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Extending properties to categories of partial maps

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by

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LFCS Report Series
ECS-LFCS-90-107

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February 1990

Extending properties to categories of partial maps

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February 7, 1990

Abstract

Properties can be extended from categories of total maps to partial maps in a uniform way, e.g. cartesian products are lifted to *lax* cartesian products.

The partial maps of a category $\mathcal{A}$ (equipped with a dominion $\mathcal{M}$) are ordered by their extent of definition, thus forming an *ordered* category $\mathbf{Ptl}(\mathcal{A}, \mathcal{M})$. The **Ptl** functor preserves adjunctions, including those that define products, etc. It has a coreflection **Tot** that picks out the total maps of an arbitrary ordered category, and a reflection **Dom** which constructs a category of domains for its morphisms. Each of these adjunctions yields a characterisation of categories of partial maps, without assuming any further structures on the categories.

*The author is supported by SERC grants GR/E 78487 and GR/F 07866, and NSERC operating grant OGPIN 016.

Contents

1 Introduction

The partial maps of a category $\mathcal{A}$ (equipped with a dominion $\mathcal{M}$ which specifies their domains) are naturally ordered by their extension: if thought of as programs then $p \leq q$ iff whenever p terminates then q does and takes the same value. Thus, the order relates programs which meet the same specification, but where one is better than the other in terms of some property, i.e. extent of definition. Hence the partial maps form an ordered category $\mathbf{Ptl}(\mathcal{A}, \mathcal{M}) = \mathcal{A}_p$. This ordering should be considered explicitly since improving the components of a large program should not change its specification. This can be guaranteed by demanding that program constructors, e.g. products, preserve the order.

Previous characterisations of categories of partial morphisms (surveyed in [18]), whether or not they exploited the ordering, axiomatised the partiality of the morphisms simultaneously with other structures, such as the product structure in Rosolini's p-categories [19] or Carboni's bicategories of partial maps [3], or the existence of a 'terminal' object [5]. These structures were used to define the domain of a morphism, either as a subobject or as an endomorphism [6]. These approaches lead to long lists of ad hoc equations mixing the various structures. Here, no additional structures are assumed on $\mathcal{A}$. Rather, if they exist and are compatible with the dominion then they are extended to $\mathcal{A}_p$ in a uniform way.

The structures on $\mathcal{A}_p$ are generally lax, that is, some of the usual equations are replaced by inequalities, with the consequent weakening of universal properties. For example, $\mathcal{A}$ has all cartesian products iff the diagonal functor $\Delta : \mathcal{A} \rightarrow \mathcal{A}^2$ has a right adjoint $\times : \mathcal{A}^2 \rightarrow \mathcal{A}$. They occur in many computing contexts, e.g. to represent pairs of types or their values, or to model parallelism (e.g. [15,24]). The product extends to a lax functor $\times_p$ which is lax right adjoint to the diagonal of $\mathcal{A}_p$ and hence a lax product functor. Its universal property is weaker than than that of the usual cartesian product, which may not exist or may fail to preserve the order. Lax natural transformations have appeared, at least implicitly, in abstraction techniques, both for strictness analysis [2] and data refinement [8,16], in predicate transformation [9], and in modelling lambda-calculus [21].

In order to extend structures and properties from total morphisms to partial morphisms uniformly it is necessary to study the properties of $\mathbf{Ptl}$ as a construction. It is a 2-functor (acts on categories, functors and natural transformations) and so preserves many structures of $\mathcal{A}$, in particular, those defined by adjunctions between dominion-preserving functors.

Ptl has a coreflection (right adjoint) **Tot** which maps an arbitrary ordered category $\mathcal{O}$ to its subcategory of total morphisms, with a dominion given by the stable embeddings (of embedding-projection pairs). Thus, we have:

$$\begin{array}{rcl} \mathbf{Tot}(\mathbf{Ptl}(\mathcal{A},\mathcal{M})) & \cong & (\mathcal{A},\mathcal{M}) \\ \mathbf{Ptl}(\mathbf{Tot}(\mathcal{O})) & \xrightarrow{\varepsilon} & \mathcal{O} \end{array}$$

Hence $\mathcal{O}$ is a category of partial maps iff ε is an isomorphism, which yields a characterisation of categories of partial maps.

Rosolini showed how every p-category $\mathcal{C}$ arises as a full sub-category of a category of partial maps, by constructing a category of domains for $\mathcal{C}$. Here, an arbitrary ordered category $\mathcal{O}$ is given a category $\mathbf{Dom}(\mathcal{O}) = \mathcal{O}_t$ of domains, again without reference to any product structure, etc. If the ordering on $\mathcal{O}$ is extensional, then there is an embedding $\rho : \mathcal{O} \rightarrow \mathcal{O}_{tp}$ and **Dom** is a reflection for **Ptl**. This restriction on $\mathcal{O}$ can probably be removed by considering partial maps for categories which are already ordered, e.g. those of domain theory. In any event, the reflection yields a second characterisation of categories of partial maps.

Many of the examples are framed in terms of the category **Pos** of partially ordered sets for generality, though the same results hold in various categories of domains, or O-categories [20,23]. Although an elementary familiarity with 2-categories [8,10,13,21] is useful, it is not essential to appreciate the argument, as the ordered categories are discussed in detail, while all other 2-categories considered are of categories, functors and transformations.

1.1 Acknowledgements

I would like to thank E. Moggi for many helpful conversations.

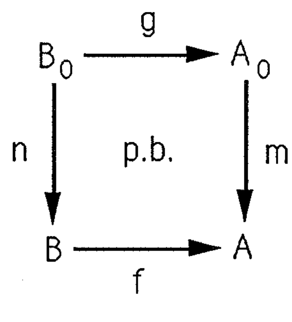
2 Partial maps and ordered categories

2.1 Partial morphisms

A partial function from a set A to a set B is given by a subset A_0 of A and a (total) function $A_0 \rightarrow B$. In general, we do not wish to allow A_0 to be arbitrary, but rather belong to some suitable class of subsets of A which are, say, computable. Thus, to define partial maps or morphisms in a category we must first describe their admissible subobjects.

Let A be an object in a category $\mathcal{A}$. A monomorphism $m : A_0 \rightarrow A$ can be thought of as representing a subobject of A (its 'image'). They can be preordered by $m \leq m' : A_1 \rightarrow A$ if there is a morphism $p : A_0 \rightarrow A_1$ such that $m'p = m$. It follows that p is unique since m' is a monomorphism, and is itself a monomorphism since m is (thus the 'image' of m is smaller than the 'image' of m_1). The equivalence classes for this preorder are the ***subobjects*** of A. The *intersection* of the subobjects represented by m and m' (if it exists) is represented by the monomorphism $m \cap m' = mm'' : A_2 \rightarrow A$ where $m'' : A_2 \rightarrow A_0$ is the pullback of m' along m.

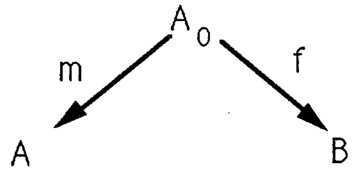
Definition 2.1 *Consider a family of monomorphisms $\mathcal{M}$ in a category $\mathcal{A}$ which is closed under composition and isomorphisms. A monomorphism $m : A_0 \rightarrow A$ in $\mathcal{M}$ is* stable *if, for every $f : B \rightarrow A$ in $\mathcal{A}$ there is a pullback:*



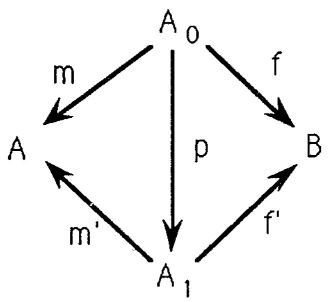
with $n \in \mathcal{M}$. If every monomorphism in $\mathcal{M}$ is stable then it is a dominion.

The set of all isomorphisms of $\mathcal{A}$ is always a dominion, as is the set of all stable monomorphisms. Other examples arise from the open inclusions of topological spaces, e.g. the Scott opens of domain theory, or the recursively enumerable subsets of natural numbers. Fix a category $\mathcal{A}$ equipped with a dominion $\mathcal{M}$ for the rest of the paper.

