Relevance Logic and Concurrent Composition

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Abstract

Compositionality, i.e. that properties of composite systems are deduced in terms of those of their immediate constituents, is crucial to the tractability and practical usefulness of program logics. A general technique for obtaining this for parallel composition appeals to a relativisation of properties with respect to properties of parallel environments. This induces a notion of consequence on properties which will in general be a relevant one. Based on this observation we suggest using modal or temporal extensions of relevance logics to build compositional logics for processes.

We investigate the general model theory of propositional relevance logic and introduce a notion of model based on semilattices with an inf-preserving binary operation. We present a number of correspondence and completeness results, investigate the relationship to Sylvan and Meyer's ternary relation model, and present concrete models based on Milner's SCCS.

To account for dynamic behaviour a modal extension of linear logic is introduced, interpreted over models extended by prefixing in the style of CCS/SCCS. We show a variety of characterisation results, relating models to processes under testing preorders, and obtain completeness results, first for the general algebraically based interpretations and next for the process-based ones giving, for the latter, procedures for deciding validity and satisfiability of formulas.

From an computational point of view the processes considered are unacceptably weak in that they lack a suitable notion of external, or controllable, choice. To remedy this we consider indexed modal models under weak preorders, generalising notions of process equivalence such as testing and failures equivalence. We give characterisations of these in terms of modal logic and axiomatise the logics obtained. Relevant extensions of these logics are introduced, interpreted over model classes on which a parallel composition is defined. We axiomatise the logics obtained, giving decision procedures as before, and conclude by specialising the results to testing equivalence and synchronous parallel composition.

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Declaration

The thesis has been composed by myself and the research reported is my own with minor exceptions as indicated in the text. Chapter 3 is a revised and extended version of [25].

Contents

Chapter 1

Introduction

One of the basic aims of theoretical computer science is to provide ways of reasoning about computing systems such that their correctness may be subjected to a formal proof. It has long been realised that structure is a highly desirable feature of systems, and one which our theories must reflect. The principle of compositionality, or syntax-directedness, i.e. that properties of composite systems are deduced from those of their immediate constituents, is crucial to the tractability and practical usefulness of theories.

In the context of parallel programming the need for powerful reasoning and structuring tools is all the more acute. Compared to their sequential counterparts, the building of concurrent systems is an activity far more prone to errors and subtle bugs, and the detection of errors when they arise may be exceedingly difficult using informal reasoning alone (see e.g. [46]). In view of this it is not surprising that the problem of developing theories for concurrency that combines a suitable level of expressiveness with compositionality has turned out to be a difficult one. The basic problem that must be addressed is the potentiality of interaction between processes. This means that

i) determinism is lost (at least as a reasonably abstract modelling concept), as account must be taken of the usually diverse ways in which interaction is possible,

ii) modelling programs as functions from initial to terminal values is no longer adequate—the intermediate states a process may pass through is of importance to the final outcome.

Modal and temporal logics are general and useful tools for expressing and verifying interesting properties of concurrent programs. Compositionality, however, is problematic. Modal and temporal properties focus on the global states of programs and transitions, or paths, through them, and there is in general no direct way of composing properties of parallel subcomponents into nontrivial properties of their parallel composition.

For instance, in Owicki-Gries's extension of Hoare-logics to accomodate parallellism [84], proofs for parallel compositions involve proofs for the subcomponents as well as checking a complicated side-condition to the effect that these subproofs are interference-free. In this case, however, some form of side-condition is evidently needed, as the input-output behaviour of programs expressed by Hoare-triples is inadequate for capturing the behaviour of programs in parallel contexts, as is well known. Expressiveness of formulas may be increased such as to carry the information of the properties of potential intermediate states needed in various ways—by adding history-variables as in [103], by reinterpreting Hoare-triples completely, turning the precondition into an invariant as in [62], or—more elegantly—by including the invariance properties needed to prove interference freedom directly in the formulas themselves [105].

Modal and temporal logics, on the other hand, do not suffer from this problem of insufficient expressive power. For path-oriented logics Abrahamson [1, 2] suggests that in operationally interpreting concurrent programs, allowance must be made for "phantom-transitions" due to the environment, thus enabling a compositional account of systems in terms of their sets of computation sequences, or paths. This idea has direct bearing on linear time temporal logic [87], based as it is on computation sequences, or paths, rather than states. By enriching such a logic with atomic propositions keeping track of whether a transition is genuinely due to the program under consideration or it is a "phantom move", Barringer et al [12] succeeds in providing a compositional account of both a shared-memory and a communications-based system of processes. Very similar ideas may be used for giving compositional linear-time accounts of concurrent networks (c.f. [113, 117]), and they also underpin Stirling's compositional Hoare-logic in [105].

In state- rather than path-oriented logics such as modal logic or branching-time temporal logic [14] no such direct solution seems possible. Instead, Stirling [106] and Winskel [115] consider manipulating the deductive structure of program logics to provide a logical handle on the parallel combinator. In [115] Winskel suggests enriching a modal logic by pointwise extensions of the process combinators—in the actual case those of SCCS [78]. He defines the connective $\circ$

on properties by

$$\phi \circ \psi = \{p \times q \mid p \models \phi, q \models \psi\},$$

where ϕ, ψ are used to denote properties, p, q to denote processes, $\times$ SCCS-type parallel composition and $p \models \phi$ the satisfaction of property ϕ by p. For this to work properly one has to require that such "intensionally combined" properties occur only as assumptions—this, though, may be overcome by quotienting, i.e. taking instead

$$\phi \circ \psi = \{[p \times q] \mid [p] \models \phi, [q] \models \psi\},$$

where $[p]$ denotes the equivalence class of p under some suitable congruence. This is the approach taken in e.g. [45] for the CCS $+$-combinator.

Stirling [106] considers axiomatising an intuitionistic version of Hennessy-Milner logic [51] with an added ternary relation $\phi, \psi \models' \gamma$ on properties defined by

$$\phi, \psi \models' \gamma \text{ iff for all } p, q, \text{ if } p \models \phi \text{ and } q \models \psi \text{ then } p \times q \models \gamma,$$

corresponding, in terms of the set-up of [115] to the assertion $\phi \circ \psi \models \gamma$. One can think of $\phi, \psi \models' \gamma$ as stating that γ is a consequence of assuming ϕ and ψ respectively of each of the parallel subcomponents.

What this amounts to is a transition from regarding structural proof rules as "axioms" to objects that themselves require proof. Let the proof-theoretic correlates of $\models$, $\models'$ be $\vdash$, $\vdash'$ respectively. To obtain compositionality w.r.t. $\times$ we are looking for a finite set of rules, or more precisely rule-schemas, of the form

$$\text{from } p \vdash \phi \text{ and } q \vdash \psi \text{ infer } p \times q \vdash \gamma \tag{1.1}$$

which are sound and complete, for instance in the sense that $p \times q \models \gamma$ iff for some ϕ, ψ, $p \vdash \phi$ and $q \vdash \psi$ are provable, $p \times q \vdash \gamma'$ follows from $p \vdash \phi$ and $q \vdash \psi$ by one of the compositional proof rules, and γ is a consequence of γ'. Note that every triple $\phi, \psi \models' \gamma$ encodes a compositional proof rule of the form 1.1 which is independent of the structure of p and q.

In this sense compositionality w.r.t. $\times$ amounts to the requirement that $\models'$ be axiomatisable by just a finite number of axiom schemas of the form $\phi, \psi \models' \gamma$ (plus a rule of consequence). Such a requirement does, however, seem overly harsh, and the idea underpinning Stirling's suggestion is essentially to allow instances of $\models'$ to be subject to proof not only by the instantiation of axioms but also by the application of proof rules.

In logical terms $\models'$ is a consequence relation. Let us assume, what is very reasonable, that parallel composition is associative and commutative. We can then generalise $\models'$ to allow arbitrary finite, nonempty sequences on the left hand side by:

$$\overline{\phi} \models' \psi \text{ iff for all } \overline{p}, \text{ if } \overline{p} \models \overline{\phi} \text{ then } \prod(\overline{p}) \models \psi,$$

where we use $\overline{p}, \overline{\phi}$ etc. to denote sequences and for $n \geq 1$ let

$$\prod(p_1, \ldots, p_n) = p_1 \times \cdots \times p_n.$$

This relation $\models'$ has the following basic properties:

i) Reflexivity: $\phi \models' \phi$,

ii) Permutation: $\overline{\phi} \models' \gamma$ only if $\overline{\psi} \models' \gamma$, $\overline{\psi}$ a permutation of $\overline{\phi}$,

iii) Cut: $\overline{\phi} \models' \psi$ and $\overline{\gamma}, \psi, \overline{\delta} \models' \theta$ only if $\overline{\gamma}, \overline{\phi}, \overline{\delta} \models' \theta$,

as in fact $\models'$ will for any commutative and associative operation $\times$. It will, however, in general fail

i) Weakening: $\overline{\phi} \models' \gamma$ only if $\overline{\phi}, \psi \models' \gamma$, for all ψ,

ii) Contraction: $\overline{\phi}, \psi, \psi \models' \gamma$ only if $\overline{\phi}, \psi \models' \gamma$.

The failure of weakening marks our notion of consequence as *relevant* in the sense of [7, 34]. Characteristic of relevance logics is the idea that conclusions must in some way depend on the assumptions being made. This notion of *dependency* can be fleshed out in a number of ways; common, however, to all of them is the rejection of weakening. In the present context this corresponds to the fact that properties of processes are not in general inherited outwards through parallel environments. With the failure of contraction as well we are moreover dealing with a *linear* consequence relation [43, 8]. In linear logic not only the usage/nonusage of assumptions is significant, but also the number of times an assumption is being used, i.e. assumptions may be thought of as coming in multisets [8]. Here this is reflected by the fact that in parallel contexts it is the number of occurrences, and not just the existence, of subcomponents that matters.

Our suggestion, now, is to consider program logics where this notion of consequence has been *internalised*, or made expressible within the logic itself, in the sense that there is an operation $\to$ on formulas with the property that

$$\overline{\phi}, \psi \models' \gamma \text{ iff } \overline{\phi} \models' \psi \to \gamma,$$

for all $\overline{\phi}$, ψ and γ. A natural candidate for $\rightarrow$ is obtained by taking

$$p \models \phi \rightarrow \psi \text{ iff for all } q, \text{ if } q \models \phi \text{ then } p \times q \models \psi. \tag{1.2}$$

This $\rightarrow$ may be understood as an operation of relativisation of properties with respect to parallel environments, and one can read $\phi \rightarrow \psi$ as e.g. "in every parallel ϕ-context, ψ (will hold)". With this operation compositionality with respect to $\times$ is trivial: To prove $p \times q \models \psi$ prove for some ϕ that $p \models \phi \rightarrow \psi$ and $q \models \phi$, or—as $\times$ is assumed to be commutative—symmetrically that $q \models \phi \rightarrow \psi$ and $p \models \phi$. As the notation suggests we can think of the $\rightarrow$ as an implication, albeit a highly nonstandard one. In the terminology of [9] $\rightarrow$ is an intensional, or internal implication just as the operation $\circ$ of [115] in this context is an intensional conjunction, i.e.

$$\overline{\phi}, \psi, \gamma \models' \delta \text{ iff } \overline{\phi}, \psi \circ \gamma \models' \delta,$$

for all $\overline{\phi}, \psi, \gamma$ and δ. This connective is also known as fusion, or cotenability [7, 34] or as "times" in [43].

If parallel composition admit a neutral element 1 we obtain further a natural notion of validity. In CCS [76, 79] 1 will be NIL, and in TCSP [15], when $\times$ is $\|$ ($|||$) 1 will be RUN(STOP). Then $\models'$ is naturally extended to arbitrary finite sequences on the left, by stipulating that $\prod(\varepsilon) = 1$, for ε the empty sequence. Now we may define ϕ to be *valid*, $\models' \phi$, iff $\varepsilon \models' \phi$ iff $1 \models \phi$. It is not hard to check that then $\phi \rightarrow \psi$ is valid just in case whenever $p \models \phi$ then $p \models \psi$.

If one assumes just the algebraic properties indicated for $\times$ and 1 the logic we have just defined (i.e. the set of valid implicative formulas) is just the implicative fragment of linear logic [43], and the notions of model, satisfaction and validity is but a slight variation on the semilattice models of relevance logics due to Urquhart [109]—the semilattice operation being idempotent which $\times$ is not in general.

The question we raise in the present thesis is to what extent these ideas can be used to build compositional logics for concrete systems of processes. The essential problem that must be faced is how to combine a logical account of the static structure of processes based on the ideas outlined above with a logical account of their dynamic behaviour, for instance in terms of modal or temporal operators. We investigate these issues within the framework of process calculi such as CCS, SCCS and TCSP.

1.1 Applications

Although this remains to be investigated in practice, we believe such relativised properties to be natural and potentially highly useful in expressing and proving the appropriate properties of parallel programs.

An interesting class of properties could be those that are "invariant under parallel composition", i.e. properties ϕ such that whenever $p \models \phi$ and $q \models \phi$ then also $p \times q \models \phi$, or equivalently s.t. $\phi \rightarrow (\phi \rightarrow \phi)$ is valid. Consider for instance the problem of communication protocol specification. Such protocols are usually organised in a layered fashion, lower layers providing primitive services to higher ones. A protocol specification must ensure that any number of entities communicating according to the rules laid down provides a service to the higher layers of the required quality when some appropriate lower level service, or communication medium, is assumed. If ψ_n is the service specification for layer n and ϕ the protocol specification for layer $n+1$ then the validity of $\phi \rightarrow (\phi \rightarrow (\psi_n \rightarrow \psi_{n+1}))$ expresses that any two entities conforming to the protocol specification in parallel with an entity providing the service of layer n conforms with the service specification for layer $n+1$. But this does not in general entail that then for instance also $\phi \rightarrow (\phi \rightarrow (\phi \rightarrow (\psi_n \rightarrow \psi_{n+1})))$ is valid, although we should clearly like this to be the case. This will, however, be ensured if ϕ is invariant under parallel composition; i.e. if the parallel composition of any two entities conforming to the protocol specification is itself an entity conforming to the specification.

Another potential application could be in systems inductive in the number of parallel subcomponents such as systolic arrays [61]. We illustrate this by a little example:

Example 1.1 We consider an n-bit adder producing the modulo 2^n sum of input bitstrings $\overline{a} = a_1, \ldots, a_n$ and $\overline{b} = b_1, \ldots, b_n$ as a bitstring $\overline{s} = s_1, \ldots, s_n$. Let ϕ_n denote this desired property, namely that

$$\textstyle\sum_{i=1}^{n} s_i 2^{i-1} = \sum_{i=1}^{n} a_i 2^{i-1} + \sum_{i=1}^{n} b_i 2^{i-1} [\text{mod } 2^n],$$

and ψ_n, $n \geq 1$, denote the property that c_n is the carry of this sum—i.e. that $c_n = 1$ iff $\sum_{i=1}^{n} a_i 2^{i-1} + \sum_{i=1}^{n} b_i 2^{i-1} \geq 2^n$. Let in addition $\phi'_n(\psi'_n)$ denote the property that for each $i > n$, $s_i(c_i)$ is undefined, and γ_n the conjunction $\phi_n \wedge \psi_n \wedge \phi'_n \wedge \psi'_n$.

Now a 1-bit full adder p_i, $i \geq 0$ is a system with three binary inputs a_i, b_i and c_{i-1} and two binary outputs s_i and c_i s.t. s_i is the modulo 2 sum of a_i, b_i and

c_{i-1} and c_i is the carry of this sum. Then an n-bit adder is built in the obvious way from n 1-bit full adders $p_1, \ldots, p_n$ where for each i, $1 < i \leq n$, the output c_{i-1} of p_{i-1} is connected to input c_{i-1} of p_i (we assume generally that identically labelled ports are identified) and input c_0 of p_1 is constant 0 (we use the notation $p_1[c_0 = 0]$ to denote this). In proving this fact we need just to prove that

i) $p_1[c_0 = 0] \models \gamma_1$, and

ii) for all $n > 1$, $p_n \models \gamma_{n-1} \to \gamma_n$.

To go a little further, let a 1-bit carry generator k_i be a 1-bit full adder without any sum output s_i, and similarly a 1-bit half adder be a 1-bit full adder minus carry. Then the inductive property for carry generators can (in the present context) be stated as

$$k_n \models (((\phi_{n-1} \wedge \psi_{n-1}) \to \phi_n) \wedge \phi'_n \wedge \psi'_n) \to (\gamma_{n-1} \to \gamma_n),$$

that is, intuitively, when put in any context that acts as a 1-bit half adder with respect to inputs a_n, b_n and c_{n-1} and output s_n, and that leaves s_i and c_j undefined for all $i > n$ and $j \geq n$, then k_n in that context acts as a 1-bit full adder.

1.2 Summary

We base our investigations on the general model theory of relevance logics. In **chapter 2** we review a number of well known relevance logics and introduce a notion of model based on semilattice ordered sets with an inf-preserving binary operation. A natural minimal logic, **G**, is introduced, based on which we show a variety of correspondence and completeness results for the most standard relevance logics—both distributive and nondistributive ones. The relation between our notion of model and Sylvan and Meyer's ternary relation model [97, 98] is investigated, and we conclude the chapter by giving a first example of the application of relevance logic to concurrency by turning fragments of Milner's SCCS [78] into frames.

The remainder of the thesis consists of the detailed development of two related examples. In the first we remain as close as possible to the relevance logic framework, and in the second we adopt a more "standard" semantical approach, based on indexed modal models and testing equivalence.

In **chapter 3** we consider extensions of models by prefixing in the style of CCS/SCCS to account for dynamic behaviour. The result we name "synchronous algebras": (in-) equational classes of algebras, akin to the "process algebras" of Bergstra and Klop (see e.g. [13]), extending the notion of model for positive linear logic introduced in chapter 2. We obtain a quite straightforward completeness result for a conservative extension of positive linear logic by indexed forwards and backwards modalities.

For this to be of interest beyond the technicalities involved, the computational interpretation of models and formulas is essential. Terms in the language of synchronous algebras are interpreted as processes by giving them an operational semantics in the style of SCCS. The initial synchronous algebras may then be characterised as coinciding with processes under suitable (somewhat nonstandard) notions of testing preorders [30]. Additionally, satisfaction on the initial algebras can be given in syntactical/operational terms, thus obtaining a logical characterisation of the testing preorders themselves. We provide complete axiomatisations of the logics obtained when restricting attention to the initial algebras, obtaining as spin-offs from the rewriting-based completeness proofs procedures for deciding satisfiability and validity of formulas. We conclude the chapter by discussing possible variations and extensions, and the difficulties involved in generalising the approach to richer process systems—operationally the expressiveness of the process language is very poor, lacking in particular some suitable notion of external, or controllable, choice.

In **chapter 4** we then set up the semantical framework to be used in the remainder of the thesis. We consider indexed modal models with a notion of weak preorder generalising a number of well known process equivalences such as testing equivalence [30], failures and "improved" failures equivalence [15, 16], readiness semantics [82] and refusal testing equivalence [86]. This extends results of De Nicola [28, 29] who observed the similarity between testing and failures equivalence, and (a modified version of) Kennaway's weak equivalence [57]. The weak preorders are defined in terms of bisimulation-like orderings [85, 53, 77] applied to frames that have been suitably transformed, or linearised, by shifting emphasis from individual states to sets of states. On this basis the weak preorders can be characterised in terms of indexed modal logic by interpreting formulas over transformed models. We give, as the main result of the chapter, sound and complete axiomatisations of the two basic modal logics obtained.

We then in **chapter 5** adapt the basic ideas of the introduction to the setting of chapter 4 and consider modal relevance logics interpreted over classes of models closed under a parallel composition. This results in interpretations quite similar to the "initial algebra" interpretations of chapter 3. The key to axiomatising the full logics lies, as in chapter 3, in providing the necessary axiomatic tools for decomposing implications into simpler formulas. A number of such decomposition properties can be proved in quite general terms; for some requiring an appeal to additional closure properties of model classes such as closure under disjoint union (corresponding to internal choice) and the existence of least models satisfying certain formulas, corresponding—intuitively—to closure under guarded choice. On the basis of these decomposition properties we obtain complete axiomatisations for the full logics using the from chapter 3 familiar rewriting technique—relativised w.r.t. complete axiomatisations of the ground fragments for the given model class. We conclude the chapter by giving a detailed example based on "testing models", i.e. frames extended by certain atomic propositions needed for the general preorders induced by our logics to coincide with the corresponding testing preorders. The main part of the work consist in showing that testing models provide an admissible model class, i.e. that atomic propositions can be suitably decomposed, that least testing models satisfying certain formulas exist, etc. It is then a relatively straightforward matter to obtain (unrelativised) complete axiomatisations—this amounts basically to specialising the ground fragment axiomatisations to the class of testing models. As in chapter 3 the completeness proof yield procedures for checking the validity and satisfiability of formulas as a spin-off.

Finally, in **chapter 6**, we evaluate the results obtained and discuss possible directions for future work. We suggest that no definite answer to the viability of the basic idea introduced has been given. Such an answer will depend on a further development, probably focusing on the work described in chapter 3, to the stage where full process systems such as CCS can be captured—at least on a semantical level. Also it is important to relax the emphasis on testing-related equivalences made here, and consider in particular relevant extensions of Hennessy-Milner Logic [51].

Chapter 2

On Relevance Logics and their Models

In the introduction we saw how a linear implication was suitable for capturing the purely structural properties of parallel composition. Here we ask what happens when more connectives—primarily extensional ones—are added.

After a few preliminary definitions in section 1 we review in section 2 some well known relevance logics. We introduce in section 3 a notion of model based on semilattices with an inf-preserving binary operation. This gives rise to a (very) minimal logic, **G**, which we introduce and prove sound and complete in section 4. On this basis we exhibit in section 5 a variety of correspondence and completeness results for some standard relevance logics, and show in section 6 that the axiomatisations obtained are correct.

In section 7 we discuss the relationship between our notion of model and Sylvan and Meyer's ternary relation model [97, 98], and in section 8, we give a first example of the application of relevance logic to concurrency by turning Milner's SCCS [78] under bisimulation equivalence into a model for positive linear logic and under reverse simulation into a model for the standard system $\mathbf{R}^+$. Concluding the chapter we discuss briefly relations to other models such as Dunn's algebraic models (c.f. [7]) and suggest directions for future work.

2.1 Preliminaries

We start by fixing the basic notions of propositional formula, deduction, proof and logic.

2.1.1 Formulas

The language of propositional *formulas* $\phi \in \mathrm{Fm}$, is given by the abstract syntax

$$\phi ::= \alpha \mid \top \mid \bot \mid \mathrm{t} \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \circ \phi \mid \neg \phi$$

where $\alpha \in \mathrm{Ap}$—some countably infinite set of *atomic proposition symbols* assumed fixed for the remainder of the chapter. $\to$ is the implication, $\wedge/\vee$ is extensional, or truthfunctional conjunction/disjunction, $\top/\bot$ is the extensional truth/falsehood constant, $\circ$ is intensional conjunction, or fusion, t is the intensional truthhood constant, and $\neg$ is negation. We adopt the following precedence convention: $\neg > \circ > \wedge > \vee > \to$. Thus for instance

$$\phi \circ \psi \wedge \gamma \to \delta \vee \theta \wedge \gamma$$

denotes the formula

$$((\phi \circ \psi) \wedge \gamma) \to (\delta \vee (\theta \wedge \gamma)).$$

This terminology follows standard usage in relevance logic, widely differing from that of Girard for linear logic [43]. In Girard's notation replace $\wedge$ by $\&$, $\vee$ by $\oplus$, $\circ$ by $\otimes$, $\to$ by $\multimap$, $\bot$ by 0, and t by 1.

If X is a language of formulas and A a set of connectives then X_A denotes the *A-fragment* of X, i.e. the sublanguage of all $\phi \in X$ built exclusively from $\alpha \in \mathrm{Ap}$ and connectives in A. We abbreviate $X_{A\cup\{\mathrm{t},\wedge,\vee,\to\}}$ by X_A^+, for A a singleton $\{f\}$, $X_{\{f\}}^{(+)}$ by $X_f^{(+)}$ and $X_\emptyset^+$ by X^+. Fm^+ is the fragment of primary interest to us.

2.1.2 Deduction, proof, logic

We assume standard notions of axiom and (finitary) rule (schema), substitution and (substitution-) instance. Deductions and proofs are given in a standard Hilbert-type format: Given a set C of axioms and rules over Fm, a *deduction of ψ from the sequence of assumptions* $\Phi = \phi_1, \ldots, \phi_n$ in C is a finite sequence $\psi_1, \ldots, \psi_m$ of formulas s.t. $\psi_m = \psi$ and for each i, ψ_i is either

i) one of the assumptions ϕ_j or

ii) an instance of an axiom in C or

iii) the result of an application of a rule in C applied to some of $\psi_1, \ldots, \psi_{i-1}$ as premisses.

Note that a formula may occur more than once in an assumption sequence, and that assumptions may not necessarily be used. If it is unclear which of i) to iii) applies to ψ_i we assume this to be resolved by an appropriate annotation of ψ_i. In particular if ψ_i is an assumption ϕ_j and ϕ_j occurs more than once among $\phi_1, \ldots, \phi_n$ we assume ψ_i to be annotated such as to identify a unique ϕ_j of which ψ_i is an occurrence. The existence of a deduction of ϕ from assumptions Φ in C is denoted by $\Phi \vdash_C \phi$, or by $\vdash_C \phi$ if Φ is the empty sequence ε.

A *logic* (over Fm) is a subset of Fm. The logic *axiomatised by* C is the set $\mathbf{L}(C) = \{\phi \mid \vdash_C \phi\}$. We use boldface characters such as **R**, **E** to refer to logics with fixed concrete axiomatisations, and **L** as a variable ranging over such logics. These can be sub- and superscripted as for sublanguages of Fm. Thus $\mathbf{L}_A$ refers to the logic axiomatised by the set of axioms and rules in **L** that only mention connectives in Fm_A. Note that only when **L** is a conservative extension of $\mathbf{L}_A$ is $\mathbf{L} \cap \mathrm{Fm}_A = \mathbf{L}_A$.

These notions apply equally well to languages other than Fm. In particular we have occasion to consider *sequents*, or consecutions [7]. These are expressions of the form $\Phi \vdash \phi$, and the provability of such a sequent in C is denoted alternatively by $\Phi \vdash_C \phi$. The notational ambiguity here is resolved by C.

2.2 Some relevance logics

The purpose of the present section is to obtain some initial familiarity with relevance logics, and to establish some terminology. The section follows to a certain extent section 1.3 of Dunn's survey [34] to which—together with Anderson and Belnap's volume [7]—we refer for additional material.

2.2.1 The implicative fragment

The subject matter of relevance logic is the notion of relevant deduction and its companion, relevant implication. The task is to grasp the idea of relevance between assumption, or antecedent, and conclusion. The deduction theorem is the tool relating deduction and implication, and so provides an excellent setting for discussing the issues involved.

The *deduction theorem* for a logic $\mathbf{L} \subseteq \mathrm{Fm}_{\to}$ states that whenever $\Phi, \phi \vdash_{\mathbf{L}} \psi$ then $\Phi \vdash_{\mathbf{L}} \phi \to \psi$. The deduction theorem holds in particular for the implicative fragment, $\mathbf{H}_{\to}$, of intuitionistic propositional logic. $\mathbf{H}_{\to}$ may be axiomatised by the axioms

$$\begin{array}{lll} \text{S:} & (\phi \to (\psi \to \gamma)) \to ((\phi \to \psi) \to (\phi \to \gamma)) & \text{(Self-distribution)} \\ \text{K:} & \phi \to (\psi \to \phi) & \text{(Positive paradox)} \end{array}$$

together with modus ponens:

m.p.: From $\phi \to \psi$ and ϕ infer ψ.

The deduction theorem for $\mathbf{H}_{\to}$ is proved by induction on deductions and shows that whenever $\mathbf{H}_{\to} \subseteq \mathbf{L}$ then the deduction theorem holds for $\mathbf{L}$. Moreover if the deduction theorem holds for $\mathbf{L}$ then it contains S and K as theorems. Thus $\mathbf{H}_{\to}$ is minimal among the logics containing m.p. as a derived rule and satisfying the deduction theorem.

The deduction theorem, however, allows for irrelevance—the canonical example is the standard derivation of K. For ϕ is a deduction of ϕ from assumptions ϕ, ψ for arbitrary ψ, hence if the deduction theorem holds for $\mathbf{L}$ then $\vdash_{\mathbf{L}} \phi \to (\psi \to \phi)$. But the assumption ψ was never used and the deduction is thus rejected by relevantists as *irrelevant*.

This situation may be remedied by restricting attention to deductions that have some desired property R of relevance. Then the R *deduction theorem* for $\mathbf{L}$ states that if there is an R deduction in $\mathbf{L}$ of ψ from assumptions Φ, ϕ then there is an R deduction in $\mathbf{L}$ of $\phi \to \psi$ from assumptions Φ. Further, if $\mathbf{L}$ is minimal with this property then we say that the R deduction theorem *characterises* $\mathbf{L}$.

We use this perspective to introduce a number of relevance logics. We restrict for the time being attention to logics with m.p. as their sole rule of inference.

Relevance and linearity

We first investigate the idea of relating relevance to the usage of *occurrences* of assumptions. We can make this precise by introducing the notion of the i'th assumption being used n times in a deduction $\psi_1, \ldots, \psi_m$. This is defined inductively as the least relation s.t.

i) if ψ_m is an occurrence of the i'th assumption then that assumption occurs once in $\psi_1, \ldots, \psi_m$,

ii) if ψ_m is an application of m.p. to ψ_{m_1}, ψ_{m_2} and the i'th assumption occurs n_j times in $\psi_1, \ldots, \psi_{m_j}$, $j \in \{1, 2\}$, then that assumption occurs $n_1 + n_2$ times in $\psi_1, \ldots, \psi_m$.

Then $\psi_1, \ldots, \psi_m$ is *relevant*, if each occurrence of an assumption is used at least once in $\psi_1, \ldots, \psi_m$, and it is *linear*, if each occurrence of an assumption is used exactly once.

The logic $\mathbf{R}_\rightarrow$, characterised by the relevant deduction theorem [24, 81], is axiomatised by m.p. together with the axioms

I:	$\phi \rightarrow \phi$	(Reflexivity)
B:	$(\psi \rightarrow \gamma) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \gamma))$	(Prefixing)
C:	$(\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\psi \rightarrow (\phi \rightarrow \gamma))$	(Permutation)
W:	$(\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$	(Contraction)

The logic $\mathbf{LL}_\rightarrow$ axiomatised by I, B, C and m.p. (i.e. $\mathbf{R}_\rightarrow$ without Contraction) is similarly characterised by the linear deduction theorem [8].

Mingle

The appropriate conception of assumptions for relevant and linear deduction is as *multisets*—if Φ and Ψ are equal as multisets then ϕ is relevantly/linearly deducible from Φ iff it is from Ψ. Alternatively assumptions may be considered as sets. Say a formula ϕ is used in the deduction $\psi_1, \ldots, \psi_m$ if ψ_m is either an occurrence of assumption ϕ, or if it is an application of m.p. to ψ_{m_1}, ψ_{m_2} and ϕ is used in either of the subdeductions $\psi_1, \ldots, \psi_{m_j}$, $j \in \{1, 2\}$. Then a deduction from assumptions Φ is *mingle* if it uses each formula in Φ. There is a straightforward mingle deduction in particular of

$$\text{Mingle: } \phi \rightarrow (\phi \rightarrow \phi)$$

and the logic $\mathbf{RM}_\rightarrow$, characterised by the mingle deduction theorem (c.f. [8, 34]) is axiomatised by adding Mingle to the axiomatisation above of $\mathbf{R}_\rightarrow$ [1].

P-W and tickets

A couple of less important variants should also be mentioned.

[1] It should noted that the full logic $\mathbf{RM}$ to be introduced below is *not* a conservative extension of $\mathbf{RM}_\rightarrow$. The formula

$$((((\phi \rightarrow \psi) \rightarrow \psi) \rightarrow \phi) \rightarrow \gamma) \rightarrow (((((\psi \rightarrow \phi) \rightarrow \phi) \rightarrow \psi) \rightarrow \gamma) \rightarrow \gamma)$$

in particular is an example of a formula provable in $\mathbf{RM}$ but not in $\mathbf{RM}_\rightarrow$ [71, 7].

The very weak logic **P-W** [2] has received some attention due to a conjecture of Belnap proved by Martin and Meyer [66] that $\phi \to \psi$ and $\psi \to \phi$ are theorems of **P-W** only when ϕ and ψ are syntactically identical. **P-W** is axiomatised by I, B, Suffixing and m.p. (i.e. $\mathbf{LL}_{\to}$ minus Permutation plus Suffixing). Thus we should expect the appropriate conception of assumptions in **P-W** to be as sequences.

The logic $\mathbf{T}_{\to}$ of *ticket entailment* is obtained by adding Contraction to **P-W**. $\mathbf{T}_{\to}$ is based on a conception of implications as "tickets of inference". An adaptation of Anderson and Belnaps motivation could go something like this [7]: Statements can be "law-like"—i.e. tickets—or they can be factual; which of these roles a statement takes is dependent on its use in the deduction as either a major or a minor premiss to m.p. Provable implications, in particular, are tickets and should only be used as a minor premiss for the purpose of deriving new tickets.

It is possible to formulate deduction theorems for **P-W** and $\mathbf{T}_{\to}$ along the above lines using a "relevance relation" R between deductions and *sequences* of assumption indices [3]. Going into details, however, will take us too far afield.

Strict implication

This distinction between law-like and factual resembles the modal distinction between necessity and contingency. As a strict implication the $\to$ expresses the impossibility of the antecedent holding while the consequent does not. In the context of the deduction theorem we should expect necessities—and hence implications in particular—to be deducible from necessities only. A deduction of ϕ from assumptions $\phi_1, \ldots, \phi_m$ is S4-*strict*, if either $m = 0$ or for all j s.t. $1 \leq j \leq m-1$, ϕ_j is an implication—i.e. of the form $\psi \to \gamma$. Thus full permutation is not admissible.

The logic **C4** of S4-type strict implication, characterised by the S4-strict deduction theorem [10], is axiomatised by I, S,

$$\text{KStrict: } (\phi \to \psi) \to (\gamma \to (\phi \to \psi))$$

and m.p. [47].

Thus **C4** does not evade completely the paradoxes of implication. In particular, if ψ is any provable implication then so is $\phi \to \psi$ for any ϕ. More generally

[2] We omit the $\to$-subscript here, as we consider **P-W** only in its implicative fragment.

[3] See also Kron [58, 59].

for any of the Lewis systems $\mathbf{L}$, if ψ is provable in classical propositional calculus then $\phi \to \psi$ will be provable in $\mathbf{L}$.

Strictness and relevance

Anderson and Belnap's preferred implicative logic $\mathbf{E}_{\to}$ is characterised by the S4-strict and relevant deduction theorem [7]. In relation to $\mathbf{R}_{\to}$ the "modal culprit" is C, and in $\mathbf{E}_{\to}$, C is replaced by the weaker

$$\text{Restr. Perm.: } (\phi \to ((\psi \to \gamma) \to \delta)) \to ((\psi \to \gamma) \to (\phi \to \delta)).$$

The logic $\mathbf{B}_{\to}$

With $\mathbf{E}_{\to}$ we have by no means exhausted the range of relevance logics. As a final example we mention the very weak logic $\mathbf{B}_{\to}$ (B supposedly for basic). $\mathbf{B}_{\to}$ is too weak to be characterised by a deduction theorem based on sequences as above. It is introduced by Sylvan and Meyer as the weakest logic falling naturally under their semantical approach in [98, 97]—we return to this later. $\mathbf{B}_{\to}$ is axiomatised by I and m.p. together with weakenings of B and Suffixing to

$$\begin{array}{ll} \text{Suffix-rule:} & \text{From } \phi \to \psi \text{ infer } (\psi \to \gamma) \to (\phi \to \gamma) \\ \text{Prefix-rule:} & \text{From } \psi \to \gamma \text{ infer } (\phi \to \psi) \to (\phi \to \gamma) \end{array}$$

2.2.2 Extensional connectives

The consideration of extensional conjunction and disjunction adds a level of complexity to the story told in the previous section. We start with conjunction.

Conjunction

In the deductive setting conjunction is added by the rules adjunction, $\wedge$-elim-rule-1 and -2:

$$\begin{array}{ll} \text{adj.:} & \text{From } \phi \text{ and } \psi \text{ infer } \phi \wedge \psi, \\ \wedge\text{-elim-rule-1} & \text{From } \phi \wedge \psi \text{ infer } \phi, \\ \wedge\text{-elim-rule-2} & \text{From } \phi \wedge \psi \text{ infer } \psi. \end{array}$$

We explain how the properties of "relevance" in the previous section are extended to cover deductions involving these rules. Some care must be taken, for if adj. is taken to be an R deduction of $\phi \wedge \psi$ from ϕ, ψ and the R deduction theorem holds then K is easily proved by $\wedge$-elim-rule-1 and m.p. The solution, intuitively speaking, is to require assumptions to be relevant to *both* premisses in applications of adj. We consider first only linear deduction. The notion of using the i'th assumption n times is extended to deductions involving the above three rules by taking the i'th assumption to be used n times in an application of adj. if it is used n times in each of the premisses, and the obvious extensions for (the unary) $\wedge$-elim-rule-1 and -2. So *dependency* seems actually a better term in this connection than the more traditional usage. Axiomatically $\mathbf{LL}_{\{\to,\wedge\}}$ is obtained from $\mathbf{LL}_{\to}$ by adding

$$\begin{array}{ll} \wedge\text{-intro:} & (\phi \to \psi) \wedge (\phi \to \gamma) \to (\phi \to \psi \wedge \gamma) \\ \wedge\text{-elim-1:} & \phi \wedge \psi \to \phi \\ \wedge\text{-elim-2:} & \phi \wedge \psi \to \psi \end{array}$$

together with adj. It is straightforward then to extend the characterisation result above to $\mathbf{LL}_{\{\to,\wedge\}}$.

Disjunction

Similar concerns arise in the consideration of disjunction. We add the rules [4]

$$\begin{array}{ll} \vee\text{-intro-rule-1:} & \text{From } \phi \text{ infer } \phi \vee \psi \\ \vee\text{-intro-rule-2:} & \text{From } \psi \text{ infer } \phi \vee \psi \\ \vee\text{-elim-rule:} & \text{From } \phi \vee \psi,\ \phi \to \gamma \text{ and } \psi \to \gamma \text{ infer } \gamma. \end{array}$$

Extending the usage-relation to $\vee$-intro-rule-1 and -2 is straightforward. For $\vee$-elim-rule, if ψ_m is obtained from $\psi_{m_1} = \phi \vee \psi$, $\psi_{m_2} = \phi \to \gamma$ and $\psi_{m_3} = \psi \to \gamma$ then the i'th assumption is used n times in $\psi_1, \ldots, \psi_m$ if $n = n_1 + n_2$, the i'th assumption is used n_1 times in $\psi_1, \ldots, \psi_{m_1}$ and n_2 times in both $\psi_1, \ldots, \psi_{m_2}$ and $\psi_1, \ldots, \psi_{m_3}$. To obtain the $\{\to, \wedge, \vee\}$-fragment, $\mathbf{LL}_{\{\to,\wedge,\vee\}}$, we add the following three axioms to $\mathbf{LL}_{\{\to,\wedge\}}$:

$$\begin{array}{ll} \vee\text{-intro-1} & \phi \to \phi \vee \psi \\ \vee\text{-intro-2} & \psi \to \phi \vee \psi \\ \vee\text{-elim} & (\phi \to \gamma) \wedge (\psi \to \gamma) \to (\phi \vee \psi \to \gamma) \end{array}$$

Again we can extend the characterisation to $\mathbf{LL}_{\{\to,\wedge,\vee\}}$.

