma'}(X')$, where $X' = X; \phi_1$, which then gives us $\psi: TERM_X(\Omega) \rightarrow TERM_{X'}(\Omega')$. Now define $WFF_X(\phi): WFF_X(\Omega) \rightarrow WFF_{X'}(\Omega')$ by its effect on the generators of $WFF_X(\Omega)$, which are the atomic formulae, by

$$WFF_X(\phi)(\pi(t_1, \dots, t_n)) = \phi_3(\pi)(\psi(t_1), \dots, \psi(t_n)).$$

Finally, define $FoSen_X(\phi)$ to be the restriction of $WFF_X(\phi)$ to $FoSen_X(\Omega) \subseteq WFF_X(\Omega)$. For this to work, it must be checked that $WFF_X(\phi)$ carries closed Ω -formulae to closed Ω' -formulae; but this is easy. $FoSen(\phi)$ is then defined to be the union of the functions $FoSen_X(\phi)$ on the union of the sets $FoSen_X(\Omega)$.

It remains to define satisfaction. This corresponds to the usual “semantic definition of truth” (originally due to Tarski [90]). If A is a first order model, let $Asgn_X(A)$ denote the set of all assignments of values in A to variables in X , i.e., $[X \rightarrow A]$, the set of all S -indexed functions $f: X \rightarrow A$.

Definition 62 Given a sentence P , define $Asgn_X(A, P)$, the set of assignments in A for which P is true, inductively as follows:

1. if $P = \pi(t_1, \dots, t_n)$ then $f \in Asgn_X(A, P)$ iff $(f^\#(t_1), \dots, f^\#(t_n)) \in \beta(\pi)$, where $f^\#(t)$ denotes the evaluation of the Σ -term t in the Σ -algebra part of A with the values of variables given by the assignment f using initiality:
2. $Asgn_X(A, true) = Asgn_X(A)$.
3. $Asgn_X(A, \neg P) = Asgn_X(A) - Asgn_X(A, P)$.
4. $Asgn_X(A, P \& Q) = Asgn_X(A, P) \cap Asgn_X(A, Q)$.
5. $Asgn_X(A, (\forall x)P) = \{f \mid Asgn_X(A, f, x) \subseteq Asgn_X(A, P)\}$, where $Asgn_X(A, f, x)$ is the set of all assignments f' that agree with f except possibly on the variable x from X .

Then a model A satisfies a sentence P with variables from X , written $A \models P$, iff $Asgn_X(A, P) = Asgn_X(A)$. \square

Finally, we must verify the satisfaction condition. This follows from an argument much like that used for the equational case, and is omitted here; the result can also be obtained using the method of “charters” [39]. Thus, summarizing the above, we have

Example 63 (Many-Sorted) First Order Logic is an institution; let us denote it FOL . \square

Example 64 (Many-Sorted) First Order Logic with Equality. This institution is closely related to that of Example 63. A signature for first order logic with equality is a first order signature $\Omega = \langle S, \Sigma, \Pi \rangle$ that has a particular predicate symbol \equiv_s in $\Pi_{s,s}$ for each $s \in S$. A morphism of signatures for first order logic with equality is a morphism of signatures for first order logic that preserves these predicate symbols, i.e., that satisfies $\phi_3(\equiv_s) = \equiv_{\phi_1(s)}$. This gives a category $FoSigEq$ of signatures for first order logic with equality.

If Ω is a signature for first order logic with equality, then a model for it is just an Ω -model A in the usual first order sense satisfying the additional condition that

$$a \equiv_s a' \text{ iff } a = a'$$

for all $s \in S$ and for all $a, a' \in A_s$.

A homomorphism of first order Ω -models with equality is just a first order Ω -homomorphism (in the sense of Definition 60), and we get a category $\mathbf{FoModEq}(\Omega)$ of Ω -models for each signature Ω in $|\mathbf{FoSigEq}|$, and $\mathbf{FoModEq}$ is a functor on $\mathbf{FoSigEq}$. Ω -sentences are defined just as in Example 63, and so is satisfaction. We thus get a functor $FoSenEq: \mathbf{FoSigEq} \rightarrow \mathbf{Set}$. The Satisfaction Condition follows immediately from that of first order logic without equality. Let us denote this institution by \mathcal{FOEQ} . \square

It is interesting to note that many liberal institutions that are familiar from logic have served as bases for programming languages, including those mentioned below.

Example 65 (Many-Sorted) Horn Clause Logic with Equality. We may specialize the previous example by limiting the form that sentences can take, but without restricting either the predicate or operator symbols that can enter into them. In particular, we maintain the equality symbol with its fixed interpretation from Example 64; but we require that all sentences be of the form

$$(\forall X) A_1 \& A_2 \& \dots \& A_n \Rightarrow A,$$

where each A_i is an atomic formula $\pi(t_1, \dots, t_m)$. In particular, disjunction, negation and existential quantifiers are excluded. That this is an institution follows from the facts that first order logic with equality is an institution, and that the class of Horn clauses is closed under translation by signature morphisms. Let us denote this institution \mathcal{HCLLEQ} ; it is the basis¹⁸ for the logical programming language \mathbf{Eqlog} [42]. \square

Example 66 (Many-Sorted) Conditional Equational Logic. As a specialization of Example 65, let equality be the only predicate symbol. This gives the institution often called conditional equational logic. It is the basis¹⁸ of the logical programming language \mathbf{OBJ} [27, 47]. \square

Example 67 (Many-Sorted) Horn Clause Logic without Equality. We can also restrict Example 65 by dropping equality with its fixed interpretation. This too is obviously an institution because it is just a restriction of ordinary first order logic; it is the basis for the logical programming language (pure) \mathbf{Prolog} [62]. ([28] discusses a more general logical system and contains results that show that it is liberal.) Let us denote this institution \mathcal{HCL} . \square

It seems clear that many-sorted temporal or modal logic can be treated in much the same way, by adding the appropriate modal operators to the signature and defining their correct interpretation in all models, which will be “Kripke” or “alternative world” structures. Higher order equational logic is also presumably an institution; the development should again follow that of equational logic, using higher order sorts and operator symbols¹⁹. “Inequational logic” [11], order-sorted equational logic [44], and various kinds of infinitary equational logic, such as continuous algebras [46, 93], are also institutions [87]. We further conjecture that in general, mechanical theorem provers, such as the Boyer-Moore prover [12], Aubin’s system [3] and STP [83], are based on logical systems that are institutions (or if they are not, they should be modified so that they are!). Clearly, it would be helpful to have some general results to help establish that various logical systems are institutions. This motivates the study of “charters” and “parchments” in [39].

A way to generate many other examples [64] is to let the sentences over a signature Ω be all the specifications using Ω written in some specification language (such as \mathbf{Clear}); we could think of such a specification as a convenient abbreviation for a (possibly very large) conjunction

¹⁸More precisely, the order-sorted variant of this logic is the basis.

¹⁹The sorts involved will be the objects of the free Cartesian closed category on the basic sort set [71, 80].

of simpler sentences. However, this seems inappropriate under the view that the purpose of a specification language is to assist with programming-in-the-large, by providing facilities for building new specifications by reusing old specifications as a whole, e.g., by applying a parameterized specification, as opposed to local operations on sentences, such as conjunction.

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