A *partial morphism* $A \rightharpoonup B$ of $\mathcal{A}$ w.r.t. $\mathcal{M}$ is represented by a span (m, f)

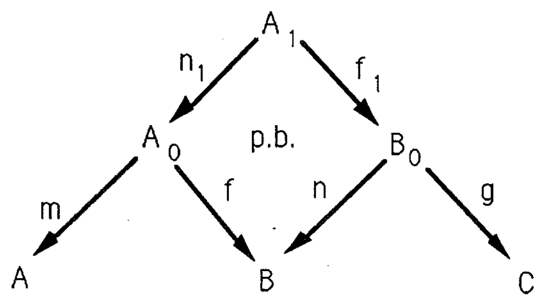


whose *domain* is the admissible subobject represented by $m \in \mathcal{M}$. An inequality $(m, f) \leq (m', f') : A \rightharpoonup B$ is realised by a morphism p which makes the following diagram commute:



Then the domain of (m', f') contains that of (m, f), on which they agree.

Given another partial morphism $(n, g) : B \rightharpoonup C$ then their *composite* is represented by the span $A \rightharpoonup C$ obtained by pulling back:

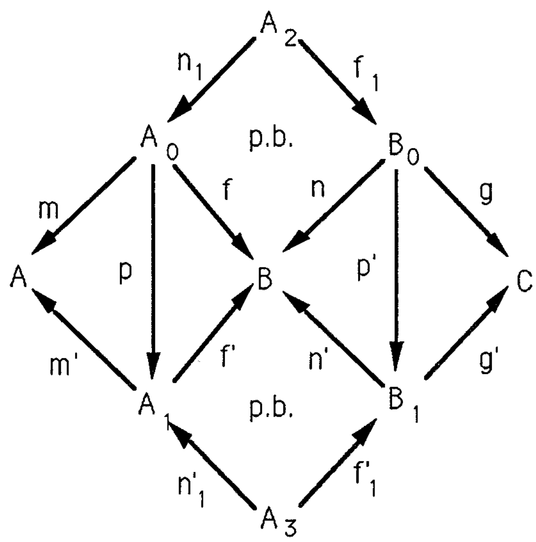


A_1 is the largest admissible subobject on which f is defined and takes values in the domain of g. That composition is associative follows from the universal property of pullbacks. The identity on A is represented by the span (id_A, id_A). Thus, the partial morphisms form a category $\mathbf{Ptl}(\mathcal{A}, \mathcal{M})$ which is denoted $\mathcal{A}_p$ when $\mathcal{M}$ is understood.

Lemma 2.2 *Composition in $\mathcal{A}_p$ preserves the ordering of morphisms: if $(m, f) \leq (m', f') : A \to B$ and $(n, g) \leq (n', g') : B \to C$ then*

$$(n, g)(m, f) \leq (n', g')(m', f')$$

Proof Let p and p' realise the two inequalities. Then the commutativity of



induces the desired morphism $A_2 \to A_3$. //

2.2 Ordered categories

An *ordered category* $\mathcal{O}$ is a category whose homsets are ordered, with the order preserved by composition, i.e. $f \leq f' : A \to B$ and $g \leq g' : B \to C$ implies $gf \leq g'f'$.

Examples 2.3 (i) Categories of partial maps.

(ii) Every category $\mathcal{A}$ may be thought of as a *discrete* ordered category, i.e. one whose homsets are discretely ordered ($f \leq g$ iff $f = g$).

(iii) The category **Pos** of partially ordered sets and order-preserving morphisms, and all its subcategories, e.g. Scott domains or O-categories.

(iv) A typed rewrite system [16] yields an ordered category whose objects and morphisms are the types and terms with $s \geq$
Composition is given by substitution.

(v) The category **Rel** of sets and relations is an ordered category with $R \leq S : B \rightarrow A$ iff $R \subset S$ as subsets of $A \times B$. More generally, if $\mathcal{E}$ is a regular category [1] then $\mathbf{Rel}(\mathcal{E})$ is an ordered category. When relations are used to model non-determinism then $R \leq S : B \rightarrow A$ iff every result of R is a result of S.

(vi) $\mathbf{Rel}_0$ has the same underlying category as **Rel** but with $R \leq S : B \rightarrow A$ iff $Rb \neq \phi$ implies $Rb = Sb$. That is, $\mathbf{Rel}_0 = \mathcal{A}_p$ where $\mathcal{A}$ is the category of universally defined relations ($Rb \neq \phi$ for all b) and $\mathcal{M}$ is the dominion of monomorphic functions.

(vii) An ordered monoid is a poset with an associative, binary, monotonic operation (multiplication) with a two-sided unit. Thus it is an ordered category with one object whose morphisms are the elements of the set, with composition given by multiplication.

Ordered categories may be thought of as 2-categories in which the hom-categories are pre-orders, or as categories enriched over the category of pre-orders [12,14]. Much of the terminology defined below derives from these subjects. Fix an ordered category $\mathcal{O}$ for the rest of the paper.

2.3 Products in ordered categories

In ordered categories, the usual notion of the cartesian product is not appropriate, as can be seen in the category $\mathbf{Sets}_p$ of sets and partial functions. The cartesian product (in the unordered category) of sets A and B is given by:

$$A \otimes B = A + (A \times B) + B$$

where $A \times B$ is their product in **Sets**. Given partial functions $f : C \rightharpoonup A$ and $g : C \rightharpoonup B$ the induced morphism $C \rightharpoonup A \otimes B$ is given by

$$\langle f,g\rangle_p(c) = \begin{cases} f(c) & \text{if } f(c)\downarrow \text{ and } g(c)\uparrow \\ (f(c),g(c)) & \text{if } f(c)\downarrow \text{ and } g(c)\downarrow \\ g(c) & \text{if } f(c)\uparrow \text{ and } g(c)\downarrow \\ \uparrow & \text{if } f(c)\uparrow \text{ and } g(c)\uparrow \end{cases}$$

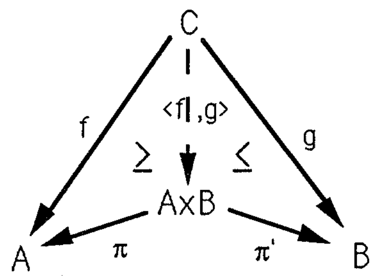
where $f(c)\downarrow$ (respectively, $f(c)\uparrow$) denotes that $f(c)$ is *defined* (*undefined*).

Now, the definition of $A\otimes B$ is dependent on the presence of sums (unions) and complements, which latter prevent $\otimes$ from preserving the order; from $f \leq f' : C \rightharpoonup A$ and $g \leq g' : C \rightharpoonup B$ as above it does not follow that $\langle f,g\rangle_p \leq \langle f',g'\rangle_p$. For example, if f and $g' : C \rightharpoonup B$ are total functions and g is nowhere defined, then the image of $\langle f,g\rangle_p$ is in the A-component of $A\otimes B$ while that of $\langle f,g'\rangle_p$ is in the $A\times B$-component, which makes them incomparable. Thus, the pairing is not stable under an increase in the domain of definition, i.e. 'improving' the components of a program could change its specification! Consequently, $\otimes$ is inadequate as a product for the ordered category.