[4] In natural deduction settings the rule $\vee$-elim-rule is usually stated using assumptions rather than implications as here. But remember that we do not have available the formal machinery for discharging assumptions in the Hilbert-type setting here.

Distribution

Completely analogous results can be obtained for the other implicative logics we have considered. It turns out, however, that of these, only in the extension of $\mathbf{H}_{\rightarrow}$ by the above 6 axioms and 1 rule [5], $\mathbf{H}_{\{\rightarrow,\wedge,\vee\}}$, is *distribution*—the formula

$$\text{D: } \phi \wedge (\psi \vee \gamma) \rightarrow (\phi \wedge \psi) \vee (\phi \wedge \gamma)$$

provable. As distribution has traditionally been assumed for the $\{\rightarrow, \wedge, \vee\}$-fragments of all the logics introduced above except linear logic, in $\mathbf{R}_{\{\rightarrow,\wedge,\vee\}}$, $\mathbf{RM}_{\{\rightarrow,\wedge,\vee\}}$, $\mathbf{T}_{\{\rightarrow,\wedge,\vee\}}$, and $\mathbf{E}_{\{\rightarrow,\wedge,\vee\}}$ D is added as an extra axiom. Moreover, $\mathbf{E}_{\{\rightarrow,\wedge,\vee\}}$ adds also the axiom

$$\wedge\text{-}\Box\text{-distribution: } (\Box\phi) \wedge (\Box\psi) \rightarrow \Box(\phi \wedge \psi),$$

where $\Box\phi =_{\text{def}} (\phi \rightarrow \phi) \rightarrow \phi$.

Concerning the role of D it should be added that by appealing to nested sequences of assumptions it is possible to set up the deductive machinery in a way that allows characterisation results to be proved for distributive logics as well (c.f. [32, 34]).

As regards the extensional truth- and falsehood constants $\top$ and $\bot$ one just adds the standard axioms $\top$-intro: $\phi \rightarrow \top$ and $\bot$-elim: $\bot \rightarrow \phi$, although for obvious reasons (i.e. "fallacies of relevance") these are not usually considered in the relevance literature.

2.2.3 Intensional connectives

The structural properties of assumptions, as we have seen, are of central importance when relevance is considered. It is possible to introduce connectives that allows this structure to be directly expressible in the logic itself. First, the intensional conjunction, or fusion, $\circ$, serves as an internalisation of the operation of concatenating assumption sequences, and secondly the intensional truthhood constant, t, internalises the empty sequence of assumptions as a unit for concatenation. One may think of t, in particular, as "the conjunction of all logical truths".

[5] $\mathbf{H}_{\{\rightarrow,\wedge,\vee\}}$ may be given an equivalent and more standard axiomatisation by replacing adj. and $\wedge$-intro by the single axiom $\phi \rightarrow (\psi \rightarrow \phi \wedge \psi)$. This, of course, is not true for the weaker logics.

This intuition is mirrored in their axiomatisations. They are added to the logics previously introduced by the axiom

$$\text{t-intro: t}$$

and the rules

Residuation:	From $(\phi \circ \psi) \rightarrow \gamma$ infer $\phi \rightarrow (\psi \rightarrow \gamma)$
Fusion:	From $\phi \rightarrow (\psi \rightarrow \gamma)$ infer $(\phi \circ \psi) \rightarrow \gamma$
t-elim:	From ϕ infer $\mathrm{t} \rightarrow \phi$

It is quite straightforward to see that with respect to all the deduction predicates R introduced above extended in the obvious way to these three (unary) rules, if the R deduction theorem holds for **L** then

i) there is an R deduction in **L** of ϕ from assumptions $\phi_1, \phi_2, \ldots, \phi_{n-1}, \phi_n$ iff there is one from assumptions $(\ldots(\phi_1 \circ \phi_2) \circ \ldots \circ \phi_{n-1}) \circ \phi_n$ iff there is one from assumptions $\phi_1 \circ (\phi_2 \circ \ldots \circ (\phi_{n-1} \circ \phi_n) \ldots)$, and

ii) there is an R deduction in **L** of ϕ from assumptions Φ iff there is one from assumptions Φ, t iff there is one from assumptions t, Φ.

In logics above $\mathbf{LL}_{\{\rightarrow,\mathrm{t},\circ\}}$ the axiomatisation of t and $\circ$ may be simplified to the two axioms $\phi \leftrightarrow (\mathrm{t} \rightarrow \phi)$ and $(\phi \rightarrow (\psi \rightarrow \gamma)) \leftrightarrow (\phi \circ \psi \rightarrow \gamma)$. Here $\leftrightarrow$ is as usual defined by $\phi \leftrightarrow \psi =_{\mathrm{def}} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. Finally in $\mathbf{H}^+_\circ$ the intensional and extensional conjunctions and truthhood constants collapse—i.e. we can prove $(\phi \circ \psi) \leftrightarrow (\phi \wedge \psi)$ and $\mathrm{t} \leftrightarrow \top$.

2.2.4 Negation

As we move to negation, things become less settled. At least three different notions have been considered in the literature.

First, the full propositional logics **LL**, **R**, **RM**, **T**, **E** are obtained by adding a "DeMorgan" negation axiomatised by

CP:	$(\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\phi)$	(Contraposition)
DN:	$\neg\neg\phi \rightarrow \phi$	(Double negation)
RAA:	$(\phi \rightarrow \neg\phi) \rightarrow \neg\phi$	(Reductio ad absurdum)

With this negation we can prove the usual De-Morgan laws such as

$$\neg(\phi \wedge \psi) \leftrightarrow \neg\phi \vee \neg\psi$$

etc.—yet avoiding "fallacies of relevance" such as $\phi \wedge \neg\phi \rightarrow \psi$. Using this negation we can define an intensional disjunction $+$ by $\phi + \psi \; =_{\mathrm{def}} \neg\phi \rightarrow \psi$ and an intensional falsehood constant f by f $=_{\mathrm{def}} \neg$t. Then De-Morgan laws such as $\neg(\phi \circ \psi) \leftrightarrow \neg\phi + \neg\psi$ hold here as well.

Alternatively, an intuitionistic negation can be obtained by defining $\neg$ in terms of $\perp$ by the axiom Neg. Def.: $\neg\phi \leftrightarrow (\phi \rightarrow \perp)$.

Finally, Meyer and Sylvan consider "classical relevance logics" [72, 73, 69] obtained by adding a relevant implication to classical propositional logic. We shall have more to say along these lines in the latter part of the thesis, where we consider relevant extensions of various modal logics containing for instance both $\top$ and $\perp$.

2.3 Semantics of positive relevance logics

We take as our starting point the semilattice model of Urquhart [109].

Definition 2.1 (Urquhart's model).

i) An "Urquhart-frame" (*U-frame*) is a structure $\mathcal{F} = \langle S, \cdot, 1\rangle$ with $\cdot$ a binary operation on S and $1 \in S$.

ii) An "Urquhart-model" (*U-model*) is a structure $\mathcal{M} = \langle \mathcal{F}, V\rangle$ with $\mathcal{F}$ a U-frame and V a *valuation* mapping atomic propositions α into subsets of $S_{\mathcal{F}}$.

For a U-model $\mathcal{M} = \langle \mathcal{F}, V\rangle$ the relation $\models_{\mathcal{M}} \subseteq S_{\mathcal{F}} \times \mathrm{Fm}_{\rightarrow}$ of *satisfaction* is defined inductively by

$x \models_{\mathcal{M}} \alpha$ iff $x \in V(\alpha)$,

$x \models_{\mathcal{M}} \phi \rightarrow \psi$ iff for all $y \in S$, if $y \models_{\mathcal{M}} \phi$ then $x \cdot y \models_{\mathcal{M}} \psi$.

Then a formula ϕ is *valid* (in the sense of Urquhart) in a U-model $\mathcal{M}$, if $1_{\mathcal{M}} \models_{\mathcal{M}} \phi$.

We obtain a range of soundness and completeness results for the purely implicative fragment.

Theorem 2.2 (Urquhart [109]).

i) $\vdash_{\mathbf{LL}_{\rightarrow}} \phi$ *iff* ϕ *is valid in all U-models* $\mathcal{M} = \langle \mathcal{F}, V\rangle$ *with* $\mathcal{F}$ *a commutative monoid (i.e. with* $\cdot_{\mathcal{F}}$ *commutative and associative and* $1_{\mathcal{F}}$ *neutral).*

ii) $\vdash_{\mathbf{R}_\to} \phi$ *iff* ϕ *is valid in all U-models* $\mathcal{M} = \langle \mathcal{F}, V \rangle$ *with* $\mathcal{F}$ *a unital semilattice (i.e. with* $\mathcal{F}$ *a commutative monoid and* $\cdot_{\mathcal{F}}$ *idempotent).*

iii) $\vdash_{\mathbf{RM}_\to} \phi$ *iff* ϕ *is valid in all U-models* $\mathcal{M} = \langle \mathcal{F}, V \rangle$ *with* $\mathcal{F}$ *a unital semilattice and* V *satisfying* $x, y \in V(\alpha)$ *only if* $x \cdot y \in V(\alpha)$.

iv) $\vdash_{\mathbf{H}_\to} \phi$ *iff* ϕ *is valid in all U-models* $\mathcal{M} = \langle \mathcal{F}, V \rangle$ *with* $\mathcal{F}$ *a unital semilattice and* V *satisfying* $x \in V(\alpha)$ *only if* $x \cdot y \in V(\alpha)$ *for all* $y \in S_{\mathcal{F}}$. □

Problems arise, however, when conjunction and—in particular—disjunction is added. Urquhart [109] considers the standard satisfaction conditions:

$x \models \phi \wedge \psi$ iff $x \models \phi$ and $x \models \psi$,

$x \models \phi \vee \psi$ iff $x \models \phi$ or $x \models \psi$.

Soundness for the models of 2.2.i)-iv) is no problem. Completeness, however, for $\mathbf{LL}_{\{\to,\wedge,\vee\}}$ is out of the question, as all models based on commutative monoids will validate distribution, D, which fails to be provable.

For the distributive logics the situation is similar. An example, due to Dunn and Meyer, of a formula valid in all models based on unital semilattices but not provable in $\mathbf{R}_{\{\to,\wedge,\vee\}}$ is

$$(\phi \to \psi \vee \gamma) \wedge (\psi \to \gamma) \to (\phi \to \gamma).$$

We propose to solve this problem by equipping models with an auxilary semilattice structure, thus allowing the interpretation of $\vee$ in particular to be relaxed.

Definition 2.3 (Frame, model).

i) A *frame* is a structure $\mathcal{F} = \langle S, \sqcap, \cdot, 1 \rangle$ where

a) $1 \in S$,

b) $\langle S, \sqcap \rangle$ is a semilattice, i.e. $\sqcap$ is an associative, commutative and idempotent binary operation on S,

c) $\cdot$ is a binary operation on S preserving inf's in both arguments, i.e. for all $x, y, z \in S$, $(x \sqcap y) \cdot z = (x \cdot z) \sqcap (y \cdot z)$ and $x \cdot (y \sqcap z) = (x \cdot y) \sqcap (x \cdot z)$.

ii) A subset ∇ of S is a *filter*, if for all $x, y \in S$, $x, y \in \nabla$ iff $x \sqcap y \in \nabla$ [6].

[6] Our terminology is slightly nonstandard in that we allow filters to be empty and also improper.

iii) A *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is a frame and V is a valuation s.t. for each $\alpha \in \mathrm{Ap}$, $V(\alpha)$ is a filter. Then $\mathcal{M}$ is said to be *based on* $\mathcal{F}$.

iv) Let $\boldsymbol{\mathcal{F}}$ ($\boldsymbol{\mathcal{M}}$) range over classes of frames (models), and $\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{F}}}$ denote the class of all models based on some $\mathcal{F} \in \boldsymbol{\mathcal{F}}$.

We often denote "multiplication" $\cdot$ by juxtaposition and assume $\cdot$ to bind more strongly than $\sqcap$. The partial ordering $\leq$ on frames is derived from the semilattice operation by $x \leq y$ iff $x \sqcap y = x$. Note that monotonicity of $\cdot$ w.r.t. $\leq$ in both arguments follow from inf-preservation.

The satisfaction relation $\models_{\mathcal{M}} \subseteq S_{\mathcal{M}} \times \mathrm{Fm}^{+}_{\circ}$ is defined by

$x \models_{\mathcal{M}} \alpha$ iff $x \in V(\alpha)$,

$x \models_{\mathcal{M}} \mathrm{t}$ iff $1 \leq x$,

$x \models_{\mathcal{M}} \phi \wedge \psi$ iff $x \models_{\mathcal{M}} \phi$ and $x \models_{\mathcal{M}} \psi$,

$x \models_{\mathcal{M}} \phi \vee \psi$ iff $x \models_{\mathcal{M}} \phi$ or $x \models_{\mathcal{M}} \psi$ or there are $x_1, x_2 \in S$ s.t. $x_1 \sqcap x_2 \leq x$, $x_1 \models_{\mathcal{M}} \phi$ and $x_2 \models_{\mathcal{M}} \psi$,

$x \models_{\mathcal{M}} \phi \to \psi$ iff for all $y \in S$, $y \models_{\mathcal{M}} \phi$ only if $xy \models_{\mathcal{M}} \psi$,

$x \models_{\mathcal{M}} \phi \circ \psi$ iff there are $x_1, x_2 \in S$ s.t. $x_1 x_2 \leq x$, $x_1 \models_{\mathcal{M}} \phi$ and $x_2 \models_{\mathcal{M}} \psi$.

Intuitively we may think of elements as possible worlds, set-ups, or better, carriers of information, and $x \leq y$ as: y carries at least as much information as x. Given x and y their meet, $x \sqcap y$, exists, and $\cdot$ preserves it. This ensures the wellbehavedness of implication w.r.t. the extensional connectives. An element x may, in particular, carry the information $\phi \vee \psi$ without it necessarily carrying the information ϕ nor ψ. There is nothing mysterious in this. This is the case, for instance, in Beth models for intuitionistic logic (c.f. [112]), where x may carry the information $\phi \vee \psi$ because any maximal path from x will contain an element carrying either the information ϕ or ψ. Here, x carries the information $\phi \vee \psi$ if there are x_1, x_2 s.t. x_1 carries ϕ and x_2 carries ψ and any information common to x_1 and x_2 is carried by x as well—this includes in particular all logical consequences of both ϕ and ψ.

The interpretation of $\to$ is as in Urquhart's model. An x carries the information $\phi \to \psi$ if the "intensional combination" to the right with any element carrying the information ϕ results in an element carrying ψ. For $\circ$, x carries the information $\phi \circ \psi$ if x_1, x_2 can be found s.t. all information carried by their "intensional combination" $x_1 x_2$ is also carried by x.

Note that the filter property for valuations extends to the whole of $\mathrm{Fm}_\circ^+$:

Proposition 2.4 (The filter property). *For all models $\mathcal{M}$ and $\phi \in \mathrm{Fm}_\circ^+$,*

$$[\![\phi]\!]_{\mathcal{M}} = \{x \in S_{\mathcal{M}} \mid x \models_{\mathcal{M}} \phi\}$$

is a filter.

PROOF: Straightforward induction on the structure of ϕ. $\square$

For the definition of validity assume $\mathcal{M}$ to be a model with 1 left-unit for multiplication. Then ϕ is valid in the sense of Urquhart iff $1_{\mathcal{M}} \models_{\mathcal{M}} \phi$ iff $[\![\mathrm{t}]\!]_{\mathcal{M}} \subseteq [\![\phi]\!]_{\mathcal{M}}$. This motivates the following definition:

Definition 2.5 (Entailment).

i) Let $\mathcal{M}$ be a model, $\phi, \psi \in \mathrm{Fm}_\circ^+$. Then $\phi \models_{\mathcal{M}} \psi$ iff $[\![\phi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

ii) Let $\boldsymbol{\mathcal{M}}$ be a class of models. Then $\phi \models_{\boldsymbol{\mathcal{M}}} \psi$ iff for all $\mathcal{M} \in \boldsymbol{\mathcal{M}}$, $\phi \models_{\mathcal{M}} \psi$, and for $\boldsymbol{\mathcal{F}}$ a class of frames, $\phi \models_{\boldsymbol{\mathcal{F}}} \psi$ iff $\phi \models_{\boldsymbol{\mathcal{M}}_{\boldsymbol{\mathcal{F}}}} \psi$.

iii) Let $\phi \models \psi$ (ϕ *entails* ψ) iff $\phi \models_{\boldsymbol{\mathcal{F}}} \psi$ where $\boldsymbol{\mathcal{F}}$ is the class of all frames.

2.4 The minimal logic G

In this section we axiomatise the logic **G** of valid entailments $\phi \models \psi$. This logic is far weaker than any of the logics we have considered above. This is due to the fact that the only assumptions made concerning $\cdot$ and 1 is that $\cdot$ be wellbehaved w.r.t. the semilattice structure. The result is that $\rightarrow$ is stripped completely of its deductive power and all that remains is that it behaves as expected w.r.t. the extensional connectives—it is in this sense that it is minimal. It thus provides an excellent setting for studying correspondences between properties of models and logics in the subsequent sections.

2.4.1 Axiomatisation

For the $\{\wedge, \vee\}$-fragment of **G** we have the familiar lattice axioms and rules:

A1:	$\phi \vdash \phi$	(Reflexivity)
A2:	$\phi \wedge \psi \vdash \phi$	($\wedge$-elim-1)
A3:	$\phi \wedge \psi \vdash \psi$	($\wedge$-elim-2)
A4:	$\phi \vdash \phi \vee \psi$	($\vee$-intro-1)
A5:	$\psi \vdash \phi \vee \psi$	($\vee$-intro-2)
R1:	From $\phi \vdash \psi$, $\psi \vdash \gamma$ infer $\phi \vdash \gamma$	(Transitivity)
R2:	From $\phi \vdash \psi$, $\phi \vdash \gamma$ infer $\phi \vdash \psi \wedge \gamma$	($\wedge$-intro)
R3:	From $\phi \vdash \gamma$, $\psi \vdash \gamma$ infer $\phi \vee \psi \vdash \gamma$	($\vee$-elim)

G is then axiomatised by adding:

A6:	$(\phi \to \psi) \wedge (\phi \to \gamma) \vdash \phi \to (\psi \wedge \gamma)$	($\to$-$\wedge$-intro)
A7:	$(\phi \to \gamma) \wedge (\psi \to \gamma) \vdash (\phi \vee \psi) \to \gamma$	($\to$-$\vee$-elim)
R4:	From $\psi \vdash \gamma$ infer $\phi \to \psi \vdash \phi \to \gamma$	(Covariance)
R5:	From $\psi \vdash \phi$ infer $\phi \to \gamma \vdash \psi \to \gamma$	(Contravariance)

A *normal logic* is any logic **L** extending **G**—i.e. such that all axioms of **G** are theorems of **L** and all rules of **G** are derived rules of **L**. **L** may contain connectives other than those in Fm^+. Fusion, in particular, can be added by

A8:	$\phi \vdash \psi \to \phi \circ \psi$	($\to$-$\circ$-intro)
R6:	From $\phi \vdash \psi \to \gamma$ infer $\phi \circ \psi \vdash \gamma$	($\circ$-elim)

Let $\mathbf{G}_\circ$ be the extension of **G** thus obtained, and in any fragment containing $\circ$ let a normal logic extend $\mathbf{G}_\circ$.

This presentation emphasises the implication. Alternatively we can emphasise the fusion:

Proposition 2.6 *Let $\mathbf{G}'$ be the logic axiomatised by axioms A1–A5, R1–R3 plus the axioms*

$$\textit{A9:} \quad (\phi \vee \psi) \circ \gamma \vdash (\phi \circ \gamma) \vee (\psi \circ \gamma)$$

$$\textit{A10:} \quad \phi \circ (\psi \vee \gamma) \vdash (\phi \circ \psi) \vee (\phi \circ \gamma)$$

plus rules R6 and its converse

$$\textit{R7:} \quad \text{From } \phi \circ \psi \vdash \gamma \text{ infer } \phi \vdash \psi \to \gamma,$$

Then $\mathbf{G}_\circ = \mathbf{G}'$.

Proof: We have to show that every axiom of $\mathbf{G}_\circ$ is a theorem, and every rule of $\mathbf{G}_\circ$ a derived rule of $\mathbf{G}'$ and conversely. This is straightforward given the two derived rules

$$\text{D1:}\quad \text{From } \phi \vdash \psi \text{ infer } \phi \circ \gamma \vdash \psi \circ \gamma$$

$$\text{D2:}\quad \text{From } \psi \vdash \gamma \text{ infer } \phi \circ \psi \vdash \phi \circ \gamma$$

We show that D1 is a derived rule of both $\mathbf{G}_\circ$ and $\mathbf{G}'$. First for $\mathbf{G}_\circ$:

1.	$\phi \vdash \psi$	Ass.
2.	$\psi \circ \gamma \vdash \psi \circ \gamma$	A1
3.	$\psi \vdash \gamma \to \psi \circ \gamma$	2,R7
4.	$\phi \vdash \gamma \to \psi \circ \gamma$	1,3,R1
5.	$\phi \circ \gamma \vdash \psi \circ \gamma$	4,R6

The proof for $\mathbf{G}'$ is similar, just replace steps 2,3 with an instance of A8. Next for D2 in $\mathbf{G}'$:

1.	$\psi \vdash \gamma$	Ass.
2.	$\phi \vdash \gamma \to \phi \circ \gamma$	A8
3.	$\gamma \to \phi \circ \gamma \vdash \psi \to \phi \circ \gamma$	1,R5
4.	$\phi \vdash \psi \to \phi \circ \gamma$	1,3,R1
5.	$\phi \circ \psi \vdash \phi \circ \gamma$	4,R6

The proof in $\mathbf{G}_\circ$ uses A10. □

2.4.2 Soundness and completeness

We proceed to prove soundness and completeness.

Theorem 2.7 (Soundness of $\mathbf{G}$). *For all $\phi, \psi \in \mathrm{Fm}^+_\circ$, if $\phi \vdash_{\mathbf{G}_\circ} \psi$ then $\phi \models \psi$.*

PROOF: Straightforward. We show all axioms valid and all rules validity preserving. To show validity of A7, for instance, let $\mathcal{M}$ be any model, $x \models_{\mathcal{M}} \phi \to \gamma$ and $x \models_{\mathcal{M}} \psi \to \gamma$. To show $x \models_{\mathcal{M}} (\phi \vee \psi) \to \gamma$ assume that $y \models_{\mathcal{M}} \phi \vee \psi$. If $y \models_{\mathcal{M}} \phi$ or $y \models_{\mathcal{M}} \psi$ then $xy \models_{\mathcal{M}} \gamma$ is immediate. Otherwise there are $y_1, y_2 \in S_{\mathcal{M}}$ s.t. $y_1 \models_{\mathcal{M}} \phi$, $y_2 \models_{\mathcal{M}} \psi$ and $y_1 \sqcap y_2 \leq y$. Then $xy_1 \sqcap xy_2 \models_{\mathcal{M}} \gamma$ by the filter property and as $\cdot$ preserves $\sqcap$, so $x(y_1 \sqcap y_2) \models_{\mathcal{M}} \gamma$ as well. But then by monotonicity of $\cdot$ and the filter property also $xy \models_{\mathcal{M}} \gamma$ as desired. □

The completeness proof appeals to a standard Henkin-type construction, building for each normal logic $\mathbf{L}$ a canonical model $\mathrm{C}(\mathbf{L})$ from theories—the syntactical equivalent of filters.

So let $\mathbf{L}$ be normal. An $\mathbf{L}$-*theory* is a subset $\nabla \subseteq \mathrm{Fm}$ s.t.

i) whenever $\phi \in \nabla$ and $\phi \vdash_{\mathbf{L}} \psi$ then $\psi \in \nabla$, and

ii) if $\phi, \psi \in \nabla$ then $\phi \wedge \psi \in \nabla$.

Let $\mathrm{th}(\mathbf{L})$ denote the set of all $\mathbf{L}$-theories. Note that (as for filters) we allow theories both to be empty and to be improper. We note some standard properties of theories:

i) $\mathrm{th}(\mathbf{L})$ is closed under arbitrary intersections.

ii) If $\nabla_1, \nabla_2 \in \mathrm{th}(\mathbf{L})$, $\nabla_1 \neq \emptyset$, $\nabla_2 \neq \emptyset$ then $\nabla_1 \cap \nabla_2 \neq \emptyset$.

iii) For every $A \subseteq \mathrm{Fm}$ there is a least theory $\mathrm{th}_{\mathbf{L}}(A) \in \mathrm{th}(\mathbf{L})$ s.t. $A \subseteq \mathrm{th}_{\mathbf{L}}(A)$.

iv) For any $\phi \in \mathrm{Fm}$, $\mathrm{th}_{\mathbf{L}}(\{\phi\}) = \{\psi \mid \phi \vdash_{\mathbf{L}} \psi\}$.

v) For any $A \subseteq \mathrm{Fm}$, $\mathrm{th}_{\mathbf{L}}(A) = \{\psi \mid \exists n \geq 1, \phi_1, \ldots, \phi_n \in A. \phi_1 \wedge \ldots \wedge \phi_n \vdash_{\mathbf{L}} \psi\}$.

Let in particular $\uparrow \phi = \mathrm{th}_{\mathbf{L}}(\{\phi\})$.

Definition 2.8 (Canonical model). For $\mathbf{L}$ a normal logic, let the *canonical* $\mathbf{L}$-*model* be the structure $\mathrm{C}(\mathbf{L}) = \langle \mathrm{th}(\mathbf{L}), \cap, \cdot, \uparrow \mathrm{t}, V \rangle$, where

$$\nabla_1 \cdot \nabla_2 = \{\psi \mid \exists \phi \in \nabla_2 \text{ s.t. } \phi \to \psi \in \nabla_1\},$$

and $V(\alpha) = \{\nabla \in \mathrm{th}(\mathbf{L}) \mid \alpha \in \nabla\}$.

Lemma 2.9 *For any normal* $\mathbf{L}$*,* $\mathrm{C}(\mathbf{L})$ *is a model.*

Proof: It is clear that $\sqcap_{\mathrm{C}(\mathbf{L})}$ and $1_{\mathrm{C}(\mathbf{L})}$ are well-defined and that $\{\nabla \mid \alpha \in \nabla\}$ is a filter for each $\alpha \in \mathrm{Ap}$. We check that $\nabla_1 \cdot \nabla_2 \in \mathrm{th}(\mathbf{L})$.
If $\phi \to \psi \in \nabla_1$ and $\psi \vdash_{\mathbf{L}} \gamma$ then $\phi \to \gamma \in \nabla_1$ by R4. Assume then that $\phi \to \psi$, $\gamma \to \delta \in \nabla_1$ and we have to show $\phi \wedge \gamma \to \psi \wedge \delta \in \nabla_1$. Now

$$\phi \to \psi \vdash_{\mathbf{G}} \phi \wedge \gamma \to \psi \text{ and } \gamma \to \delta \vdash_{\mathbf{G}} \phi \wedge \gamma \to \delta$$

by A2,A3 and R5. Then

$$(\phi \to \psi) \wedge (\gamma \to \delta) \vdash_{\mathbf{G}} (\phi \wedge \gamma \to \psi) \wedge (\phi \wedge \gamma \to \delta)$$

by A2,A3 and R2, but then by R1,A6

$$(\phi \to \psi) \wedge (\gamma \to \delta) \vdash_{\mathbf{G}} \phi \wedge \gamma \to \psi \wedge \delta$$

and the result follows.
We then just need to check that $\circ$ preserves $\sqcap$. If $\phi \to \psi \in \nabla_1$ and $\phi \in \nabla_2 \cap \nabla_3$ then $\psi \in (\nabla_1 \cdot \nabla_2) \sqcap (\nabla_1 \cdot \nabla_3)$. Conversely if $\phi \to \gamma, \psi \to \gamma \in \nabla_1$, $\phi \in \nabla_2$ and

$\psi \in \nabla_3$ then by A7, $(\phi \vee \psi) \to \gamma \in \nabla_1$, and by A4, A5, $\phi \vee \psi \in \nabla_2 \cap \nabla_3$, so $\gamma \in \nabla_1 \cdot (\nabla_2 \cap \nabla_3)$. For the second argument place, if $\phi \to \psi \in \nabla_1 \cap \nabla_2$ and $\phi \in \nabla_3$ then also $\psi \in (\nabla_1 \cdot \nabla_3) \sqcap (\nabla_2 \cdot \nabla_3)$. Conversely if $\phi \to \gamma \in \nabla_1$, $\psi \to \gamma \in \nabla_2$, $\phi, \psi \in \nabla_3$ then $\phi \wedge \psi \in \nabla_3$ and by A2, A3 and R5, $\phi \wedge \psi \to \gamma \in \nabla_1 \cap \nabla_2$, so $\gamma \in (\nabla_1 \cap \nabla_2) \cdot \nabla_3$. □

If $\mathbf{L}$ is over some fragment containing $\circ$, $\cdot$ can alternatively be defined by $\nabla_1 \cdot \nabla_2 = \mathrm{th}_{\mathbf{L}}\{\phi \circ \psi \mid \phi \in \nabla_1, \psi \in \nabla_2\}$. To see this let $\phi \in \nabla_1$ and $\psi \in \nabla_2$. Then $\psi \to \phi \circ \psi \in \nabla_1$ by A8 and then $\phi \circ \psi \in \nabla_1 \cdot \nabla_2$. This shows that

$$\mathrm{th}_{\mathbf{L}}\{\phi \circ \psi \mid \phi \in \nabla_1, \psi \in \nabla_2\} \subseteq \nabla_1 \cdot \nabla_2$$

Conversely if $\phi \to \gamma \in \nabla_1$ and $\phi \in \nabla_2$ then $(\phi \to \gamma) \circ \phi \in \nabla_1 \cdot \nabla_2$ and then, as $(\phi \to \gamma) \circ \phi \vdash_{\mathbf{G}} \gamma$ we obtain $\psi \in \mathrm{th}_{\mathbf{L}}\{\phi \circ \psi \mid \phi \in \nabla_1, \psi \in \nabla_2\}$.

Lemma 2.10 *For any normal* $\mathbf{L}$*,* $\phi \in \mathrm{Fm}_{\circ}^{+}$ *and* $\nabla \in S_{\mathrm{C}(\mathbf{L})}$*,* $\nabla \models_{\mathrm{C}(\mathbf{L})} \phi$ *iff* $\phi \in \nabla$*.*

Proof: By induction on the structure of ϕ. The cases for atomic propositions α, t and conjunction are immediate.
$\phi = \phi_1 \vee \phi_2$. Assume $\nabla \models \phi$. If $\nabla \models \phi_1$ then by the induction hypothesis, $\phi_1 \in \nabla$, and then by A4, $\phi \in \nabla$—similarly for ϕ_2. So assume instead there are ∇_1, ∇_2 s.t. $\nabla_1 \models \phi_1$, $\nabla_2 \models \phi_2$ and $\nabla_1 \cap \nabla_2 \subseteq \nabla$. By the induction hypothesis, $\phi_1 \in \nabla_1$ and $\phi_2 \in \nabla_2$ and then as previously, $\phi \in \nabla$. For the converse direction assume $\phi \in \nabla$. If $\phi_1 \in \nabla$ or $\phi_2 \in \nabla$ we are done by the induction hypothesis. Now $\uparrow \phi_1 \cap \uparrow \phi_2 \subseteq \nabla$ for if $\psi \in \uparrow \phi_1 \cap \uparrow \phi_2$ then $\phi_1 \vdash_{\mathbf{L}} \psi$ and $\phi_2 \vdash_{\mathbf{L}} \psi$ and then by R3, $\phi \vdash_{\mathbf{L}} \psi$ and then $\psi \in \nabla$. Then we are done by the induction hypothesis.
$\phi = \phi_1 \to \phi_2$. Assume that $\nabla \models \phi$. Now $\phi_1 \in \uparrow \phi_1$ so by the induction hypothesis, $\uparrow \phi_1 \models \phi_1$ and then $\nabla \cdot \uparrow \phi_1 \models \phi_2$. Then by the induction hypothesis again, $\phi_2 \in \nabla \cdot \uparrow \phi_1$, and then there is some ψ s.t. $\psi \to \phi_2 \in \nabla$ and $\psi \in \uparrow \phi_1$. Then $\phi_1 \vdash_{\mathbf{L}} \psi$ and then by R5, $\phi_1 \to \phi_2 \in \nabla$. Conversely assume that $\phi \in \nabla$. Let $\nabla_1 \models \phi_1$—i.e. $\phi_1 \in \nabla_1$ by the induction hypothesis. Then $\phi_2 \in \nabla \cdot \nabla_1$ and then by the induction hypothesis again, as ∇_1 was arbitrary, we can conclude that $\nabla \models \phi$.
$\phi = \phi_1 \circ \phi_2$. Assume that $\nabla \models \phi$. Then there are ∇_1, ∇_2 s.t. $\nabla_1 \cdot \nabla_2 \subseteq \nabla$, $\nabla_1 \models \phi_1$ and $\nabla_2 \models \phi_2$. By the induction hypothesis, $\phi_1 \in \nabla_1$ and $\phi_2 \in \nabla_2$, but then $\phi \in \nabla_1 \cdot \nabla_2$ so $\phi \in \nabla$. Conversely assume that $\phi \in \nabla$. Then $\uparrow \phi_1 \cdot \uparrow \phi_2 \subseteq \nabla$ and by the induction hypothesis, $\uparrow \phi_1 \models \phi_1$, $\uparrow \phi_2 \models \phi_2$ and then $\nabla \models \phi_1 \circ \phi_2$ and we are done. □

Completeness is now essentially proved.

Corollary 2.11 (Completeness of $\mathbf{G}$, $\mathbf{G}_\circ$). *For all $\phi, \psi \in \mathrm{Fm}^+$ ($\phi, \psi \in \mathrm{Fm}_\circ^+$), if $\phi \models \psi$ then $\phi \vdash_{\mathbf{G}} \psi$ ($\phi \vdash_{\mathbf{G}_\circ} \psi$).*

PROOF: If $\phi \nvdash_{\mathbf{G}} \psi$ ($\phi \nvdash_{\mathbf{G}_\circ} \psi$) then $\psi \notin \uparrow \phi$, and then by 2.10 $\uparrow \phi \models_{\mathrm{C}(\mathbf{G})} \phi$ ($\uparrow \phi \models_{\mathrm{C}(\mathbf{G}_\circ)} \phi$) and $\uparrow \phi \not\models_{\mathrm{C}(\mathbf{G})} \psi$ ($\uparrow \phi \not\models_{\mathrm{C}(\mathbf{G}_\circ)} \psi$). □

One way of adding $\top$ and $\bot$ governed by the axioms $\phi \vdash \top$ and $\bot \vdash \phi$ is to introduce corresponding bottom- and top-elements $\bot_{\mathcal{M}}$, $\top_{\mathcal{M}}$ in the model with satisfaction extended by $x \models_{\mathcal{M}} \top$ iff $\bot_{\mathcal{M}} \leq x$ and $x \models_{\mathcal{M}} \bot$ iff $\top_{\mathcal{M}} \leq x$; or equivalently: $x \models_{\mathcal{M}} \top$ for all $x \in S$ and $x \models_{\mathcal{M}} \bot$ iff $x = \top_{\mathcal{M}}$.

Soundness is immediate. For the canonical model construction we have to restrict attention to nonempty theories. For the canonical model construction we just have to insure that inhabitation of ∇_1 and ∇_2 entails inhabitation of $\nabla_1 \cdot \nabla_2$. But we shall have $\phi \vdash_{\mathbf{G}_{\{\top,\bot\}}} \top \to \top$ by R7 and the above axiom for $\top$, thus $\top \to \top \in \nabla_1$ and $\top \in \nabla_2$, whence $\top \in \nabla_1 \cdot \nabla_2$. Then the least element becomes $\uparrow \top$ and the greatest $\uparrow \bot$.

2.5 Correspondence results

The main use for the minimal logic is to provide a setting for the investigation of correspondences between logical properties and model properties.

We identify the frame conditions defined by a range of axioms and proof rules which are characteristic for some of the logics exhibited in section 2.2.

The axioms and proof rules are

A11: $\phi \vdash \mathrm{t} \to \phi$
A12: $\mathrm{t} \vdash \phi \to \phi$
A13: $\psi \to \gamma \vdash (\phi \to \psi) \to (\phi \to \gamma)$
A14: $\phi \vdash (\phi \to \psi) \to \psi$
A15: $\phi \wedge (\psi \vee \gamma) \vdash (\phi \wedge \psi) \vee (\phi \wedge \gamma)$
A16: $\phi \to (\phi \to \psi) \vdash \phi \to \psi$
A17: $\phi \vdash \psi \to \phi$
A18: $\phi \wedge (\phi \to \psi) \vdash \psi$
A19: $(\phi \to \psi) \wedge (\psi \to \gamma) \vdash \phi \to \gamma$
A20: $\phi \to \psi \vdash (\psi \to \gamma) \to (\phi \to \gamma)$
A21: $\phi \to (\psi \to \gamma) \vdash \psi \to (\phi \to \gamma)$
R8: From $\phi \vdash \mathrm{t} \to \psi$ infer $\phi \vdash \psi$

R9: From $\mathrm{t} \vdash \phi \to \psi$ infer $\phi \vdash \psi$

The corresponding frame conditions are

CA11: $\forall x.x \leq x1$

CA12: $\forall x.x \leq 1x$

CA13: $\forall x, y, z.x(yz) \leq (xy)z$

CA14: $\forall x, y.xy = yx$

CA15: $\forall x, y, z.$ if $x \sqcap y \leq z$, $x \not\leq z$ then there are x', y' s.t. $x \leq x'$, $y \leq y'$ and $x' \sqcap y' = z$

CA16: $\forall x, y.(xy)y \leq xy$

CA17: $\forall x, y.x \leq xy$

CA18: $\forall x.xx \leq x$

CA19: $\forall x, y.x(xy) \leq xy$

CA20: $\forall x, y, z.y(xz) \leq (xy)z$

CA21: $\forall x, y, z.(xz)y \leq (xy)z$

CR8: $\forall x.x1 \leq x$

CR9: $\forall x.1x \leq x$

If a semilattice satisfies CA15, in particular, we say it is *distributive.* This terminology is slightly nonstandard. Usually one takes a semilattice to be distributive, if whenever $x \sqcap y \leq z$, whether $x \leq z$ or $y \leq z$ or not, there are x', y' s.t. $x \leq x'$, $y \leq y'$ and $x' \sqcap y' = z$. Clearly, however, for unital semilattices the two notions coincide. Moreover, a binary operation $\cdot$ that satisfies CA18 is said to be *semi-idempotent.*

Theorem 2.12 *For all classes $\boldsymbol{\mathcal{F}}$ of frames, any i, j s.t. $11 \leq i \leq 21$, $8 \leq j \leq 9$, $\boldsymbol{\mathcal{F}}$ validates* Ai (Rj) *iff $\boldsymbol{\mathcal{F}}$ satisfies* CAi (CRj).

PROOF: We go through some of the cases:

$i = 11$. If $\boldsymbol{\mathcal{F}}$ satisfies CA11, $\mathcal{M} \in \boldsymbol{\mathcal{M}}_{\mathcal{F}}$, $x, y \in S_{\mathcal{M}}$, $x \models_{\mathcal{M}} \phi$ and $y \models_{\mathcal{M}} \mathrm{t}$ then $1 \leq y$, thus $x \leq x1 \leq xy$, and then by theorem 2.4 $xy \models \phi$ (we start dropping subscripts of $\models$ here). Conversely if $\boldsymbol{\mathcal{F}}$ fails to satisfy CA11 let $\mathcal{F} \in \boldsymbol{\mathcal{F}}$, $x \in S_{\mathcal{F}}$ and $x \not\leq x1$. Pick any V with

$$V(\alpha_x) = \uparrow x = \{x' \mid x' \geq x\}$$

for some $\alpha_x \in \mathrm{Ap}$, where $\uparrow x$ is the upper closure of x, i.e. the set $\{y \mid x \leq y\}$. Then $\mathcal{M} = \langle \mathcal{F}, V \rangle$ is a model, $x \models \alpha_x$ but $x \not\models \mathrm{t} \to \alpha_x$.

$i = 13$. If $\boldsymbol{\mathcal{F}}$ satisfies CA13, $\mathcal{M} \in \boldsymbol{\mathcal{M}}_{\mathcal{F}}$, $x, y, z \in S$, $x \models \psi \to \gamma$, $y \models \phi \to \psi$ and $z \models \phi$ then $x(yz) \models \gamma$, and by CA13 $x(yz) \leq (xy)z$ so $(xy)z \models \gamma$ as well.

Conversely assume that $x(yz) \not\leq (xy)z$. Fix $V(\alpha_x) = \uparrow x(yz)$, $V(\alpha_y) = \uparrow yz$ and $V(\alpha_z) = \uparrow z$. Then $x \models \alpha_y \to \alpha_x$, $y \models \alpha_z \to \alpha_y$ and $z \models \alpha_z$, but $(xy)z \not\models \alpha_x$.
$i = 15$. If $\mathcal{F}$ satisfies CA15, $\mathcal{M} \in \mathcal{M}_{\mathcal{F}}$, $x \in S$ and $x \models \phi \wedge (\psi \vee \gamma)$ then $x \models \phi$ and either $x \models \psi$ or $x \models \gamma$ or there are $x_1, x_2 \in S$ s.t. $x_1 \sqcap x_2 \leq x$, $x_1 \models \psi$ and $x_2 \models \gamma$. In the first and second case $x \models (\phi \wedge \psi) \vee (\phi \wedge \gamma)$ is immediate, so assume the third. If $x_1 \leq x$ or $x_2 \leq x$ we are done so assume not. Then there are $x_1', x_2' \in S$ s.t. $x_1 \leq x_1'$, $x_2 \leq x_2'$ and $x_1' \sqcap x_2' = x$. Then $x_1' \models \phi \wedge \psi$ and $x_2' \models \phi \wedge \gamma$ and we are done. Conversely, if CA15 fails for $\mathcal{F}$, let $x \sqcap y \leq z$, $x \not\leq z$ and $y \not\leq z$ and assume there are no x', y' s.t. $x \leq x'$, $y \leq y'$ and $x' \sqcap y' = z$. Let $\mathcal{M}$ be any $\mathcal{F}$-based model satisfying $V(\alpha_x) = \uparrow x$, $V(\alpha_y) = \uparrow y$ and $V(\alpha_z) = \uparrow z$. Then $z \models \alpha_z \wedge (\alpha_x \vee \alpha_y)$. Suppose $z \models (\alpha_z \wedge \alpha_x) \vee (\alpha_z \wedge \alpha_y)$. We cannot have $z \models \alpha_z \wedge \alpha_x$ or $z \models \alpha_z \wedge \alpha_y$. So let $z_1, z_2 \in S$, $z_1 \sqcap z_2 \leq z$, $z_1 \models \alpha_z \wedge \alpha_x$ and $z_2 \models \alpha_z \wedge \alpha_y$. Then $z \leq z_1$ and $z \leq z_2$ s.t. $z \leq z_1 \sqcap z_2$ and then $z = z_1 \sqcap z_2$—a contradiction.
$i = 16$. If $\mathcal{F}$ satisfies CA16, $\mathcal{M} \in \mathcal{M}_{\mathcal{F}}$, $x, y \in S$, $x \models \phi \to (\phi \to \psi)$ and $y \models \phi$ then $(xy)y \models \psi$ and so by CA16, $xy \models \psi$. Conversely, let $\mathcal{F}$ fail CA16 with $x, y \in S$, $(xy)y \not\leq xy$. Let $\mathcal{M}$ be any $\mathcal{F}$-based model with $V(\alpha_x) = \uparrow (xy)y$ and $V(\alpha_y) = \uparrow y$. Then $x \models \alpha_y \to (\alpha_y \to \alpha_x)$ and $x \not\models \alpha_y \to \alpha_x$.
$i = 18$. If $\mathcal{F}$ satisfies CA18, $\mathcal{M} \in \mathcal{M}_{\mathcal{F}}$, $x \in S$, $x \models \phi$ and $x \models \phi \to \psi$. Then $xx \models \psi$ and then $x \models \psi$. Conversely if $\mathcal{F}$ fails CA18 let $x \in S$ with $xx \not\leq x$ and $\mathcal{M}$ be any $\mathcal{F}$-based model with $V(\alpha_x) = \uparrow x$, $V(\alpha_{xx}) = \uparrow xx$. Then $x \models \alpha_x \wedge (\alpha_x \to \alpha_{xx})$ but $x \not\models \alpha_{xx}$. □

Note that we have no axiom or rule defining the dual of CA13. This can be defined in fragments with $\circ$ by the rule

$$\text{From } \delta \vdash \psi \circ \gamma \text{ and } \phi \vdash \psi \to (\gamma \to \theta) \text{ infer } \phi \vdash \delta \to \theta.$$

It seems impossible to replace this rule with an axiom or rule in the Fm^+-fragment (of course in the presence of A14, this rule is derived), suggesting what may be natural anyway, namely that for noncommutative operations $\cdot$ one really needs both left and right versions of the $\to$ (c.f. [6]).

To extend theorem 2.12 to cover completeness as well we need to consider also the properties of canonical models.

Lemma 2.13 *For any normal* $\mathbf{L}$*, any* i, j *s.t.* $11 \leq i \leq 21$, $8 \leq j \leq 9$, *if* Ai(Rj) *is a theorem (derived rule) of* $\mathbf{L}$ *then* C($\mathbf{L}$) *satisfies condition* CAi(CRj).

Proof: We go through some of the cases:
$i = 11$. Let $\phi \vdash_{\mathbf{L}} \mathrm{t} \to \phi$ for all ϕ. Let $\nabla \in S_{\mathrm{C}(\mathbf{L})}$ and $\phi \in \nabla$. Then $\mathrm{t} \to \phi \in \nabla$ and $\mathrm{t} \in 1_{\mathrm{C}(\mathbf{L})}$, thus $\nabla \subseteq \nabla \cdot 1_{\mathrm{C}(\mathbf{L})}$.
$i = 13$. Assume that $(\psi \to \gamma) \vdash_{\mathbf{L}} (\phi \to \psi) \to (\phi \to \gamma)$ for all ϕ, ψ, γ. Let $\nabla_1, \nabla_2, \nabla_3 \in S_{\mathrm{C}(\mathbf{L})}$ and $\gamma \in \nabla_1 \cdot (\nabla_2 \cdot \nabla_3)$. Then there are ϕ and ψ s.t. $\psi \to \gamma \in \nabla_1$, $\phi \to \psi \in \nabla_2$ and $\phi \in \nabla_3$. By the assumption, $(\phi \to \psi) \to (\phi \to \gamma) \in \nabla_1$, so $\phi \to \gamma \in \nabla_1 \cdot \nabla_2$ and then $\gamma \in (\nabla_1 \cdot \nabla_2) \cdot \nabla_3$, and we are done.
$i = 15$. Assume that $\phi \wedge (\psi \vee \gamma) \vdash_{\mathbf{L}} (\phi \wedge \psi) \vee (\phi \wedge \gamma)$ for all ϕ, ψ, γ. Let $\nabla_1 \cap \nabla_2 \subseteq \nabla$, $\nabla_1 \not\subseteq \nabla$ and $\nabla_2 \not\subseteq \nabla$. Let now $\nabla_1' = \nabla_1 \sqcup \nabla$ and $\nabla_2' = \nabla_2 \sqcup \nabla$. Then $\nabla_1 \subseteq \nabla_1'$, $\nabla_2 \subseteq \nabla_2'$ and $\nabla \subseteq \nabla_1' \cap \nabla_2'$, and we must show $\nabla_1' \cap \nabla_2' \subseteq \nabla$. So let $\phi \in \nabla_1' \cap \nabla_2'$. Then there are $\phi_1 \in \nabla_1$, $\phi_2 \in \nabla_2$, $\psi_1, \psi_2 \in \nabla$ s.t. $\phi_1 \wedge \psi_1 \vdash_{\mathbf{L}} \phi$ and $\phi_2 \wedge \psi_2 \vdash_{\mathbf{L}} \phi$. Let $\psi = \psi_1 \wedge \psi_2$. Then $\psi \in \nabla$, $\phi_1 \wedge \psi \vdash_{\mathbf{L}} \phi$ and $\phi_2 \wedge \psi \vdash_{\mathbf{L}} \phi$, thus $(\phi_1 \wedge \psi) \vee (\phi_2 \wedge \psi) \vdash_{\mathbf{L}} \phi$. But then $(\phi_1 \vee \phi_2) \wedge \psi \vdash_{\mathbf{L}} \phi$ and $\phi_1 \vee \phi_2 \in \nabla$, so $(\phi_1 \vee \phi_2) \wedge \psi \in \nabla$ and then $\phi \in \nabla$ as desired.
$i = 18$. Let $\phi \wedge (\phi \to \psi) \vdash_{\mathbf{L}} \psi$ for all ϕ, ψ. Let $\nabla \in S_{\mathrm{C}(\mathbf{L})}$ and let $\psi \in \nabla \cdot \nabla$. Then there is a $\phi \in \nabla$ s.t. $\phi \to \psi \in \nabla$. Then $\psi \in \nabla$, so $\nabla \cdot \nabla \subseteq \nabla$.
$j = 8$. Suppose that for all ϕ, ψ, if $\phi \vdash_{\mathbf{L}} \mathrm{t} \to \psi$ then $\phi \vdash_{\mathbf{L}} \psi$. Let $\nabla \in S_{\mathrm{C}(\mathbf{L})}$ and $\psi \in \nabla \cdot 1_{\mathrm{C}(\mathbf{L})}$. Then there is a $\gamma \in 1_{\mathrm{C}(\mathbf{L})}$ s.t. $\gamma \to \psi \in \nabla$. Then, by R5, $\mathrm{t} \to \psi \in \nabla$ and by the definition, $\mathrm{t} \in 1_{\mathrm{C}(\mathbf{L})}$. Now $\mathrm{t} \to \psi \vdash_{\mathbf{L}} \mathrm{t} \to \psi$ by A1 and then $\mathrm{t} \to \psi \vdash_{\mathbf{L}} \psi$, so $\psi \in \nabla$, and then $\nabla \cdot 1_{\mathrm{C}(\mathbf{L})} \subseteq \nabla$. □

Theorem 2.12 and lemma 2.13 together gives us a range of soundness and completeness theorems in one go.

Corollary 2.14 (Soundness and completeness of **G**-extensions). *Let* **L** *be axiomatised by* **G** *and any selection of axioms and rules from* A11–A21, R8–R9. *Then for all* $\phi, \psi \in \mathrm{Fm}^+$, $\phi \vdash_{\mathbf{L}} \psi$ *iff for all frames* $\mathcal{F}$ *satisfying the corresponding conditions among* CA11–CA21, CR8–CR9, $\phi \models_{\mathcal{F}} \psi$.

Proof: Let **L** be any such logic and $\mathcal{F}$ any frame satisfying the conditions. Then all axioms of **L** are valid, and all rules preserve validity by 2.7 and 2.12. Hence the only-if direction holds. Suppose conversely that $\phi \nvdash_{\mathbf{L}} \psi$. Then in $\mathrm{C}(\mathbf{L})$, $\uparrow \phi \models_{\mathrm{C}(\mathbf{L})} \phi$ and $\uparrow \phi \not\models_{\mathrm{C}(\mathbf{L})} \psi$ by 2.10, so $\phi \not\models_{\mathrm{C}(\mathbf{L})} \psi$, and by lemma 2.13 the frame underlying $\mathrm{C}(\mathbf{L})$ satisfies the conditions. □

2.6 Correctness of axiomatisations

We shall next relate these logics to the ones introduced earlier in section 2.2 motivating much of the enterprise.

Definition 2.15 Let the logics $\mathbf{GB}^+$, $\mathbf{GT}^+$, $\mathbf{GLL}^+$, $\mathbf{GR}^+$ and $\mathbf{GH}^+$ be axiomatised as follows:

i) $\mathbf{GB}^+ = \mathbf{G}$+A12+A15+R9,

ii) $\mathbf{GT}^+ = \mathbf{GB}^+$+A13+A16,

iii) $\mathbf{GLL}^+ = \mathbf{G}$+A12+A13+A14+R9,

iv) $\mathbf{GR}^+ = \mathbf{GT}^+$+A14,

v) $\mathbf{GH}^+ = \mathbf{GR}^+$+A17.

These logics correspond to their Hilbert-type versions of section 2.2 in the following way:

Theorem 2.16 (Correctness of $\mathbf{G}$-extensions). *For* $\mathbf{L} \in \{\mathbf{B}^+, \mathbf{T}^+, \mathbf{LL}^+, \mathbf{R}^+, \mathbf{H}^+\}$, $\phi \in \mathrm{Fm}^+$, $\vdash_{\mathbf{L}} \phi$ *according to the axiomatisation of* $\mathbf{L}$ *in section 2.2 iff* $\mathrm{t} \vdash_{\mathbf{GL}} \phi$.

Proof: For one direction we show that if $\vdash_{\mathbf{L}} \phi$ then $\mathrm{t} \vdash_{\mathbf{GL}} \phi$ and for the other that if $\phi \vdash_{\mathbf{GL}} \psi$ then $\vdash_{\mathbf{L}} \phi \to \psi$. It suffices to check that each axiom of $\mathbf{L}$ corresponds to a theorem of $\mathbf{GL}$ and each rule of $\mathbf{L}$ to a derivable rule of $\mathbf{GL}$ and conversely.

The only difficult case to prove is that the correlate of C, $\phi \to (\psi \to \gamma) \vdash \psi \to (\phi \to \gamma)$, is provable in $\mathbf{GLL}^+$. We prove first three derived rules:

D3: From $\phi \vdash \psi \to \gamma$ infer $\psi \vdash \phi \to \gamma$.

Proof:	1.	$\phi \vdash \psi \to \gamma$	Ass.
	2.	$(\psi \to \gamma) \to \gamma \vdash \phi \to \gamma$	1,R5
	3.	$\psi \vdash (\psi \to \gamma) \to \gamma$	A14
	4.	$\psi \vdash \phi \to \gamma$	2,3,R1

D4: From $\phi \vdash \psi \to \gamma$ infer $\gamma \to \delta \vdash \phi \to (\psi \to \delta)$.

Proof:	1.	$\phi \vdash \psi \to \gamma$	Ass.
	2.	$\gamma \to \delta \vdash (\psi \to \gamma) \to (\psi \to \delta)$	A13
	3.	$\psi \to \gamma \vdash (\gamma \to \delta) \to (\psi \to \delta)$	2,D3
	4.	$\phi \vdash (\gamma \to \delta) \to (\psi \to \delta)$	1,3,R1
	5.	$\gamma \to \delta \vdash \phi \to (\psi \to \delta)$	4,D3

D5: From $\phi \vdash \psi \to (\gamma \to \delta)$ infer $\phi \vdash \gamma \to (\psi \to \delta)$.

$$\begin{array}{lll} \text{Proof:} & 1. & \phi \vdash \psi \to (\gamma \to \delta) & \text{Ass.} \\ & 2. & (\gamma \to \delta) \to \delta \vdash \phi \to (\psi \to \delta) & \text{1,D4} \\ & 3. & \gamma \vdash (\gamma \to \delta) \to \delta & \text{A14} \\ & 4. & \gamma \vdash \phi \to (\psi \to \delta) & \text{2,3,R1} \\ & 5. & \phi \vdash \gamma \to (\psi \to \delta) & \text{4,D3} \end{array}$$

Now finally we have $\phi \to (\psi \to \gamma) \vdash_{\mathbf{GLL}^+} \phi \to (\psi \to \gamma)$ by A1 and then by D5, $\phi \to (\psi \to \gamma) \vdash_{\mathbf{GLL}^+} \psi \to (\phi \to \gamma)$. $\square$

In view of this and corollary 2.14 we shall for $\mathbf{L} \in \{\mathbf{B}^+, \mathbf{T}^+, \mathbf{LL}^+, \mathbf{R}^+, \mathbf{H}^+\}$ call any frame satisfying the conditions corresponding to the axioms and rules for **GL** of def. 2.15 an **L**-*frame*.

There is one notable omission here: Curiously it does not seem possible to capture the mingle-axiom $\phi \vdash \phi \to \phi$ in the present setting. One candidate could be the condition

$$\forall xy.x \leq xy \text{ or } y \leq xy.$$

This certainly makes $\phi \vdash \phi \to \phi$ go through, but the converse directions of theorem 2.12 and lemma 2.13 fails.

2.7 The ternary relation model

Having established the soundness and completeness of our notion of model we discuss how our notion of model relates to the classical ternary relation model of Sylvan and Meyer [97, 98].

In view of the correspondence results established above, and the corresponding ones of [97] it suffices to consider models for the minimal logic $\mathbf{B}^+$ considered by Sylvan and Meyer. So let a $\mathbf{B}^+$-*frame* be any frame satisfying conditions CA12, CA15 and CR9—i.e. which is distributive and has 1 as a left unit, and let a $\mathbf{B}^+$-*model* be any model based on some $\mathbf{B}^+$-frame. The corresponding notions of Sylvan and Meyer are:

Definition 2.17 (The ternary relation model [98]).

i) Let $\mathcal{F} = \langle K, R, 1\rangle$ with K a set, $1 \in K$ and R a ternary relation on K [7]. Define for $a, b, c, d \in K$ the derived relations

[7] In [98] 0 is used instead of 1.

i) $a \leq b$ iff $R1ab$, and

ii) R^2abcd iff for some $e \in K$, $Rabe$ and $Recd$.

Then $\mathcal{F}$ is a *ternary relation frame*—or *3R-frame* for short—if $\leq$ is a pre-order, and for all $a, b, c \in K$, if $R^2 1abc$ then $Rabc$.

ii) A *3R-model* is any pair $\langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is a 3R-frame and V a valuation of Ap into $K_{\mathcal{F}}$ such that for all $a, b \in K$ and $\alpha \in \mathrm{Ap}$, if $a \in V(\alpha)$ and $a \leq b$ then $b \in V(\alpha)$ as well.

The satisfaction relation $\models_{\mathcal{M}} \subseteq K_{\mathcal{M}} \times \mathrm{Fm}^+$ for $\mathcal{M}$ a 3R-model is defined by

$a \models_{\mathcal{M}} \alpha$ iff $a \in V(\alpha)$,

$a \models_{\mathcal{M}} \mathrm{t}$ iff $1_{\mathcal{M}} \leq a$,

$a \models_{\mathcal{M}} \phi \wedge \psi$ iff $a \models_{\mathcal{M}} \phi$ and $a \models_{\mathcal{M}} \psi$,

$a \models_{\mathcal{M}} \phi \vee \psi$ iff $a \models_{\mathcal{M}} \phi$ or $a \models_{\mathcal{M}} \psi$, and

$a \models_{\mathcal{M}} \phi \to \psi$ iff for all $b, c \in K$, if $b \models_{\mathcal{M}} \phi$ and $Rabc$ then $c \models_{\mathcal{M}} \psi$.

The intuitive understanding of $Rabc$ is: "the piece of information c contains at least as much information as the combination of the pieces of information a and b". So in our setting we should wish to render $Rabc$ as $a \cdot b \leq c$.

Sylvan and Meyer do not consider fusion; it may be added, however, using the condition (c.f. [34])

$$a \models_{\mathcal{M}} \phi \circ \psi \text{ iff there are } a', b, c \in K \text{ s.t. } Rbca',\ a' \leq a,\ b \models_{\mathcal{M}} \phi \text{ and } c \models_{\mathcal{M}} \psi.$$

Analogous to the filter property 2.4 we obtain here

Proposition 2.18 (Sylvan and Meyer, [98]). *For all 3R-models $\mathcal{M}$, $a, b \in K_{\mathcal{M}}$ and $\phi \in \mathrm{Fm}^+$, if $a \models_{\mathcal{M}} \phi$ and $a \leq b$ then $b \models_{\mathcal{M}} \phi$.* □

We state soundness and completeness without proof—this can also easily be derived from later results:

Theorem 2.19 (Sylvan and Meyer, [98]). *For all $\phi \in \mathrm{Fm}^+$, $\vdash_{\mathbf{B}^+} \phi$ iff for all 3R-models $\mathcal{M}$, $1_{\mathcal{M}} \models_{\mathcal{M}} \phi$.* □

2.7.1 From 3R-models to models

There is a natural way of embedding each ternary relation model into a model (of our type). Let $\mathcal{M} = \langle K, R, 1, V\rangle$ be a 3R-model. Say a subset $\xi \subseteq K$ is *upper*, if whenever $a \in \xi$ and $a \leq b$ then $b \in \xi$. Then we obtain a $\mathbf{B}^+$-model $(\mathcal{M})^* = \langle S, \sqcap, \cdot, 1', V'\rangle$ by taking

$$S = \{\xi \subseteq K \mid \xi \text{ upper}\},$$

$$\xi_1 \sqcap \xi_2 = \xi_1 \cup \xi_2,$$

$$\xi_1 \cdot \xi_2 = \{c' \in K \mid \exists a \in \xi_1, b \in \xi_2, c \in K.Rabc, c \leq c'\},$$

$$1' = \uparrow 1 = \{a \in K \mid 1 \leq a\}, \text{ and}$$

$$V'(\alpha) = \{\xi \in S \mid \xi \subseteq V(\alpha)\}.$$

The embedding of $\mathcal{M}$ into $(\mathcal{M})^*$ is the map $\uparrow: a \mapsto \uparrow a$. This embedding preserves satisfaction in both directions; this may be seen using induction on formulas. The proof, moreover, goes through for $\circ$ without difficulties.

2.7.2 From models to 3R-models

Can we conversely turn each $\mathbf{B}^+$-model into a 3R-model? One idea could be to appeal to the completeness proof construction: First we take filters of the $\mathbf{B}^+$-model obtaining a "$\mathbf{B}^+$-algebra"—i.e. an algebra with the signature of Fm^+ satisfying the $\mathbf{B}^+$ axioms and rules. This $\mathbf{B}^+$-algebra is then turned into a 3R-model using the construction of theorem 2.19 of [98]. However, such an approach won't work because in the construction of theorem 2.19, the canonical unit 1, as in our canonical model construction, is taken to be the collection of all $\mathbf{B}^+$-theorems. Moreover, the canonical 1 is required to be prime, and this, as shown by Meyer [69] and also below, turns out to hold for $\mathbf{B}^+$ but *not* in general, of course.

The first problem is to tackle the diverging satisfaction conditions for the disjunction, and we solve this by picking out the special elements of $\mathbf{B}^+$-models—the primes—for which the two conditions coincide. We shall thus be looking at conditions for which such special elements exist in sufficient number. Much of this is a variation on standard results, found for instance in [42].

Definition 2.20 (Order-prime, irreducible).

i) Let L be a semilattice. An $x \in L$ is (*order-*) *prime* if for all $x_1, x_2 \in L$, if $x_1 \sqcap x_2 \leq x$ then $x_1 \leq x$ or $x_2 \leq x$.

ii) x is *irreducible*, if for all $x_1, x_2 \in L$, if $x_1 \sqcap x_2 = x$ then $x_1 = x$ or $x_2 = x$.

PRIME(L) (IRR(L)) denotes the set of primes (irreducibles) of L. It is not hard to check that if x is prime then x is irreducible and if L is distributive then the converse holds as well. Our interest in primes is supported by the following observation:

Proposition 2.21 *Let $\mathcal{F}$ be any $\mathbf{B}^+$-frame and $x \in S_{\mathcal{F}}$. Then x is prime iff for all $\mathcal{F}$-based models $\mathcal{M}$, $x \models_{\mathcal{M}} \phi \vee \psi$ iff $x \models_{\mathcal{M}} \phi$ or $x \models_{\mathcal{M}} \psi$.*

Proof: If x is prime and $x \models_{\mathcal{M}} \phi \vee \psi$ then either $x \models_{\mathcal{M}} \phi$ or $x \models_{\mathcal{M}} \psi$—and we are done—or for some x_1, x_2, $x_1 \models_{\mathcal{M}} \phi$, $x_2 \models_{\mathcal{M}} \psi$ and $x_1 \sqcap x_2 \leq x$. Then $x_1 \leq x$ or $x_2 \leq x$, so by 2.4 $x \models_{\mathcal{M}} \phi$ or $x \models_{\mathcal{M}} \psi$. If conversely x is not prime let $x_1 \sqcap x_2 \leq x$, $x_1 \not\leq x$ and $x_2 \not\leq x$. Let $\mathcal{M}$ be any $\mathcal{F}$-based model with $V(\alpha_1) = \uparrow x_1$, $V(\alpha_2) = \uparrow x_2$. Then $x \models_{\mathcal{M}} \alpha_1 \vee \alpha_2$, $x \not\models_{\mathcal{M}} \alpha_1$ and $x \not\models_{\mathcal{M}} \alpha_2$. □

So the special elements we seek are primes. We intend to represent an $x \in S_{\mathcal{M}}$ as the set of primes above x, and we need to seek out conditions for which it is the case that $x \models \phi$ iff for all prime $x' \geq x$, $x' \models \phi$. We ensure this by letting the primes be order-generating and assuring properties to be appropriately closed.

We say that a subset A of a semilattice L is *order-generating* iff for all $x \in L$, $\inf(\uparrow x \cap A)$ is defined and equal to x. This is the same as having the *separation property*, i.e. that whenever $y \not\leq x$ then there is an $a \in A$ s.t. $x \leq a$ and $y \not\leq a$. To ensure order-generation we appeal to algebraic semilattices.

Definition 2.22 (Algebraic semilattice, Scott topology).

i) Let L be a semilattice closed under directed sup's (*up-complete* for short). An $x \in L$ is *compact* (or isolated, or finite), if for all directed $\Delta \subseteq L$ s.t. $x \leq \sup\Delta$ there is an $x' \in \Delta$ s.t. $x \leq x'$. Let $\mathrm{K}(L)$ denote the set of compact elements of L and for $x \in L$, $\mathrm{K}(x) = \{x' \in \mathrm{K}(L) \mid x' \leq x\}$.

ii) An up-complete semilattice L is *algebraic*, if for all $x \in L$, $\mathrm{K}(x)$ is directed and $x = \sup(\mathrm{K}(x))$.

iii) For L an up-complete poset, a set $U \subseteq L$ is (*Scott-*) *open* iff U is upper and for all directed $\Delta \subseteq L$, if $\sup\Delta \in U$ then for some $x \in \Delta$, $x \in U$.

It is well known (c.f. [42]) that for functions $f : L_1 \to L_2$, where L_1 and L_2 are up-complete semilattices, f is continuous w.r.t. the Scott topology iff for all directed $\Delta \subseteq L_1$, $f(\sup\Delta) = \sup f(\Delta) = \sup\{f(x) \mid x \in \Delta\}$.

Proposition 2.23 (c.f. [42]). *Let L be an algebraic semilattice.*

i) If U is open in L then for any $x \in L \setminus U$ there is an $m \in L \setminus U$ with $x \leq m$ and m maximal in $L \setminus U$.

ii) If $x \in L$ is maximal in $L \setminus \nabla$ for ∇ a filter then x is irreducible.

iii) If L is distributive then PRIME(L) is order-generating.

Proof: i) By Zorn's lemma. Let $A = \{y \in L \setminus U \mid x \leq y\}$ for $x \in L \setminus U$. Let C be any chain in A and let $z = \sup C$ (note that $\Delta = \{x \mid \exists y \in C. x \leq y\}$ is directed and $\sup\Delta = \sup C$ so $\sup C$ exists). Suppose that $z \in U$. Now $\mathrm{K}(z)$ is directed and $z = \sup(\mathrm{K}(z))$ and hence as U is open there is some compact $z_0 \in U$ s.t. $z_0 \leq z$. But then $z_0 \leq c$ for some $c \in C$—a contradiction. Hence $\sup C \in A$ and then A has a maximal member.
ii) We get that $\uparrow x \setminus \{x\} = \uparrow x \cap \nabla$ which is a filter. But then x is irreducible.
iii) It suffices to check that IRR(L) has the separation property. If $x \not\leq y$ then there is some compact $a \in L$ s.t. $a \leq x$ but $a \not\leq y$. Clearly $\uparrow a$ is open, $x \in \uparrow a$ and $\uparrow a$ a filter. By i) there is some maximal $z \in L \setminus \uparrow a$, $y \leq z$ and $x \not\leq z$. By ii) z is irreducible. □

We then introduce the class of $\mathbf{B}^+$-models of special interest. In the inductive case for $\to$ in the proof of theorem 2.26 we need not only to ensure that each element is the inf of the set of primes above it, but additionally to ensure that if $A' \subseteq A \subseteq$PRIME(L) and $\inf A$ exists (i.e. A is determined as the set of primes above some element $x = \inf A$) then also $\inf A'$ exists. If L has this property for all $A \subseteq L$ we say it is *conditionally prime-complete*.

Definition 2.24 Let a *normal* $\mathbf{B}^+$*-model* be any $\mathbf{B}^+$-model $\mathcal{M} = \langle S, \sqcap, \cdot, 1, V\rangle$, where

i) $\langle S, \sqcap\rangle$ is conditionally prime-complete and algebraic,

ii) $1 \in$PRIME(S),

iii) if for each prime $x' \geq x$, $x' \in V(\alpha)$ then $x \in V(\alpha)$.

Say in addition that $\mathcal{M}$ is *continuous* if $\cdot$ is continuous in both arguments.

Note that definition 2.24.iii) together with the upper closure of $V(\alpha)$ implies the filter property of $V(\alpha)$. We can now define the desired embedding. Let $\mathcal{M} = \langle S, \sqcap, \cdot, 1, V\rangle$ be a normal $\mathbf{B}^+$-model. Then $(\mathcal{M})^{3R} = \langle K, R, 1', V'\rangle$ is the 3R-model given by

$K = \text{PRIME}(S)$,

$Rabc$ iff $a \cdot b \leq c$,

$1' = 1$, and

$V'(\alpha) = V(\alpha) \cap K$ for all $\alpha \in \text{Ap}$.

To check that satisfaction is preserved we need in particular to verify our correlate of the "squeeze"-lemma of [34]. Say a model $\mathcal{M}$ has the *squeeze property* if whenever $x \cdot y \leq c$ and c is prime then there are primes a, b s.t. $x \leq a$, $y \leq b$, $a \cdot y \leq c$, and $x \cdot b \leq c$.

Lemma 2.25 (The squeeze lemma). *Any continuous normal* $\mathbf{B}^+$*-model (in fact any distributive, continuous, algebraic model) has the squeeze property.*

Proof: Let $x \cdot y \leq c$ and c be prime. Let $A = \{x' \mid x' \cdot y \not\leq c\}$. Then A is upper and if $x_1, x_2 \in A$ then $(x_1 \sqcap x_2) \cdot y \not\leq c$ as $\cdot$ preserves binary inf's and c is prime. So A is a filter. Let further $\Delta \subseteq S_{\mathcal{M}}$ be directed and $\sup\Delta \in A$, i.e. $(\sup\Delta) \cdot y \not\leq c$. Now $(\sup\Delta) \cdot y = \sup(\Delta \cdot y)$ by continuity of $\cdot$ so $\sup(\Delta \cdot y) \not\leq c$, and then for some $d \in \Delta$ does $d \cdot y \not\leq c$. Hence A is open and then by 2.23 there is some maximal $m \in L \setminus A = \{x' \mid x' \cdot y \leq c\}$ s.t. $x \leq m$. By 2.23 m is irreducible and then by distributivity m is prime. The other half follows by a symmetrical argument. □

We can now show that for continuous normal $\mathbf{B}^+$-models satisfaction is preserved on primes, and satisfaction on arbitrary elements are characterised by satisfaction on the dominating primes.

Theorem 2.26 *Let $\mathcal{M}$ be a continuous normal $\mathbf{B}^+$-model. Then*

i) $x \models_{\mathcal{M}} \phi$ iff for all prime $a \geq x$, $a \models_{\mathcal{M}} \phi$.

ii) $x \models_{\mathcal{M}} \phi$ iff for all prime $a \geq x$, $a \models_{(\mathcal{M})^{3R}} \phi$. □

The proof of i) and ii) is again by induction on ϕ. Curiously i) does not go through for $\circ$: There is no way from the assumption that $x \models \phi_1 \circ \phi_2$ and for any prime $a \geq x$ that there are primes a_1, a_2 s.t. $a_1 \cdot a_2 \leq a$, $a_1 \models \phi_1$ and $a_2 \models \phi_2$ to establish the existence of x_1, x_2 s.t. $x_1 \models \phi_1$, $x_2 \models \phi_2$ and $x_1 \cdot x_2 \leq x$.

2.7.3 Continuous models

Now we have a rendition of 3R-models as $\mathbf{B}^+$-models and of normal $\mathbf{B}^+$-models as 3R-models. How do these relate?

It is not too hard to check that $(\mathcal{M})^*$ is a normal $\mathbf{B}^+$-model for any 3R-model $\mathcal{M}$. An upper set ξ, in particular, is compact iff $\xi = K \backslash \downarrow A$ for some finite $A \subseteq K$, where $\downarrow A = \{a' \in K \mid \exists a \in A.\ a' \leq a\}$. Note, however, that $(\mathcal{M})^*$ does not in general seem to be continuous. This amounts to showing that for all $\xi_1, \xi_2 \in S_{(\mathcal{M})^*}$,

$$\begin{aligned} \xi_1 \cdot \xi_2 &= \sup\{\xi \cdot \xi_2 \mid \xi \leq \xi_1, \xi \text{ compact }\} \\ &= \sup\{\xi_1 \cdot \xi \mid \xi \leq \xi_2, \xi \text{ compact }\}, \end{aligned}$$

i.e. that whenever $c' \notin \xi_1 \cdot \xi_2$ then we should be able to extend $\xi_1(\xi_2)$ to some compact ξ s.t. $\xi_1 \subseteq \xi$ $(\xi_2 \subseteq \xi)$ and $c' \notin \xi \cdot \xi_2$ $(c' \notin \xi_1 \cdot \xi)$. But there is no way to ensure this with the given notion of 3R-model. So the representation theorem almost within range at this stage evades us on this point.

So the continuous models are in need of justification outside their relations to the 3R-models in general. As it turns out the class of continuous normal $\mathbf{B}^+$-models is quite a useful one, in that it is complete for $\mathbf{B}^+$, such that restricting attention to such models as regards the relationship to the logics is harmless.

It is not hard to check that for any normal logic $\mathbf{L}$, $\mathrm{C}(\mathbf{L})$ is conditionally prime-complete and algebraic as a semilattice, and that $\cdot_{\mathrm{C}(\mathbf{L})}$ is continuous in both arguments. To show further that $\mathrm{C}(\mathbf{B}^+)$ is a continuous, normal $\mathbf{B}^+$-model we need to check additionally that conditions 2.24.ii) and iii) are satisfied.

For condition iii) say a theory ∇ is *prime*, if whenever $\phi \vee \psi \in \nabla$ then $\phi \in \nabla$ or $\psi \in \nabla$. Clearly for any normal $\mathbf{L}$ and $\nabla \in \mathrm{C}(\mathbf{L})$, ∇ is prime as a theory iff ∇ is order-prime in $\mathrm{C}(\mathbf{L})$. Then we obtain:

Lemma 2.27 *For any normal, distributive* $\mathbf{L}$ *(i.e. normal* $\mathbf{L}$ *s.t.* A15 *is a theorem of* $\mathbf{L}$*),* $\mathrm{C}(\mathbf{L})$ *satisfies condition 2.24.iii).*

PROOF: We noted that $\mathrm{C}(\mathbf{L})$ is algebraic. By 2.13 $\mathrm{C}(\mathbf{L})$ is distributive. By 2.23, $\mathrm{C}(\mathbf{L})$ is generated by its order-primes, hence by its prime theories. Now if in $\mathrm{C}(\mathbf{L})$, $\nabla \in V(\alpha)$ and $\nabla \subseteq \nabla'$ then also $\nabla' \in V(\alpha)$. If conversely $\nabla \notin V(\alpha)$ then $\mathrm{th}_{\mathbf{L}}(\nabla \cup \{\alpha\}) \nsubseteq \nabla$ so we find a prime theory ∇' s.t. $\nabla \subseteq \nabla'$ and $\mathrm{th}_{\mathbf{L}}(\nabla \cup \{\alpha\} \nsubseteq \nabla'$. Then $\alpha \notin \nabla'$ so $\nabla' \notin V(\alpha)$ and we are done. □

So it only remains to be checked that the unit, 1, of $\mathrm{C}(\mathbf{B}^+)$ is a prime theory. This amounts, in view of the definition of 1, to showing that $\mathbf{B}^+$ is prime as a logic (or, equivalently, that it has the disjunctive property)—i.e. if $\vdash_{\mathbf{B}^+} \phi \vee \psi$ then $\vdash_{\mathbf{B}^+} \phi$ or $\vdash_{\mathbf{B}^+} \psi$. The primeness of the logics we have so far considered, i.e. $\mathbf{B}^+$, $\mathbf{T}^+$, $\mathbf{R}^+$ etc., is proved by Meyer in [70] (see also §22 of [7]).

We outline for the sake of completeness a proof here along similar lines. The basic idea is to give a truth-functional analysis of the logic, by giving a property of formulas invariant under the application of axioms and rules which imply that the logic is prime. It is more convenient to carry out the proof in the Hilbert-type formulation of section 2.2 than the sequent-based one. This, of course, is harmless in view of theorem 2.16. Given a logic $\mathbf{L}$ we define the predicate Admissible $\subseteq \mathrm{Fm}^+$ inductively by

Admissible α for all $\alpha \in \mathrm{Ap}$,

Admissible t,

Admissible $\phi \wedge \psi$ iff Admissible ϕ and Admissible ψ,

Admissible $\phi \vee \psi$ iff either ($\vdash_{\mathbf{L}} \phi$ and Admissible ϕ) or ($\vdash_{\mathbf{L}} \psi$ and Admissible ψ),

Admissible $\phi \to \psi$ iff whenever $\vdash_{\mathbf{L}} \phi$ and Admissible ϕ then Admissible ψ.

and say $\mathbf{L}$ is *coherent*, if whenever $\vdash_{\mathbf{L}} \phi$ then Admissible ϕ. Note that coherence entails primeness, for if $\mathbf{L}$ is coherent and $\vdash_{\mathbf{L}} \phi \vee \psi$ then Admissible $\phi \vee \psi$, thus either $\vdash_{\mathbf{L}} \phi$ or $\vdash_{\mathbf{L}} \psi$.