By contrast, the usual product from **Sets** does preserve the order with $\langle f,g\rangle : C \rightharpoonup A\times B$ defined by

$$\langle f,g\rangle(c) = \begin{cases} (f(c),g(c)) & \text{if } f(c)\downarrow \text{ and } g(c)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

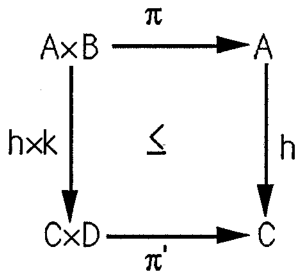
The price to be paid is that the usual commuting diagram is replaced by



since $\langle f,g\rangle$ is defined only when f and g both are:

$$\begin{aligned} \pi\langle f,g\rangle &\leq f \\ \pi'\langle f,g\rangle &\leq g \end{aligned} \tag{1}$$

Thus, $A\times B$ is not a cartesian product in the usual sense, but has a universal property expressed in terms of the ordering, i.e. $\langle f,g\rangle$ is maximal among the morphisms satisfying (1). In particular, the projections are not natural. Rather, there is an inequality in the following square



Definition 2.4 *A* lax product *of objects A and B in an ordered category $\mathcal{O}$ is an object P equipped with a pair of projections $\pi : P \to A$ and $\pi' : P \to B$ which satisfy the following universal property: given morphisms $f : C \to A$ and $g : C \to B$ there is a morphism $h = \langle f, g \rangle : C \to P$ which is maximum among those satisfying*

$$\begin{array}{rcl} \pi h & \leq & f \\ \pi' h & \leq & g \end{array} \tag{2}$$

The maximality condition above ensures that $\langle -, - \rangle$ preserves the order, i.e. if $f \leq f'$ and $g \leq g'$ the $\langle f, g \rangle \leq \langle f', g' \rangle$.

Lemma 2.5 *Products in $\mathcal{A}$ are lax products in $\mathcal{A}_p$.*
Proof Let $(m, f) : C \rightharpoonup A$ and $(n, g) : C \rightharpoonup B$ be partial morphisms and let $A \times B$ be a product in $\mathcal{A}$. Define

$$\langle f, g \rangle = (m \cap n, h) : C \rightharpoonup A \times B$$

where h is the total map into $A \times B$ induced by the restrictions of f and g to $m \cap n$ (which is in $\mathcal{M}$ by stability). //

Unfortunately, there may be non-isomorphic (even non-equivalent) products of a given pair of objects in an ordered category! For example, any admissible subobject $m : X \to A \times B$ (e.g. in $\mathbf{Sets}_p$) is also a product of A and B with projections given by πm and $\pi' m$. The pairing of f and g is then the pullback of $\langle f, g \rangle$ along m.

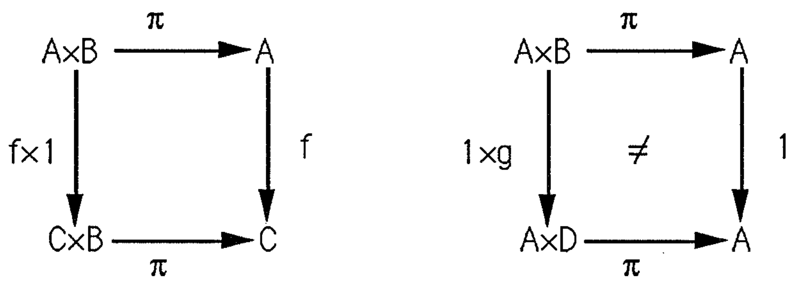
The complexity of the situation is made even clearer in **Rel**. As in $\mathbf{Sets}_p$ the cartesian product of **Sets** lifts to a lax cartesian product. However, the sum of **Sets** does, too! Its first projection $\pi_{A,B} : A + B \rightarrow A$ is given by

$$\pi(x) = \begin{cases} x & \text{if } x \in A \\ \uparrow & \text{if } x \in B \end{cases}$$

Given relations $R : C \rightarrow A$ and $S : C \rightarrow B$ then $\langle R, S \rangle_+(c) = Rc + Sc$. This lax product also satisfies the usual universal property: its projections are natural in the usual sense. How then, is one to choose between them? I believe the correct approach is to keep both, since each has additional properties which specify it uniquely: the sum satisfies the usual universal property while the product ($\times$) extends the usual product of **Sets**. This latter condition will be generalised below. Other lax products may arise with further special properties.

2.4 p-categories

p-categories [18,19] were motivated by the study of the partial maps of a category $\mathcal{A}$ with cartesian products $\times$, and can be characterised as the full sub-categories of $\mathcal{A}_p$ closed under this product. Roughly, all the usual equations for products hold, except that the projections are only natural in one variable each instead of two, e.g.



The *domain* of a morphism $f : A \rightarrow B$ in a p-category $\mathcal{C}$ is given by the following idempotent endomorphism of A.

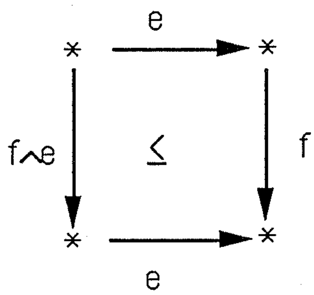
$$dom(f) = \pi_{A,B}(1 \times f)\Delta_A : A \rightarrow A$$

$dom(f)$ is to be thought of as defined only when f is, and then acting as the identity. Domains can be used to order the homsets by

$$f \leq g : A \rightarrow B \text{ iff } g\, dom(f) = f$$

making $\mathcal{C}$ an ordered category. Then $\pi(1 \times g) \leq \pi$ above and $\times$ yields lax products.

However, not every ordered category with lax products is a p-category since the diagonal and projections may fail to be natural in any variable. For example, if M is an ordered monoid where the order has meets ($\wedge$) then they yield lax products for M as an ordered category (whose sole object is $*$) with projections and diagonals given by the unit e of the monoid. The laxness of the first projection is shown by:



so that M is a p-category iff e is its largest element.

More fundamentally, seeking a p-category structure may obscure the partial map structure. The underlying category of $\mathbf{Rel}_0$ is not a p-category with respect to its lax product $\times$ since the diagonal fails to be natural; if $R : B \rightarrow A$ is a relation then $\Delta Rb \subseteq Rb \times Rb$ is its diagonal. It is a p-category with respect to its cartesian product $+$ but then every morphism has domain the identity! Thus, neither product yields a p-category where the domain of a relation $R : B \rightarrow A$ is the set of elements of B on which its image is non-empty, i.e. the order structure of $\mathbf{Rel}_0$.

Thus, even when trying to characterise categories of partial maps with products, it is better to retain the order structure and consider lax products.

3 Total morphisms

Every ordered category has a sub-category of total maps (defined in terms of its deflations) which is equipped with a dominion of stable embeddings. For $\mathcal{A}_p$ this category with dominion is isomorphic to $(\mathcal{A}, \mathcal{M})$. The definition has been modified since [11] was written, so that now it agrees with the standard usage for p-categories.

3.1 Embedding-projection pairs and deflations

Let $f : A \to B$ and $f_* : B \to A$ be morphisms of an ordered category $\mathcal{O}$. Recall [13,20] that $f \dashv f_*$ is an *adjunction* if

$$1_A \leq f_* f \tag{3}$$

$$f f_* \leq 1_B \tag{4}$$

Then f_* is a *right adjoint* to f which, if it exists, is unique. Similarly, f is *left adjoint* to f_*. If (3) is an equality then f is an *embedding* with *projection* f_*. Since $(mn)_* = n_* m_*$ the embeddings form a subcategory $\mathcal{O}^E$ of monomorphisms.

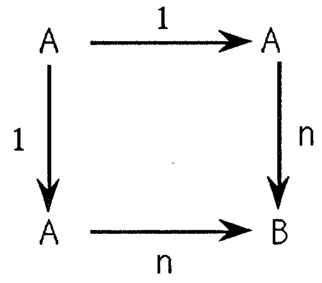
Examples 3.1 (i) In a discretely ordered category the adjunctions are exactly the isomorphisms.

(ii) In an ordered category of domains, the embedding-projection pairs are defined as usual.

(iii) In **Rel** and $\mathbf{Rel}_0$ the adjunctions are all of the form $(1, f) \dashv (f, 1)$ and are embedding-projection pairs iff f is a monomorphism. This is proved similarly to the next result.