Now the problem is just to show $\mathbf{B}^+$ coherent—essentially amounting to a soundness theorem of the "Admissible"-interpretation of $\mathbf{L}$. Indeed we can just as well show the $\{\to, \wedge, \vee, \mathrm{t}\}$-fragment of *all* the logics we have considered coherent.

Theorem 2.28 (See also [70]). *Let* $\mathbf{L}$ *be any of the logics over* Fm^+ *introduced in section 2.2. Then* $\mathbf{L}$ *is prime.* □

The proof is tedious, but easy. It suffices to show Admissible ϕ for all axioms ϕ of $\mathbf{L}$ and that the set of formulas s.t. $\vdash_{\mathbf{L}} \phi$ and Admissible ϕ is closed under the proof rules. It should be noted that the technique is easily extended to other connectives such as $\circ$—and indeed also to modal logics—see Meyer's section of Anderson and Belnap referred to above.

Corollary 2.29 *If* $\mathbf{L}$ *is a logic over* Fm^+ *satisfying the conditions of theorem 2.28 and containing* $\mathbf{B}^+$ *then* $\mathrm{C}(\mathbf{L})$ *is a continuous, normal* $\mathbf{B}^+$*-model.* □

Corollary 2.30 (Soundness and completeness for continuous, normal $\mathbf{B}^+$-models). *Let* $\mathbf{L}$ *be any of the logics* $\mathbf{B}^+$, $\mathbf{T}^+$, $\mathbf{R}^+$, $\mathbf{H}^+$ *and* $\phi \in \mathrm{Fm}^+$. *Then* $\vdash_{\mathbf{L}} \phi$ *iff for all continuous, normal* $\mathbf{B}^+$*-models* $\mathcal{M}$ *satisfying the conditions corresponding to the axioms and rules for* $\mathbf{GL}$ *as stated in definition 2.15,* $1_{\mathcal{M}} \models_{\mathcal{M}} \phi$. □

2.8 Examples: Synchronous processes as models

It is not too early to present an example showing that we indeed do come across structures like the above in practice—moreover, it will serve as a useful indication of the direction later parts of the thesis will take.

We build $\mathbf{LL}^+$- and $\mathbf{R}^+$-frames based on a fragment of Milner's SCCS [78], by taking $\cdot$ to be synchronous parallel composition, 1 to be an idling process, and $\sqcap$ to be the choice operator $+$. The appropriate fragment of *SCCS-processes* $p \in \mathrm{Pc}^+$ is given by the abstract syntax

$$p ::= 1 \mid \lambda(p) \mid p + p \mid p \times p$$

where $\lambda \in \mathcal{L}$, some commutative monoid with unit e. We adopt the precedence convention $\lambda() > \times\ > +$.

The idea is that $\mathcal{L}$ is the set of atomic, uninterpreted actions, or labels, a process may perform. The set $\mathcal{L}$ is equipped with a binary operation $\cdot$ usually denoted by juxtaposition, to account for the simultaneous occurrence of actions. It is natural to assume, as in SCCS, that $\mathcal{L}$ form a commutative monoid—i.e. that $\cdot$ is commutative and associative and that there is a label e neutral with respect to simultaneous occurrence. In SCCS it is moreover assumed that $\mathcal{L}$ forms a group—we have no need of inverses, and consequently omit this assumption.

The idea of processes performing actions is formalised by equipping Pc^+ with a $\mathcal{L}$-indexed family of transition relations $\xrightarrow{\lambda}$, the holding of $p \xrightarrow{\lambda} p'$ signifying that the process p may perform the action λ, and after doing so behave like p'. The relations $\xrightarrow{\lambda}$ are defined as the least s.t.

$1 \xrightarrow{e} 1$,

$\lambda(p) \xrightarrow{\lambda} p$,

$p \xrightarrow{\lambda} p'$ only if $p + q \xrightarrow{\lambda} p'$ and $q + p \xrightarrow{\lambda} p'$,

$p \xrightarrow{\lambda} p'$ and $q \xrightarrow{\mu} q'$ only if $p \times q \xrightarrow{\lambda\mu} p' \times q'$.

Note that for each $\lambda \in \mathcal{L}$, $\xrightarrow{\lambda}$ is image-finite—i.e. for each $p \in \mathrm{Pc}^+$, $\{p' \mid p \xrightarrow{\lambda} p'\}$ is finite.

The crucial question is how to order processes. We consider first an ordering based on Park's notion of bisimulation equivalence [85] leading to an $\mathbf{LL}^+$-frame, and secondly we consider the strictly weaker simulation ordering [51], leading to $\mathbf{R}^+$—at least when $\mathcal{L}$ in addition is idempotent, i.e. forms a semilattice.

Definition 2.31 A relation $R \subseteq \mathrm{Pc}^+ \times \mathrm{Pc}^+$ is a *simulation* [8], if whenever pRq then

i) for all $\lambda \in \mathcal{L}$ and $q' \in \mathrm{Pc}^+$, if $q \xrightarrow{\lambda} q'$ then there is some $p' \in \mathrm{Pc}^+$ s.t. $p \xrightarrow{\lambda} p'$ and $p'Rq'$.

If whenever pRq then i) holds and in addition its converse

ii) for all $\lambda \in \mathcal{L}$, $p' \in \mathrm{Pc}^+$, if $p \xrightarrow{\lambda} p'$ then there is some $q' \in \mathrm{Pc}^+$ s.t. $q \xrightarrow{\lambda} q'$ and $p'Rq'$,

then R is a *bisimulation*. Then p *simulates* q, $p \sqsubseteq_s q$, (p and q are *bisimilar*, $p \simeq_b q$), if there is some (bi-) simulation R s.t. pRq; and p and q are *simulation equivalent*, $p \simeq_s q$, if $p \sqsubseteq_s q$ and $q \sqsubseteq_s p$.

[8] In fact, according to [51] a *reverse* simulation.

It is well known that bisimulation equivalence is strictly finer than simulation equivalence [51]. That containment holds is trivial. To see that containment is strict consider the processes

$$p = \lambda(\mu(1) + \eta(1))$$

$$q = (\lambda(\mu(1)) + \lambda(\eta(1))) + \lambda(\mu(1) + \eta(1)).$$

with $\mu \neq \eta$.

Both simulation and bisimulation are wellbehaved w.r.t. the process constructors:

Proposition 2.32 *With respect to the operations on* Pc^+*,*

i) $\sqsubseteq_s$ *is a precongruence.*

ii) $\simeq_b$ *is a congruence.*

PROOF: That $\sqsubseteq_s$ is a preorder and $\simeq_b$ an equivalence is clear. We must in addition show that if $p \sqsubseteq_s q$ ($p \simeq_b q$) then for all $r \in \mathrm{Pc}^+$, $\lambda \in \mathcal{L}$, i) $p+r \sqsubseteq_s q+r$, ii) $r+p \sqsubseteq_s r+q$, iii) $p \times r \sqsubseteq_s q \times r$, iv) $r \times p \sqsubseteq_s r \times q$ and v) $\lambda(p) \sqsubseteq_s \lambda(q)$ and similarly for $\simeq_b$.
Cases ii) and iv) follows by the commutativity of $+$ and $\times$ proved below. For case i) show that if pRq and R is a (bi-)simulation then $R \cup I \cup \{\langle p+r, q+r\rangle\}$ where I is the identity is a (bi-)simulation. For iii) show the same for

$$\{\langle p' \times r', q' \times r'\rangle \mid r' \in \mathrm{Pc}^+, p'Rq'\}$$

and for v) show the same for $R \cup \{\langle \lambda(p), \lambda(q)\rangle\}$. □

We build an $\mathbf{LL}^+$-frame from processes under bisimulation equivalence. Moreover, assuming that the label monoid forms a semilattice—which is justifiable in many practical applications—with respect to simulation we obtain an $\mathbf{R}^+$-frame.

Theorem 2.33

i) $\langle \mathrm{Pc}^+/\simeq_b, +/\simeq_b, \times/\simeq_b, [1]_{\simeq_b}\rangle$ *is an* $\mathbf{LL}^+$*-frame.*

ii) For $\mathcal{L}$ *a semilattice,* $\langle \mathrm{Pc}^+/\simeq_s, +/\simeq_s, \times/\simeq_s, [1]_{\simeq_s}\rangle$ *is an* $\mathbf{R}^+$*-frame.*

PROOF: i). We must check that

a) $\langle \mathrm{Pc}^+/\simeq_b, +/\simeq_b\rangle$ is a semilattice,

b) $\langle \mathrm{Pc}^+/\simeq_b, \times/\simeq_b, [1]_{\simeq_b}\rangle$ is a commutative monoid, and that

c) $\times$ distributes over $+$, i.e. that for all $p, q, r \in \mathrm{Pc}^+$, $(p+q)\times r \simeq_b (p\times r) + (q\times r)$.

a) is straightforward, c.f. [76, 51]. For b) show

$$R_1 = \{\langle p\times q, q\times p\rangle \mid p, q \in \mathrm{Pc}^+\},$$

$$R_2 = \{\langle (p\times q)\times r, p\times (q\times r) \in \mathrm{Pc}^+\}, \text{ and}$$

$$R_3 = \{\langle p\times 1, p\rangle \mid p \in \mathrm{Pc}^+\}$$

to be bisimulations. For c), assume $(p+q)\times r \xrightarrow{\lambda} s$. Then $s = s'\times r'$ and for some $\mu, \eta \in \mathcal{L}$, $p+q \xrightarrow{\mu} s'$ and $r \xrightarrow{\eta} r'$. Then either $p \xrightarrow{\mu} s'$ or $q \xrightarrow{\mu} s'$ and then $(p\times r)+(q\times r) \xrightarrow{\lambda} s$. The converse direction is straightforward. Note that the synchronous nature of the parallel composition is crucial here.
ii). First we note that $\sqsubseteq_s$ coincides with the induced semilattice ordering, for we always have $p+q \sqsubseteq_s p$; moreover $p \sqsubseteq_s p+q$ iff $p \sqsubseteq_s q$.

Now in addition to i) we need only show that

a) For $\mathcal{L}$ a semilattice and $p \in \mathrm{Pc}^+(\mathcal{L})$, $p\times p \sqsubseteq_s p$, and that

b) $\langle \mathrm{Pc}^+/\simeq_s, +/\simeq_s\rangle$ is distributive.

For a) check that $\{\langle p\times p, p\rangle \mid p \in \mathrm{Pc}^+\}$ is a simulation: If $p \xrightarrow{\lambda} p'$ then $p\times p \xrightarrow{\lambda} p'\times p'$. For b) first define the sets $\mathrm{init}(p) = \{\lambda \mid \exists p'.p \xrightarrow{\lambda} p'\}$ and $p/\lambda = \{p' \mid p \xrightarrow{\lambda} p'\}$. Assume that $p+q \sqsubseteq_s r$, $p \not\sqsubseteq_s r$ and $q \not\sqsubseteq_s r$. Then

i) either $\mathrm{init}(r) \not\subseteq \mathrm{init}(p)$ or $\mathrm{init}(r) \subseteq \mathrm{init}(p)$ and there is some $\lambda \in \mathrm{init}(r)$ and $r' \in r/\lambda$ s.t. for all $p' \in p/\lambda$, $p' \not\sqsubseteq_s r'$, and

ii) the same for q.

Then $\mathrm{init}(r)\cap\mathrm{init}(p) \neq \emptyset$ and $\mathrm{init}(r)\cap\mathrm{init}(q) \neq \emptyset$, for $\mathrm{init}(r) \neq \emptyset$ and if for instance $\mathrm{init}(r)\cap\mathrm{init}(p) = \emptyset$ then $q \sqsubseteq_s r$. By the semilattice properties of $+$ we can use the $\sum$-notation for finite, nonempty sums. Let now for $\lambda \in \mathrm{init}(p)\cap\mathrm{init}(r)$

$$p_\lambda = \sum\{\lambda(r') \mid r' \in r/\lambda \text{ and for some } p'' \in p/\lambda,\ p'' \sqsubseteq_s r'\}$$

and then $p' = \sum_{\lambda \in \mathrm{init}(p) \cap \mathrm{init}(r)} p_\lambda$ and define q' similarly. Clearly the sums involved are finite. Furthermore assume that for all $\lambda \in \mathrm{init}(p) \cap \mathrm{init}(r)$ and for all $r' \in r/\lambda$ there is no $p'' \in p/\lambda$ s.t. $p'' \sqsubseteq_s r'$. Then, as $p + q \sqsubseteq_s r$, $\mathrm{init}(p) \cap \mathrm{init}(r) \subseteq \mathrm{init}(q)$ and for all $r' \in r/\lambda$ there is some $q' \in q/\lambda$ s.t. $q' \sqsubseteq_s r'$. Moreover, whenever $\lambda \in \mathrm{init}(r) \setminus \mathrm{init}(p)$, $\lambda \in \mathrm{init}(q)$ and the same thing holds, as $p + q \sqsubseteq_s r$. But then $q \sqsubseteq_s r$—a contradiction. Hence the sums are also nonempty, and p', q' are well-defined. Clearly $p \sqsubseteq_s p'$ and $q \sqsubseteq_s q'$. Also $p' + q' \simeq_s r$, for if $r \xrightarrow{\lambda} r'$ then $p' + q' \xrightarrow{\lambda} r'$ and if $p' + q' \xrightarrow{\lambda} r'$ then $r \xrightarrow{\lambda} r'$. □

2.9 Concluding remarks

We have introduced general algebraic models of relevance logics based on semilattices with an inf-preserving binary operation. This model, in particular, is capable of dealing with both distributive and non-distributive logics in a uniform way. We have investigated the relationship to the ternary relation-based model of Sylvan and Meyer [98], and we finally exhibited two examples of process-based models. Concluding this chapter we briefly discuss a few outstanding issues and provide some pointers towards future work.

2.9.1 Dunn's algebraic models and quantales

In view of the strongly algebraic nature of the soundness and completeness proofs it is not surprising that there are strong connections also to the algebraic models of Dunn (c.f. [7], § 28).

The appropriate algebraic model for **G**, a **G**-*algebra*, is a structure

$$\mathcal{A} = \langle A, \wedge, \vee, \circ, e \rangle,$$

where

i) $\langle A, \wedge, \vee \rangle$ is a lattice with $\leq$ the induced order,

ii) $\circ$ is a binary operation on A preserving $\vee$ in both arguments,

iii) $e \in A$, and

iv) there is an operation $\rightarrow$ s.t. $a \circ b \leq c$ iff $a \leq b \rightarrow c$.

Clearly this is just an "algebraized" version of **G** itself—proposition 2.6 can be used for seeing this.

The completeness proof for **G** provides a way of turning any **G**-algebra $\mathcal{A}$ into a frame $\mathcal{A}^\dagger$ (by taking filters instead of their syntactical correlates, theories). Conversely any frame $\mathcal{F} = \langle S, \sqcap, \cdot, 1\rangle$ is turned into a **G**-algebra $\mathcal{F}^\dagger = \langle A, \wedge, \vee, \circ, e\rangle$ by letting A be the set of all filters in $\mathcal{F}$, $\wedge = \cap$, $\nabla_1 \vee \nabla_2$ be the least filter containing $\nabla_1 \cup \nabla_2$, $\nabla_1 \circ \nabla_2 = \{xy \mid x \in \nabla_1,\ y \in \nabla_2\}$ and $e = \uparrow 1$. We can then take

$$\nabla_1 \to \nabla_2 = \{x \mid \forall y \in \nabla_1.\ xy \in \nabla_2\}. \tag{2.1}$$

It is not hard to check now that $\mathcal{F}$ is embedded in $\mathcal{F}^{\dagger\dagger}$ and $\mathcal{A}$ in $\mathcal{A}^{\dagger\dagger}$, where we take an embedding of **G**-algebras to preserve all operations except possibly $\to$. To obtain preservation of $\to$ as well we need to assume that for any x, $y \in S$ there is a least z s.t. $y \leq z \cdot x$. We can then restrict attention to principal filters (i.e. those of the form $\uparrow x$ for $x \in S$) and define implication by $\uparrow x \to \uparrow y = \uparrow z$, where z is least s.t. $y \leq z \cdot x$.

Alternatively we may consider quantale-like structures [6], by assuming not only binary, but arbitrary sup's in $\mathcal{A}$ preserved by $\cdot$ in both arguments. Call such an $\mathcal{A}$ a *prequantale*. In a prequantale implication may be defined by $a \to b = \bigvee\{c \mid c \circ a \leq b\}$. It is not hard to see that $\mathcal{F}^\dagger$ is a prequantale for $\mathcal{F}$ a frame. Moreover, in $\mathcal{F}^\dagger$ this last definition of $\to$ and that of 2.1 coincides.

2.9.2 The process-based models

The SCCS-based models of section 2.8 deals only with the static structure of processes. An obvious way of capturing also dynamic behaviour is to add modal operators. For the case of simulation we can add an $\mathcal{L}$-indexed family of operators $\{[\lambda]\}_{\lambda\in\mathcal{L}}$ with the interpretation [9]

$$p \models [\lambda]\phi \text{ iff for all } p' \text{ s.t. } p \xrightarrow{\lambda} p',\ p' \models \phi.$$

It is easy to see that if $p \models [\lambda]\phi$ and $p \sqsubseteq_s p'$ then also $p' \models [\lambda]\phi$ and conversely that if $p \models [\lambda]\phi$, $q \models [\lambda]\phi$ then $p + q \models [\lambda]\phi$ (as part of an inductive argument, as in the proof of theorem 2.4) thus maintaining the filter property. Similarly, for the case of bisimulation, we can add the dual $\{\langle\lambda\rangle\}_{\lambda\in\mathcal{L}}$ *provided* all occurrences of $\langle\lambda\rangle$ is in the scope of some $[\lambda']$—recall that here processes are ordered by $p \sqsubseteq_b q$ iff $p + q \simeq_b p$.

[9] Strictly speaking satisfaction as in section 2.8 is on $\simeq_s$-congruence classes of processes.

Moreover, it is not difficult to see that with the addition of these modalities, the characterisation of simulation and bisimulation equivalence on image-finite transition systems in terms of Hennessy-Milner logic [51] will in fact extend to the present setting.

In the following chapters we return to this the main topic of the thesis, of giving process-based interpretations of modal extensions of relevance logics.

2.9.3 Future work

There is a lot of scope for extending the work reported in this chapter. We list some of the important issues:

i) Concerning the deduction theorems it seems plausible that a much more systematic exposition can be obtained, possibly along the lines of Slaney and Read [101, 95] who uses assumption algebras.

ii) What is the connection between our notion of model and that of Ono and Komori [83] for BCK- and related logics? Their interest is primarily in logics that reject contraction rather than weakening. Their models, however, have essential features in common with the ones we consider, in particular (for their total models) the interpretation of $\rightarrow$ and $\vee$.

iii) It is clear that the work on finding frame conditions corresponding to logical axioms and rules can be carried a great deal further. A number of examples relating to the ternary relation model can be found, for instance, in [98, 95]. What are the limitations of this? Can we for instance characterise the relevantly definable first order frame conditions along the lines of similar results in modal logic (c.f. [111])? What are the appropriate validity preserving constructions on frames? Can we characterise those formulas that define first order frame conditions?

   We have, for instance, been unable to find a general first-order condition on frames such that the correspondence results go through for Mingle, $\phi \vdash \phi \rightarrow \phi$. With respect to continuous, normal $\mathbf{B}^+$-models we can adopt the condition of Dunn [33]: for all x, y and prime $a \geq x \cdot y$, either $x \leq a$ or $y \leq a$.

iv) What happens when the constants $\top$ and $\bot$ with the interpretations $x \models \top$, $x \not\models \bot$ for all x are added? Then we obtain for instance

$$x \models_{\mathcal{M}} \phi \to \bot \text{ iff for all } y \in S,\ y \not\models_{\mathcal{M}} \phi,$$

and

$$x \models_{\mathcal{M}} (\phi \to \bot) \to \bot \text{ iff for some } y \in S,\ y \models_{\mathcal{M}} \phi.$$

Curiously this seems to be rather pathological in the general setting, although we deal with the problem in later chapters where models are free structures, thus giving a handle on the existence or nonexistence of elements. The general consequences of such a definition remains, however, to be worked out.

v) How does our model cope with the De-Morgan negation? The standard solution (c.f. [97]) is to stipulate an involution $(\cdot)^\star$ on frames satisfying

i) $x^{\star\star} = x$, and

ii) $x \leq y$ iff $y^\star \leq x^\star$,

and then letting $x \models_{\mathcal{M}} \neg\phi$ iff $x^\star \not\models_{\mathcal{M}} \phi$. This, however, will fail to make the filter property go through. One possibility is to appeal to continuous, normal $\mathbf{B}^+$-models and take instead

$$x \models_{\mathcal{M}} \neg\phi \text{ iff for all prime } a \geq x, a^\star \not\models_{\mathcal{M}} \phi.$$

A similar technique may be adopted for fusion to sidestep the problems encountered following theorem 2.26.

Chapter 3

Relevance logics of processes

In the previous chapter we saw how the basic static structure of synchronous processes gave rise to frames with multiplication interpreted as parallel composition. Here we consider extensions of linear logic by modal operators to account for the dynamic behaviour of processes, corresponding to CCS/SCCS-type prefixing on the level of frames.

In section 1 we introduce processes, their operational semantics, and the notion of ordering processes according to their ability to accept suitable tests [30]. This gives rise to (in-) equational classes of algebras, akin to the process algebras of Bergstra and Klop (c.f. [13]), which we call *synchronous algebras*. These extend $\mathbf{LL}^+$-frames by the addition of prefixing and are introduced in section 2. We characterise the initial synchronous algebras, and show these to coincide with processes under the corresponding testing preorders.

We then introduce in section 6 extensions of linear logic by indexed forwards and backwards modalities and their interpretations over synchronous algebras. In section 7 we characterise the corresponding interpretations on processes in syntactical/operational terms and show that they induce the testing preorders on processes.

We next consider axiomatisations. In section 8 we provide axiomatisations for the fragment without extensional constants that are sound and complete with respect to validity in all synchronous algebra models, and in section 9 we discuss the difficulties involved in extending these results.

In sections 10 and 11 we provide sound and complete axiomatisations for the "initial" interpretations which in view of their computational characterisations are of particular interest. The (quite cumbersome) rewriting technique used in the completeness proofs provide procedures for deciding properties of formulas such as validity and satisfiability w.r.t. the free interpretations.

We conclude the chapter with a discussion of two areas where further work is needed. The first is the problem of model checking, and the second is the problem of extending the work to richer process languages.

3.1 Synchronous processes with internal choice

We start by introducing the process system and its operational semantics. It is very similar to the system Pc^{+} introduced in section 2.8, except that we replace the $+$-operator by the internal choice operator $\oplus$, and add a deadlock-operator 0. It is a fragment of a more expressive system considered by Hennessy in [49], and our presentation differs from his only in inessential details.

3.1.1 Processes

Let $\mathcal{L}$ denote an Abelian group with unit e, multiplication denoted by juxtaposition, and inverse $(\cdot)^{-1}$, and let λ, μ, η, ν range over $\mathcal{L}$. We refer to $\mathcal{L}$ as the *action group*, and assume it to be fixed for the remainder of the chapter. The set of *processes*, $\mathrm{Pc}^{\oplus}$, ranged over by p, q, r, s, is given by the abstract syntax

$$p ::= 0 \mid 1 \mid \lambda(p) \mid p \oplus p \mid p \times p$$

and we adopt the precedence convention $\lambda() > \times > \oplus$. It turns out often to be convenient to work with finite nonempty sets $P, Q \subseteq \mathrm{Pc}^{\oplus}$ rather than the processes themselves. The operations on $\mathrm{Pc}^{\oplus}$ are extended to such sets by pointwise extensions, e.g.

$$P \times Q = \{p \times q \mid p \in P, q \in Q\}.$$

As usual we account for the dynamic behaviour of processes by equipping them with a transition structure. In order to capture internal nondeterminism in a synchronous setting, we split up the notion of transition into stabilisation and the action performance, similar in spirit to [49]. The sublanguage of *stable processes* $\mathrm{St} \subseteq \mathrm{Pc}^{\oplus}$ is given by

$$p ::= 0 \mid 1 \mid \lambda(p) \mid p \times p$$

and the *stabilisation relation*, $\twoheadrightarrow \subseteq \mathrm{Pc}^{\oplus} \times \mathrm{St}$, is the least relation s.t.

$0 \twoheadrightarrow 0$,

$1 \twoheadrightarrow 1$,

$\lambda(p) \twoheadrightarrow \lambda(p)$,

$p \twoheadrightarrow r$ only if $p \oplus q \twoheadrightarrow r$ and $q \oplus p \twoheadrightarrow r$,

$p \twoheadrightarrow p'$, $q \twoheadrightarrow q'$ only if $p \times q \twoheadrightarrow p' \times q'$.

We let $\mathrm{st}(P) = \{q \mid \exists p \in P.\ p \twoheadrightarrow q\}$; this is clearly a finite and nonempty set. The application of st to single processes is obtained by specialisation, i.e. $\mathrm{st}(p) = \mathrm{st}(\{p\})$. This procedure is applied to all operations and relations on sets P, Q to be introduced below.

The $\mathcal{L}$-indexed family of *transition relations* $\xrightarrow{\lambda} \subseteq \mathrm{St} \times \mathrm{Pc}^{\oplus}$, then, is the least s.t.

$1 \xrightarrow{e} 1$,

$\lambda(p) \xrightarrow{\lambda} p$,

$p \xrightarrow{\lambda} p'$, $q \xrightarrow{\mu} q'$ only if $p \times q \xrightarrow{\lambda\mu} p' \times q'$.

One can think of a process p as a "virtual state" and the set $\mathrm{st}(p)$ as the set of "physical states" realising p. The ability of performing actions is a property belonging to physical states and one may read $p \xrightarrow{\lambda} p'$ as "p can become p' by performing the action λ". The transition structure is very simple in that we do not consider any form of external choice. In particular note that if there are λ, p' s.t. $p \xrightarrow{\lambda} p'$ then λ and p' are unique with this property, and we let $\mathrm{succ}(p)$ denote this p' if defined.

3.1.2 Tests and test acceptance

Processes are identified according to the set of potential outcomes when running them with a test; an outcome being either failure or success depending on whether or not the test is brought to termination. The notions of test and test acceptance differs somewhat from De Nicola and Hennessy's original account [30, 49, 28] in that their notion takes divergence into account. We return to this issue and the relationship between testing and other notions of process equivalence in the following chapter.

Here *tests* $t \in \mathcal{T}$ are simply finitely branching trees generated by the abstract syntax

$$t ::= 0 \mid \surd \mid \sum_{i \in I} \lambda_i t_i$$

where I is a finite, nonempty set. For a finite and nonempty set $\Lambda \subseteq \mathcal{L}$ we abbreviate $\sum_{\lambda \in \Lambda} \lambda \surd$ by Λ itself.

Intuitively, 0 is the test that always fails, $\surd$ the test that always succeeds, and sums explore the action performing capabilities of processes. As usual a test is applied to a process by running them both in parallel. This idea is formalised by equipping pairs $\langle p, t \rangle$ with a transition structure. We define the binary relation $\rightarrow$ by

i) if $p_1 \in \mathrm{Pc}^{\oplus} \setminus \mathrm{St}$ then $\langle p_1, t_1 \rangle \rightarrow \langle p_2, t_2 \rangle$ iff $p_1 \twoheadrightarrow p_2$ and $t_1 = t_2$,

ii) if $p_1 \in \mathrm{St}$ then $\langle p_1, t_1 \rangle \rightarrow \langle p_2, t_2 \rangle$ iff $t_1 = \sum_{i \in I} \lambda_i t'_i$, $p_1 \xrightarrow{\lambda_i} p_2$ and $t_2 = t'_i$ for some $i \in I$.

Let then $\rho = \langle p_0, t_0 \rangle \rightarrow \ldots \rightarrow \langle p_n, t_n \rangle$ be any maximal ($\rightarrow$-) derivation from $\langle p, t \rangle$; that is, s.t. $p = p_0$, $t = t_0$ and for no p', t' does $\langle p_n, t_n \rangle \rightarrow \langle p', t' \rangle$. ρ is said to be *successful*, if $t_n = \surd$. We then introduce the test acceptance relations may and must by

i) P may t iff there is a $p \in P$ and a successful derivation from $\langle p, t \rangle$,

ii) P must t iff for all $p \in P$, all maximal derivations from $\langle p, t \rangle$ are successful,

and the may and must relation on individual processes is as usual obtained by specialisation. Thus an outcome of applying a test t to a process p, corresponding to a maximal derivation from $\langle p, t \rangle$, may be either successful or unsuccessful, and sets of outcomes are distinguished in two different ways, according to whether or not they are sometimes successful and never unsuccessful. This gives rise to natural preorders on processes, the *testing preorders*.

Definition 3.1

i) $P \sqsubseteq_1 Q$ iff for all $t \in \mathcal{T}$, if P may t then Q may t,

ii) $P \sqsubseteq_2 Q$ iff for all $t \in \mathcal{T}$, if P must t then Q must t,

iii) $\sqsubseteq_3 = \sqsubseteq_1 \cap \sqsubseteq_2$.

We refer to $\sqsubseteq_1$ as the may- and $\sqsubseteq_2$ as the must-preorder, and derive the testing equivalences $\simeq_i$, $i \in \{1,2,3\}$, by $\simeq_i = \sqsubseteq_i \cap \sqsupseteq_i$.

Similar to the characterisation of testing equivalence shown by De Nicola in his thesis [28], these preorders have recursive characterisations much like Kennaway's weak equivalence [57]—this, incidentally, is a topic we return to in the next chapter. To present this we introduce a few derived notions:

P can λ iff for some $p \in \mathrm{st}(P)$ and q, $p \xrightarrow{\lambda} q$.

$P/\lambda = \{q \mid \exists p \in \mathrm{st}(P).\ p \xrightarrow{\lambda} q\}$.

P live iff for all $p \in \mathrm{st}(P)$ there is some λ s.t. p can λ.

Definition 3.2 The functions F_1, F_2 on binary relations R on sets P, Q are defined by

i) $P\ (F_1(R))\ Q$ iff for all λ, if P can λ then Q can λ and $(P/\lambda)R(Q/\lambda)$,

ii) $P\ (F_2(R))Q$ iff for all λ, Λ,

a) P must Λ only if Q must Λ, and

b) if P live and Q can λ then P can λ and $(P/\lambda)R(Q/\lambda)$.

Let $P \sqsubseteq^0 Q$ for all P, Q, and $P \sqsubseteq_i^{n+1} Q$ iff $P(F_i(\sqsubseteq_i^n))Q$. We can then show the greatest fixed point of F_i to coincide with $\sqsubseteq_i$:

Theorem 3.3 (Recursive characterisation of $\sqsubseteq_i$). *Let $i \in \{1,2\}$.*

i) F_i preserves arbitrary intersections and hence possesses a greatest fixed point νF_i given by $\nu F_i = \bigcap_{n \geq 0} \sqsubseteq_i^n$.

ii) $\sqsubseteq_i = \nu F_i$.

Proof: i) The first part follows easily from the definitions. For the second check first that if R is any fixed point of F_i then $R \subseteq \sqsubseteq_i^n$ for all $n \geq 0$. Check secondly that νF_i is actually a fixed point of F_i.
ii) We only prove it for the case $i = 2$—the case for $i = 1$ is similar and simpler. For $\subseteq$ we show first by induction on n that $\sqsubseteq_i \subseteq \sqsubseteq_2^n$ for all $n \geq 0$. The base case is trivial. So assume $P \sqsubseteq_2 Q$. Then if P must Λ also Q must Λ. Assume that P live and Q can λ. Let $\Lambda' = \{\lambda \mid P \text{ can } \lambda\}$. Then P must Λ' and hence Q must Λ'

as well. If $\lambda \notin \Lambda'$ then P must $t = \sum_{\lambda' \in \Lambda' \cup \{\lambda\}} \lambda' t_{\lambda'}$, where $t_\lambda = 0$ and $t_{\lambda'} = \surd$ when $\lambda' \neq \lambda$. But then Q $\not{m}$ust t—a contradiction. Thus $\lambda \in \Lambda'$ and P can λ. Also $P/\lambda \sqsubseteq_2 Q/\lambda$. For if not we find a t_λ s.t. P/λ must t_λ and Q/λ $\not{m}$ust t_λ. Let Λ' be as above and $t = \sum_{\lambda' \in \Lambda'} \lambda' t'_{\lambda'}$ where $t'_\lambda = t_\lambda$ and $t'_{\lambda'} = \surd$ for $\lambda' \neq \lambda$. Then P must t and Q $\not{m}$ust t—a contradiction. By the induction hypothesis, $P/\lambda \sqsubseteq_2^n Q/\lambda$ and thus we have shown $P \sqsubseteq_2^{n+1} Q$.

For the converse containment assume that P must t and Q $\not{m}$ust t. Again t must have the form $t = \sum_{i \in I} \lambda_i t_i$. Let $\Lambda = \{\lambda_i \mid i \in I\}$. Then P must Λ. If Q $\not{m}$ust Λ then $P \not\sqsubseteq_2^1 Q$ and we are done, so assume instead that Q must Λ. Then there is some $q \in Q$, $i \in I$, $q' \in \mathrm{st}(q)$ s.t. q' can λ_i and $\mathrm{succ}(q')$ $\not{m}$ust t_i. Then Q can λ_i and as P must Λ, P live. So if P $\not{c}$an λ_i then $P \not\sqsubseteq_2^1 Q$ and we are done. Otherwise P/λ_i must t_i and Q/λ_i $\not{m}$ust t_i. By the induction hypothesis we find an n s.t. $P/\lambda_i \not\sqsubseteq_2^n Q/\lambda_i$, but then $P \not\sqsubseteq_2^{n+1} Q$ and we are done. □

It is now a straightforward matter to check that the preorders $\sqsubseteq_i$ are well-behaved with respect to the structure of processes.

Theorem 3.4 *For $i \in \{1, 2, 3\}$, $\sqsubseteq_i$ is substitutive w.r.t. the operations on* $\mathrm{Pc}^\oplus$.

PROOF: It suffices to show $\sqsubseteq_i^n$ substitutive for all $n \geq 0$ and $i \in \{1, 2\}$. We take the case for $i = 2$ and $\times$. So assume that $P_1 \sqsubseteq_2^{n+1} P_2$. If $P_1 \times Q$ must Λ then

$$P_1 \text{ must } \{\lambda_1 \mid P_1 \text{ can } \lambda_1, \exists \lambda_Q . Q \text{ can } \lambda_Q, \lambda_1 \lambda_Q \in \Lambda\} = \Lambda_1.$$

Then P_2 must Λ_1, whence $P_2 \times Q$ must Λ. Suppose that $P_1 \times Q$ live and $P_2 \times Q$ can λ. Then P_1, Q live and P_2 can λ_1, Q can λ_2 for some λ_1, λ_2 s.t. $\lambda = \lambda_1 \lambda_2$. Then P_1 can λ_1 whence $P_1 \times Q$ can λ. Also whenever P_2 can λ_1, Q can λ_2 and $\lambda = \lambda_1 \lambda_2$ then

$$(P_1/\lambda_1) \times (Q/\lambda_2) \sqsubseteq_2^n (P_2/\lambda_1) \times (Q/\lambda_2)$$

by the induction hypothesis. Note here the following monotonicity properties of $\sqsubseteq_1^n$ and $\sqsubseteq_2^n$:

i) if $P \sqsubseteq_1^n Q$, $P' \subseteq P$, $Q \subseteq Q'$ then $P' \sqsubseteq_1^n Q'$,

ii) if $P \sqsubseteq_2^n Q$, $P \subseteq P'$, $Q' \subseteq Q$ then $P' \sqsubseteq_2^n Q'$.

Using ii) we then obtain $(P_1 \times Q)/\lambda \sqsubseteq_2^n (P_2 \times Q)/\lambda$, whence $P_1 \times Q \sqsubseteq_2^{n+1} P_2 \times Q$ and we are done. □

3.2 Synchronous algebras

In this section we investigate the (in-) equational theories of processes under the testing preorders. They extend in the "may" and "must" cases the notion of $\mathbf{LL}^+$-frame by the addition of prefixing, and serve as frames for the modal relevant logics of processes to follow.

Definition 3.5 A *synchronous algebra* (over the action group $\mathcal{L}$) is any structure $\mathcal{A} = \langle S, \leq, 0, \oplus, 1, \cdot, \mathcal{L}\rangle$ where

i) $\langle S, \leq, 0\rangle$ is a poset with $0 \in S$ least,

ii) $\langle S, \oplus\rangle$ is a semilattice with $\oplus$ preserving $\leq$,

iii) $\langle S, \cdot, 1\rangle$ is a commutative monoid with $\cdot$ preserving $\oplus$, $\leq$ and 0, and

iv) Each $\lambda \in \mathcal{L}$ is an operator on S preserving $\leq$ and $\oplus$ s.t. for all $\lambda, \mu \in \mathcal{L}$ and $x, y \in S$,

   a) $\lambda(x) \oplus 0 = \lambda(x \oplus 0) \oplus 0$,

   b) $\lambda(x) \cdot \mu(y) = \lambda\mu(x \cdot y)$,

   c) $e(1) = 1$.

Due to the semilattice properties of the internal sum $\oplus$ we can use the "big $\oplus$" notation $\bigoplus$ for finite, nonempty sums. If in addition to the above properties $\oplus$ is the inf (sup) w.r.t. $\leq$ we say that $\mathcal{A}$ is a *must- (may-) algebra* and denote the $\oplus$ by $\sqcap$ ($\sqcup$). Notice that in both these cases equation iv.a) is redundant. Let $\mathcal{C}_3$ denote the class of all synchronous algebras and $\mathcal{C}_1$ ($\mathcal{C}_2$) the class of all may- (must-) algebras. In all three cases $\langle S, \oplus, \cdot, 1\rangle$ is an $\mathbf{LL}^+$-frame with 0 unit for $\oplus$ in the may-case and zero in the must-case.

Clearly $\mathcal{C}_3$ forms an inequational class, and $\mathcal{C}_1$ and $\mathcal{C}_2$ equational classes of algebras—in the latter cases define $\leq$ in the standard way by $x \leq y$ iff $x \oplus y = y$ for $\mathcal{C}_1$ and for $\mathcal{C}_2$ $x \leq y$ iff $x \oplus y = x$. Thus by standard universal algebra (c.f. [21, 44]) all these classes admit free algebras. For $i \in \{1, 2, 3\}$ let $\mathcal{F}_i$ denote the initial algebra in $\mathcal{C}_i$—i.e. the algebra freely generated by the empty set of generators.

3.2.1 The initial synchronous algebras

These initial algebras coincide (up to isomorphism) with quotients of $\mathrm{Pc}^{\oplus}$ by the appropriate testing preorders. This is not a result of great novelty. A number of authors have provided equational characterisations of process systems that are more expressive than ours (c.f. [51, 30]). On the other hand we have a specific use for our algebraic structures, and their operational justification is indispensable for the interest of subsequent results concerning logics for synchronous algebras. For these reasons we have generally referred proofs to the appendix.

The proof falls into two parts. In this section, we state representation theorems for the free algebras in terms of sets of "normal paths". These take the role of the *normal forms* usually appealed to, and we rely fairly heavily on them in the remainder of the chapter. These representations provide a *denotational semantics* for process terms. In the following section, the second half of the proof boils down to a proof that these semantics are *fully abstract*, i.e. that the orderings on the representations induce the corresponding testing preorders on process terms.

We consequently start by introducing the representations. In the may-case elements are represented by downwards closed sets of paths ordered by the *Hoare*-ordering, as it is known in the theory of powerdomains [91]. The must case and the general case is obtained by taking instead upwards/convex closure and the *Smyth/Egli–Milner* ordering respectively. Paths are built from strings $\overline{\lambda} = \lambda_1 \cdots \lambda_n \in \mathcal{L}^*$. Juxtaposition $\overline{\lambda}_1\overline{\lambda}_2$ denotes the concatenation of $\overline{\lambda}_1$ and $\overline{\lambda}_2$, $[\lambda]$, or if no confusion results just λ, denotes the singleton string containing λ and ε denotes the empty string.

Definition 3.6

i) A *path* is a pair $s = \langle \overline{\lambda}, j \rangle$ with $\overline{\lambda} \in \mathcal{L}^*$ and $j \in \{0, 1\}$.

ii) Paths are ordered by $\langle \overline{\lambda}_1, j_1 \rangle \leq \langle \overline{\lambda}_2, j_2 \rangle$ iff either

   a) $j_1 = j_2 = 1$ and either $\overline{\lambda}_1 = \overline{\lambda}_2[e]^n$ or $\overline{\lambda}_2 = \overline{\lambda}_1[e]^n$,

   b) $j_1 = j_2 = 0$ and $\overline{\lambda}_1$ is a prefix of $\overline{\lambda}_2$, or

   c) $j_1 = 0$, $j_2 = 1$ and $\overline{\lambda}_1$ is a prefix of $\overline{\lambda}_2[e]^n$ for some $n \geq 0$.

iii) A set S of paths is

   a) *lower*- (1-) closed if whenever $s \in S$ and $s' \leq s$ then $s' \in S$,

b) *upper* (2-) closed, if whenever $s \in S$ and $s \leq s'$ then $s' \in S$,

c) *convex* (3-) closed, if whenever $s_1, s_2 \in S$ and $s_1 \leq s \leq s_2$ then $s \in S$.

iv) The orderings $\leq_i$, $i \in \{1,2,3\}$ on sets of paths are defined by

a) $S_1 \leq_1 S_2$ iff for all $s_1 \in S_1$ there is an $s_2 \in S_2$ s.t. $s_1 \leq s_2$,

b) $S_1 \leq_2 S_2$ iff for all $s_2 \in S_2$ there is an $s_1 \in S_1$ s.t. $s_1 \leq s_2$,

c) $\leq_3 = \leq_1 \cap \leq_2$.

The path $\langle \varepsilon, 1\rangle$, in particular, represents the unit 1 for multiplication and thus satisfies $\langle \varepsilon, 1\rangle \leq \langle e, 1\rangle \leq \langle \varepsilon, 1\rangle$. The technical development is somewhat simplified if we pick out as canonical representatives paths $\langle \overline{\lambda}, j\rangle$ satisfying the condition that e is a suffix of $\overline{\lambda}$ only when $j = 0$. We call such a path *normal*, or an n-path, and let NP denote the set of all n-paths.

Associated with the notion of i-closure, $i \in \{1,2,3\}$, is the closure operator cl_i satisfying $\mathrm{cl}_i(S) = \bigcap\{S' \mid S \subseteq S' \text{ and } S' \text{ is } i\text{-closed}\}$. If $S = \mathrm{cl}_i(S')$ then S' *(i-)generates* S, and S is *(i-)finitely generated* (f.g.) if some finite S' generates S. Note that if S is i-f.g. then there is a least set $\mathrm{gen}_i(S)$ i-generating S. This is shown in appendix A. We use f.g., closed and nonempty sets of n-paths to build the free algebras.

We need next to define the operations of *path prefixing* and *multiplication*. Path prefixing is defined by $\lambda(\langle \overline{\lambda}, j\rangle) = \langle \lambda\overline{\lambda}, j\rangle$ whenever $\lambda \neq e$ or $\langle \overline{\lambda}, j\rangle \neq \langle \varepsilon, 1\rangle$ and $e(\langle \varepsilon, 1\rangle) = \langle \varepsilon, 1\rangle$. Multiplication is defined inductively by

$$\langle \lambda_1\overline{\lambda}_1, j_1\rangle \cdot \langle \lambda_2\overline{\lambda}_2, j_2\rangle = (\lambda_1\lambda_2)(\langle \overline{\lambda}_1, j_1\rangle \cdot \langle \overline{\lambda}_2, j_2\rangle),$$

$$\langle \overline{\lambda}_1, j_1\rangle \cdot \langle \varepsilon, 1\rangle = \langle \varepsilon, 1\rangle \cdot \langle \overline{\lambda}_1, j_1\rangle = \langle \overline{\lambda}_1, j_1\rangle,$$

$$\langle \overline{\lambda}_1, j_1\rangle \cdot \langle \varepsilon, 0\rangle = \langle \varepsilon, 0\rangle \cdot \langle \overline{\lambda}_1, j_1\rangle = \langle \varepsilon, 0\rangle.$$

The constants and operations on sets of paths is now given by

$$0_i = \mathrm{cl}_i(\langle \varepsilon, 0\rangle),$$

$$1_i = \mathrm{cl}_i(\langle \varepsilon, 1\rangle),$$

$$\lambda_i(S) = \mathrm{cl}_i\{\lambda(s) \mid s \in S\},$$

$$S_1 \oplus_i S_2 = \mathrm{cl}_i(S_1 \cup S_2),$$

$S_1 \cdot_i S_2 = \mathrm{cl}_i\{s_1 \cdot s_2 \mid s_1 \in S_1, s_2 \in S_2\}.$

Let then

$$D_i = \{S \mid S \text{ is an } i\text{-f.g. and nonempty set of normal paths}\},$$

and $\mathcal{D}_i$ denote the structure $\mathcal{D}_i = \langle D_i, \leq_i, 0_i, \oplus_i, 1_i, \cdot_i, \{\lambda_i \mid \lambda \in \mathcal{L}\}\rangle$. We then obtain the following representation result where $\cong$ denotes isomorphism of synchronous algebras.

Theorem 3.7 (Representation of the initial synchronous algebras). *For $i \in \{1,2,3\}$, $\mathcal{F}_i \cong \mathcal{D}_i$.*

Proof: See appendix A. □

3.2.2 The algebraic characterisation of processes

We then go on to show that processes modulo the testing preorders give yet another presentation of the initial algebras. Again most proofs are to be found in the appendix. Note that, as $\mathrm{Pc}^{\oplus}$ is the algebra of terms, or words, over the $\mathrm{Pc}^{\oplus}$-signature, for every algebra $\mathcal{A}$ also over that signature, there is a unique homomorphism $f_{\mathcal{A}} : \mathrm{Pc}^{\oplus} \to \mathcal{A}$. We let $[\![\cdot]\!]_i$ denote $f_{\mathcal{D}_i}$, for $i \in \{1,2,3\}$, and extend $[\![\cdot]\!]_i$ to finite, nonempty sets $P \subseteq \mathrm{Pc}^{\oplus}$ by $[\![P]\!]_i = \sum_i\{[\![p]\!]_i \mid p \in P\}$. Thus the mappings $[\![\cdot]\!]_i$ provides semantic mappings into $\mathcal{D}_i$. We use the $[\![\cdot]\!]_i$ to establish an isomorphism between $\mathrm{Pc}^{\oplus}/\sqsubseteq_i$ and $\mathcal{D}_i$, noting that $\mathrm{Pc}^{\oplus}/\sqsubseteq_i$ is well-defined due to theorem 3.4. The full abstraction of $[\![\cdot]\!]_i$, i.e. that $p \sqsubseteq_i q$ iff $[\![p]\!]_i \leq_i [\![q]\!]_i$ for all $p, q \in \mathrm{Pc}^{\oplus}$, is in fact sufficient to obtain this result.

Theorem 3.8 *If $[\![\cdot]\!]_i$ is fully abstract w.r.t. $\sqsubseteq_i$ then $\mathrm{Pc}^{\oplus}/\sqsubseteq_i \cong \mathcal{D}_i$.*

Proof: We establish a correspondence $f : \mathrm{Pc}^{\oplus}/\sqsubseteq_i \to \mathcal{D}_i$ by $f : [p]_{\simeq_i} \mapsto [\![p]\!]_i$ (where as usual $[p]_{\simeq_i}$ denotes the $\simeq_i$ congruence class of p). Because of (the only-if direction of) full abstraction, f is independent of the choice of representatives—i.e. whenever $p \simeq_i q$ then $f([p]_{\simeq_i}) = f([q]_{\simeq_i})$. Evidently f is surjective, and it is injective because of (the if-direction of) full abstraction. We thus only need to check that f is a homomorphism, but this is trivial by the definitions—for instance for prefixing we obtain $f((\lambda)_{\mathrm{Pc}^{\oplus}/\sqsubseteq_i}([p]_{\simeq_i})) = f([\lambda(p)]_{\simeq_i}) = [\![\lambda(p)]\!]_i = \lambda_i[\![p]\!]_i$. □

We prove full abstraction by mimicing the operational structure of processes on the algebras $\mathcal{D}_i$ themselves. Define $\{\xrightarrow{\lambda}\}_{\lambda \in \mathcal{L}}$ as the least family of relations s.t.

$\langle \varepsilon, 1\rangle \xrightarrow{e} \langle \varepsilon, 1\rangle$, and

$\langle \lambda\overline{\lambda}, j\rangle \xrightarrow{\lambda} \langle \overline{\lambda}, j\rangle$.

For a set $S \subseteq \mathrm{NP}$ we then let

$S\ \mathrm{can}^{\mathrm{D}}\ \lambda$ iff $s \xrightarrow{\lambda} s'$ for some $s \in S$ and $s' \in \mathrm{NP}$,

$S\ \mathrm{live}^{\mathrm{D}}$ iff for all $s \in S$, there is some $\lambda \in \mathcal{L}$ and $s' \in \mathrm{NP}$ s.t. $s \xrightarrow{\lambda} s'$,

$S\ \mathrm{must}^{\mathrm{D}}\ \Lambda$ iff for all $s \in S$ there is some $\lambda \in \Lambda$ s.t. $s \xrightarrow{\lambda} s'$ for some $s' \in \mathrm{NP}$,

$S/_{\mathrm{D}}\lambda = \{s' \mid \exists s \in S, s \xrightarrow{\lambda} s'\}$.

The operational structure on processes and their representations are related in the following way:

Lemma 3.9

i) $[\![p]\!]_i = \sum_i\{[\![p']\!]_i \mid p \twoheadrightarrow p'\}$,

ii) $P\ \mathrm{can}\ \lambda$ *iff* $[\![P]\!]_i\ \mathrm{can}^{\mathrm{D}}\ \lambda$, $i \in \{1,3\}$, *iff* $[\![P]\!]_2\ \mathrm{live}^{\mathrm{D}}$ *and* $[\![P]\!]_2\ \mathrm{can}^{\mathrm{D}}\ \lambda$,

iii) *For* $i \in \{2,3\}$, $P\ \mathrm{live}$ *iff* $[\![P]\!]_i\ \mathrm{live}^{\mathrm{D}}$,

iv) *For* $i \in \{2,3\}$, $P\ \mathrm{must}\ \Lambda$ *iff* $[\![P]\!]_i\ \mathrm{must}^{\mathrm{D}}\ \Lambda$,

v) *If* $P\ \mathrm{can}\ \lambda$ *then* $[\![P/\lambda]\!]_i = [\![P]\!]_i/_{\mathrm{D}}\lambda$.

PROOF: This is proved in appendix B. □

We can use def. 3.2 to adapt the testing preorders to the representations $\mathcal{D}_i$, $i \in \{1,2\}$, obtaining the maps F_i^D, and preorders $\preceq_i^n$ and $\preceq_i = \bigcap_{n\geq 0} \preceq_i^n$, the greatest fixed point of F_i^D. As usual we let $\preceq_3 = \preceq_1 \cap \preceq_2$. The full abstraction result is thus split in two. The next lemma states that the testing preorders matches the computationally motivated orderings on the representations:

Lemma 3.10 $P \sqsubseteq_i Q$ *iff* $[\![P]\!]_i \preceq_i [\![Q]\!]_i$.

PROOF: We show for $i \in \{1,2\}$ (this suffices) using induction on n that for all $n \geq 0$, $P \sqsubseteq_i^n Q$ iff $[\![P]\!]_i \preceq_i^n [\![Q]\!]_i$, from which the result follows. The base case is trivial, so let $n = n'+1$. Then':
$i = 1$: $P \sqsubseteq_1^n Q$ iff for all λ, if P can λ then Q can λ and $P/\lambda \sqsubseteq_1^{n'} Q/\lambda$ iff for all λ, if $[\![P]\!]_1\ \mathrm{can}^{\mathrm{D}}\ \lambda$ then $[\![Q]\!]_1\ \mathrm{can}^{\mathrm{D}}\ \lambda$ and $[\![P/\lambda]\!]_1 \preceq_i^{n'} [\![Q/\lambda]\!]_1$ (by 3.9.ii)

and the induction hypothesis) iff for all λ, if $[\![P]\!]_1$ can$^{\mathrm{D}}$ λ then $[\![Q]\!]_1$ can$^{\mathrm{D}}$ λ and $[\![P]\!]_1/_{\mathrm{D}}\lambda \preceq_i^{n'} [\![Q]\!]_1/_{\mathrm{D}}\lambda$ (by 3.9.v)) iff $[\![P]\!]_1 \preceq_i^n [\![Q]\!]_1$.
The case for $i = 2$ is completely similar. □

Additionally we need to verify that for each $i \in \{1, 2, 3\}$, the algebraic ordering $\leq_i$ coincides with the "computational" ordering $\preceq_i$.

Lemma 3.11 *For all $i \in \{1, 2, 3\}$, $\leq_i = \preceq_i$.*

Proof: See appendix B. □

We thus arrive at the desired conclusion—the characterisation of processes modulo the testing preorders as the initial synchronous algebras.

Corollary 3.12 (Algebraic characterisation theorem). *For all $i \in \{1, 2, 3\}$, $\mathrm{Pc}^{\oplus}/\sqsubseteq_i \cong \mathcal{D}_i \cong \mathcal{F}_i$.*

Proof: The statement follows, by 3.8, from the full abstraction of the $[\![\cdot]\!]_i$. But this follows from 3.10 and 3.11. □

3.3 Modal relevant logics of processes

In this section we present modal extensions of linear logic interpreted over may- and must-algebras. We add forwards and backwards indexed modalities to $\mathrm{Fm}^{+}_{\top,\perp}$ as well as a constant $\underline{0}$ whose interpretation is tied up with the constant 0 in the algebras. This choice of connectives allow us to give simple and elegant logical accounts of the structure of may- and must-algebras, in particular the interplay between the static operations of multiplication and internal choice, and prefixing, expressing the dynamic capabilities of processes.

Modalities have had some attention in relevance logic in general. We mentioned the logic **E** of relevant and strict implication in section 2.2.1. Sylvan and Meyer [98] added an **S4**-type modality to **R** and considered ternary relation models with an auxillary binary accessibility relation, and in the same spirit Chidgey [23] considered relevant tense logics. In linear logic an **S4**-type modality on which weakening and contraction is allowed is added, thus greatly increasing expressive power and making a simple translation from $\mathbf{H}^{+}$ possible. In comparison our aims are a lot more specialised: we are interested in frames with a very specific relationship between dynamic and static properties.

The language $\mathrm{Fm}^{+}_{\top,\bot}$ of formulas is extended by a new constant $\underline{0}$ and for each $\lambda \in \mathcal{L}$ unary operators (λ) and $\overline{(\lambda)}$. We denote this extension by $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$, the corresponding extension of Fm^{+} by $\mathrm{PFm}_{\underline{0}}$, and let PFm denote the sublanguage of all $\underline{0}$-free $\phi \in \mathrm{PFm}_{\underline{0}}$. (λ) is the indexed forwards modality and $\overline{(\lambda)}$ the indexed backwards, or reverse, modality. We use a neutral notation for these as they are highly nonstandard and as their computational interpretation is very different in the may- and must-cases. In fragments with $\bot$ negation is introduced by $\neg\phi =_{\mathrm{def}} \phi \to \bot$, and the constant (λ) by $(\lambda) =_{\mathrm{def}} (\lambda)\top$. We assume the modal operators (λ) and $\overline{(\lambda)}$ to have strongest binding power.

We first introduce the algebraic interpretations. These can be given uniformly by appealing to the converse of the ordering $\leq$ in the may-case. For this purpose let a *2-set* be a filter and a *1-set* the dual of a 2-set—i.e. an *ideal*: a set Δ s.t. for all x, y, $x, y \in \Delta$ iff $x \sqcup y \in \Delta$; let R_2 abbreviate $\leq$, R_1 abbreviate $\geq$, and let i range over the set $\{1,2\}$. An *i-model* (or *may- (must-) model* in the case $i = 1$ $(i = 2)$), now, is a pair $\mathcal{M} = \langle \mathcal{A}, V \rangle$ with $\mathcal{A} \in \boldsymbol{\mathcal{C}}_i$ and V a valuation s.t. for all $\alpha \in \mathrm{Ap}$, $V(\alpha)$ is an i-set in $\mathcal{A}$; let $\boldsymbol{\mathcal{M}}_i$ denote the class of all i-models. The satisfaction relation $\models^i_{\mathcal{M}} \subseteq \mathcal{A}_{\mathcal{M}} \times \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ for $\mathcal{M}$ an i-model is defined inductively by

$x \models^i_{\mathcal{M}} \alpha$ iff $x \in V(\alpha)$,

$x \models^i_{\mathcal{M}} \phi \to \psi$ iff for all $y \in \mathcal{A}$, if $y \models^i_{\mathcal{M}} \phi$ then $x \cdot y \models^i_{\mathcal{M}} \psi$,

$x \models^i_{\mathcal{M}} \phi \wedge \psi$ iff $x \models^i_{\mathcal{M}} \phi$ and $x \models^i_{\mathcal{M}} \psi$,

$x \models^i_{\mathcal{M}} \phi \vee \psi$ iff $x \models^i_{\mathcal{M}} \phi$ or $x \models^i_{\mathcal{M}} \psi$ or there are $x_1, x_2 \in \mathcal{A}$ s.t. $(x_1 \oplus x_2) R_i x$, $x_1 \models^i_{\mathcal{M}} \phi$ and $x_2 \models^i_{\mathcal{M}} \psi$,

$x \models^i_{\mathcal{M}} (\lambda)\phi$ iff there is an $x' \in \mathcal{A}$ s.t. $\lambda(x') R_i x$ and $x' \models^i_{\mathcal{M}} \phi$,

$x \models^i_{\mathcal{M}} \overline{(\lambda)}\phi$ iff $\lambda(x) \models^i_{\mathcal{M}} \phi$,

$x \models^i_{\mathcal{M}} \mathrm{t}$ iff $1 R_i x$,

$x \models^i_{\mathcal{M}} \underline{0}$ iff $0 R_i x$,

$x \models^i_{\mathcal{M}} \top$ for all $x \in \mathcal{A}$, and

$x \not\models^i_{\mathcal{M}} \bot$ for all $x \in \mathcal{A}$, The interpretation of non-modal formulas coincide with the interpretation given in chapter 2—dualised in the may-case. For the modalities, according to the intuition underpinning our general models an element x satisfies $(\lambda)\phi$ just in case it contains at least as much information as some $\lambda x'$ with x' satisfying ϕ, and x satisfies $\overline{(\lambda)}\phi$ if λx satisfies ϕ. The interpretation of $\underline{0}$ is tied up with the corresponding 0 in models. Thus, for $i = 1$, $x \models^i_{\mathcal{M}} \underline{0}$ iff $x = 0$ and for $i = 2$, $x \models^i_{\mathcal{M}} \underline{0}$ for all $x \in \mathcal{M}$; so $\underline{0}$ and $\top$ in this case are synonymous. With respect to notions of validity and entailment we define as in chapter 2:

Definition 3.13 (Entailment, validity).

i) $\phi \models^i_{\mathcal{M}} \psi$ iff for all $x \in \mathcal{M}$, if $x \models^i_{\mathcal{M}} \phi$ then $x \models^i_{\mathcal{M}} \psi$,

ii) $\phi \models^i \psi$ (ϕ *i-entails* ψ) iff for all $\mathcal{M} \in \boldsymbol{\mathcal{M}}_i$, $\phi \models^i_{\mathcal{M}} \psi$,

iii) $\models^i_{\mathcal{M}} \phi$ iff $1_{\mathcal{M}} \models^i_{\mathcal{M}} \phi$, and

iv) $\models^i \phi$ (ϕ *i-valid*) iff for all $\mathcal{M} \in \boldsymbol{\mathcal{M}}_i$, $\models^i_{\mathcal{M}} \phi$.

Note that, as in chapter 2, we obtain $\phi \models^i \psi$ iff $\models^i \phi \to \psi$, and t $\models^i \phi$ iff $\models^i \phi$. Generalising the filter property, let $[\![\phi]\!]^i_{\mathcal{M}} = \{x \in \mathcal{M} \mid x \models^i_{\mathcal{M}} \phi\}$. Then

Proposition 3.14 (The i-set property) *For all* $i \in \{1, 2\}$, $\mathcal{M} \in \boldsymbol{\mathcal{M}}_i$ *and* $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$, $[\![\phi]\!]^i_M$ *is an i-set.*

Proof: We need only extend the proof of 2.4 by considering the modal operators. For the forwards modalities $x_1, x_2 \models^i_{\mathcal{M}} (\lambda)\phi$ iff there are x_1', x_2' s.t. $\lambda(x_1')R_i x_1$, $\lambda(x_2')R_i x_2$, $x_1', x_2' \models^i_{\mathcal{M}} \phi$ iff there are x_1', x_2' s.t.

$$\lambda(x_1') \oplus \lambda(x_2') = \lambda(x_1' \oplus x_2')R_i(x_1 \oplus x_2)$$

and $x_1' \oplus x_2' \models^i_{\mathcal{M}} \phi$ (by the induction hypothesis) iff $x_1 \oplus x_2 \models^i_{\mathcal{M}} (\lambda)\phi$. For the reverse modalities $x_1, x_2 \models^i_{\mathcal{M}} \overline{(\lambda)}\phi$ iff $\lambda(x_1), \lambda(x_2) \models^i_{\mathcal{M}} \phi$ iff $\lambda(x_1) \oplus \lambda(x_2) \models^i_{\mathcal{M}} \phi$ (by the induction hypothesis) iff $\lambda(x_1 \oplus x_2) \models^i_{\mathcal{M}} \phi$ iff $x_1 \oplus x_2 \models^i_{\mathcal{M}} \phi$. □

3.4 Process interpretations

In this section we investigate the satisfaction relation on the initial algebras, and through those the induced satisfaction relation on processes. Notice first that for the initial algebras we can in many cases forget about the orderings R_i and use equality instead:

Proposition 3.15 *For $i \in \{1,2\}$ and all models $\mathcal{M}$ based on $\mathcal{D}_i$,*

i) $S \models^i_{\mathcal{M}} \phi \vee \psi$ iff $S \models^i_{\mathcal{M}} \phi$ or $S \models^i_{\mathcal{M}} \psi$ or there are $S_1, S_2 \in D_i$ s.t. $S = S_1 \oplus_i S_2$, $S_1 \models^i_{\mathcal{M}} \phi$ and $S_2 \models^i_{\mathcal{M}} \psi$.

ii) $S \models^2_{\mathcal{M}} (\lambda)\phi$ iff there is $S' \in D_2$ s.t. $S' \models^2_{\mathcal{M}} \phi$ and $S = (\lambda)_2(S')$,

iii) $S \models^2_{\mathcal{M}} \mathrm{t}$ iff $S = 1_2$.

Proof: i) The if-direction is trivial. Let $S \models^i_{\mathcal{M}} \phi \vee \psi$. If $S \models^i_{\mathcal{M}} \phi$ or $S \models^i_{\mathcal{M}} \psi$ we are done, so assume not and let $S_1 \oplus S_2 R_i S$, $S_1 \models^i_{\mathcal{M}} \phi$ and $S_2 \models^i_{\mathcal{M}} \psi$. Then $S \subseteq S_1 \cup S_2$. Now $S \cap S_1$, $S \cap S_2 \in D_i$, for if either of $S \cap S_1$, $S \cap S_2 \in D_i$ are empty then either $S \subseteq S_1$ or $S \subseteq S_2$ and then by 3.14 either $S \models^i_{\mathcal{M}} \phi$ or $S \models^i_{\mathcal{M}} \psi$. Also $S \cap S_1$, $S \cap S_2$ are i-closed and generated by $S \cap (\mathrm{gen}_i(S_1))$, $S \cap (\mathrm{gen}_i(S_2))$ respectively. But then by 3.14, $S \cap S_1 \models^i_{\mathcal{M}} \phi$, $S \cap S_2 \models^i_{\mathcal{M}} \psi$ and $S = (S \cap S_1) \cup (S \cap S_2)$, and we are done.
ii) Again the if-direction is trivial and we assume $S \models^2_{\mathcal{M}} (\lambda)\phi$. Then there is $S' \in D_2$ s.t. $S' \models^2_{\mathcal{M}} \phi$ and $S \subseteq (\lambda)_2(S')$. Let $S_1 = \{s \mid \lambda(s) \in S\}$. Then $S_1 \in D_2$ and $S = (\lambda)_2(S_1)$ and $S_1 \subseteq S'$. Then by 3.14 $S_1 \models^2_{\mathcal{M}} \phi$ and we are done.
iii) Assume $S \models^2_{\mathcal{M}} \mathrm{t}$. Then $S \subseteq 1_2 = \{\langle \varepsilon, 1\rangle\}$, so as $S \neq \emptyset$, $S = 1_2$. □

So for the must-interpretation for the initial algebras we can forget about the orderings completely and use only equality. Notice that for the initial algebras the models are distributive in the sense of chapter 2 (the proof of 3.15.i) shows this), whereas this of course does not hold for models in general. Thus any sound axiomatisation for algebras will be incomplete for the initial ones, in that for the latter $\wedge/\vee$–distribution will hold, but not so in general (see chapter 2). The complete axiomatisation for the initial algebras below provides lots of other examples.

We can use the structure of the initial algebras to further simplify the satisfaction condition for $\vee$. The notion of an element in a supsemilattice being *coprime* is just the dual of primeness in a semilattice (re. ch. 2). Now

Proposition 3.16 *For $i \in \{1,2\}$, $\mathcal{M}$ a model based on $\mathcal{D}_i$ and $S \in D_i$,*

i) S is coprime (w.r.t. $\subseteq$) iff $S = \mathrm{cl}_i\{s\}$ for some $s \in \mathrm{NP}$.

ii) $S \models^i_{\mathcal{M}} \phi$ iff for all coprime $S' \subseteq S$, $S' \models^i_{\mathcal{M}} \phi$,

iii) If S is coprime then $S \models^i_{\mathcal{M}} \phi \vee \psi$ iff $S \models^i_{\mathcal{M}} \phi$ or $S \models^i_{\mathcal{M}} \psi$.

PROOF: i) and iii) are immediate.
ii) Use 3.14.i) and the fact that all $S \in D_i$ are f.g. $\square$

We next show that the interpretations induce the ordering $\leq_i$ on the initial algebra representations. For this purpose we define the mappings r_i giving for each $S \in D_i$ a formula $r_i(S)$ representing S. First the mapping $r : \mathrm{NP} \to \mathrm{PFm}_{\underline{0}}$ is defined inductively by

$$r(\langle \varepsilon, 0\rangle) = \underline{0},$$

$$r(\langle \varepsilon, 1\rangle) = \mathrm{t},$$

$$r(\langle \lambda\overline{\lambda}, j\rangle) = (\lambda) r(\langle \overline{\lambda}, j\rangle).$$

We then extend r to maps r_i, $i \in \{1, 2\}$, by

$$r_i(S) = \bigvee\{r(s) \mid s \in \mathrm{gen}_i(S)\}.$$

Note that by the obvious associativity, commutativity and idempotency of $\vee$ w.r.t. $\models^i$, the use of the $\bigvee$-notation is justified. Note also that any $S \in D_i$ is nonempty, so r_i is well-defined. We can now show that the satisfaction relation for variable-free formulas induces the appropriate orderings on the initial algebras:

Theorem 3.17 *For $i \in \{1, 2\}$ and all models $\mathcal{M}$ based on $\mathcal{D}_i$ the following are equivalent:*

i) $S_2 \models^i_{\mathcal{M}} r_i(S_1)$,

ii) $S_1 R_i S_2$, and

iii) for all variable-free $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$, if $S_1 \models^i_{\mathcal{M}} \phi$ then $S_2 \models^i_{\mathcal{M}} \phi$,

PROOF: i) implies ii). Assume that $S_2 \models^i_{\mathcal{M}} r_i(S_1)$. This is the case, by 3.16, iff for all $s_2 \in \mathrm{gen}_i(S_2)$, $\mathrm{cl}_i\{s_2\} \models^i_{\mathcal{M}} r_i(S_1)$, which holds iff for all $s_2 \in \mathrm{gen}_i(S_2)$ there is some $s_1 \in \mathrm{gen}_i(S_1)$ s.t. $\mathrm{cl}_i\{s_2\} \models^i_{\mathcal{M}} r(s_1)$, by 3.16. We then just need to check that for $i = 1$, $\mathrm{cl}_1\{s_2\} \models^1_{\mathcal{M}} r(s_1)$ iff $s_2 \leq s_1$ and for $i = 2$ the dual, namely that $\mathrm{cl}_2\{s_2\} \models^2_{\mathcal{M}} r(s_1)$ iff $s_1 \leq s_2$.
We do this in both cases by induction on the length of s_1.
$i = 1$. If $s_1 = \langle \varepsilon, 0\rangle$ then $\mathrm{cl}_1\{s_2\} \models^1_{\mathcal{M}} r(s_1)$ iff $\mathrm{cl}_1\{s_2\} \models^1_{\mathcal{M}} \underline{0}$ iff $\mathrm{cl}_1\{s_2\} \subseteq 0_1$ iff $s_2 = s_1$.
If $s_1 = \langle \varepsilon, 1\rangle$ we obtain $\mathrm{cl}_1\{s_2\} \models^1_{\mathcal{M}} r(s_1)$ iff $\mathrm{cl}_1\{s_2\} \subseteq 1_1$ iff $s_2 \leq s_1$.
Finally if $s_1 = \langle \lambda\overline{\lambda}, j\rangle$ we obtain

$\mathrm{cl}_1\{s_2\} \models^1_{\mathcal{M}} r(s_1)$ iff

$\mathrm{cl}_1\{s_2\} \models^1_{\mathcal{M}} (\lambda)r(\langle\overline{\lambda}, j\rangle)$ iff

there is an $S_2' \in D_i$ s.t. $\mathrm{cl}_1\{s_2\} \leq (\lambda)_1(S_2')$ and $S_2' \models^1_{\mathcal{M}} r(\langle\overline{\lambda}, j\rangle)$ iff

there is an $S_2' \in D_i$ s.t. $\mathrm{cl}_1\{s_2\} \leq (\lambda)_1(S_2')$ and for all $s_2' \in S_2$, $\mathrm{cl}_1\{s_2\} \models^1_{\mathcal{M}} r(\langle\overline{\lambda}, j\rangle)$ (by 3.16) iff

there is an $S_2 \in D_i$ s.t. $\mathrm{cl}_1\{s_2\} \subseteq (\lambda)_1(S_2')$ and for all $s_2' \in S_2'$, $s_2' \leq \langle\overline{\lambda}, j\rangle$ by the induction hypothesis, iff

$s_2 \leq \langle\lambda\overline{\lambda}, j\rangle$.

$i = 2$. If $s_1 = \langle\varepsilon, 0\rangle$ then $\mathrm{cl}_2\{s_2\} \models^2_{\mathcal{M}} r(s_1)$ iff $\mathrm{cl}_2\{s_2\} \models^2_{\mathcal{M}} \underline{0}$ iff $s_1 \leq s_2$.
If $s_1 = \langle\varepsilon, 1\rangle$ then $\mathrm{cl}_2\{s_2\} \models^2_{\mathcal{M}} r(s_1)$ iff $\mathrm{cl}_2\{s_2\} \models^2_{\mathcal{M}} \mathrm{t}$ iff $1_2 \leq_2 \mathrm{cl}_2\{s_2\}$ iff $\langle\varepsilon, 1\rangle \leq s_2$.
So assume that $s_1 = \langle\lambda\overline{\lambda}, j\rangle$ and we obtain

$\mathrm{cl}_2\{s_2\} \models^2_{\mathcal{M}} r(s_1) = (\lambda)r(\langle\overline{\lambda}, j\rangle)$ iff

there is some $S_2' \in D_2$ s.t. $S_2' \models^2_{\mathcal{M}} r(\langle\overline{\lambda}, j\rangle)$ and $(\lambda)_2(S_2') \leq_2 \mathrm{cl}_2\{s_2\}$ iff

there is some $S_2' \in D_2$ s.t. $S_2' \models^2_{\mathcal{M}} r(\langle\overline{\lambda}, j\rangle)$ and $s_2 \in (\lambda)_2(S_2')$ iff

there is some $S_2 \in D_2$ s.t. $S_2' \models^2_{\mathcal{M}} r(\langle\overline{\lambda}, j\rangle)$ and $s_2 = \lambda(s_2')$ where $s_2' \in S_2'$ iff

$s_2 = \lambda(s_2')$ and $\mathrm{cl}_2\{s_2'\} \models^2_{\mathcal{M}} r(\langle\overline{\lambda}, j\rangle)$ by 3.16 iff

$s_2 = \lambda(s_2')$ and $\langle\overline{\lambda}, j\rangle \leq s_2'$ by the induction hypothesis iff

$s_1 \leq s_2$.

ii) implies iii). By 3.14.
iii) implies i). Show first by induction on the maximal length of paths in $\mathrm{gen}_i(S_1)$, that for all S_1, $S_1 \models^i_{\mathcal{M}} r(S_1)$—this is easy. Then the result follows by 3.14. □

Going on to consider process based interpretations we introduce, for the interpretation of disjunctions, the notion of traces of a process; this corresponds semantically to the notion of a path or a coprime set. For $p \in \mathrm{Pc}^{\oplus}$ the set $\mathrm{traces}(p)$ of *traces of* p is defined inductively by

$$\mathrm{traces}(0) = \{0\},$$

$$\mathrm{traces}(1) = \{1\},$$

$$\mathrm{traces}(\lambda(p)) = \{\lambda(q) \mid q \in \mathrm{traces}(p)\},$$

$$\mathrm{traces}(p \oplus q) = \mathrm{traces}(p) \cup \mathrm{traces}(q),$$

$$\mathrm{traces}(p \times q) = \{p_1 \times q_1 \mid p_1 \in \mathrm{traces}(p), q_1 \in \mathrm{traces}(q)\},$$

and for $P \subseteq \mathrm{Pc}^{\oplus}$ finite and nonempty, let $\mathrm{traces}(P) = \bigcup_{p \in P} \mathrm{traces}(p)$. Say a $p \in \mathrm{Pc}^{\oplus}$ is a *trace*, if $\mathrm{traces}(p) = \{p\}$. We verify that traces have the expected properties:

Lemma 3.18 *For $i \in \{1, 2, 3\}$,*

i) For all traces $p \in \mathrm{Pc}^{\oplus}$, $[\![p]\!]_i = \mathrm{cl}_i\{s\}$ for some $s \in \mathrm{NP}$,

ii) For all finite, nonempty $P \subseteq \mathrm{Pc}^{\oplus}$, $[\![P]\!]_i = [\![\mathrm{traces}(P)]\!]_i$.

PROOF: i) By induction on the structure of p.
ii) By 3.12 it suffices to show that $P \simeq_3 \sum(\mathrm{traces}(P))$. We show this by induction on the structure of P using the synchronous algebra equations. □

Through the algebraic characterisation of processes, satisfaction on the initial algebras for variable-free formulas induce a corresponding relation of satisfaction on the process terms, by letting $P \models^i_{\mathrm{P}} \phi$ iff $[\![P]\!]_i \models^i_{\mathrm{Pc}^{\oplus}/\sqsubseteq_i} \phi$ for all finite, nonempty $P \subseteq \mathrm{Pc}^{\oplus}$ and variable-free ϕ. We then arrive at the main result of this section, namely the characterisation of satisfaction on processes in operational/syntactical terms.

Theorem 3.19 *For all variable-free $\phi, \psi \in \mathrm{PFm}_{\{\underline{0}, \top, \bot\}}$:*

i) $P \models^i_{\mathrm{P}} \phi \vee \psi$ iff for all traces p of P, $p \models^i_{\mathrm{P}} \phi$ or $p \models^i_{\mathrm{P}} \psi$,

ii) $P \models^1_{\mathrm{P}} (\lambda)\phi$ iff for some Q, $Q \models^1_{\mathrm{P}} \phi$, and whenever P can μ then $\lambda = \mu$ and if P can λ then $P/\lambda \models^1_{\mathrm{P}} \phi$,

iii) $P \models^2_{\mathrm{P}} (\lambda)\phi$ iff P must $\{\lambda\}$ and $P/\lambda \models^2_{\mathrm{P}} \phi$,

iv) $P \models^1_{\mathrm{P}} \mathrm{t}$ iff whenever P can λ then $\lambda = e$ and if P can e then $P/e \models^1_{\mathrm{P}} \mathrm{t}$,

v) $P \models^2_{\mathrm{P}} \mathrm{t}$ iff P must $\{e\}$ and $P/e \models^2_{\mathrm{P}} \mathrm{t}$, *vi)* $P \models^1_{\mathrm{P}} \underline{0}$ *iff* P $\mathrm{can}\!\!\!/\,$ λ *for all* λ,

vii) $P \models^2_{\mathrm{P}} \underline{0}$ *for all* P.