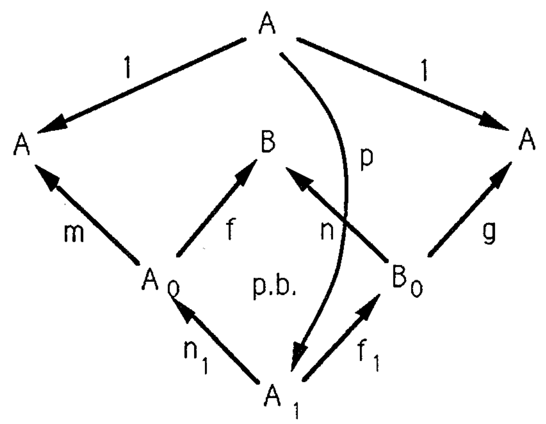
Lemma 3.2 *If $n \in \mathcal{M}$ then $(1, n) \dashv (n, 1)$ is an embedding-projection pair. Conversely, every adjunction in $\mathcal{A}_p$ is of this form.*

Proof If $n \in \mathcal{M}$ then $(n,1)(1,n) = (1,1)$ since n is a monomorphism iff the following diagram is a pullback:



Also, $(1,n)(n,1) = (n,n) \leq (1,1)$ is realised by n.

Conversely, if $(m,f) \dashv (n,g) : B \to A$ then $(1,1) \leq (n,g)(m,f)$



Thus, $mn_1p = 1$ implies mn_1 is both a split epimorphism and a monomorphism, and so an isomorphism. Thus m, n_1 and p are all isomorphisms and, without loss of generality, identities. Consequently, the adjunction is an embedding-projection pair. Further, $gf_1 = 1$ shows that g is a split epimorphism. Now, $(1,f)(n,g) = (n,fg) \leq (1,1)$ implies $fg = n$ whence g is also a monomorphism. Thus, g is an isomorphism and, without loss of generality, the identity. Consequently, $f = n \in \mathcal{M}$ and the adjunction is $(1,n) \dashv (n,1)$. //

A morphism $\alpha : A \to A$ in $\mathcal{O}$ is a *deflation* on A if

$$\alpha^2 = \alpha \leq 1_A$$

α is to be thought of as implicitly specifying its image as a subobject of A. Thus, if $f : A \to B$ is a morphism then $f\alpha$ is the restriction of f to this 'image'.

Lemma 3.3 *If m is an embedding in $\mathcal{O}$ then mm_* is a deflation. In $\mathcal{A}_p$ all deflations are of this form.*
Proof That $m_*m = 1$ implies $mm_*mm_* = mm_* \leq 1$. If $(m, f) \leq 1_A$ in $\mathcal{A}_p$ is realised by p then $f = p = m$. //

3.2 Total morphisms

A morphism $f : B \to A$ in $\mathcal{O}$ is *total* if, for each morphism $g : C \to B$ and each deflation $\gamma : C \to C$ we have

$$fg\gamma = fg \Longrightarrow g\gamma = g$$

Taking $g = 1$ as a special case we see that any restriction of f to a proper subobject of B is strictly smaller than it.

Proposition 3.4 *A morphism $f : B \to A$ in a p-category $\mathcal{C}$ (regarded as an ordered category) is total iff $dom(f) = 1_B$ (i.e. total in the sense of [19]). In particular, the totals in $\mathcal{A}_p$ are the morphisms of the form $(1, f)$.*
Proof The domains of morphisms in a p-category are deflations. If f is total then $f dom(f) = f1_B$ implies $dom(f) = 1$. Conversely, if γ and g are as above, and $fg\gamma = fg$ then

$$dom(g\gamma) = dom(fg\gamma) = dom(fg) = dom(g)$$

which implies that $g\gamma = g$. The proof for $\mathcal{A}_p$ follows by direct calculation, or by considering it as a p-category. //

Examples 3.5 (i) If $\mathcal{O}$ is discrete then the only deflations are the identities and every morphism is total.

(ii) In **Pos** an order-preserving map $f : P \to Q$ is total iff

$$x \leq y \ \wedge \ fx = fy \quad \Longrightarrow \quad x = y$$

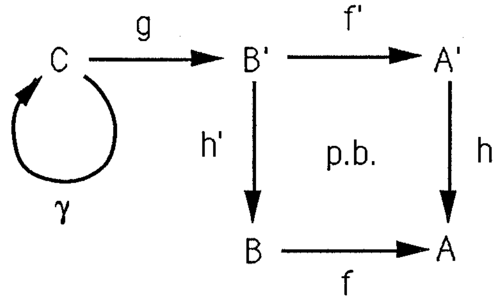
To see this, let $\mathbf{2} = \{\bot \leq \top\}$ with deflation $\alpha : \mathbf{2} \to \mathbf{2}$ which is constantly $\bot$. If f is total and $fx = fy$ with $x \leq y$ then define $g : \mathbf{2} \to P$ by $g(\bot) = x$ and $g(\top) = y$. Then $fg\alpha = fg$ which implies that

$$x = g\alpha(\top) = g(\top) = y$$

Conversely, assume that $fg\gamma = fg$ for some $g : R \to P$ and deflation $\gamma : R \to R$. Given $z \in R$ then $g\gamma(z) \leq g(z)$ and have the same image under f. Hence they are equal and so $g\gamma = g$.

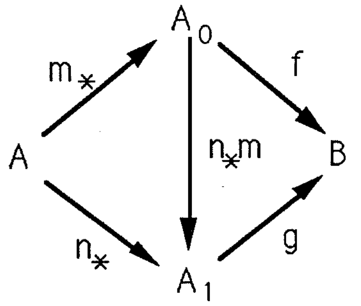
(iii) The deflations on a set A in **Rel** (or $\mathbf{Rel}_0$) correspond to the subsets of A. A relation $R : B \to A$ is total iff it is universally defined, i.e. $Rb \neq \phi$ for all $b \in B$.

Lemma 3.6 *The total morphisms form a category* $\mathbf{Tot}(\mathcal{O}) = \mathcal{O}_t$ *(with embedding* $\iota : \mathcal{O}_t \to \mathcal{O}$*) which contains all monomorphisms and is closed under pullbacks, whenever they exist. Further, if* gf *is total then* f *is, too.*
Proof The hardest step is proved here: the others are left to the reader. Consider the following diagram in which the square is a pullback and γ is a deflation.



If $f'g\gamma = f'g$ then $fh'g\gamma = fh'g$. Thus $h'g\gamma = h'g$ since f is total. Now $g\gamma$ and g are equated by both f' and h', and so are equal. //

Lemma 3.7 *If* f *and* g *are total and* m *and* n *are embeddings in* $\mathcal{O}$ *such that* $fm_* = gn_*$ *then* n_*m *and* m_*n *are inverse, i.e. each morphism of* $\mathcal{O}$ *has at most one factorisation (up to isomorphism) as a projection followed* *by a total morphism.*



Proof The assumption implies

$$gn_*mm_* = fm_*mm_* = fm_* = gn_*$$

Thus $n_*mm_* = n_*$ since mm_* is a deflation and g is total, and $gn_*m = f$ since m_* is an epimorphism. Hence the diagram commutes. Similarly, replacing n_*m by $m_*n : A_1 \rightarrow A_0$ yields another commuting diagram. This shows that n_*m and m_*n are inverse since m_* and n_* are epimorphisms. //

3.3 First characterisation theorem

Since embeddings are monomorphisms they are all total. Thus, the stable embeddings in $\mathcal{O}_t$ form a dominion. The *stable projections* are those corresponding to stable embeddings.

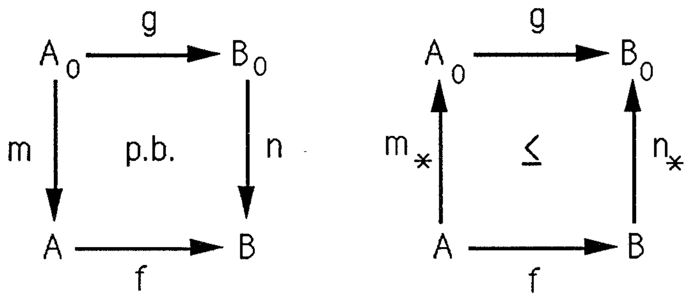
Theorem 3.8 *All embeddings in $\mathcal{A}_p$ are stable in $\mathcal{A}_{pt}$. Thus,*

$$\mathbf{Tot}(\mathbf{Ptl}(\mathcal{A}, \mathcal{M})) \cong (\mathcal{A}, \mathcal{M})$$

Also, an ordered category $\mathcal{O}$ is a category of partial maps iff it satisfies the following three conditions:

(i) (Unique factorisation) Every morphism has a unique factorisation as a stable projection followed by a total morphism.