Proof: i) We obtain

$P \models^i_{\mathrm{P}} \phi \vee \psi$ iff

$[\![P]\!]_i \models^i_{\mathrm{Pc}^{\oplus}/\sqsubseteq_i} \phi \vee \psi$ by 3.18 iff

for all $s \in [\![\mathrm{traces}(P)]\!]_i$, $\mathrm{cl}_i\{s\} \models^i_{\mathrm{Pc}^{\oplus}/\sqsubseteq_i} \phi$ or $\mathrm{cl}_i\{s\} \models^i_{\mathrm{Pc}^{\oplus}/\sqsubseteq_i} \psi$ by 3.16 iff

for all traces p of P, $[\![p]\!]_i \models^i_{\mathrm{Pc}^{\oplus}/\sqsubseteq_i} \phi$ or $[\![p]\!]_i \models^i_{\mathrm{Pc}^{\oplus}/\sqsubseteq_i} \psi$ iff

for all traces p of P, $p \models^i_{\mathrm{P}} \phi$ or $p \models^i_{\mathrm{P}} \psi$.

ii) We obtain

$P \models^1_{\mathrm{P}} (\lambda)\phi$ iff

$[\![P]\!]_1 \models^1_{\mathrm{Pc}^{\oplus}/\sqsubseteq_1} (\lambda)\phi$ iff

there is an $S_1 \in D_1$ s.t. $S_1 \models^1_{\mathrm{Pc}^{\oplus}/\sqsubseteq_1} \phi$ and $[\![P]\!]_1 \leq_1 (\lambda)_1(S_1)$ iff

there is an $S_1 \in D_1$ s.t. $S_1 \models^1_{\mathrm{Pc}^{\oplus}/\sqsubseteq_1} \phi$ and if $[\![P]\!]_1\ \mathrm{can}^{\mathrm{D}}\ \mu$ then $\lambda = \mu$ and if $[\![P]\!]_1\ \mathrm{can}^{\mathrm{D}}\ \lambda$ then $[\![P]\!]_1/_{\mathrm{D}}\lambda \leq_1 S_1$ iff

for some Q, $[\![Q]\!]_1 \models^1_{\mathrm{Pc}^{\oplus}/\sqsubseteq_1} \phi$ and if $[\![P]\!]_1\ \mathrm{can}^{\mathrm{D}}\ \mu$ then $\lambda = \mu$ and if $[\![P]\!]_1\ \mathrm{can}^{\mathrm{D}}\ \lambda$ then $[\![P/\lambda]\!]_1 \models^1_{\mathrm{Pc}^{\oplus}/\sqsubseteq_1} \phi$, by 3.14 iff

for some Q, $Q \models^1_{\mathrm{P}} \phi$, and whenever P can μ then $\lambda = \mu$ and if P can λ then $P/\lambda \models^1_{\mathrm{P}} \phi$.

iii)–vii) are similar. □

Note that a completely similar characterisation of satisfaction on the initial algebras $\mathcal{D}_i$ can be given, appealing to paths instead of traces. Note finally that these two interpretations induce the proper testing orderings on processes:

Corollary 3.20 *For* $i \in \{1,2\}$, $P \sqsubseteq' Q$ *iff for all variable-free* $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\perp\}}$, *if* $P \models^i_{\mathrm{P}} \phi$ *then* $Q \models^i_{\mathrm{P}} \phi$, *where if* $i = 1$, $\sqsubseteq' = \sqsupseteq_i$ *and if* $i = 2$, $\sqsubseteq' = \sqsubseteq_i$.

Proof: By 3.10 and 3.11, $P \sqsubseteq' Q$ iff $[\![P]\!]_i R_i [\![Q]\!]_i$ whence the conclusion follows by theorem 3.17. □

3.5 Axiomatising the may- and must-logics

In this section we exhibit a Hilbert-type axiomatisation for the fragment PFm without the logical constants $\underline{0}$, $\top$ and $\bot$ which is complete w.r.t validity in all models for both the may- and must-interpretations. That such an axiomatisation is at all possible is not so surprising in view of the fact that the only difference between (the dual of the) may-algebras and must-algebras is the role of 0.

3.5.1 Axiomatisation

The axiomatisation is an extension of positive linear logic, $\mathbf{LL}^+$, as presented in section 2.2. The logic **PL** over PFm is axiomatised by the the $\mathbf{LL}^+$ axioms and rules I (1), B (2), C (3), $\wedge$-intro (4), $\wedge$-elim-1 (5), $\wedge$-elim-2 (6), $\vee$-elim (7), $\vee$-intro-1 (8), $\vee$-intro-2 (9), the axiom

$$10 : \ \phi \leftrightarrow (\mathrm{t} \rightarrow \phi)$$

and the rules m.p. and adj. plus the axioms

$$11: \quad (\lambda)(\phi \vee \psi) \rightarrow (\lambda)\phi \vee (\lambda)\psi$$

$$12: \quad \overline{(\lambda)}\phi \wedge \overline{(\lambda)}\psi \rightarrow \overline{(\lambda)}(\phi \wedge \psi)$$

$$13: \quad \phi \rightarrow \overline{(\lambda)}((\lambda)\phi)$$

$$14: \quad (\lambda)(\overline{(\lambda)}\phi) \rightarrow \phi$$

$$15: \quad \overline{(\lambda)}((\mu)\phi \rightarrow \psi) \leftrightarrow (\phi \rightarrow \overline{(\lambda\mu)}\psi)$$

and the rules

(e)-nec.	Fromϕ infer $(e)\phi$
$\overline{(e)}$-nec.	From ϕ infer$\overline{(e)}\phi$
(λ)-mon.	From $\phi \rightarrow \psi$ infer $(\lambda)\phi \rightarrow (\lambda)\psi$
$\overline{(\lambda)}$-mon.	From $\phi \rightarrow \psi$ infer $\overline{(\lambda)}\phi \rightarrow \overline{(\lambda)}\psi$

Notice the $\Diamond$-like nature of (λ) and similarly the $\Box$-like nature of $\overline{(\lambda)}$, brought out by 11. and 12. The dual nature of (λ) and $\overline{(\lambda)}$ is brought out by 13. and 14.—note, however, that when we pass in the following sections to the initial algebras this duality will be broken up. Axiom 15 is the axiom accounting for the synchronous nature of the parallel composition. We have necessitation rules for (e) and $\overline{(e)}$ tied up with the interpretation of validity and the expected monotonicity properties for modal operators. We note a few theorems of **PL**:

Proposition 3.21

i) $\vdash_{\mathbf{PL}} (\lambda)(\phi \vee \psi) \leftrightarrow (\lambda)\phi \vee (\lambda)\psi$,

ii) $\vdash_{\mathbf{PL}} (\lambda)(\phi \wedge \psi) \rightarrow (\lambda)\phi \wedge (\lambda)\psi$,

iii) $\vdash_{\mathbf{PL}} \overline{(\lambda)}(\phi \wedge \psi) \leftrightarrow \overline{(\lambda)}\phi \wedge \overline{(\lambda)}\psi$,

iv) $\vdash_{\mathbf{PL}} \overline{(\lambda)}\phi \vee \overline{(\lambda)}\psi \rightarrow \overline{(\lambda)}(\phi \vee \psi)$,

v) $\vdash_{\mathbf{PL}} \mathrm{t} \leftrightarrow (e)\mathrm{t}$,

vi) $\vdash_{\mathbf{PL}} \mathrm{t} \rightarrow \overline{(e)}\mathrm{t}$,

vii) $\vdash_{\mathbf{PL}} (\phi \vee \psi \rightarrow \gamma) \leftrightarrow ((\phi \rightarrow \gamma) \wedge (\psi \rightarrow \gamma))$,

viii) $\vdash_{\mathbf{PL}} ((\phi \rightarrow \psi) \vee (\phi \rightarrow \gamma)) \rightarrow (\phi \rightarrow \psi \vee \gamma)$,

ix) $\vdash_{\mathbf{PL}} (\lambda^{-1}\mu)(\phi \rightarrow \psi) \rightarrow ((\lambda)\phi \rightarrow (\mu)\psi)$.

PROOF: The only difficult cases are v), vi) and ix).
v), vi) First:

$$\begin{array}{lll}
1. & \mathrm{t} \rightarrow \mathrm{t} & \text{Axiom 1} \\
2. & (\mathrm{t} \rightarrow \mathrm{t}) \rightarrow \mathrm{t} & \text{Axiom 10} \\
3. & \mathrm{t} & \text{1,2,m.p.} \\
4. & \overline{(e)}\mathrm{t} & 3,\overline{(e)}\text{-nec.} \\
5. & \overline{(e)}\mathrm{t} \rightarrow (\mathrm{t} \rightarrow \overline{(e)}\mathrm{t}) & \text{axiom 10} \\
6. & \mathrm{t} \rightarrow \overline{(e)}\mathrm{t} & \text{4,5,m.p.}
\end{array}$$

Hence $\vdash_{\mathbf{PL}} \mathrm{t} \rightarrow \overline{(e)}\mathrm{t}$ and similarly $\vdash_{\mathbf{PL}} \mathrm{t} \rightarrow (e)\mathrm{t}$ by (e)-nec. in step 4. Now:

$$\begin{array}{lll}
7. & (e)\mathrm{t} \rightarrow (e)\overline{(e)}\mathrm{t} & 6,(e)\text{-mon} \\
8. & (e)\overline{(e)}\mathrm{t} \rightarrow \mathrm{t} & \text{Axiom 14} \\
9. & (((e)\mathrm{t} \rightarrow (e)\overline{(e)}\mathrm{t})) \rightarrow ((e)\mathrm{t} \rightarrow \mathrm{t}) & \text{8,axiom 2,m.p.} \\
10. & (e)\mathrm{t} \rightarrow \mathrm{t} & \text{7,9,m.p.}
\end{array}$$

Thus $\vdash_{\mathbf{PL}} (e)\mathrm{t} \rightarrow \mathrm{t}$, and then $\vdash_{\mathbf{PL}} (e)\mathrm{t} \leftrightarrow \mathrm{t}$ by adj.
ix) By 13 and transitivity we obtain $\vdash_{\mathbf{PL}} (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \overline{(\mu)}(\mu)\psi)$.
Now $\mu = \lambda^{-1}\mu\lambda$ so by 15, $\vdash_{\mathbf{PL}} (\phi \rightarrow \overline{(\mu)}(\mu)\psi) \rightarrow \overline{(\lambda^{-1}\mu)}((\lambda)\phi \rightarrow (\mu)\psi)$
and then by transitivity and m.p. $\vdash_{\mathbf{PL}} (\phi \rightarrow \psi) \rightarrow \overline{(\lambda^{-1}\mu)}((\lambda)\phi \rightarrow (\mu)\psi)$
and then by (λ)-mon. $\vdash_{\mathbf{PL}} (\lambda^{-1}\mu)(\phi \rightarrow \psi) \rightarrow (\lambda^{-1}\mu)\overline{(\lambda^{-1}\mu)}((\lambda)\phi \rightarrow (\mu)\psi)$
hence by 14, transitivity and m.p. $\vdash_{\mathbf{PL}} (\lambda^{-1}\lambda)(\phi \rightarrow \psi) \rightarrow ((\lambda)\phi \rightarrow (\mu)\psi)$. □

The converse of vi) $\overline{(e)}\mathrm{t} \rightarrow \mathrm{t}$, turns out to be unsound. For $\mathbf{PL}'$ any extension of $\mathbf{PL}$ the equivalence relation $\equiv_{\mathbf{L}}$ on formulas is given by $\phi \equiv_{\mathbf{L}} \psi$ iff $\vdash_{\mathbf{L}} \phi \leftrightarrow \psi$.

Proposition 3.22 (Replacement). *For any logic* $\mathbf{PL}'$ *over* $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ *extending* $\mathbf{PL}$, $\equiv_{\mathbf{PL}'}$ *is a congruence.*

PROOF: By the results of chapter 2 it is clear that $\equiv_{\mathbf{PL}'}$ is preserved under $\rightarrow$, $\wedge$ and $\vee$. The preservation of $\equiv_{\mathbf{PL}'}$ under (λ) and $\overline{(\lambda)}$ follow by (λ)-mon and $\overline{(\lambda)}$-mon. □

3.5.2 Soundness

For the soundness and completeness theorems we refer the parts concerning the $\{\rightarrow, \wedge, \vee, \mathrm{t}\}$-fragment of $\mathbf{PL}$ to the proofs of the relevant theorems in chapter 2.

Theorem 3.23 (Soundness of $\mathbf{PL}$). *For all* $\phi \in \mathrm{PFm}$, *if* $\vdash_{\mathbf{PL}} \phi$ *then* $\models^1 \phi$ *and* $\models^2 \phi$.

PROOF: As usual we show all axioms to be valid and all rules to preserve validity. For the $\{\rightarrow, \wedge, \vee, \mathrm{t}\}$-fragment refer to chapter 2. For the new axioms and rules we take only 11 and 15—the rest are equally straightforward:
11. Suppose $x \models^i_{\mathcal{M}} (\lambda)(\phi \vee \psi)$ with $\mathcal{M} \in \boldsymbol{\mathcal{M}}_i$. Then there is an x' s.t. $x' \models^i_{\mathcal{M}} \phi \vee \psi$ and $\lambda(x')R_i x$. If $x' \models^i_{\mathcal{M}} \phi$ or $x' \models^i_{\mathcal{M}} \psi$ then we are done. Otherwise let $x'_1 \oplus x'_2 R_i x'$, $x'_1 \models^i_{\mathcal{M}} \phi$ and $x'_2 \models^i_{\mathcal{M}} \psi$. Then $\lambda(x'_1) \models^i_{\mathcal{M}} (\lambda)\phi$ and $\lambda(x'_2) \models^i_{\mathcal{M}} (\lambda)\psi$ so $\lambda(x'_1) \oplus \lambda(x'_2) \models^i_{\mathcal{M}} (\lambda)\phi \vee (\lambda)\psi$ and then $x \models^i_{\mathcal{M}} (\lambda)\phi \vee (\lambda)\psi$ by theorem 3.14 as $\lambda(x'_1) \oplus \lambda(x'_2) = \lambda(x'_1 \oplus x'_2)R_i\lambda(x')R_i x$.
15. Suppose first that $x \models^i_{\mathcal{M}} \overline{(\lambda)}((\mu)\phi \rightarrow \psi)$. Then $\lambda(x) \models^i_{\mathcal{M}} (\mu)\phi \rightarrow \psi$. Let $y \models^i_{\mathcal{M}} \phi$. Then $\mu(y) \models^i_{\mathcal{M}} (\mu)\phi$ and then

$$\lambda(x) \cdot \mu(y) = (\lambda\mu)(x \cdot y) \models^i_{\mathcal{M}} \psi.$$

Thus $x \cdot y \models^i_{\mathcal{M}} \overline{(\lambda\mu)}\psi$ as was to be shown. Next assume that $x \models^i_{\mathcal{M}} \phi \rightarrow \overline{(\lambda\mu)}\psi$. Let $y \models^i_{\mathcal{M}} (\mu)\phi$. Then there is a y' s.t. $y' \models^i_{\mathcal{M}} \phi$ and $\lambda(y')R_i y$. We obtain $x \cdot y' \models^i_{\mathcal{M}} \overline{(\lambda\mu)}\psi$; i.e. $(\lambda\mu)(x \cdot y') \models^i_{\mathcal{M}} \psi$. Now

$$(\lambda\mu)(x \cdot y') = \lambda(x) \cdot \mu(y')R_i\lambda(x) \cdot y$$

thus $\lambda(x) \cdot y \models^i_{\mathcal{M}} \psi$ by 3.14 as desired. □

3.5.3 Completeness

We go on then to prove completeness by extending the completeness proofs of section 2.4 and 2.5.

Theorem 3.24 (Completeness of **PL**). *For all* $\phi \in \mathrm{PFm}$ *and* $i \in \{1,2\}$*, if* $\models^i \phi$ *then* $\vdash_{\mathbf{PL}} \phi$*.*

PROOF: We extend the canonical model construction of section 2.4. Recall that there a **PL**-*theory* $\nabla \in \mathrm{th}(\mathbf{PL})$ is a set ∇ s.t. whenever $\phi \in \nabla$ and $\vdash_{\mathbf{PL}} \phi \to \psi$ then $\psi \in \nabla$ and if $\phi, \psi \in \nabla$ then $\phi \wedge \psi \in \nabla$. We first turn to the case for $i = 1$ and define a may-model built out of *nonempty* theories. Let then

$$\mathrm{C}_1 = \langle\langle \mathrm{th}(\mathbf{PL}) \setminus \{\emptyset\}, \leq_1, 0_1, \oplus_1, \cdot, 1, \mathcal{L}\rangle, V\rangle,$$

where

i) $\nabla_1 \leq_1 \nabla_2$ iff $\nabla_2 \subseteq \nabla_1$,

ii) $0_1 = \mathrm{PFm}$,

iii) $\oplus_1 = \cap$,

iv) $\nabla_1 \cdot \nabla_2 = \{\psi \in \mathrm{PFm} \mid \exists \phi \in \nabla_2 . \phi \to \psi \in \nabla_1\}$,

v) $1 = \{\phi \mid \vdash_{\mathbf{PL}} \phi\}$,

vi) $\lambda(\nabla) = \{\phi \mid \overline{(\lambda)}\phi \in \nabla\}$,

vii) $V(\alpha) = \{\nabla \mid \alpha \in \nabla\}$.

It is not hard to check that the operations are indeed well-defined. For $\oplus_1$, let ∇_1, ∇_2 be nonempty **PL**-theories and let $\phi \in \nabla_1$, $\psi \in \nabla_2$. Then $\phi \vee \psi \in \nabla_1 \cap \nabla_2$, so $\nabla_1 \oplus_1 \nabla_2$ is indeed nonempty (in addition to being a **PL**-theory).
For $\cdot$ we saw $\nabla_1 \cdot \nabla_2$ to be a theory in section 2.4. To see that $\nabla_1 \cdot \nabla_2$ is nonempty if ∇_1, ∇_2 are let $\phi \in \nabla_1$, $\psi \in \nabla_2$. By

$$\vdash_{\mathbf{PL}} \phi \to (\psi \to ((\phi \to (\psi \to \gamma)) \to \gamma))$$

we obtain

$$\psi \to ((\phi \to (\psi \to \gamma)) \to \gamma) \in \nabla_1$$

and then

$$((\phi \to (\psi \to \gamma)) \to \gamma) \in \nabla_1 \cdot \nabla_2,$$

so $\nabla_1 \cdot \nabla_2 \neq \emptyset$.

For $\lambda \in \mathcal{L}$, let ∇ be a nonempty **PL**-theory and assume that $\phi \in \lambda(\nabla)$ and $\vdash_{\mathbf{PL}} \phi \to \psi$. Then $\overline{(\lambda)}\phi \in \nabla$ and by $\overline{(\lambda)}$-mon and the theory property of ∇, $\overline{(\lambda)}\psi \in \nabla$, so $\psi \in \lambda(\nabla)$ as desired. Also if $\phi, \psi \in \lambda(\nabla)$ then $\overline{(\lambda)}\phi, \overline{(\lambda)}\psi \in \nabla$ so $\overline{(\lambda)}\phi \wedge \overline{(\lambda)}\psi \in \nabla$ and then by 12, $\overline{(\lambda)}(\phi \wedge \psi) \in \nabla$. But then $\phi \wedge \psi \in \lambda(\nabla)$.

Note that $\lambda(\nabla) = \mathrm{th}\{(\lambda)\phi \mid \phi \in \nabla\}$. For $\subseteq$ assume that $\phi \in \lambda(\nabla)$, i.e. $\overline{(\lambda)}\phi \in \nabla$. Then $(\lambda)\overline{(\lambda)}\phi \in \mathrm{th}\{(\lambda)\phi \mid \phi \in \nabla\}$ and by 14. we obtain $\phi \in \mathrm{th}\{(\lambda)\phi \mid \phi \in \nabla\}$. For the converse inclusion assume $n \geq 1$ and $\phi_1, \ldots \phi_n \in \nabla$ and $\vdash_{\mathbf{PL}} (\lambda)\phi_1 \wedge \cdots \wedge (\lambda)\phi_n \to \psi$ i.e. $\psi \in \mathrm{th}\{(\lambda)\phi \mid \phi \in \nabla\}$. Now

$$\vdash_{\mathbf{PL}} (\lambda)(\phi_1 \wedge \cdots \wedge \phi_n) \to (\lambda)\phi_1 \wedge \cdots \wedge (\lambda)\phi_n$$

by $\wedge$-elim-1/-2, (λ)-mon, $\wedge$-intro and m.p., so $\vdash_{\mathbf{PL}} (\lambda)(\phi_1 \wedge \cdots \wedge \phi_n) \to \psi$, moreover $\phi_1 \wedge \cdots \wedge \phi_n \in \nabla$. By $\overline{(\lambda)}$-mon,

$$\vdash_{\mathbf{PL}} \overline{(\lambda)}(\lambda)(\phi_1 \wedge \cdots \wedge \phi_n) \to \overline{(\lambda)}\psi.$$

But then $\overline{(\lambda)}\psi \in \nabla$ and then $\psi \in \lambda(\nabla)$ as desired. Now inhabitation of $\lambda(\nabla)$ follows, for if $\phi \in \nabla$ then $(\lambda)\phi \in \lambda(\nabla)$.

We need then to check that C_1 indeed forms a model.

Certainly $\langle \mathrm{th}(\mathbf{PL}) \setminus \{\emptyset\}, \leq_1, 0_1\rangle$ forms a poset with 0_1 least and $\oplus_1$ the sup w.r.t. $\leq_1$.

For the check that $\langle \mathrm{th}(\mathbf{PL}) \setminus \{\emptyset\}, \cdot, 1\rangle$ forms a commutative monoid with $\cdot$ preserving $\oplus_1$ see lemmas 2.9 and 2.13 of chapter 2. To see that $\cdot$ preserves 0_1 let $\phi \in \mathrm{PFm}$ and ∇ be a nonempty **PL**-theory, and we must show $\phi \in \nabla \cdot \mathrm{PFm}$. Now let $\psi \in \nabla$ be arbitrary—ψ exists. Then $\psi \to \phi \in \mathrm{PFm}$ and then $\phi \in \nabla \cdot \mathrm{PFm}$—notice it is here we need theories to be nonempty.

So it only remains to check the properties relating to actions $\lambda \in \mathcal{L}$. For preservation of $\oplus_1$ we obtain $\phi \in (\lambda)(\nabla_1 \oplus_1 \nabla_2)$ iff $(\lambda)\phi \in \nabla_1$ and $(\lambda)\phi \in \nabla_2$ iff $\phi \in (\lambda)(\nabla_1) \oplus (\lambda)(\nabla_2)$.

For the synchronous law assume first that $\psi \in (\lambda)(\nabla_1) \cdot (\mu)(\nabla_2)$. Then for some $\phi \in (\mu)(\nabla_2)$, $\phi \to \psi \in (\lambda)(\nabla_1)$. Then $(\lambda) \in \nabla_2$. By 14, transitivity and m.p. we obtain

$$\vdash_{\mathbf{PL}} (\phi \to \psi) \to ((\mu)\overline{(\mu)}\phi \to \psi),$$

so $(\mu)\overline{(\mu)}\phi \to \psi \in (\lambda)(\nabla_1)$, thus $\overline{(\lambda)}((\mu)\overline{(\mu)}\phi \to \psi) \in \nabla_1$. Then by 15, $\overline{(\lambda)}\phi \to \overline{(\lambda\mu)}\psi \in \nabla_1$ as well, thus $\overline{(\lambda\mu)}\psi \in \nabla_1 \cdot \nabla_2$. But then $\psi \in (\lambda\mu)(\nabla_1 \cdot \nabla_2)$ as desired. For the converse inclusion, assume that this holds, thus $\overline{(\lambda\mu)}\psi \in \nabla_1 \cdot \nabla_2$. Then for some $\phi \in \nabla_2$ does $\phi \to \overline{(\lambda\mu)}\psi \in \nabla_1$, and then by 15, $\overline{(\lambda)}((\mu)\phi \to \psi) \in \nabla_1$ as well, and then $(\mu)\phi \to \psi \in (\lambda)(\nabla_1)$. Also $(\mu)\phi \in (\mu)(\nabla_2)$ as we saw above and thus $\psi \in (\lambda)(\nabla_1) \cdot (\mu)(\nabla_2)$ as wanted.

Finally to check that $(e)(1) = 1$ we obtain $\phi \in (e)(1)$ iff $\overline{(e)}\phi \in 1$ iff $\vdash_{\mathbf{PL}} \mathrm{t} \to \overline{(e)}\phi$ iff $\vdash_{\mathbf{PL}} \overline{(e)}\phi$. Then by (e)-nec, $\vdash_{\mathbf{PL}} (e)\overline{(e)}\phi$ so by 14, $\vdash_{\mathbf{PL}} \phi$, i.e. $\phi \in 1$. Conversely if $\phi \in 1$ then $\vdash_{\mathbf{PL}} \phi$, thus $\vdash_{\mathbf{PL}} \overline{(e)}\phi$ by $\overline{(e)}$-nec and we are done.

We have thus completed the proof that the canonical model C_1 is indeed a model. We then need to show that for all $\nabla \in \mathrm{C}_1$, $\nabla \models^1_{\mathrm{C}_1} \phi$ iff $\phi \in \nabla$. We proceed by induction on the structure of ϕ. For the cases of ϕ not an outer-level occurrence of a modal operator we refer to the corresponding cases in the proof of 2.10. For the rest:

$\phi = (\lambda)\phi'$. If $\nabla \models^1 (\lambda)\phi'$ then there is some nonempty $\mathbf{PL}$-theory ∇' s.t. $\nabla' \models^1 \phi'$ and $(\lambda)(\nabla') \geq_1 \nabla$. By the induction hypothesis $\phi' \in \nabla'$ thus $(\lambda)\phi' \in (\lambda)(\nabla')$ by the above observation and then $(\lambda)\phi' \in \nabla$. Conversely, if $(\lambda)\phi' \in \nabla$ then $(\lambda)(\mathrm{th}\{\phi'\}) \geq_1 \nabla$ and by the induction hypothesis, $\mathrm{th}\{\phi'\} \models^1 \phi'$. Thus $\nabla \models^1 (\lambda)\phi'$.

$\phi = \overline{(\lambda)}\phi'$. If $\nabla \models^1 \overline{(\lambda)}\phi'$ then $(\lambda)(\nabla) \models^1 \phi'$ and by the induction hypothesis, $\phi' \in (\lambda)(\nabla)$, thus $\overline{(\lambda)}\phi' \in \nabla$. Conversely, if $\overline{(\lambda)}\phi' \in \nabla$ then $\phi' \in (\lambda)(\nabla)$ and by the induction hypothesis $(\lambda)(\nabla) \models^1 \phi'$—i.e. $\nabla \models^1 \overline{(\lambda)}\phi'$.

Now the proof for $i = 1$ is complete, for if $\not\vdash_{\mathbf{PL}} \phi$ then $\phi \notin 1$, thus $1 \not\models^1 \phi$.

The proof for $i = 2$ is very similar, and if anything simpler in that we do not have to require theories to be nonempty. Here define

$$\mathrm{C}_2 = \langle\langle \mathrm{th}(\mathbf{PL}), \subseteq, \emptyset, \cap, \cdot, 1, \mathcal{L}\rangle, V\rangle$$

where $\cdot$, 1, $(\cdot)$ and V are as for C_1. We have already seen the operations to be well-defined. Further that $\langle \mathrm{th}(\mathbf{PL}), \subseteq, \emptyset\rangle$ is a poset with $\emptyset$ least and $\cap$ the inf is trivial, and so is the preservation of $\emptyset$ by $\cdot$. All the other properties have already been checked for $i = 1$ and the proof that $\nabla \models^2 \phi$ iff $\phi \in \nabla$ is identical to the proof for $i = 1$, and thus the result follows. □

We can show **PL** to be a conservative extension of positive linear logic by embedding the class of general models of chapter 2 into the class of must-models in a way that preserves satisfaction.

Theorem 3.25 **PL** *is a conservative extension of* $\mathbf{LL}^+$*.*

PROOF: If $\phi \in \mathrm{Fm}^+$ and $\not\vdash_{\mathbf{LL}^+} \phi$ then we find a linear model $\mathcal{M}$ s.t. $1_{\mathcal{M}} \not\models \phi$, by 2.14. Moreover we may assume $\mathcal{M}$ to be *pointed*—i.e. contain an element 0 s.t. $0 \leq x$ for all $x \in \mathcal{M}$ and moreover s.t. for all $x \in \mathcal{M}$, $x \cdot 0 = 0$. This is seen by reference to the canonical model construction in which $\emptyset$ has this property, exactly as in the proof of 3.24 for $i = 2$. We can then turn $\mathcal{M}$ into a 2-model, $\mathcal{M}'$, by simply defining $(\lambda)(x) = x$ for all $\lambda \in \mathcal{L}$, and then it is a matter of a trivial induction on the structure of ϕ to check that then for all $x \in \mathcal{M}$, $x \models_{\mathcal{M}} \phi$ iff $x \models^2_{\mathcal{M}'} \phi$. But then $1_{\mathcal{M}'} \not\models^2_{\mathcal{M}'} \phi$ and then by 3.23 $\not\vdash_{\mathbf{PL}} \phi$ and we are done. □

3.6 Adding extensional constants

Let us then turn to the issue of adding the constants $\underline{0}$, $\top$ and $\bot$ to the axiomatisation. The basic problem is that these constants together with the implication increases expressiveness in rather a pathological way. For instance for $\bot$ we obtain (remember we defined $\neg$ by $\rightarrow$ and $\bot$)

$$x \models^i_{\mathcal{M}} \neg\phi \text{ iff for all } y \in \mathcal{M}, y \not\models^i_{\mathcal{M}} \phi,$$

and similarly

$$x \models^i_{\mathcal{M}} \neg\neg\phi \text{ iff for some } y \in \mathcal{M}, y \models^i_{\mathcal{M}} \phi.$$

This means that we are not in general going to obtain completeness by a canonical model construction. For let C' be such a canonical model and $1_{\mathrm{C}'} = \{\phi \mid \vdash_{\mathbf{PL}'} \phi\}$ the set of all theorems in a suitable axiomatisation $\mathbf{PL}'$, and suppose further that $1_{\mathbf{PL}'}$ has the for the purpose of completeness essential property that $1_{\mathbf{PL}'} \models_{\mathrm{C}'} \phi$ iff $\phi \in 1_{\mathrm{C}'}$ iff $\vdash_{\mathbf{PL}'} \phi$. Then we obtain for all ϕ, that either $\vdash_{\mathbf{PL}'} \neg\phi$ or $\vdash_{\mathbf{PL}'} \neg\neg\phi$. But for this to be sound it must be the case for all ϕ that either for all models $\mathcal{M}$ there are no $x \in \mathcal{M}$ s.t. $x \models_{\mathcal{M}} \phi$ or for all models $\mathcal{M}$ there is some $x \in \mathcal{M}$ with $x \models_{\mathcal{M}} \phi$—or in other words: If ϕ is satisfiable in some model then ϕ is satisfiable in all models. This is a very strong property that certainly fail for our model classes. Indeed it—apart from its intrinsic interest due to its operational motivation—is another reason for our interest in axiomatising the logics resulting from restricting attention to only the initial algebras, as there we have a good handle on which formulas are satisfiable and which are not. Nonetheless it may be possible to get quite far without being quite so restrictive. This is in particular true for the may-interpretation due to the fact that for all $\mathcal{M} \in \boldsymbol{\mathcal{M}}_1$, $0_{\mathcal{M}} \models^1_{\mathcal{M}} \phi$ iff for some $x \in \mathcal{M}$, $x \models^1_{\mathcal{M}} \phi$. In view of this it might be possible to achieve the desired property (i.e. that satisfiability in one model entails satisfiability in all models) anyhow because of the algebraic structure of models. The essential culprit is the reverse modalities:

Theorem 3.26 *Say a model* $\mathcal{M} \in \boldsymbol{\mathcal{M}}_1$ *is* OK, *if for all* $\alpha \in \mathrm{Ap}$, $V(\alpha) \neq \emptyset$. *For a model* $\mathcal{M} \in \boldsymbol{\mathcal{M}}_1$ *say a* $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ *is* ($\mathcal{M}$-)satisfiable *if* $x \models^1_{\mathcal{M}} \phi$ *for some* $x \in \mathcal{M}$ *and say* ϕ *is* OK-satisfiable, *if* ϕ *is* $\mathcal{M}$*-satisfiable for some* OK $\mathcal{M}$. *Then for all* $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ *that does not contain reverse modalities, i.e. does not contain any subformula of the form* $\overline{(\lambda)}\phi'$, *if* ϕ *is* OK*-satisfiable then for all* OK $\mathcal{M} \in \boldsymbol{\mathcal{M}}_1$, ϕ *is* $\mathcal{M}$*-satisfiable.*

Proof: It suffices to show that if $0_{\mathcal{M}} \models^1_{\mathcal{M}} \phi$ for some OK $\mathcal{M}$ then $0_{\mathcal{M}'} \models^1_{\mathcal{M}'} \phi$ for all OK $\mathcal{M}'$. This is proved by induction on the structure of ϕ, assuming ϕ does not contain reverse modalities, using the i-set property, proposition 3.14. □

On the other hand it is clear that the addition of reverse modalities makes the induction in this proof fail. Moreover, for the must-interpretation the situation is more difficult in that we there have neither a top-element corresponding to the $\underline{0}$ in may-models nor a sup, and it is therefore hard to see how a must-correlate of theorem 3.26 could be made to go through.

3.6.1 Zero in the may-interpretation

We can nevertheless get quite far, in particular for the may-case, even though completeness for the full model classes is out of reach. First for the $\underline{0}$ in the may-interpretation we consider the logic $\mathbf{PL}^1_{\underline{0}}$ over $\mathrm{PFm}_{\underline{0}}$ axiomatised by the **PL** axioms and rules plus the four axioms:

16. $\underline{0} \to \mathrm{t}$
17. $(\underline{0} \to (\phi \to \psi)) \to ((\underline{0} \to \phi) \to (\underline{0} \to \psi))$
18. $(\underline{0} \to \psi) \to (\phi \to (\underline{0} \to \psi))$
19. $(\underline{0} \to \overline{(\lambda)}\phi) \to (\underline{0} \to \phi)$

The purpose of 16, 17 (an instance of S) and 19 is to capture the fact that in any 1-model $\mathcal{M}$, $0_{\mathcal{M}}$ is greatest w.r.t. the inverse of $\leq_{\mathcal{M}}$, and 18 (an instance of K) captures the preservation of $0_{\mathcal{M}}$ under $\cdot$. Note that in the fragment without reverse modalities and $\bot$ we could replace 16–19 by simply the axiom $\underline{0} \to \phi$.

Theorem 3.27 (Soundness of $\mathbf{PL}^1_{\underline{0}}$). *For all $\phi \in \mathrm{PFm}_{\underline{0}}$, if $\vdash_{\mathbf{PL}^1_{\underline{0}}} \phi$ then $\models^1 \phi$.*

PROOF: Note first that the proof of theorem 3.23 goes through in the enriched language. For the new axioms:
16. If $x \models^1_{\mathcal{M}} \underline{0}$ then $x = 0 \leq 1$ so $x \models^1_{\mathcal{M}} \mathrm{t}$.
17. Let $x \models^1_{\mathcal{M}} \underline{0} \to (\phi \to \psi)$, $y \models^1_{\mathcal{M}} \underline{0} \to \phi$ and $z \models^1_{\mathcal{M}} \underline{0}$. Then $z = 0$ so $x \cdot z = 0 \models^1_{\mathcal{M}} \phi \to \psi$ and $y \cdot z = 0 \models^1_{\mathcal{M}} \phi$, hence $x \cdot y \cdot z = 0 \models^1_{\mathcal{M}} \psi$ as was to be shown.
18. Assume $x \models^1_{\mathcal{M}} \underline{0} \to \phi$ and let y be arbitrary. Let $z \models^1_{\mathcal{M}} \underline{0}$—i.e. $z = 0$. Then $x \cdot z = x \cdot y \cdot z = 0 \models^1_{\mathcal{M}} \phi$.
19. Assume $x \models^1_{\mathcal{M}} \underline{0} \to \overline{(\lambda)}\phi$ and let $y \models^1_{\mathcal{M}} \underline{0}$—i.e. $y = 0$. Then $x \cdot y = 0 \models^1_{\mathcal{M}} \overline{(\lambda)}\phi$, so $\lambda(0) \models^1_{\mathcal{M}} \phi$. But $0 \leq \lambda(0)$ and then $0 \models^1_{\mathcal{M}} \phi$. □

Using 16–19 we can even get a modified canonical model construction to go through. For this purpose let

$$\boldsymbol{\nabla}_0 = \{\nabla \in \mathrm{th}(\mathbf{PL}^1_{\underline{0}}) \mid \nabla \neq \emptyset, \nabla \subseteq \mathrm{th}_{\mathbf{PL}^1_{\underline{0}}}\{\underline{0}\}\},$$

and then $\mathrm{C}'_1 = \langle\langle \boldsymbol{\nabla}_0, \leq_1, 0_1, \oplus_1, \cdot, 1, (\cdot)\rangle, V\rangle$ where $0_1 = \mathrm{th}_{\mathbf{PL}^1_{\underline{0}}}\{\underline{0}\}$ and the rest of the operations $\leq_1$, $\oplus_1$, $\cdot$ etc. are as for C_1 in the proof of theorem 3.24 (except that the range of V is restricted to $\boldsymbol{\nabla}_0$).

It is not too hard to check that C'_1 is indeed a 1-model. We need first to check the operations etc. to be well-defined. This is trivial for 0_1 and $\oplus_1$. Let $\nabla_1, \nabla_2 \in \boldsymbol{\nabla}_0$ and we show $\nabla_1 \cdot \nabla_2 \in \boldsymbol{\nabla}_0$. We have already seen $\nabla_1 \cdot \nabla_2$ to be nonempty. So let $\psi \in \nabla_1 \cdot \nabla_2$ and we show $\psi \in 0_1$. For some $\phi \in \nabla_2$, $\phi \to \psi \in \nabla_1$. Now $\phi, \phi \to \psi \in 0_1$ so $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to (\phi \to \psi)$ and $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to \phi$ and then by 17, $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to \psi$, i.e. $\psi \in 0_1$. For 1 suppose $\vdash_{\mathbf{PL}^1_{\underline{0}}} \phi$. Then $\vdash_{\mathbf{PL}^1_{\underline{0}}} \mathrm{t} \to \phi$ by 10 and then by 16 we obtain $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to \phi$, i.e. $\phi \in 0_1$. For (λ) assume $\nabla \in \boldsymbol{\nabla}_0$ and $\phi \in (\lambda)(\nabla)$, i.e. $\overline{(\lambda)}\phi \in \nabla$. Then $\overline{(\lambda)}\phi \in 0_1$ thus $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to \overline{(\lambda)}\phi$, but then $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to \phi$ by 19 and thus $\phi \in 0_1$.

The remainder of the proof that C'_1 indeed forms a 1-model is identical to the corresponding part of the proof of theorem 3.24—except for those properties relating to 0_1. The only nontrivial among these is the preservation of 0_1 under multiplication. So let $\nabla \in \boldsymbol{\nabla}_0$ and we must show $\nabla \cdot 0_1 = 0_1$. We have already shown $\subseteq$, and for the converse assume that $\phi \in 0_1$—i.e. that $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to \phi$. Then, by 18, $\vdash_{\mathbf{PL}^1_{\underline{0}}} \psi \to (\underline{0} \to \phi)$ for all $\psi \in \mathrm{PFm}_{\underline{0}}$. Now ∇ is nonempty so $\psi \in \nabla$ for some ψ, so $\underline{0} \to \phi \in \nabla$. But $\underline{0} \in 0_1$ and thus $\phi \in \nabla \cdot 0_1$.