(ii) (BC) If the left-hand square below is a pullback with n a stable embedding and f total then the right-hand square actually commutes



(5)

(This is reminiscent of the Beck-Chevalley condition [1]).

(iii) Total morphisms are maximal in their order.

Proof The first result follows directly. For the second, we begin by showing that $\mathcal{A}_p$ has these properties.

(i) $(m, f) = (m, 1)(1, f)$. Uniqueness follows by Lemma 3.7.

(ii) In general $gm_* = n_*ngm_* = n_*fmm_* \leq n_*f$. However, in $\mathcal{A}_p$ we further have $(n, 1)(1, f) = (m, g)$.

(iii) Trivial.

Conversely, assume that $\mathcal{O}$ satisfies these conditions. Define a functor $\varepsilon : \mathcal{O}_{tp} \to \mathcal{O}$ to be the identity on objects with

$$\varepsilon(m, f) = fm_*$$

Then (BC) makes ε a functor, which is fully faithful by unique factorisation. It remains to show that the ordering of the two categories are the same, i.e. ε preserves the order and is locally full.

If $(m, f) \leq (n, g)$ is realised by $p : A_0 \to A_1$ then $fm_* = gpp_*n_* \leq gn_*$ which shows that ε preserves the order. Conversely, if $fm_* \leq gn_*$ then $f \leq gn_*m$ which is forced to be an equality by maximality of the total morphisms. Thus n_*m is total by Lemma 3.6. Hence $nn_*m \leq m$ is an equality of total morphisms which shows that $(m, f) \leq (n, g)$ in $\mathcal{O}_{tp}$ is realised by n_*m. //

Examples 3.9 (i) **Rel** has the unique factorisations and satisfies (BC) since $\mathbf{Rel}_0$ is a category of partial maps. Maximality fails since one totally-defined relation may be strictly smaller than another.

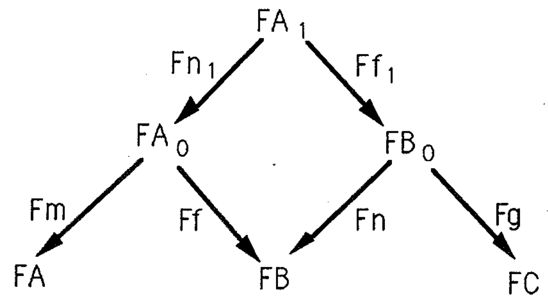
(ii) **Pos** fails to satisfy any of the three conditions. If $f : P \rightarrow Q$ is order-preserving then it factorises as $f = hg$ where $h : R \rightarrow Q$ is total and g is the quotient of P in which, for each $q \in Q$ the connected components of $f^{-1}(q)$ are identified. This is a projection iff each connected component has a least element. That (BC) fails can be seen by examining the pullback of the two distinct strict monomorphisms $\mathbf{2} \rightarrow \mathbf{3} = \{\bot \leq 1 \leq 2\}$.

4 Lax adjunctions

Many computationally important structures on a category, e.g. products and exponentials, are expressed in terms of functors, natural transformations and adjunctions. How can they be extended to categories of partial maps?

4.1 Lax fuctors and transformations

Let $(\mathcal{B}, \mathcal{N})$ be a category with dominion and let $F : (\mathcal{A}, \mathcal{M}) \rightarrow (\mathcal{B}, \mathcal{N})$ be a functor which *preserves dominions*, i.e. maps morphisms in $\mathcal{M}$ to those in $\mathcal{N}$. Define $\mathbf{Ptl}(F) = F_p : \mathcal{A}_p \rightarrow \mathcal{B}_p$ to agree with F on objects, and satisfy $F_p(m, f) = (Fm, Ff)$. Clearly F_p preserves the local order, identities, embeddings, projections and total morphisms. However, consider F_p applied to a composite:



Since the diamond above commutes there is a morphism from FA_1 to the pullback of Ff and Fn which establishes that

$$F_p((n,g)(m,f)) \leq F_p(n,g)F_p(m,f) \tag{6}$$

This is always an equality iff F preserves all pullbacks of admissible subobjects, e.g. if F is a right adjoint. Thus F_p is not always a functor in the usual sense.

Let $\mathcal{O}'$ be an ordered category. A *lax functor* $F : \mathcal{O} \rightarrow \mathcal{O}'$ consists of

(i) an object assignment $ob(\mathcal{O}) \rightarrow ob(\mathcal{O}')$ also called F and

(ii) for each pair X, Y of objects of $\mathcal{O}$ an order-preserving function

$$F_{X,Y} : \mathcal{O}(X,Y) \longrightarrow \mathcal{O}'(FX, FY)$$

which satisfies

(iii) $F(gf) \leq FgFf$ for f and g a composable pair of morphisms, and

(iv) $F(id_X) \leq id_{FX}$ for each object X of $\mathcal{O}$.

If (iv) is an equality then F is *normal.* If (iii) is also an equality then F is a *rigid functor.* If (iii) and (iv) are reversed then F is an oplax functor.

For example, a *behaviour* for a category $\mathcal{L}$ representing a typed programming language is a lax functor $\mathcal{L} \rightarrow \mathcal{O}$ where $\mathcal{O}$ is an ordered category of properties [22].

A normal, lax functor $F : \mathcal{O} \rightarrow \mathcal{O}'$ is a *total functor* if it preserves total morphisms, and whenever gf is a composable pair of morphisms then

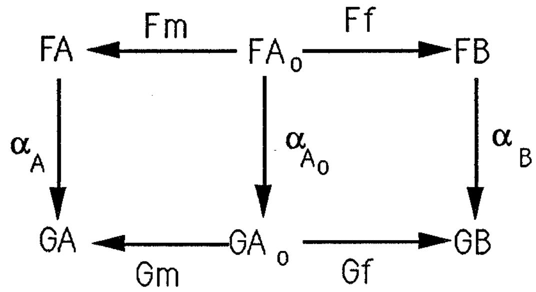
$$g \text{ total } \Longrightarrow F(gf) = FgFf$$

Tot$(F) = F_t : \mathcal{O}_t \rightarrow \mathcal{O}'_t$ is the functor obtained by restricting F to $\mathcal{O}_t$. Such an F is *stable* if it preserves stable embeddings, i.e. F_t is dominion-preserving. Clearly, if $F : (\mathcal{A}, \mathcal{M}) \rightarrow (\mathcal{B}, \mathcal{N})$ is dominion-preserving then F_p is stable and total.

Lemma 4.1 *Total functors preserve adjunctions in which the left adjoint is total, e.g. embeddings are mapped to adjunctions.*

Proof Let $F : \mathcal{O} \rightarrow \mathcal{O}'$ be a total functor and let $f \dashv f_* : A \rightarrow B$ be an adjunction in $\mathcal{O}$ with f total. Then $1_{FB} = F1_B \leq F(f_*f) \leq F(f_*)Ff$ since F is normal. Also $FfF(f_*) = F(ff_*) \leq F1_A \leq 1_{FA}$ since f is total. //

Let $\alpha : F \Rightarrow G$ be a natural transformation. Define $\mathbf{Ptl}(\alpha) = \alpha_p$ by $\mathbf{Ptl}(\alpha)_A = (1, \alpha_A) : FA \rightarrow GA$. The commutativity of



induces a morphism from FA_0 to the pullback of Gm and α_A so that

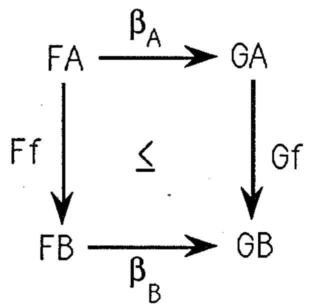
$$\alpha.F(m, f) \leq G(m, f).\alpha$$

It is always an equality iff for each $m \in \mathcal{M}$ the left square above is a pullback. For example, if $\mathcal{A}$ has cartesian products then this holds for the diagonal $\delta_A : A \rightarrow A \times A$ but not for the projections from the product $\times$ (unless $\mathcal{M}$ is trivial).

Let $F, G : \mathcal{O} \rightarrow \mathcal{O}'$ be lax functors. A *lax natural transformation* $\alpha : F \Rightarrow G$ consists of a family of morphisms $\alpha_X : FX \rightarrow GX$ of $\mathcal{O}$ which satisfy

$$Gf.\alpha_X \leq \alpha_Y Ff : FX \rightarrow GY$$

Dually, an *oplax natural transformation* or *optransformation* $\beta : F \Rightarrow G$ is given by morphisms $\beta_A : FA \rightarrow GA$ called its *components* which satisfy



(7)

If these inequalities are equalities then β is a *rigid natural transformation.*

An optransformation $\beta : F \Rightarrow G$ is a *total transformation* if its components are all total, and f total implies (7) commutes. If F and G are total functors then $\mathbf{Tot}(\beta) = \beta_t$ is then the (ordinary) natural transformation obtained by restricting β to $\mathcal{O}_t$. Clearly, if α is a natural transformation between dominion-preserving functors then α_p is a total transformation. Other examples of optransformations are the *simulations* of [9] and the *abstractions* of [2].

4.2 2-categories and lax adjunctions

2-categories were introduced to isolate the composition rules for functors and natural transformations. A 2-category $\mathcal{B}$ consists of some objects (e.g. categories), each pair A, B of which is equipped with a hom-*category* $\mathcal{B}(A, B)$ whose objects are the *1-cells* or *morphisms* of $\mathcal{B}$ (e.g. functors $A \rightarrow B$), and whose morphisms are the *2-cells* of $\mathcal{B}$ (e.g. natural transformations). Horizontal and vertical compositions of 1- and 2-cells, and the corresponding equations, are defined as for functors and natural transformations. For example, the (small) categories, functors and transformations form a 2-category **Cat** which has a sub-2-category **DomCat** of categories with dominions, dominion-preserving functors and all natural transformations.

Adjunctions can be defined within any 2-category. A 1-cell $f : A \rightarrow B$ is left adjoint to $g : B \rightarrow A$ in $\mathcal{B}$ with unit and counit given by 2-cells $\eta : 1 \Rightarrow gf$ and $\varepsilon : fg \Rightarrow 1$ respectively, if the usual triangle laws hold:

$$\begin{array}{rcl} 1 = \varepsilon_f . f\eta & : & f \rightarrow fgf \rightarrow f \\ 1 = g\varepsilon . \eta_g & : & g \rightarrow gfg \rightarrow g \end{array} \tag{8}$$

As in **Cat**, a right adjoint to f, if it exists, is unique. Thus, an adjunction in **DomCat** is just an adjunction of the usual kind in **Cat** where the adjoints are both dominion-preserving.

2-functors and 2-natural transformations are defined so as to preserve all the structure. Consequently, 2-functors preserve adjunctions, and all the structures defined in their terms. In this paper, the only 2-categories explicitly considered consist of categories, functors and transformations, so that a formal study is not necessary to follow the argument.

Despite all the expectations encouraged by an elementary introduction to category theory, the ordered categories, lax functors and optransformations do not form a 2-category! They fail only in that applying a lax functor $H : \mathcal{B} \to \mathcal{C}$ to an optransformation β as above does not always yield an optransformation since $H(\beta.Ff) \leq H(Gf.\beta) \leq HGf.H\beta$ but in general

$$H\beta.HFf \geq H(\beta.Ff) \tag{9}$$

One of the consequences is that a general definition of an adjunction between lax functors, a *lax adjunction*, is not determined by any 2-category structure. Various suggestions have been made (e.g. [7,9,10,21], but the primary difficulty is that there can be non-isomorphic right adjoints to a given lax functor, (see [10]) just as there may be non-isomorphic lax products. One solution is to so restrict the lax functors and optransformations that they do form a 2-category and then use its notion of adjunction.

Theorem 4.2 *The ordered categories, total functors and total transformations form a 2-category called* **TotOrdCat**. *It has a sub-2-category* **TotOrdCat**$_*$ *consisting of the ordered categories with stable, total functors and total transformations. Adjunctions in* **TotOrdCat** *are called* total adjunctions.

Proof If H and β are a total functor and transformation, then (9) is an equality and $H\beta$ is a total transformation. //

Another method of creating a 2-category containing lax functors and transformations is being developed by Carboni et al. [4]. They consider ordered categories equipped with a given sub-category of left adjoints, with 'functors' and 'transformations' that have special properties with respect to these left adjoints. The general approach is similar to, without including, that taken here, since total morphisms are not always left adjoints.

4.3 A coreflection for Ptl

Theorem 4.3 **Ptl** $\dashv$ **Tot** : **TotOrdCat**$_*$$\to$**DomCat** *is an adjunction between 2-functors, with* **Tot** *a coreflection for* **Ptl**.

Proof We have established that **Ptl** and **Tot** preserve the 2-categorical data. The remaining details required to establish that **Ptl** and **Tot** are 2-functors are left to the reader.

The stable, total functors $\varepsilon : \mathcal{O}_{tp} \to \mathcal{O}$ which form the counit of the adjunction are defined by $\varepsilon(m, f) = fm_*$ as in Theorem 3.8. The inequality in

(5) makes ε a lax functor, which is faithful by the uniqueness of embedding-total factorisations (if they exist) though not necessarily full. It preserves the order since $(m, f) \leq (n, g)$ implies $fm_* \leq gn_*$. That it is a stable, total functor follows trivially.

Now ε is natural since if $F : \mathcal{O} \rightarrow \mathcal{O}'$ is a stable, total functor then, for (m, f) as above, $F(fm_*) = FfF(m_*) = Ff(Fm)_*$ while if $\alpha : F \Rightarrow G$ is a stable, total transformation then $\varepsilon\alpha_{tp} = \alpha\varepsilon$ since they both have the same components as α.

The unit of the adjunction is given by the natural isomorphisms between $(\mathcal{A},\mathcal{M})_{pt}$ and $(\mathcal{A}, \mathcal{M})$. Thus **Tot** is a coreflection. //

Corollary 4.4 Ptl *extends adjunctions between dominion-preserving functors to total adjunctions.* //

Thus, structures defined in terms of adjunctions between dominion-preserving functors extend automatically to total adjunctions between the corresponding categories of partial maps, e.g. products, terminal objects and initial objects are extended to the corresponding total (lax) structures. For example, $\mathcal{A}$ has cartesian products iff the diagonal $\Delta : \mathcal{A} \rightarrow \mathcal{A}^2$ has a right adjoint $\times$. They are both dominion-preserving ($\mathcal{A}^2$ has $\mathcal{M}^2$ as dominion). Thus $\times_p$ is total right adjoint to Δ_p which is the diagonal. Thus $\mathcal{A}_p$ has total (lax) products.

By contrast, sums extend to lax coproducts iff they preserve the dominion, i.e. if $m, n \in \mathcal{M}$ then $m + n \in \mathcal{M}$, which is the case, for example, in **Sets** and **Pos**.

If $\mathcal{A}$ is cartesian closed then each exponential $B \rightarrow (-)$ preserves monomorphisms since it is a right adjoint. If, further, it is dominion-preserving then $\mathcal{A}_p$ is total cartesian closed. This is the case with **Sets** where all monomorphisms are in the dominion. Note that the exponential object is the same in both **Sets** and **Sets**$_p$ and not the 'object of partial morphisms' $(B \rightarrow ((-) + 1))$ which should be another form of lax adjoint.

5 Categories of domains

5.1 Domains

To each ordered category $\mathcal{O}$ can be assigned a category in which each morphism of $\mathcal{O}$ has a domain, i.e. the largest admissible subobject of its source on which it is total. If $\mathcal{O}$ is extensionally ordered (see below) then it can be embedded into the partial maps of this category.

If $f : B \rightarrow A$ and $g : C \rightarrow B$ are morphisms in $\mathcal{O}$ then f is *total relative to g* or *g-total* if, for each morphism $h : D \rightarrow C$ and each deflation $\delta : D \rightarrow D$ we have

$$fgh\delta = fgh \Longrightarrow gh\delta = gh$$

This is to be interpreted as saying that f is defined on the image of g though this image may not be represented by any subobject of B. Clearly f is total iff it is 1_A-total. More generally, if $\varphi = \{\varphi_i : B \rightarrow B_i\}$ is a set of morphisms of $\mathcal{O}$ then φ is g-total if each φ_i is. It will be shown in Proposition 5.6 that (m, f) is g-total in $\mathcal{A}_p$ iff g factors through $(1, m)$.