But we cannot use this construction to prove completeness as the essential property $\phi \in \nabla$ iff $\nabla \models^1 \phi$ fails to hold in general. For if $\not\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to \phi$ ($\phi = \overline{(\lambda)\underline{0}}$ is an example) then it might be the case that for some $\nabla \in \boldsymbol{\nabla}_0$ and $\psi \in \mathrm{PFm}_{\underline{0}}$, $\nabla \models^1 \phi \to \psi$ and $\phi \to \psi \notin \nabla$.

3.6.2 Extensional truth- and falsehood

Let us then consider adding $\top$ and $\bot$. The logic $\mathbf{PL}_{\{\underline{0},\top,\bot\}}$ over $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ is axiomatised by the **PL** axioms and rules plus axioms 20, 21 below. Similarly, $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$ is the logic over $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ axiomatised by the $\mathbf{PL}^1_{\underline{0}}$ axioms and rules plus axioms 20–23:

20. $\phi \to \top$

21. $\bot \to \phi$

22. $\neg\neg\underline{0}$

23. $(\neg\neg\phi) \to (\underline{0} \to \phi)$

Note that in the may-case, $\neg\neg\phi$ and $\underline{0} \to \phi$ are synonymous. Axioms 22 and 23 are added in order to enable us to prove this. Of course in the must-interpretation, 22 is redundant; this does not seem to be the case, however, for the may-interpretation. The soundness of these axioms is not hard to check:

Theorem 3.28 (Soundness of $\mathbf{PL}_{\{\underline{0},\top,\bot\}}$ and $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$). *Let* $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$.

i) If $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \phi$ *then* $\models^1 \phi$ *and* $\models^2 \phi$.

ii) If $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \phi$ *then* $\models^1 \phi$.

PROOF: Straightforward. □

Proofs in $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$ are often quite hard to find, due to the possibly roundabout way in which proofs of $\underline{0}$-free theorems may depend on intermediate steps involving the $\underline{0}$. Thus $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$ does not seem to be a conservative extension of **PL** plus axioms 20 and 21—although we again have not yet proved this. We note a number of theorems and derived rules of the various extensions of **PL** we have introduced, partly to illustrate this point and partly for later use.

Theorem 3.29

i) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} (\phi \to \neg\psi) \to (\psi \to \neg\phi)$

ii) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \phi \to \neg\neg\phi$

iii) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \neg\neg\neg\phi \to \neg\phi$

iv) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \top \leftrightarrow (\bot \to \bot)$

v) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \phi \to (\bot \to \psi)$

vi) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \neg\phi \to (\psi \to \neg\phi)$

vii) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \phi \to \overline{(\lambda)}\top$

viii) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} (\lambda)\bot \to \phi$

ix) $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \top \to (\phi \to \top)$

x) If $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \top \to \phi$ *and* $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \phi \wedge \psi \to \gamma$ *then* $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \psi \to \gamma$

xi) If $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \neg\neg\phi$ *and* $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} ((\neg\neg\phi) \wedge \psi) \to \gamma$ *then* $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \psi \to \gamma$

xii) $\vdash_{\mathbf{PL}^1_{\underline{0}}} \underline{0} \to (\lambda)\underline{0}$

xiii) $\vdash_{\mathbf{PL}^1_{\underline{0}}} \phi \to (\underline{0} \to \phi)$

xiv) $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\underline{0} \to \phi) \leftrightarrow (\neg\neg\phi)$

xv) $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\overline{(\lambda)}\bot$

xvi) $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\neg\phi) \leftrightarrow (\neg(\lambda)\phi)$

xvii) $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\neg\phi) \to (\neg\overline{(\lambda)}\phi)$

PROOF: Most of the cases are completely straightforward. We prove only the cases that are not.
vii) By 13, $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \phi \to \overline{(\lambda)}(\lambda)\phi$ and by 20, $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} (\lambda)\phi \to \top$ so by $\overline{(\lambda)}$-mon, $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \overline{(\lambda)}(\lambda)\phi \to \overline{(\lambda)}\top$ and thus by transitivity, $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \phi \to \overline{(\lambda)}\top$.
viii) Similar. By 14, $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} (\lambda)\overline{(\lambda)}\phi \to \phi$ so by 21 and (λ)-mon $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} (\lambda)\bot \to (\lambda)\overline{(\lambda)}\phi$ and then $\vdash_{\mathbf{PL}_{\{\underline{0},\top,\bot\}}} \overline{(\lambda)}\bot \to \phi$.
xv) By 19, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\underline{0} \to \overline{(\lambda)}\bot) \to (\underline{0} \to \bot)$ so by 22, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\underline{0} \to \overline{(\lambda)}) \to \bot$. Now by xi) $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\neg\overline{(\lambda)}\bot \to (\underline{0} \to \overline{(\lambda)}\bot)$ and thus by transitivity, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\neg\neg\overline{(\lambda)}\bot$ hence by iii), $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\overline{(\lambda)}\bot$.

xvi) First for $\rightarrow$: By 21, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \bot \rightarrow \overline{(\lambda)}\bot$ and then by reflexivity and transitivity, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\phi \rightarrow (\phi \rightarrow \overline{(\lambda)}\bot)$. Now by 15, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\phi \rightarrow \overline{(\lambda)}\bot) \rightarrow \overline{(e)}((\lambda)\phi \rightarrow \bot)$ and thus by transitivity, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\phi \rightarrow \overline{(e)}((\lambda)\phi \rightarrow \bot)$. Then by 16 and permutation, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\phi \rightarrow (\bot \rightarrow \overline{(e)}((\lambda)\phi \rightarrow \bot)$ whence by 19, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\phi \rightarrow (\underline{0} \rightarrow ((\lambda)\phi \rightarrow \bot))$. But then by xi), iii) and transitivity we are done. For the converse implication, by 13 and transitivity we obtain $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\underline{0} \rightarrow \phi) \rightarrow (\underline{0} \rightarrow \overline{(\lambda)}(\lambda)\phi)$, and then by 19, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\underline{0} \rightarrow \phi) \rightarrow (\underline{0} \rightarrow (\lambda)\phi)$. By xi) we then obtain $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\neg\phi \rightarrow \neg\neg(\lambda)\phi$ and by i), $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg(\lambda)\phi \rightarrow \neg\neg\neg\phi$ and then the result obtains by iii).

xvii) By 19, $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} (\underline{0} \rightarrow \overline{(\lambda)}\phi) \rightarrow (\underline{0} \rightarrow \phi)$ so by xi) $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\neg\overline{(\lambda)}\phi \rightarrow \neg\neg\phi$, thus $\vdash_{\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}} \neg\phi \rightarrow \neg\overline{(\lambda)}\phi$ by i) and iii). □

3.7 The initial algebra interpretations

We thus go on to investigate what happens when the class of models is restricted to just the initial algebras, and only variable-free formulas are considered. Recall from 3.20 that this fragment is sufficiently expressive to induce the may- and must testing preorders respectively on processes.

3.7.1 Axiomatisation

The initial algebras differ from the canonical ones constructed in the previous section in quite a number of respects. To name but a few examples, note that in $\mathcal{D}_1$,

i) whenever $\lambda(S_1) \leq \mu(S_2)$ then $\lambda = \mu$,

ii) whenever $S_1 \cdot S_2 = 0$ then $S_1 = 0$ or $S_2 = 0$,

iii) for no S and λ does $\lambda(S) \leq 0$,

etc. etc. In axiomatising the logics with the range of interpretations restricted to the initial algebras, account must be taken of all these properties, and it should consequently not be surprising that a fair number of new axioms will have to be added.

The problems we have to deal with in the may- and must-cases are in principle the same, although the details of course are not. In particular it must be noted that a number of the theorems of the $\underline{0}$-free fragment of $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$ will be valid also in the general must-interpretation, but as their proof in $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$ depends on intermediate steps involving the $\underline{0}$ which are not generally valid in the must-interpretation, it will not suffice here to add only axioms characteristic for the initial algebra interpretation. Indeed one of our main reasons for introducing the intermediate systems $\mathbf{PL}^1_{\underline{0}}$, $\mathbf{PL}_{\{\underline{0},\top,\bot\}}$ and $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$ was just to illustrate this point.

For the may-case the logic $\mathbf{PL}^{1,D}_{\underline{0},\top,\bot}$—abbreviated as $\mathbf{PL}^{1,D}$—over variable-free formulas in $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ is axiomatised by the axioms and rules of $\mathbf{PL}^1_{\{\underline{0},\top,\bot\}}$ plus the axioms 24–33 below:

24. $\phi \wedge (\psi \vee \gamma) \rightarrow (\phi \wedge \psi) \vee \gamma$
25. $(\lambda)\phi \wedge (\lambda)\psi \rightarrow (\lambda)(\phi \wedge \psi)$
26. $(\lambda)\phi \wedge (\mu)\psi \rightarrow \underline{0}$ (whenever $\lambda \neq \mu$)
27. $\overline{(\lambda)}(\phi \vee \psi) \rightarrow \overline{(\lambda)}\phi \vee \overline{(\lambda)}\psi$
28. $\overline{(\lambda)}(\lambda)\phi \rightarrow \phi$
29. $\neg\overline{(\lambda)}(\mu)\phi$ (whenever $\lambda \neq \mu$)
30. $((\lambda)\phi \rightarrow \bigvee_{\mu\in\Lambda}(\mu)\psi_\mu) \rightarrow \bigvee_{\mu\in\Lambda}((\lambda)\phi \rightarrow (\mu)\psi_\mu)$
31. $(\neg\neg\phi \wedge \neg\neg\psi \wedge ((\lambda)\phi \rightarrow (\mu)\psi)) \rightarrow (\lambda^{-1}\mu)(\phi \rightarrow \psi)$
32. $(\top \rightarrow \bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda) \rightarrow \underline{0}$
33. $(\neg\neg\phi \wedge ((\lambda)\phi \rightarrow \underline{0})) \rightarrow \underline{0}$

For the must-case, $\mathbf{PL}^{2,D}_{\underline{0},\top,\bot}$—abbreviated as $\mathbf{PL}^{2,D}$—is the logic over variable-free $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ axiomatised by the axioms and rules of $\mathbf{PL}_{\{\underline{0},\top,\bot\}}$ plus axioms 24, 28, 29, 30 together with the following axioms 34–38:

34. $(\phi \wedge (\lambda)) \rightarrow (\lambda)\overline{(\lambda)}\phi$
35. $\neg\phi \leftrightarrow \neg(\lambda)\phi$
36. $(\neg\neg\phi \wedge ((\lambda)\phi \rightarrow (\mu)\psi)) \rightarrow (\lambda^{-1}\mu)(\phi \rightarrow \psi)$
37. $\top \rightarrow \underline{0}$
38. $\neg(\top \rightarrow \bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda)$

Recall that we assume Λ to range over finite, nonempty subsets of $\mathcal{L}$. Also, as we have already pointed out, the "big $\vee$" notation of axiom schemas 30, 32 and 38 are clearly admissible.

Axiom 24 should be no surprise in view of theorem 3.16.ii.b). Note that this axiom marks a departure from linear logic. Many of the other axioms are strengthenings of corresponding axioms or theorems encountered earlier. This goes for 25 and 27 which gives converses to theorem 3.21.ii) and iv) respectively. Axiom 28 gives a converse to axiom 13 (and to 3.21.vi) and axiom 31 gives a partial converse to 3.21.ix). Axioms 26 and 29 captures the properties of distinctly prefixed elements in the initial may-algebra. Axiom 26, for instance, captures the fact that if $S \leq \lambda(S')$ and $S \leq \mu(S'')$ with $\lambda \neq \mu$ then $S = 0$, and 29 the property that for no $S, S' \in D_1$ can it be the case that $\lambda(S) \leq \mu(S')$ if $\lambda \neq \mu$. Axiom 30 provides a kind of partial primeness property of formulas of the form $(\lambda)\phi$, and finally axioms 32 and 33 expresses the particular properties of 0 in the may-case.

For the must-case, axiom 36 in particular is just a minor strengthening of axiom 31. Axiom 37 (with axiom 20) induces the expected collapse of $\top$ and $\underline{0}$. Note also that axiom 35 is identical to theorem 3.29.xvi).

These axiomatisation may be greatly compacted, and we have not investigated possible redundancies—these can certainly not be ruled out. We note a number of theorems for later use, first of $\mathbf{PL}^{1,D}$.

Proposition 3.30

i) $\vdash_{\mathbf{PL}^{1,D}} (\lambda)(\phi \wedge \psi) \leftrightarrow (\lambda)\phi \wedge (\lambda)\psi$

ii) $\vdash_{\mathbf{PL}^{1,D}} \overline{(\lambda)}(\phi \vee \psi) \leftrightarrow \overline{(\lambda)}\phi \vee \overline{(\lambda)}\psi$

iii) $\vdash_{\mathbf{PL}^{1,D}} \phi \leftrightarrow \overline{(\lambda)}(\lambda)\phi$

iv) $\vdash_{\mathbf{PL}^{1,D}} \mathrm{t} \leftrightarrow \overline{(e)}\mathrm{t}$

v) $\vdash_{\mathbf{PL}^{1,D}} \bigwedge_{i \in I}(\bigvee_{j \in J} \phi_{i,j}) \leftrightarrow \bigvee_{f:I \to J}(\bigwedge_{i \in I} \phi_{i,f(i)})$, *for I and J finite*

vi) $\vdash_{\mathbf{PL}^{1,D}} \neg\overline{(\lambda)}\underline{0}$, *whenever $card(\mathcal{L}) \geq 1$*

vii) $\vdash_{\mathbf{PL}^{1,D}} (\top \to \underline{0}) \to \underline{0}$

viii) $\vdash_{\mathbf{PL}^{1,D}} (\top \to \mathrm{t}) \to \underline{0}$

ix) $\vdash_{\mathbf{PL}^{1,D}} (\mathrm{t} \to \underline{0}) \to \underline{0}$

PROOF: Most cases again are straightforward, or in the case of v) a standard generalisation of distributivity. We prove iv), vi) and ix).
iv) $\rightarrow$ by 3.21.vi). By 3.21.v) we obtain $\vdash_{\mathbf{PL}^{1,D}} \mathrm{t} \rightarrow (e)\mathrm{t}$ thus $\vdash_{\mathbf{PL}^{1,D}} \overline{(e)}\mathrm{t} \rightarrow \overline{(e)}(e)\mathrm{t}$ by (λ)-mon, whence $\vdash_{\mathbf{PL}^{1,D}} \overline{(e)}\mathrm{t} \rightarrow \mathrm{t}$ by 28.
vi) Assume $card(\mathcal{L}) \geq 1$. Then there is some $\mu \in \mathcal{L}$ s.t. $\lambda \neq \mu$. By 3.29.xii), $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \rightarrow (\mu)\underline{0}$. Then by $\overline{(\lambda)}$-mon, $\vdash_{\mathbf{PL}^{1,D}} \overline{(\lambda)}\underline{0} \rightarrow \overline{(\lambda)}(\mu)\underline{0}$, and then the result follows by 29 and transitivity.
ix) By 16, $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \rightarrow \mathrm{t}$, thus $\vdash_{\mathbf{PL}^{1,D}} \neg\neg\mathrm{t}$ by 3.29.xiv). Then by 3.29.xi) and 33, $\vdash_{\mathbf{PL}^{1,D}} ((e)\mathrm{t} \rightarrow \underline{0}) \rightarrow \underline{0}$ and then by 3.21.v), $\vdash_{\mathbf{PL}^{1,D}} (\mathrm{t} \rightarrow \underline{0}) \rightarrow \underline{0}$. □

Next a number of theorems of $\mathbf{PL}^{2,D}$:

Proposition 3.31

i) $\vdash_{\mathbf{PL}^{2,D}} \phi \leftrightarrow \overline{(\lambda)}(\lambda)\phi$

ii) $\vdash_{\mathbf{PL}^{2,D}} \phi \wedge (\lambda) \leftrightarrow (\lambda)\overline{(\lambda)}\phi$

iii) $\vdash_{\mathbf{PL}^{2,D}} \overline{(\lambda)}(\lambda)\phi \wedge (\lambda) \leftrightarrow (\lambda)\overline{(\lambda)}\phi$

iv) $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \leftrightarrow (\lambda)\overline{(\lambda)}(\lambda)\phi$

v) $\vdash_{\mathbf{PL}^{2,D}} \neg\phi \rightarrow \neg\overline{(\lambda)}\phi$

vi) $\vdash_{\mathbf{PL}^{2,D}} \neg\overline{(\lambda)}\phi \rightarrow \neg(\phi \wedge (\lambda))$

vii) $\vdash_{\mathbf{PL}^{2,D}} \neg\overline{(\lambda)}\bot$

viii) $\vdash_{\mathbf{PL}^{2,D}} (\lambda)(\phi \wedge \psi) \leftrightarrow (\lambda)\phi \wedge (\lambda)\psi$

ix) $\vdash_{\mathbf{PL}^{2,D}} \overline{(\lambda)}(\phi \vee \psi) \leftrightarrow \overline{(\lambda)}\phi \vee \overline{(\lambda)}\psi$

x) $\vdash_{\mathbf{PL}^{2,D}} \neg((\lambda)\phi \wedge (\mu)\psi)$, *whenever* $\lambda \neq \mu$

xi) $\vdash_{\mathbf{PL}^{2,D}} \neg(\top \rightarrow \mathrm{t})$

xii) $\vdash_{\mathbf{PL}^{2,D}} \mathrm{t} \leftrightarrow \overline{(e)}\mathrm{t}$

xiii) $\vdash_{\mathbf{PL}^{2,D}} \bigwedge_{i\in I}(\bigvee_{j\in J} \phi_{i,j}) \leftrightarrow \bigvee_{f:I\rightarrow J}(\bigwedge_{i\in I} \phi_{i,f(i)})$, *for* I *and* J *finite*

PROOF: We prove cases ii), iii), viii) and x).
ii) The $\rightarrow$-direction is axiom 34. For the converse use axioms 14 and 20, (λ)-mon and $\wedge$-intro.
iii) By axiom 28 and standard reasoning we get $\vdash_{\mathbf{PL}^{2,D}} \overline{(\lambda)}(\lambda)\phi \wedge (\lambda) \rightarrow \phi \wedge (\lambda)$ and then by ii) the $\rightarrow$-direction obtains. For the converse use ii) and axiom 13.
viii) By 20 and standard reasoning we obtain

$$\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\lambda)\psi \rightarrow (\lambda)\phi \wedge (\lambda)\psi \wedge (\lambda)$$

and hence by 34 $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\lambda)\psi \rightarrow (\lambda)\overline{(\lambda)}((\lambda)\phi \wedge (\lambda)\psi)$
whence by 3.21.iii)$\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\lambda)\psi \rightarrow (\lambda)(\overline{(\lambda)}(\lambda)\phi \wedge \overline{(\lambda)}(\lambda)\psi)$ consequently by 28 $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\lambda)\psi \rightarrow (\lambda)(\phi \wedge \psi)$. The converse follows by 3.21.i).
x) We obtain first $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\mu)\psi \rightarrow (\lambda)\phi \wedge (\mu)\psi \wedge (\lambda)$
thus by 34, $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\mu)\psi \rightarrow (\lambda)\overline{(\lambda)}((\lambda)\phi \wedge (\mu)\psi)$
and then by 3.21.iii), $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\mu)\psi \rightarrow (\lambda)(\overline{(\lambda)}(\lambda)\phi \wedge \overline{(\lambda)}(\mu)\psi)$
whence by 29 assuming $\lambda \neq \mu$, $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\mu)\psi \rightarrow (\lambda)(\overline{(\lambda)}(\lambda)\phi \wedge \bot)$
and hence $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\mu)\psi \rightarrow (\lambda)\bot$.
Now $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\bot \rightarrow \bot$ by 35 and $\vdash_{\mathbf{PL}^{2,D}} \neg\bot$, so $\vdash_{\mathbf{PL}^{2,D}} (\lambda)\phi \wedge (\mu)\psi \rightarrow \bot$ and we are done. $\square$

3.7.2 Soundness

In fact $\mathbf{PL}^{1,D}$ is unsound in general—the problem being that axiom 32 will only be sound when $\not\models^1_{\mathcal{D}_1} \top \rightarrow \bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda$, for instance because $\Lambda \neq \mathcal{L}$. Remedying this problem will yield an excessively complicated axiom system; hence in this case we shall simply assume the label universe $\mathcal{L}$ to be countably infinite.

Theorem 3.32 (Soundness of $\mathbf{PL}^{1,D}/\mathbf{PL}^{2,D}$).

i) Let $\mathcal{L}$ be countably infinite and ϕ be any variable-free formula in $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$. *Then $\vdash_{\mathbf{PL}^{1,D}} \phi$ only if $\models^1_{\mathcal{D}_1} \phi$.*

ii) For all variable-free $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$, if $\vdash_{\mathbf{PL}^{2,D}} \phi$ then $\models^2_{\mathcal{D}_2} \phi$.

PROOF: $\mathbf{PL}^{1,D}$: We need only check the new axioms 24–33.
24. By 3.16.iii).
25. Let $S \models^1 (\lambda)\phi \wedge (\lambda)\psi$ (we suppress subscripting of $\models^1$). Then there are $S_1, S_2 \in D_1$ s.t. $S_1 \models^1 \phi$, $S_2 \models^1 \psi$, $S \leq_1 \lambda(S_1)$, $S \leq_1 \lambda(S_2)$. Then $0 \models^1 \phi \wedge \psi$ by

3.14, so if $S = 0$ also $S \models^1 (\lambda)(\phi \wedge \psi)$, as then $S \leq_1 \lambda(0)$. Otherwise $S = \lambda(S')$ for some S' with $S' \leq_1 S_1$ and $S' \leq_1 S_2$. But then $S' \models^1 \phi \wedge \psi$ by 3.14 and we are done.

26. Suppose $S \models^1 (\lambda)\phi \wedge (\mu)\psi$ with $\lambda \neq \mu$. If $S = 0$ we are done. Otherwise $S = \lambda(S_1) = \mu(S_2)$ for some S_1, S_2—a contradiction.

27. Suppose $S \models^1 (\lambda)(\phi \vee \psi)$ i.e. $\lambda(S) \models^1 \phi \vee \psi$. Let now $s \in \lambda(S)$—i.e. $s = \lambda(s')$ for some $s' \in S'$. By 3.16.iii), $\mathrm{cl}_1\{s\} \models^1 \phi$ or $\mathrm{cl}_1\{s\} \models^1 \psi$. Let

$$S_\gamma = \mathrm{cl}_1\{s \mid s \in S \text{ and } \mathrm{cl}_1\{s\} \models^1 \gamma\},$$

where $\gamma \in \{\phi, \psi\}$. Then by 3.16.ii), $\lambda(S_\phi) \models^1 \phi$, $\lambda(S_\psi) \models^1 \psi$ and $S = S_\phi \oplus_1 S_\psi$, and we are done.

28. Suppose $S \models^1 \overline{(\lambda)}(\lambda)\phi$. Then $\lambda(S) \models^1 (\lambda)\phi$ so there is some $S' \in D_1$ s.t. $S' \models^1 \phi$ and $\lambda(S) \leq_1 \lambda(S')$. But then $S \leq_1 S'$ and then $S \models^1 \phi$ by 3.14.

29. Suppose $S \models^1 \overline{(\lambda)}(\mu)\phi$ with $\lambda \neq \mu$. Then there is an S' s.t. $S' \models^1 \phi$ and $\lambda(S) \leq_1 \mu(S')$—a contradiction.

30. Suppose $S \models^1 (\lambda)\phi \rightarrow \bigvee_{\mu \in \Lambda}(\mu)\psi_\mu$. First if there is no $S' \in D_1$ s.t. $S' \models^1 (\lambda)\phi$ then $S \models^1 (\lambda)\phi \rightarrow (\mu)\psi_\mu$ for all $\mu \in \Lambda$. So assume instead that such an S' exists. Let

$$\Lambda' = \{\eta \mid \exists s \in \mathrm{NP}.\eta(s) \in S\}.$$

If $\Lambda' = \emptyset$ then $S = 0$ and then $S \models^1 (\lambda)\phi \rightarrow (\mu)\psi_\mu$ for all $\mu \in \Lambda$. Otherwise for each $\eta \in \Lambda'$ let

$$S_\eta = \mathrm{cl}_1\{s \in S \mid \exists s' \in \mathrm{NP}.s = \eta(s')\}.$$

Then $S = \sum_{\eta \in \Lambda'} S_\eta$ and $S_\eta \leq_1 S$ for each $\eta \in \Lambda'$. It suffices to show that for each $\eta \in \Lambda'$ there is a $\mu \in \Lambda$ s.t. $S_\eta \models^1 (\lambda)\phi \rightarrow (\lambda)\psi_\mu$. Fix a $\eta \in \Lambda'$. Now $S_\eta \models^1 (\lambda)\phi \rightarrow \bigvee_{\mu \in \Lambda}(\mu)\psi_\mu$ by 3.14. Let $S' \models^1 \phi$—by the assumption such an S' exists. Then $\lambda(S') \models^1 (\lambda)\phi$ so $S_\eta \cdot \lambda(S') \models^1 \bigvee_{\mu \in \Lambda}(\lambda)\psi_\mu$. Then $\eta\lambda \in \Lambda$ and

$$S_\eta \models^1 (\lambda)\phi \rightarrow (\eta\lambda)\psi_{\eta\lambda}$$

and we are done.

31. Let $S \models^1 (\neg\neg\phi) \wedge (\neg\neg\psi) \wedge ((\lambda)\phi \rightarrow (\mu)\psi)$. Then there is some $S' \in D_1$ s.t. $S' \models^1 \phi$ and $S \models^1 (\lambda)\phi \rightarrow (\mu)\psi$. Pick any S' s.t. $S' \models^1 \phi$. Then $\lambda(S') \models^1 (\lambda)\phi$ so $S \cdot \lambda(S') \models^1 (\mu)\psi$. First, if $S = 0$ then $S \models^1 (\lambda^{-1}\mu)(\phi \rightarrow \psi)$. We need to show that there is some S'' s.t. $S'' \models^1 \phi \rightarrow \psi$—for this it suffices to show that $0 \models^1 \phi \rightarrow \psi$. But $0 \models^1 \psi$ as $S \models^1 \neg\neg\psi$, whence also $0 \models^1 \phi \rightarrow \psi$. Next assume that $S \neq 0$. Now if there is some $s \in S$ s.t. $s \neq (\lambda^{-1}\mu)(s')$ for all

$s' \in \mathrm{NP}$ then $s \cdot \lambda(S') \not\models^1 (\mu)\psi$. So $S = (\lambda^{-1}\mu)(S'')$ for some $S'' \in D_1$. Now $(\lambda^{-1}\mu)(S'') \cdot \lambda(S') \models^1 (\mu)\psi$ and then $\mu(S'' \cdot S') \models^1 (\mu)\psi$. Then there is some $S_1 \in D_1$ s.t. $S_1 \models^1 \psi$ and $\mu(S'' \cdot S') \leq_1 \mu(S_1)$. Then $S'' \cdot S' \leq_1 S_1$ and then $S'' \cdot S' \models^1 \psi$ by 3.14. But then, as S' was arbitrary, $S'' \models^1 \phi \to \psi$ and we are done.

32. Let $S \models^1 \top \to \bigvee_{\lambda \in \Lambda}(\lambda)\phi_\lambda$, and suppose for a contradiction that $S \neq 0$. Then for some $\mu \in \mathcal{L}$ and $s \in \mathrm{NP}$, $\mu(s) \in S$. Fix any $\eta \notin \Lambda$—here we must appeal to $\mathcal{L}$ being infinite. Now $(\mu^{-1}\eta)(0) \models^1 \top$ (as everything else) and $S \cdot (\mu^{-1}\eta)(0) \not\models^1 \bigvee_{\lambda \in \Lambda}(\lambda)\phi_\lambda$, for

$$\mathrm{cl}_1\{\mu(s) \cdot (\mu^{-1}\eta)(\langle \varepsilon, 0 \rangle)\} = \mathrm{cl}_1\{\eta(\langle \varepsilon, 0 \rangle)\} \not\models^1 (\lambda)\phi_\lambda$$

for any $\lambda \in \Lambda$. But this is a contradiction, by 3.16.ii).

33. Let $S \models^1 \neg\neg\phi$ and $S \models^1 (\lambda)\phi \to \underline{0}$. Suppose for a contradiction that $S \neq 0$—i.e. that for some $\mu \in \mathcal{L}$ and $s \in \mathrm{NP}$, $\mu(s) \in S$. Now pick some S' s.t. $S' \models^1 \phi$—such a S' exists. Then $\lambda(S') \models^1 (\lambda)\phi$ so $S \cdot \lambda(S') \models^1 \underline{0}$ i.e. $S \cdot \lambda(S') = 0$. But $\mu(s) \in S$ and for some $s' \in S'$, $\lambda(s') \in \lambda(S')$ and $\mu(s) \cdot \lambda(s') \neq \langle \varepsilon, 0 \rangle$, a contradiction.

$\mathbf{PL}^{2,D}$:

24. By 3.16.iii).

28. Suppose that $S \models^2 \overline{(\lambda)}(\lambda)\phi$. Then $\lambda(S) \models^2 (\lambda)\phi$ so there is some $S' \in D_2$ s.t. $S' \models^2 \phi$ and $\lambda(S') \leq_2 \lambda(S)$. But then $S' \leq_2 S$ and then, by 3.14, $S \models^2 \phi$ as well.

29. Suppose that $S \models^2 \overline{(\lambda)}(\mu)\phi$—i.e. $\lambda(S) \models^2 (\mu)\phi$. Then there is an $S' \in D_2$ s.t. $\mu(S') \leq_2 \lambda(S)$—but if $\lambda \neq \mu$ this is impossible.

30. Suppose that $S \models^2 (\lambda)\phi \to \bigvee_{\mu \in \Lambda_2}(\mu)\psi_\mu$. First if there is no S' s.t. $S' \models^2 \phi$ then $S \models^2 (\lambda)\phi \to (\mu)\psi_\mu$ for all $\mu \in \Lambda_2$, so assume not. Let $\Lambda_3 = \{s \in \mathrm{NP} \mid \eta(s) \in S\}$, and for each $\eta \in \Lambda_3$ let

$$S_\eta = \mathrm{cl}_2\{s \in S \mid \exists s' \in \mathrm{NP}.s = \eta(s')\}.$$

If $\Lambda_3 = \emptyset$ then $S = 0_2$ and then $S \not\models^2 (\lambda)\phi \to \bigvee_{\mu \in \Lambda_2}(\mu)\psi_\mu$.

By 3.16.ii) it suffices to establish for each $\eta \in \Lambda_3$ a $\mu \in \Lambda_2$ s.t. $S_\eta \models^2 (\lambda)\phi \to (\mu)\psi_\mu$. But $S \leq_2 S_\eta$ for each $\eta \in \Lambda_3$ and then $S_\eta \models^2 (\lambda)\phi \to (\mu)\psi_\mu$ and thus whenever $S' \models^2 (\lambda)\phi$, $S_\eta \cdot S' \models^2 \bigvee_{\mu \in \Lambda_2}(\mu)\psi_\mu$. But then it must be the case that $\eta\lambda \in \Lambda_2$ and that $S_\eta \cdot S' \models^2 (\eta\lambda)\psi_{(\eta\lambda)}$ and thus we have shown that $S_\eta \models^2 (\lambda)\phi \to (\eta\lambda)\psi_{(\eta\lambda)}$ and we are done.

34. Assume that $S \models^2 \phi$ and $S \models^2 (\lambda)$. Then there is some S' s.t. $\lambda(S') \leq_2 S$. But then there is some S'' s.t. $S = \lambda(S'')$ and $S' \leq_2 S''$. Then $S = \lambda(S'') \models^2 \phi$, so $S'' \models^2 \overline{(\lambda)}\phi$, thus $S \models^2 (\lambda)\overline{(\lambda)}\phi$ as desired.
35. For $\rightarrow$ if for no S, $S \models^2 \phi$ then for no S', $S' \models^2 (\lambda)\phi$. For the converse implication it suffices to note that if $S \models^2 \phi$ then $\lambda(S) \models^2 (\lambda)\phi$.
36. Assume that there is some S' s.t. $S' \models^2 \phi$ and that $S \models^2 (\lambda)\phi \rightarrow (\mu)\psi$. Pick any S' s.t. $S' \models^2 \phi$. Then $\lambda(S') \models^2 (\lambda)\phi$ so $S \cdot \lambda(S') \models^2 (\mu)\psi$. Pick any $s'' \in S \cdot \lambda S'$. By 3.16.ii), $\mathrm{cl}_2\{s''\} \models^2 (\mu)\psi$ and moreover for some $s \in S$, $s' \in S'$, $s'' = s \cdot \lambda(s')$. But then it must be the case that $s = (\lambda^{-1}\mu)(s_1)$ for some s_1 and thus $S = (\lambda^{-1}\mu)(S_1)$ for some S_1. But $S \cdot \lambda(S') \models^2 (\mu)\psi$ and then there is some S'' s.t. $S'' \models^2 \psi$ and $\lambda(S'') \leq_2 S \cdot \lambda(S')$. Hence $S'' \leq_2 S_1 \cdot S'$ and thus by 3.14, $S_1 \cdot S' \models^2 \psi$ and consequently—as S' was arbitrary—we obtain $S \models^2 (\lambda^{-1}\mu)(\phi \rightarrow \psi)$.
37. Trivial—$0_2 \leq_2 S$ for all S.
38. Suppose that $S \models^2 \top \rightarrow \bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda$. Then $S \cdot 0_2 \models^2 \bigvee_{\lambda\in\Lambda_1}(\lambda)\phi_\lambda$ but this is a contradiction. □

3.7.3 Completeness

The completeness proof proceeds—fundamentally different from the Henkin-style proofs we have encountered earlier—by a normal form theorem. We show that each variable-free formula in $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ can be rewritten into an $\mathbf{PL}^{1,D}/\mathbf{PL}^{2,D}$-equivalent formula in a *normal form*. Moreover it is very easy to characterise those formulas in normal form that are satisfiable (in $\mathcal{D}_1/\mathcal{D}_2$) as well as those that are valid. Thus—in addition to its role in the completeness proof—the rewriting procedure given in the proof of the normal form theorem gives an algorithm for checking for each variable-free formula, ϕ, whether it is

i) *satisfiable*, i.e. whether there is some $S \in D_1$ s.t. $S \models^1_{\mathcal{D}_1} \phi$,

ii) *valid*, i.e. whether $1_1 \models^1_{\mathcal{D}_1} \phi$, and

iii) *"universally valid"*, i.e. whether for all $S \in D_1$, $S \models^1_{\mathcal{D}_1} \phi$.

Similarly, of course, in the must-case. To complete the completeness proof given such an algorithm we just need for each valid formula in normal form to construe a proof for it, but this turns out to be a straightforward task.

We start by introducing the notions of *normal form* and *satisfiable normal form*. The latter is the least set satNF of variable-free formulas in $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ s.t.

i) t, $\top$, $\underline{0} \in$ satNF,

ii) $\bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda \in$ satNF, if for all $\lambda \in \Lambda$, $\phi_\lambda \in$ satNF.

Then the set NF of normal forms is given by NF = satNF $\cup \{\bot\}$. We check that a $\phi \in$ NF is satisfiable iff its double negation is provable iff it is a member of satNF.

Proposition 3.33 *For all $\phi \in$ NF the following are equivalent:*

i) $\phi \in$ satNF*,*

ii) $\vdash_{\mathbf{PL}^{1,D}} \neg\neg\phi$*,*

iii) $S \models^1 \phi$ *for some* $S \in D_1$*,*

iv) $\vdash_{\mathbf{PL}^{2,D}} \neg\neg\phi$*,*

v) $S \models^2_{\mathcal{D}_2} \phi$ *for some* $S \in D_2$*.*

PROOF:
i) iff ii). By theorem 3.32, $\not\vdash_{\mathbf{PL}^{1,D}} \neg\neg\bot$. So we need just check that $\vdash_{\mathbf{PL}^{1,D}} \neg\neg\phi$ whenever $\phi \in$ satNF. By theorem 3.29.xiv) it suffices to check that $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \phi$ whenever $\phi \in$ satNF. We prove this by induction on the structure of ϕ.
$\phi =$ t. We have $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \mathrm{t}$ by 16.
$\phi = \top$. Here $\vdash_{\mathbf{PL}^{1,D}} \mathrm{t} \to \top$ by 20, thus $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \top$ by 16.
$\phi = \underline{0}$. By reflexivity.
$\phi = \bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda$. By the induction hypothesis, $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \phi_\lambda$ for each $\lambda \in \Lambda$, as $\phi_\lambda \in$ satNF. Then $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to (\lambda)\phi_\lambda$ for each $\lambda \in \Lambda$, by theorem 3.29.xiv) and xvi) and then the result follows by $\vee$-intro-1/-2.
i) iff iii). It suffices to show that $0 \models^1 \phi$ whenever $\phi \in$ satNF—but this follows from 3.26.
i) iff iv). By theorem 3.32, $\not\vdash_{\mathbf{PL}^{2,D}} \neg\neg\bot$ so we need only—for the first part—check that $\vdash_{\mathbf{PL}^{2,D}} \neg\neg\phi$ whenever $\phi \in$ satNF. We continue by induction on the structure of ϕ.
$\phi =$ t. We have $\vdash_{\mathbf{PL}^{2,D}} \mathrm{t} \to \neg\neg\mathrm{t}$ and $\vdash_{\mathbf{PL}^{2,D}} \mathrm{t}$ so $\vdash_{\mathbf{PL}^{2,D}} \neg\neg\mathrm{t}$.

$\phi = \top$, $\phi = \underline{0}$. Straightforward.
$\phi = \bigvee_{\lambda\in\Lambda_1}(\lambda)\phi_\lambda$. By the induction hypothesis, $\vdash_{\mathbf{PL}^{2,D}} \neg\neg\phi_\lambda$ for each $\lambda \in \Lambda_1$ and then $\vdash_{\mathbf{PL}^{2,D}} \neg\neg(\lambda)\phi_\lambda$ by axiom 35, and the result follows by $\vee$-intro.
i) iff v). An easy induction to check that whenever $\phi \in$ satNF then $\models^2 \neg\neg\phi$. □

We next characterise the valid normal forms. The set valNF $\subseteq$ NF is the least s.t.

i) t, $\top \in$ valNF,

ii) $\bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda \in$ valNF, if ($\bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda \in$ NF, and) $e \in \Lambda$ and $\phi_e \in$ valNF.

If we further relieve us of the bother of carrying $\underline{0}$ around in the must-case we can use this characterisation for both cases. Consequently let $\mathrm{NF}_2 \subseteq$ NF denote the fragment of NF not containing any occurrences of $\underline{0}$.

Proposition 3.34 *For all $\phi \in$ NF, the following are equivalent:*

i) $\phi \in$ valNF,

ii) $1_1 \models^1_{\mathcal{D}_1} \phi$,

iii) $\vdash_{\mathbf{PL}^{1,D}} \phi$.

Moreover, if $\phi \in \mathrm{NF}_2$ then they are also equivalent to

iv) $1_2 \models^2_{\mathcal{D}_2} \phi$,

v) $\vdash_{\mathbf{PL}^{2,D}} \phi$.