Lemma 5.1 *Let f, g and h be as above. Then fg is h-total iff f is gh-total and g is h-total.*

Proof Let fg be h-total with $k : E \rightarrow D$ a morphism and $\varepsilon : E \rightarrow E$ a deflation. Then $fghk\varepsilon = fghk$ iff $hk\varepsilon = hk$ iff $ghk\varepsilon = ghk$ whence f is gh-total and g is h-total. Conversely, if $fghk\varepsilon = fghk$ then $ghk\varepsilon = ghk$ since f is gh-total, whence $hk\varepsilon = hk$ since g is h-total. //

Let A be an object of $\mathcal{O}$. The finite sets of morphisms $\varphi = \{\varphi_i : A \rightarrow A_i\}$ of $\mathcal{O}$ with source A can be pre-ordered by $\varphi' \leq \varphi$ iff

$$\varphi' \text{ is } g\text{-total} \Longrightarrow \varphi \text{ is } g\text{-total}$$

The equivalence class of φ in the preorder is a *domain* in A denoted $d(\varphi)$. Here each $d(\varphi_i)$ is to be thought of as a subobject of A with $d(\varphi)$ as their intersection. The maximal element is $d(1_A)$ (abbreviated to $d(A)$) and meets are given by unions of sets of morphisms.

Lemma 5.2 *Let $f : B \rightarrow A$ be a morphism of an ordered category $\mathcal{O}$.*

(i) If $d(\varphi') \leq d(\varphi) \leq d(A)$ are domains then $d(\varphi' f) \leq d(\varphi f)$ where $\varphi f = \{\varphi_i f \mid \varphi_i \in \varphi\}$.

(ii) If f is g-total then $d(fg) = d(g)$.

Proof Use Lemma 5.1. //

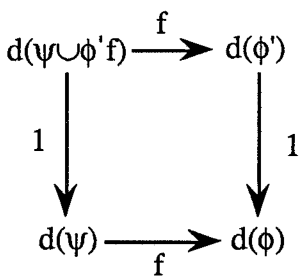
To each ordered category is associated a category $\mathbf{Dom}_0(\mathcal{O})$ whose objects are its domains. If $d(\varphi) \leq d(A)$ and $d(\psi \leq d(B)$ are domains then $f : B \rightarrow A$ is a morphism $d(\psi) \rightarrow d(\varphi)$ if

$$d(\psi) \leq d(\varphi f)$$

This can be interpreted (using Lemma 5.1) as saying that f is defined on the image of g whenever ψ is g-total (i.e. is defined on $d(\psi)$) and the image of fg is in $d(\varphi)$. Composition and identities are those of $\mathcal{O}$. If $f : d(\psi) \rightarrow d(\varphi)$ and $g : d(\theta) \rightarrow d(\psi)$ then $d(\theta) \leq d(\psi)g \leq d(\varphi f g)$ which shows that the composition is well-defined.

Lemma 5.3 *The morphisms of* $\mathbf{Dom}_0(\mathcal{O})$ *represented by identities of* $\mathcal{O}$ *form a dominion.*

Proof It suffices to show their closure under pullback. Let $f : d(\psi) \rightarrow d(\varphi)$ and $1_A : d(\varphi') \rightarrow d(\varphi)$ (that is, $d(\varphi') \leq d(\varphi)$). Then their pullback is:



since if $g : d(\theta) \rightarrow d(\psi)$ and $fg = h : d(\theta) \rightarrow d(\varphi')$ then $g : d(\theta) \rightarrow d(\varphi' f)$. //

Lemma 5.4 $\mathbf{Dom}_0(\mathcal{O})$ *is an ordered category with* $f \leq f' : d(\psi) \rightarrow d(\varphi)$ *if* ψ *is* g*-total implies* $fg \leq f'g$ *in* $\mathcal{O}$.

Proof Clearly the order is reflexive and transitive. Let $f \leq f' : d(\psi) \rightarrow d(\varphi)$ and $g \leq g' : d(\theta) \rightarrow d(\psi)$. If θ is h-total then $gh \leq g'h$. Also ψ is gh-total and so $fgh \leq f'gh \leq f'g'h$. //

The (unordered) category $\mathbf{Dom}(\mathcal{O}) = \mathcal{O}_d$ of domains of $\mathcal{O}$ is the quotient of $\mathbf{Dom}_0(\mathcal{O})$ obtained by identifying morphisms $f, f' : d(\psi) \rightarrow d(\varphi)$ which are equivalent in the preorder, i.e. ψ is h-total implies $fh = f'h$.

5.2 Extensional categories

Not every ordered category $\mathcal{O}$ can be embedded into $\mathcal{O}_{dp}$ since the ordering of partial morphisms is determined by their extent of definition, which is not generally the case in $\mathcal{O}$. An ordered category $\mathcal{O}$ is *extensional* if

$$f \leq f' : B \rightarrow A \Longrightarrow f = f' : d(f) \rightarrow A \tag{10}$$

i.e. whenever f is g-total then $fg = f'g$ (whence $d(f) \leq d(f')$). Further, $\mathcal{O}$ is *strongly extensional* if the converse of (10) also holds.

A domain $d(\varphi) \leq d(A)$ is *represented* by a subobject $m : A_0 \rightarrow A$ if

$$d(\varphi) \text{ is } g\text{-total iff } g = mg'$$

for some morphism g'. Then $m : d(A_0) \rightarrow d(\varphi)$ is a morphism in $\mathcal{O}_d$ which, if m is an embedding, is an isomorphism with m_* as its inverse. Further, if, for each $\varphi_i \in \varphi$, $d(\varphi_i)$ is represented by a stable embedding m_i then $d(\varphi)$ is represented by the joint pullback of the m_i 's.

Proposition 5.5 *let $\mathcal{O}$ be an extensional category.*

(i) If $m : A_0 \rightarrow A$ is an embedding in $\mathcal{O}$ then $d(m_*)$ is represented by m.

(ii) $\mathcal{O}$ satisfies (BC).

(iii) Total morphisms in $\mathcal{O}$ are all maximal in their orders.

Proof

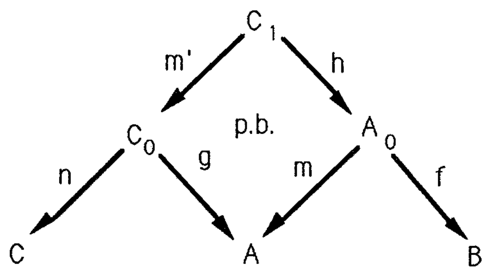
(i) If $g = mg'$ then $m_*g = g'$ which shows that m_* is g-total. Conversely, if m_* is g-total then $mm_*g \leq g$ and $d(mm_*g) = d(g)$ by Lemma 5.2 since mm_* is g-total. Thus, by extensionality, $g = mm_*g$.

(ii) If $gn_* \leq m_*f$ as in (5) then it suffices to prove that $d(gn_*) = d(m_*f)$. Now m_*f is h-total iff m_* is fh-total (since f is total) iff $fh = mk$ for some k (by (i)) iff $h = nh'$ for some h' (by the universal property of the pullback) iff gn_* is h-total.

(iii) Let $f \leq f' : B \rightarrow A$ where f is total. Then $d(B) = d(f) \leq d(f')$ shows that $d(f) = d(f')$ and so $f = f'$. //

Proposition 5.6 *Let $(m, f) : A \rightarrow B$ be a morphism in $\mathcal{A}_p$. Then the domain $d(m, f)$ is represented by $(1, m)$ and hence all domains for $\mathcal{A}_p$ are representable by stable embeddings. Consequently, $\mathcal{A}_p$ is strongly extensional.*

Proof Let $(n, g) : C \rightharpoonup A$ be a partial morphism. Then $(m, f)(n, g) = (p, fh)$ where $p = nm'$:



Now $(p, fh)(p, p) = (p, fh)$. Hence, if (m, f) is (n, g)-total then $(n, g)(p, p) = (n, g)$ which forces $m' = 1$ and $g = mh$. Thus $(n, g) = (1, m)(n, h)$. The converse is proved as in Lemma 5.5(i).

For extensionality, let $(m, f) \leq (m', f') : A \rightarrow B$ be realised by p and let (m, f) be g-total. Then $g = (1, m)g' = (1, m'p)g'$ for some g' and hence

$$(m', f')g = (m', f')(1, m'p)g' = (1, f'p)g' = (1, f)g' = (m, f)g$$

Conversely, if $(m, f) = (m', f') : d(m, f) \rightarrow B$ then

$$(m', f')(1, m) = (m, f)(1, m) = (1, f)$$

shows that $m = m'p$ and $f = f'p$ for some p whence $(m, f) \leq (m', f')$. //

5.3 Second characterisation theorem

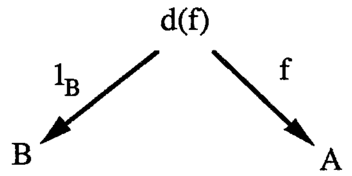
Theorem 5.7

$$\mathbf{Dom}(\mathbf{Ptl}(\mathcal{A}, \mathcal{M})) \simeq (\mathcal{A}, \mathcal{M})$$

Also, an ordered category $\mathcal{O}$ is (equivalent to) a category of partial maps iff it is strongly extensional, and each object of $\mathcal{O}_d$ is represented by a stable embedding.

Proof First, the obvious inclusion functor $(\mathcal{A},\mathcal{M})\to(\mathcal{A},\mathcal{M})_{pd}$ is fully faithful since all morphisms $d(A)\to d(B)$ are total. That it is essentially surjective on objects and dominion-preserving (and hence an equivalence) follows from Proposition 5.6, which also states half of the second result.

For its converse define $\rho : \mathcal{O}\to\mathcal{O}_{dp}$ on $f : B\to A$ by:



which is abbreviated to $(d(f), f)$. If $g : C\to B$ in $\mathcal{O}$ then

$$\rho(f)\rho(g) = (d(g)\cap d(fg), fg) = (d(fg), fg) = \rho(fg)$$

(since $d(g) \le d(fg)$) which shows that ρ is a rigid functor. The strong extensionality of $\mathcal{O}$ makes ρ order-preserving and locally full, and hence faithful. The representability of all domains shows that ρ is essentially surjective on objects, and that the dominion on $\mathcal{O}_d$ is represented by stable embeddings of $\mathcal{O}$. Thus every partial map $A\to B$ in $\mathcal{O}_{dp}$ is of the form (m, f) where $m : A_0\to A$ is an embedding in $\mathcal{O}$ and f is total. This, in turn, can be represented by $(d(fm_*), fm_*)$ since $d(fm_*) = d(m_*) \cong d(A_0)$ follows from the totality of f and Lemma 5.5. Hence ρ is full, and so an equivalence. //

5.4 Extension functors and transformations

For **Dom** to be applied to lax functors and optransformations they must not only be total, but respect the domain structure.

A normal, lax functor $F : \mathcal{O}\to\mathcal{O}'$ between extension categories is an *extension functor* if $d(\varphi) \le d(\psi)$ in $\mathcal{O}_d$ implies that $d(F\varphi) \le d(F\psi)$ in $\mathcal{O}'_d$ (i.e. $F\varphi$ is g-total implies $F\psi$ is g-total for *all* g in $\mathcal{O}'$). An optransformation $\alpha : F\Rightarrow G$ between extension functors is an *extension transformation* if for all $d(\varphi) \le d(A)$ we have $\alpha_A : d(F\varphi)\to d(G\varphi)$.

Proposition 5.8 *The collection* **ExtOrdCat** *of extension categories, functors and transformations form a sub-2-category of* **TotOrdCat**.
Proof Let $F : \mathcal{O} \to \mathcal{O}'$ be an extension functor. If f is g-total in $\mathcal{O}$ then $d(fg) = d(g)$ and so

$$\begin{array}{rcll} d(Fg) & = & dF(fg) & \text{F is an extension functor} \\ & \leq & d(FfFg) & \text{F is lax} \\ & \leq & dFg & \end{array}$$

Thus $d(FfFg) = d(Fg)$ which shows that F preserves relative totality, and hence totality since F is normal. Further $dF(fg) = d(FfFg)$ which forces $F(fg) = FfFg$ since $\mathcal{O}'$ is extensional. Hence F is a total functor.

Let $\alpha : F \Rightarrow G$ be an extension transformation. That $\alpha_A : d(FA) \to d(GA)$ exists implies that α_A is total. If f is a total morphism of $\mathcal{O}$ then the rigidity of α with respect to f follows from the extensionality of $\mathcal{O}'$. Thus α is total.

If, further, $H : \mathcal{O}' \to \mathcal{O}''$ is another extension functor then $H\alpha : HF \Rightarrow HG$ is an extension transformation since H and α are both total, which implies $H(\alpha F) = H\alpha . HF$. Clearly the extension functors and transformations are closed under all other compositions and so form a sub-2-category of **TotOrdCat**. //

5.5 A reflection for Ptl

Theorem 5.9 **Ptl** : **DomCat**$\to$**ExtOrdCat** *is a 2-functor with left 2-adjoint* **Dom** *where the counit is an equivalence, i.e.* **Dom** *is a reflection for* **Ptl**.
Proof Only steps of interest are given here. If $\alpha : F \Rightarrow G : (\mathcal{A}, \mathcal{M}) \to (\mathcal{B}, \mathcal{N})$ is a natural transformation between dominion-preserving functors then clearly F_p and G_p are extension functors. Now α_p is extensional since every domain in $\mathcal{A}_p$ is representable. Thus **Ptl** is a 2-functor into **ExtOrdCat**.

If $F : \mathcal{O} \to \mathcal{O}'$ is an extensional functor then $\mathbf{Dom}(F) = F_d : \mathcal{O}_d \to \mathcal{O}'_d$ is defined by $F_d(\varphi) = d(F\varphi)$ (where $F\varphi = \{F\varphi_i \mid \varphi_i \in \varphi\}$) and $F_d f = Ff$ for $f : d(\psi) \to d(\varphi)$. It is well-defined since $d(F\psi) \leq dF(\varphi f) \leq d(F\varphi F f)$. Also F_d preserves composition and identities since the inequalities which make functors (and transformations) lax are mapped to equalities in $\mathcal{O}'_d$ by extensionality. Finally F_d preserves the dominion since F is normal.

If $\alpha : F \Rightarrow G$ is an extension transformation then $\mathbf{Dom}(\alpha) = \alpha_d : F_d \Rightarrow G_d$ is defined by

$$\alpha_{d(\varphi)} = \alpha_A : d(F\varphi) \rightarrow d(G\varphi)$$

The counit of the adjunction is the equivalence of Theorem 5.7. The unit $\rho : \mathcal{O} \Rightarrow \mathcal{O}_{dp}$ for a extension category $\mathcal{O}$ is also defined there and must now be shown to be an extension functor.

Let $d(\varphi') \leq d(\varphi)$ be domains. Then $\rho(\varphi')$ is $(d(\theta), g)$-total iff $d(\theta) \leq d(\varphi' g)$. Now $d(\varphi' g) \leq d(\varphi g)$ by Lemma 5.2. Hence $d(\rho(\varphi')) \leq d(\rho(\varphi))$.

The triangle laws for an adjunction hold and so $\mathbf{Ptl} \dashv \mathbf{Dom}$. //

Corollary 5.10 *Let* $\mathbf{ExtOrdCat}_*$ *be the sub-2-category of* $\mathbf{ExtOrdCat}$ *of extensional, ordered categories with* stable, *extension functors and extension transformations. Then* $\mathbf{Ptl} : \mathbf{DomCat} \rightarrow \mathbf{ExtOrdCat}_*$ *is a 2-functor which has both a reflection* **Dom** *and a coreflection* **Tot**. //

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