PROOF: i) iff ii). Again by induction on the structure of ϕ.
$\phi =$ t, $\phi = \top$. We have $\phi \in$ valNF and $1 \models^1 \phi$.
$\phi = \underline{0}$, $\phi = \bot$. We have $\underline{0}, \bot \notin$ valNF and $1 \not\models^1 \underline{0}$, $1 \not\models^1 \bot$.
$\phi = \bigvee_{\lambda\in\Lambda}(\lambda)\phi_\lambda$. Assume first that $1 \models^1 \phi$. Now 1 is coprime w.r.t. $\subseteq$ by 3.16.i) so by ii), $1 \models^1 (\lambda)\phi_\lambda$ for some $\lambda \in \Lambda$. Then it is easy to see that $\lambda = e$ and $1 \models^1 \phi_e$—thus $\phi_e \in$ valNF, by the induction hypothesis, and then $\phi \in$ valNF as well. On the other hand if $\phi \in$ valNF then $e \in \Lambda$ and $\phi_e \in$ valNF, so by the induction hypothesis, $1 \models^1 \phi_e$. Then $1 \models^1 (e)\phi_e$ and then $1 \models^1 \phi$ as desired.
i) iff iii). Once more by induction on the structure of ϕ.
i) iff iv), i) iff v). As above. □

We proceed then to the normal form theorem, the by far most involved single step in the completeness proof. At first glance it may seem highly surprising that as sparse vocabulary as the constants plus only $\vee$ and (λ) suffices to express the other connectives as well—in particular the implication. On the other hand we have seen that in $\mathcal{D}_1$ (and indeed $\mathcal{D}_2/\mathcal{D}_3$ as well), all occurrences of $\cdot$ are eliminable in favour of 0, 1, $\oplus$ and operators $\lambda \in \mathcal{L}$ only; and one might reasonably suspect that the same should hold for the $\rightarrow$ in the logical context as well.

Theorem 3.35 *For each variable-free* $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$

i) there is a $\phi' \in \mathrm{NF}$ *s.t.* $\phi \equiv_{\mathbf{PL}^{1,D}} \phi'$,

ii) there is a $\phi' \in \mathrm{NF}_2$ *s.t.* $\phi \equiv_{\mathbf{PL}^{2,D}} \phi'$.

PROOF: See appendix C. □

The proof of this theorem proceed by a long and tedious double induction first on the structure and then on the modal depth of formulas. Given the normal form theorem it is a straightforward matter to prove completeness:

Corollary 3.36 (Completeness of $\mathbf{PL}^{1,D}/\mathbf{PL}^{2,D}$). *For all* $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ *variable-free,*

i) if $\models^1_{\mathcal{D}_1} \phi$ *then* $\vdash_{\mathbf{PL}^{1,D}} \phi$,

ii) if $\models^2_{\mathcal{D}_2} \phi$ *then* $\vdash_{\mathbf{PL}^{2,D}} \phi$.

PROOF: i). Assume that $\models^1 \phi$. By 3.35 we find a $\phi' \in \mathrm{NF}$ s.t. $\phi \equiv \phi'$. Then by 3.32, $\models^1 \phi'$, and then by 3.34, $\phi' \in \mathrm{valNF}$ and then by 3.34, $\vdash_{\mathbf{PL}^{1,D}} \phi'$. But then, as $\phi \equiv \phi'$, also $\vdash_{\mathbf{PL}^{1,D}} \phi$ and we are done.

ii) Similar, use 3.32, 3.34 and 3.35. □

3.8 Concluding remarks

We have demonstrated a tight correspondence between certain modal logics based on positive linear logic and a simple system of synchronous processes. We have given algebraic and logical characterisations of processes under slightly nonstandard notions of testing preorders, and presented sound and complete axiomatisations, both with respect to general algebraic models and also with respect to the process-based interpretations, in the latter case providing also decision procedures. Although we have covered the process structure, algebras and logics in a fair degree of depth, there are certainly issues that we have not touched upon. We conclude the chapter by discussing two issues of particular importance. The first is the problem of model checking, and the second the problem of adding a suitable notion of external, or controlled, choice. The latter, in particular, is likely to be essential in order to extend this work to cover also asynchronous parallel composition, for instance in the style of CCS.

3.8.1 Model checking

In practice one is often more interested in the properties satisfied by given models than in the satisfiability or validity of formulas. That is, the relations $P \models^1 \phi$ and $P \models^2 \phi$ given by

$$P \models^i \phi \text{ iff } [\![P]\!]_i \models^i_{\mathcal{D}_i} \phi,\ i \in \{1,2\}.$$

We shall briefly outline how an axiomatisation of these relations could be approached.

Let the proof-theoretic correlate of $\models^i$ be denoted by $\Vdash^i$. Note first that sound and complete axiomatisations of $\Vdash^i$ are easily derived from the logics $\mathbf{PL}^{i,D}$ of the last section. It will suffice to have the following basic axioms and rules relating formulas to process structure plus the rule of consequence:

0-int.	$0 \Vdash^i \underline{0}$
1-int.	$1 \Vdash^i \mathrm{t}$
$\lambda(\cdot)$-int.	$\dfrac{p \Vdash^i \overline{(\lambda)}\phi}{\lambda(p) \Vdash^i \phi}$
$\oplus$-int.	$\dfrac{p \Vdash^i \phi \quad q \Vdash^i \phi}{p \oplus q \Vdash^i \phi}$
$\times$-int.	$\dfrac{p \Vdash^i \phi \to \psi \quad q \Vdash^i \phi}{p \times q \Vdash^i \psi}$
Consequence-rule.	$\dfrac{p \Vdash^i \phi \quad \vdash_{\mathbf{PL}^{i,D}} \phi \to \psi}{p \Vdash^i \psi}$

The soundness and completeness for these 6 rules is not hard to establish on the basis of the above work—for completeness proceed by induction on the structure of p, appealing to the embeddings r_i of the $\mathcal{D}_i$ into $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ of section 3.3.

So in a technical sense we have solved the problem already. It would, however, also be of interest to develop axiomatisations of the $\Vdash^i$ without referring to the $\mathbf{PL}^{i,D}$. We briefly indicate what a natural deduction style formulation of such a proof system could look like. First of all we should expect the above 5 introduction rules for the process connectives, possibly with corresponding elimination rules for prefixing and internal sum. For the extensional connectives there are rules like

$$\wedge\text{-int.} \qquad \frac{p \Vdash^i \phi \qquad p \Vdash^i \psi}{p \Vdash^i \phi \wedge \psi}$$

$$\wedge\text{-elim-1.} \qquad \frac{p \Vdash^i \phi \wedge \psi}{p \Vdash^i \phi}$$

as well as $\wedge$-elim-2 and the 2 introduction rules for $\vee$. The $\vee$-elimination rule is somewhat more delicate. To formulate this rule we need to consider *open* process terms, i.e. process terms possibly containing occurrences of variables x, y, z. Let a *context*, $\mathrm{C}[\cdot]$, be any (possibly open) process term with a "slot" in it. Then the $\vee$ elimination rule could be formulated as:

$$\vee\text{-elim.} \qquad \frac{p \Vdash^i \phi \vee \psi \qquad \begin{matrix} [x \Vdash^i \phi] \\ \vdots \\ \mathrm{C}[x] \Vdash^i \gamma \end{matrix} \qquad \begin{matrix} [x \Vdash^i \psi] \\ \vdots \\ \mathrm{C}[x] \Vdash^i \gamma \end{matrix}}{\mathrm{C}[p] \Vdash^i \gamma}$$

with the side-condition that x does not occur in $\mathrm{C}[\cdot]$. It is important here that we consider contexts and not just arbitrary open terms as the conclusion of the subproofs—such a more general rule will be unsound. For the modal operators there are a number of introduction and elimination rules like

$$\overline{(\lambda)}(\lambda)\text{-int.} \qquad \frac{p \Vdash^i \phi}{p \Vdash^i \overline{(\lambda)}(\lambda)\phi}$$

$$(e)\text{-int.} \qquad \frac{p \Vdash^i \mathrm{t}}{p \Vdash^i (e)\mathrm{t}}$$

plus an elimination rule for $(\lambda)\overline{(\lambda)}$, an elimination rule for (e) and introduction/elimination rules for $\overline{(e)}$. In addition we will need monotonicity rules for the modal operators, for instance like (λ)-mon. $$\dfrac{p \Vdash^i (\lambda)\phi \qquad \begin{array}{c}[x \Vdash^i \phi] \\ \vdots \\ x \Vdash^i \psi\end{array}}{p \Vdash^i (\lambda)\psi}$$

and a corresponding rule for $\overline{(\lambda)}$. Finally for the implication we will need rules like

→-int. $$\dfrac{\begin{array}{c}[x \Vdash^i \phi] \\ \vdots \\ p \times x \Vdash^i \psi\end{array}}{p \Vdash^i \phi \to \psi}$$

with the side condition that x does not occur in p, and

(λ)-→-rule. $$\dfrac{p \Vdash^i (\lambda)(\phi \to \psi) \qquad q \Vdash^i (\mu)\phi}{p \times q \Vdash^i (\lambda\mu)\psi}$$

t-elim-1 $$\dfrac{p \Vdash^i \mathrm{t} \qquad q \Vdash^i \phi}{p \times q \Vdash^i \phi}$$

and a symmetric version of this last rule, t-elim-2.

We shall not go into the issues of soundness and completeness here—completeness, certainly, is likely to fail. The system is nevertheless powerful enough to prove interesting properties of processes.

Example 3.37 We briefly outline a proof of

$$\lambda(0) \oplus \mu(\eta(1)) \Vdash^i (\phi \to (\mu^{-1})(\eta^{-1})\mathrm{t}) \to (\phi \to (\mu^{-1}\lambda)\top \vee \mathrm{t}).$$

To prove this assume that $x \Vdash^i \phi \to (\mu^{-1})(\eta^{-1})\mathrm{t}$ and $y \Vdash^i \phi$, and we need then to show

$$(\lambda(0) \oplus \mu(\eta(1))) \times x \times y \Vdash^i (\mu^{-1}\lambda)\top \vee \mathrm{t}.$$

First we obtain $x \times y \Vdash^i (\mu^{-1})(\eta^{-1})\mathrm{t}$. We assume proofs of $\lambda(0) \Vdash^i (\lambda)(\top \to \top)$ and $\mu(\eta(1)) \Vdash^i (\mu)(\eta)(\mathrm{t} \to \mathrm{t})$ are given. Then we obtain $\lambda(0) \oplus \mu(\eta(1)) \Vdash^i (\lambda)(\top \to \top) \vee (\mu)(\eta)(\mathrm{t} \to \mathrm{t})$. Using ∨-elim we now assume first that $z \Vdash^i (\lambda)(\top \to \top)$ and secondly that $z \Vdash^i (\lambda)(\eta)(\mathrm{t} \to \mathrm{t})$, and we need to show in either case $z \times x \times y \Vdash^i (\mu^{-1}\lambda)\top \vee \mathrm{t}$. Assume proofs of $1 \Vdash^i (\eta^{-1})\mathrm{t} \to \top$ and $1 \Vdash^i (\eta)(\mathrm{t} \to \mathrm{t}) \to ((\eta^{-1})\mathrm{t} \to (e)\mathrm{t})$ to be given. Now for the first subcase $x \times y \Vdash^i (\mu^{-1})\top$ and then $z \times x \times y \Vdash^i (\mu^{-1}\lambda)\top$, and thus $z \times x \times y \Vdash^i (\mu^{-1}\lambda)\top \vee \mathrm{t}$. For the second subcase we obtain $z \Vdash^i (\mu)((\eta^{-1})\mathrm{t} \to (e)\mathrm{t})$ and hence $z \times x \times y \Vdash^i (e)(e)\mathrm{t}$. But then $z \times x \times y \Vdash^i \mathrm{t}$ and thus $z \times x \times y \Vdash^i (\mu^{-1}\lambda)\top \vee \mathrm{t}$ as desired. Now the assumptions for z can be discharged and we may conclude that

$$(\lambda(0) \oplus \mu(\eta(1))) \times x \times y \Vdash^i (\mu^{-1}\lambda)\top \vee \mathrm{t},$$

and then the desired conclusion follows by two applications of $\to$-int.

The style of proof outlined in the example is, we believe, quite natural, and the idea deserves to be further explored.

3.8.2 External choice

Another important issue is to see how and if the approach and ideas of the present chapter may be carried over in contexts where the process system is more expressive. Most urgently needed is some notion of external, or controllable, choice such as the CCS/SCCS + or the CSP $\Box$ (or perhaps the conjunctive nondeterministic operator $\wedge$ of [54]). We consider here the CSP $\Box$.

Operational and algebraic issues

Operationally the stabilisation- and transition relations are extended to processes of the form $p \Box q$ by

i) if $p \twoheadrightarrow p'$ and $q \twoheadrightarrow q'$ then $p \Box q \twoheadrightarrow p' \Box q'$,

ii) if $p \xrightarrow{\lambda} p'$ then $p \Box q \xrightarrow{\lambda} p'$, $q \Box p \xrightarrow{\lambda} p'$.

The notions of test acceptance and testing preorders $\sqsubseteq_i$, $i \in \{1,2,3\}$ extends directly to these enriched processes, and it is not hard to establish the substitutivity of $\Box$ w.r.t. these preorders. Note that the distributivity of $\mathrm{Pc}^{\oplus}$ as a semilattice under $\oplus$ modulo the must-preorder $\sqsubseteq_2$ will now fail—i.e. there will be $p_1, p_2, p \in \mathrm{Pc}^{\oplus}$ s.t. $p_1 \oplus p_2 \sqsubseteq_2 p$ but for no $p_1', p_2' \in \mathrm{Pc}^{\oplus}$ s.t. $p_1 \sqsubseteq_2 p_1'$ and $p_2 \sqsubseteq_2 p_2'$ does $p_1' \oplus p_2' \simeq_2 p$. As an example of this situation consider $\lambda(0) \oplus \mu(0) \sqsubseteq_2 \lambda(0) \Box \mu(0)$. Concerning the algebraic properties, all the laws holding for $\mathrm{Pc}^{\oplus}/\simeq_i$, $i \in \{1,2,3\}$, will clearly go through in the enriched language as well. Moreover we shall have properties such as

i) Idempotency, commutativity and associativity of $\Box$.

ii) Preservation of $\oplus$ by $\Box$ (in both arguments).

iii) The constant $\underline{0}$ will serve as a unit for $\Box$.

iv) For all $i \in \{1, 2, 3\}$, $p_1, p_2, p_3 \in \mathrm{Pc}^{\oplus}$, $\lambda \in \mathcal{L}$,

a) $p_1 \oplus p_2 \simeq_i p_1 \oplus p_2 \oplus p_1 \Box p_2$,

b) $p_1 \oplus (p_1 \Box p_2 \Box p_3) \simeq_i p_1 \oplus (p_1 \Box p_2) \oplus (p_1 \Box p_2 \Box p_3)$,

c) $\lambda(p_1) \Box \lambda(p_2) \simeq_i \lambda(p_1 \oplus p_2)$.

v) For $i = 1$ we obtain $p_1 \Box p_2 \simeq_1 p_1 \oplus p_2$.

All these properties are easily verified, using for instance theorem 3.3. They correspond to properties proved for CSP under a modified version of the failures model [15] by De Nicola [27, 28]. The semilattice properties of $\Box$ justifies the use of the "big $\Box$" notation $\sum\{p_1, \ldots, p_n\} =_{\mathrm{def}} p_1 \Box \cdots \Box p_n$ for $n \geq 1$, and $\sum \emptyset =_{\mathrm{def}} 0$, under any of the $\simeq_i$. We then have the following property generalising equation iv.b) of def. 3.5:

vi) $(\sum_{\lambda \in \Lambda_1} \lambda(p_\lambda)) \times (\sum_{\mu \in \Lambda_2} \mu(q_\mu)) = \sum_{\lambda \in \Lambda_1, \mu \in \Lambda_2} (\lambda\mu)(p_\lambda \times q_\mu)$.

We do not yet know if these equations are actually complete for $\mathrm{Pc}^{\oplus}/\simeq_i$—but this is not important to the present discussion. So let an *extended synchronous algebra* (ES-algebra) be any synchronous algebra enriched by a $\Box$ satisfying the above equations i)–vi). With these properties it may be shown without too much difficulty, that each $p \in \mathrm{Pc}^{\oplus}$ may be rewritten into a *normal form* p' generalising the "sum-of-traces-nature" of the representations $\mathcal{D}_i$ with the following recursive property

i) $p' = \bigoplus\!\!\!\!\!\sum_{i \in I} p_i$, for I a finite, nonempty set,

ii) for each $i \in I$, $p_i = \sum_{\lambda \in \Lambda_1} \lambda(p_{i,\lambda})$ with $\Lambda_1 \subseteq \mathcal{L}$ finite (and possibly empty), and each $p_{i,\lambda}$ in normal form.

The modal operators

The important issue is how to devise a logic in the spirit of **PL** allowing simultaneously reasonably natural and matching algebraic and operational interpretations and which, furthermore, induces the proper testing preorders on processes. The modal operators as they are interpreted in the present chapter are inadequate for this purpose in the presence of $\Box$. The problem with the forwards modality is that it forces the processes having a forwards modal property to being capable of performing at most one action—this clearly is too restrictive. On the other hand the reverse modality plays a crucial role in providing the link from logic to that part of the algebraic structure related to the performing of actions, and as it stands it is as incapable as the forwards modality of capturing processes with multiple, externally chosen actions possible.

Let now i range over $\{1,2\}$ and x,y,z range over elements of some *extended i-model* $\mathcal{M}$, i.e. an extended i-algebra together with a valuation satisfying the i-set property. The satisfaction conditions for all formulas except those of one of the forms $(\lambda)\phi$ or $\overline{(\lambda)}\phi$ are maintained unchanged. For the forwards modality we propose the following horrendously complicated (but as we shall see quite natural) satisfaction condition:

i) $x \models^i_{\mathcal{M}} (\lambda)\phi$ iff
$\exists n \geq 1, \Lambda_1, \ldots, \Lambda_n$ (finite, nonempty) and for all $\lambda' \in \bigcup_{j\geq 1}^n \Lambda_j = \Lambda$ some $x_{\lambda'} \in \mathcal{M}$ s.t.

  a) $\sum_{j\geq 1}^n (\sum_{\lambda' \in \Lambda_j} \lambda'(x_{\lambda'})) R_i x$ and
  b) if $\lambda \in \Lambda$ then $x_\lambda \models^i_{\mathcal{M}} \phi$,

where as usual $R_1 = \geq_1$ and $R_2 = \leq_2$. For the reverse modalities we propose to replace them by versions $\overline{(\lambda)}_\Lambda\phi$ indexed by finite, nonempty $\Lambda \subseteq \mathcal{L}$ and then take

ii) $x \models^i_{\mathcal{M}} \overline{(\lambda)}_\Lambda\phi$ iff for all $\lambda' \in \Lambda$ there is some $x_{\lambda'}$ s.t.

  a) $\sum_{\lambda' \in \Lambda} \lambda'(x_{\lambda'}) \models^i_{\mathcal{M}} \phi$, and
  b) if $\lambda \in \Lambda$ then $x_\lambda R_i x$.

Further we shall need for each finite, nonempty $\Lambda \subseteq \mathcal{L}$ an interpretation of Λ as an atomic proposition, which will for $i = 2$ correspond operationally to the "must-sets" of def. section 3.1.2. We adopt the satisfaction condition

iii) $x \models^i_{\mathcal{M}} \Lambda$ iff
$\exists n, \Lambda_1, \ldots, \Lambda_n$ as in i), and for each $\lambda \in \bigcup_{j\geq 1}^n \Lambda_j = \Lambda'$ some $x_\lambda \in \mathcal{M}$ s.t.

  a) $\sum_{j\geq 1}^n (\sum_{\lambda \in \Lambda_j} \lambda(x_\lambda)) R_i x$, and
  b) $\bigcap_{j\geq 1}^n \Lambda_j \subseteq \Lambda$.

These definitions evidently are in need of some justification. First note that we obtain the following extension of the i-set theorem, 3.14, where we let PFm' denote $\mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ with the thus indexed reverse modalities and atomic propositions Λ.

Proposition 3.38 *For all $i \in \{1,2\}$, extended i-models $\mathcal{M}$ and $\phi \in \mathrm{PFm}'$, $[\![\phi]\!]^i_M$ is an i-set.*

PROOF: We just need to check the modal operators (and constants). For the forwards modalities, let first $x, y \models^i_{\mathcal{M}} (\lambda)\phi$. This is the case just when there are $\Lambda_1, \ldots, \Lambda_n, \Lambda'_1, \ldots, \Lambda'_m \subseteq \mathcal{L}$ and for all $\lambda' \in \Lambda = \bigcup \Lambda_j$, $\mu' \in \Lambda' = \bigcup \Lambda'_k$ some $x_{\lambda'}$, $y_{\mu'}$ s.t.

a) $\sum_j (\sum_{\lambda'} \lambda'(x_{\lambda'})) R_i x$ and $\sum_k (\sum_{\mu'} \mu'(y_{\mu'})) R_i y$, and

b) if $\lambda \in \Lambda$ ($\lambda \in \Lambda'$) then $x_\lambda \models^i_{\mathcal{M}} \phi$ ($y_\lambda \models^i_{\mathcal{M}} \phi$).

Fix $n, m, \Lambda_j, \Lambda'_k$ with these properties. Then

$$\sum_j (\sum_{\lambda'} \lambda'(x_{\lambda'})) \oplus \sum_k (\sum_{\mu'} \mu'(y_{\mu'})) R_i x \oplus y.$$

Let now $\eta \in \Lambda \cup \Lambda'$. If $\eta \in \Lambda \setminus \Lambda'$ let $z_\eta = x_\eta$, if $\eta \in \Lambda' \setminus \Lambda$ set similarly $z_\eta = y_\eta$ and if $\eta \in \Lambda \cap \Lambda'$ let $z_\eta = x_\eta \oplus y_\eta$. Then we obtain

$$\sum_j (\sum_{\lambda'} \lambda'(x_{\lambda'})) \oplus \sum_k (\sum_{\mu'} \mu'(y_{\mu'})) = \sum_{l \geq 1}^{n+m} (\sum_{\eta \in C_l} \eta(z_\eta)) = z$$

where for $1 \leq l \leq n$, $C_l = \Lambda_l$ and for $n < l \leq n + m$, $C_l = \Lambda'_{l-n}$. Now z has the desired form and if $\lambda \in \Lambda \cup \Lambda'$ then $z_\lambda \models^i_{\mathcal{M}} \phi$ by the induction hypothesis, thus $x \oplus y \models^i_{\mathcal{M}} (\lambda)\phi$.

The converse direction is straightforward—it suffices to prove that if $x \models^i_{\mathcal{M}} (\lambda)\phi$ and $x R_i y$ then $y \models^i_{\mathcal{M}} (\lambda)\phi$.

For the reverse modalities, similarly assume that $x, y \models^i_{\mathcal{M}} \overline{(\lambda)}_\Lambda \phi$. Then for all $\lambda' \in \Lambda$ there are $x_{\lambda'}, y_{\lambda'}$ s.t. $\sum_{\lambda' \in \Lambda} \lambda'(x_{\lambda'}) \models^i_{\mathcal{M}} \phi$ and $\sum_{\lambda' \in \Lambda} \lambda'(y_{\lambda'}) \models^i_{\mathcal{M}} \phi$ and if $\lambda \in \Lambda$ then $x_{\lambda'} R_i x$ and $y_{\lambda'} R_i y$. Fix the $x_{\lambda'}, y_{\lambda'}$ with these properties. By the induction hypothesis,

$$\sum_{\lambda' \in \Lambda} \lambda'(x_{\lambda'}) \oplus \sum_{\lambda' \in \Lambda} \lambda'(y_{\lambda'}) = \sum_{\lambda' \in \Lambda} \lambda'(x_{\lambda'} \oplus y_{\lambda'}) \models^i_{\mathcal{M}} \phi$$

and if $\lambda \in \Lambda$ then $x_{\lambda'} \oplus y_{\lambda'} R_i x \oplus y$—thus $x \oplus y \models^i_{\mathcal{M}} \overline{(\lambda)}_\Lambda \phi$. The converse direction again is straightforward.

Finally the proof for atomic propositions Λ is very similar to that for the forwards modalities. □

Now the reverse modality is justified by being just the logical device needed to capture "guarded sums".

Theorem 3.39 *For all $i \in \{1,2\}$, extended i-models $\mathcal{M}$, $\Lambda \subseteq \mathcal{L}$ finite and non-empty, and $\phi \in \mathrm{PFm}'$ the following are equivalent:*

i) $\sum_{\lambda\in\Lambda} \lambda(x_\lambda) \models^i_{\mathcal{M}} \phi$, *and*

ii) for all $\lambda \in \Lambda$, $x_\lambda \models^i_{\mathcal{M}} \overline{(\lambda)}_\Lambda \phi$.

PROOF: First if $\sum_{\lambda\in\Lambda} \lambda(x_\lambda) \models^i_{\mathcal{M}} \phi$ then ii) follows directly from the satisfaction condition above. So assume conversely that for all $\lambda \in \Lambda$, $x_\lambda \models^i_{\mathcal{M}} \overline{(\lambda)}_\Lambda \phi$. Then for all $\lambda, \lambda' \in \Lambda$ there is some $x_{\lambda,\lambda'} \in \mathcal{M}$ s.t. $\sum_{\lambda'\in\Lambda} \lambda'(x_{\lambda,\lambda'}) \models^i_{\mathcal{M}} \phi$ and $x_{\lambda,\lambda} R x_\lambda$. Then

$$\textstyle\bigoplus_{\lambda\in\Lambda}(\sum_{\lambda'\in\Lambda} \lambda'(x_{\lambda,\lambda'})) \models^i_{\mathcal{M}} \phi$$

by 3.38. Moreover, by equational reasoning it may be seen that

$$\textstyle\bigoplus_{\lambda\in\Lambda}(\sum_{\lambda'\in\Lambda} \lambda'(x_{\lambda,\lambda'})) = \sum_{\lambda\in\Lambda} \lambda(\bigoplus_{\lambda'\in\Lambda} x_{\lambda,\lambda'})$$

which has the desired form, and when $\lambda \in \Lambda$ then $\bigoplus_{\lambda'\in\Lambda} x_{\lambda,\lambda'} R x_\lambda$, so indeed $\sum_{\lambda\in\Lambda} \lambda(x_\lambda) \models^i_{\mathcal{M}} \phi$. □

For the forwards modalities and the atomic Λ justification lies in their being just the operations needed to induce the testing preorders on processes. To see this define satisfaction $\models^i$ on finite, nonempty sets $P \subseteq \mathrm{Pc}^\oplus$ by $P \models^i \phi$ iff $[\bigoplus P]_{\simeq_i} \models^i_{\mathrm{Pc}^\oplus/\simeq_i} \phi$ for all variable-free $\phi \in \mathrm{PFm}'$.

Proposition 3.40 *For all P, λ and variable-free $\phi \in \mathrm{PFm}'$,*

i) $P \models^1 (\lambda)\phi$ *iff* P can λ *only if* $P/\lambda \models^1 \phi$,

ii) $P \models^2 (\lambda)\phi$ *iff* P live *and if* P can λ *then* $p/\lambda \models^2 \phi$,

iii) $P \models^1 \Lambda$ *for all* Λ, *and*

iv) $P \models^2 \Lambda$ *iff* P must Λ.

PROOF: i) Assume first that $P \models^1 (\lambda)\phi$, i.e. that $[\bigoplus P] \models^1 (\lambda)\phi$ (omitting subscripts). Then there is some $n \geq 1, \Lambda_1, \ldots, \Lambda_n$ and for each $\lambda' \in \Lambda = \bigcup_{j\geq 1}^n \Lambda_j$ some $p_{\lambda'}$ s.t. $[P] \leq_1 [\bigoplus_j(\sum_{\lambda'\in\Lambda_j} \lambda'(p_{\lambda'}))]$ and if $\lambda \in \Lambda$ then $[p_\lambda] \models^1 \phi$. Assume that P can λ. Then we must have $\lambda \in \Lambda$ and then $[p_\lambda] \models^1 \phi$, i.e. $p_\lambda \models^1 \phi$. But $P/\lambda \sqsubseteq_1 p_\lambda$ by 3.3 and thus $[P/\lambda] \models^1 \phi$ by 3.38.

For the converse direction let $\Lambda = \{\lambda' \mid P \text{ can } \lambda'\}$ and for each $\lambda' \in \Lambda$, $p_{\lambda'} = \sum P/\lambda'$. Then $P \simeq_1 \sum_{\lambda' \in \Lambda} \lambda'(p_{\lambda'})$ and if $\lambda \in \Lambda$ then $[p_\lambda] \models^1 \phi$—i.e. $P \models^1 (\lambda)\phi$.
ii) Assume that $P \models^2 (\lambda)\phi$, i.e. that there is some $n \geq 1, \Lambda_1, \ldots, \Lambda_n$ and for each $\lambda' \in \Lambda = \bigcup_{j \geq 1}^n \Lambda_j$ some $p_{\lambda'}$ s.t. $[\sum_j(\sum_{\lambda' \in \Lambda_j} \lambda'(p_{\lambda'}))] \leq_2 [P]$, and if $\lambda \in \Lambda$ then $[p_\lambda] \models^2 \phi$. Then p live and if P can λ then $\lambda \in \Lambda$ and $p_\lambda \sqsubseteq_2 P/\lambda$ by 3.3 and thus $P/\lambda \models^2 \phi$ by 3.38.
Conversely assume that P live and if P can λ then $P/\lambda \models^2 \lambda$. For each $p \in \mathrm{st}(P)$ let $\Lambda_p = \{\lambda' \mid p \text{ can } \lambda'\}$, let $\Lambda = \bigcup_{p \in \mathrm{st}(P)} \Lambda_p$ and for each $\lambda' \in \Lambda$ let $p_{\lambda'} = \sum P/\lambda'$. Then $\sum_{p \in \mathrm{st}(P)}(\sum_{\lambda' \in \Lambda_p} \lambda'(p_{\lambda'})) \sqsubseteq_2 P$ as is easily verified, and if $\lambda \in \Lambda$ then $p_\lambda \models^2 \phi$—thus $P \models^2 \phi$.
iii) Let $\Lambda' = \{\lambda' \mid P \text{ can } \lambda'\}$. For each $\lambda' \in \Lambda'$ let $p_{\lambda'} = \sum P/\lambda'$. Assume first that $\Lambda \not\subseteq \Lambda'$ and pick some $\lambda \in \Lambda \setminus \Lambda'$. Then

$$P \sqsubseteq_1 (\sum{}_{\lambda' \in \Lambda'} \lambda'(p_{\lambda'})) \oplus \lambda(0)$$

and $\Lambda' \cap \{\lambda\} \subseteq \Lambda$, so $P \models^1 \Lambda$. Next if $\Lambda \subseteq \Lambda'$ then $P \sqsubseteq_1 \sum_{\lambda' \in \Lambda'} \lambda'(p_{\lambda'})$ and moreover $\bigcap\{\{\lambda'\} \mid \lambda' \in \Lambda'\} \subseteq \Lambda$, so $P \models^1 \Lambda$ here as well.
iv) Assume first that $P \models^2 \Lambda$, i.e. that there is $n \geq 1$, $\Lambda_1, \ldots, \Lambda_n$ and for all $\lambda' \in \bigcup_{j \geq 1}^n \Lambda_j = \Lambda$ some $p_{\lambda'}$ s.t.

$$[\sum{}_j(\sum_{\lambda' \in \Lambda_j} \lambda'(p_{\lambda'}))] \leq_2 [P],$$

and $\bigcap_j \Lambda_j \subseteq \Lambda$. Now $\sum_j(\sum_{\lambda' \in \Lambda_j} \lambda'(p_{\lambda'}))$ must Λ and hence P must Λ as well. Conversely assume that P must Λ. We then just need the construction of ii) to prove that $P \models^2 \Lambda$. □

It is now an easy task to see that our extended logic maintains the characterisation of the testing preorders.

Theorem 3.41 *For all P, Q and $i \in \{1, 2\}$, PR_iQ iff for all variable-free formulas $\phi \in \mathrm{PFm}'$, if $P \models^i \phi$ then $Q \models^i \phi$.*

PROOF: The only-if direction follows from 3.38. For the if-direction use 3.3 and 3.40—the proof is then straightforward. □

With the aid of these modified modalities and the constants Λ we have thus succeeded in giving matching algebraic and operational interpretations for the extended process system. We leave for future investigations the problem of finding (sound and) complete axiomatisations of the resulting logics.

Chapter 4

Models, weak orderings and logics

In this chapter we set up the basic semantical framework for the remainder of the thesis, namely that of indexed modal models. In this setting we study process equivalences such as testing equivalence, and their logical characterisations. The chapter serves as the groundwork for an application in the following chapter of our ideas to a semantically much richer setting than that of chapter 3.

We start by introducing the basic notions of (weak) indexed modal models and their linearisations, define the weak preorders and equivalences on models and show some of their basic properties. The idea is very similar to the shift of emphasis from processes to sets of processes in section 3.1.2. In section 2 we present characterisations of the weak preorders in terms of tests and modal logics.

The weak equivalences generalise a number of well known process equivalences such as testing equivalence proper [30] as well as our slightly nonstandard version of chapter 3, the failures equivalence of Brookes, Hoare and Roscoe [15], the improved failures equivalence of Brookes and Roscoe [16], the readiness semantics of Olderog and Hoare [82] and the refusals testing equivalence of Phillips [86]. We go on to show this in section 3, extending corresponding results of De Nicola [28, 29].

Finally, in section 4, we present as the main result of the chapter axiomatisations of the modal logics obtained, and prove them sound and complete. We rely on the completeness proofs in the following chapter. For this reason we have to adapt a proof method which is somewhat more cumbersome than would otherwise be necessary; we use a rewriting technique similar to the proof of theorem 3.35.

4.1 Linearising and ordering models

In this section we introduce and study the basic properties of the semantical framework to be employed in the sequel, namely indexed modal frames and models, their linearisations obtained by passing from states to sets of states and transition relations to partial successor relations, and the notions of "weak" equivalences and orderings—the term "weak" originally coined by Kennaway [57] for a particular equivalence among those we cover.

4.1.1 Frames, models and their linearisations

In its most basic form an indexed modal frame, or transition system, is a set S of states, or worlds, together with a set L of labels, or actions, and an L-indexed family of transition, or accessibility, relations $\xrightarrow{\lambda}$, $\lambda \in L$. It is often convenient to consider frames with some added structure. First, as in section 3.1, we consider an auxillary, unlabelled transition relation $\twoheadrightarrow$, used for denoting internal, or unobservable transitions similar to the τ-labelled transitions of CCS [76]. Secondly we assume a primitive partiality predicate $\uparrow$ to be defined on S. This is used for specifying states such as those containing unguarded recursion in e.g. CCS that should not be regarded as fully determined (c.f. [30, 77, 106]). So we let an (extended) *frame* be a structure $\mathcal{F} = \langle S, L, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda \in L}, \uparrow\rangle$, where

i) S is a set of *states*,

ii) L is a set of *labels*,

iii) $\twoheadrightarrow \subseteq S \times S$ and for all $\lambda \in L$, $\xrightarrow{\lambda} \subseteq S \times S$ are binary relations on S, the *transition* relations, and

iv) $\uparrow \subseteq S$ is the *partiality* predicate.

A *rooted frame* is a pair $\langle \sigma, \mathcal{F}\rangle$ with $\mathcal{F}$ a frame and $\sigma \in S_{\mathcal{F}}$. We often want to consider frames $\mathcal{F}$ with $L_{\mathcal{F}}$ drawn from a specific universe $\mathcal{L}$ of labels; if this is the case we say $\mathcal{F}$ is a *frame over* $\mathcal{L}$.

Process equivalences such as bisimulation and weak equivalences such as testing equivalence provide criteria for determining when two (rooted) frames should be considered semantically the same. Hennessy-Milner logic [51] gives a logical characterisation of bisimulation equivalence; similar characterisations of testing and failures equivalence may be given by restricting attention to suitable fragments of Hennessy-Milner logic [50, 17].

Testing equivalence may also be given a recursive characterisation by applying a bisimulation-like equivalence to frames that have been suitably transformed [28, 29]. This idea is used in the Edinburgh Concurrency Workbench [20] for applying techniques developed for bisimulation-related equivalences to testing equivalence. The transformation, or *linearisation*, involves first a shift of emphasis from individual states to *sets of states*, replacing the transition relation $\xrightarrow{\lambda}$ on states with the forwards linear, partial successor relation $\stackrel{\lambda}{\Rightarrow}$ on sets, $\Sigma_1, \Sigma_2 \subseteq S$, defined by

$$\Sigma_1 \stackrel{\lambda}{\Rightarrow} \Sigma_2 \text{ iff } \Sigma_2 \neq \emptyset \text{ and } \Sigma_2 = \{\tau \in S \mid \exists \sigma \in \Sigma_1, \sigma_1, \tau_1 \in S.\ \sigma \twoheadrightarrow^* \sigma_1 \xrightarrow{\lambda} \tau_1 \twoheadrightarrow^* \tau\}.$$

Secondly it is necessary to compensate for the loss of information in passing from $\xrightarrow{\lambda}$ to $\stackrel{\lambda}{\Rightarrow}$ by considering atomic propositions such as the "must-sets" of labels of section 3.1.2. These two ingredients are reflected in the acceptance tree model for testing equivalence [50], where the accessibility relation is forwards linear and partial, and where nodes are decorated with auxillary information such as acceptance sets.

We use these observations to give alternative logical accounts of weak equivalences based on indexed modal logic applied to linearised models. However, in attempting to linearise models we need some sensible way of extending valuations from states to sets of states. Intuitively we think of a set $\Sigma \subseteq S_{\mathcal{F}}$ as a kind of "virtual state"—a state that can at any moment turn into one of its members. If we keep this interpretation in mind there are two natural monotonicity properties of valuations that ensures such an extension can be sensibly carried out: Let $V : \mathrm{Ap} \to \mathcal{P}(S)$ be a valuation. Then V is

1- or *may*, if whenever $\sigma \twoheadrightarrow \sigma'$ and $\sigma' \in V(\alpha)$ then $\sigma \in V(\alpha)$, and

2- or *must*, if whenever $\sigma \twoheadrightarrow \sigma'$ and $\sigma \in V(\alpha)$ then $\sigma' \in V(\alpha)$.

Intuitively, a 1-valuation is a valuation that is existentially, and a 2-valuation one that is universally quantified over $\twoheadrightarrow$-futures. Such valuations have obvious extensions to sets $\Sigma \subseteq S$ obtained by taking

$\Sigma \in V(\alpha)$ iff $\sigma \in V(\alpha)$ for some $\sigma \in \Sigma$, whenever V is a 1-valuation, and

$\Sigma \in V(\alpha)$ iff $\sigma \in V(\alpha)$ for all $\sigma \in \Sigma$, whenever V is a 2-valuation.

In general a model must be allowed to contain both of these two types of valuation, and we thus arrive at the following basic definitions:

Definition 4.1 (Model, model class). Assume a nonempty labeluniverse $\mathcal{L}$ and a nonempty set Ap of atomic proposition symbols to be given.

i) A *model (over* $\mathcal{L}$ *and* Ap*)* is a structure $\mathcal{M} = \langle \mathcal{F}, V_1, V_2 \rangle$, where

   a) $\mathcal{F}$ is a frame over $\mathcal{L}$, and

   b) for $i \in \{1, 2\}$, V_i is an i-valuation of Ap into $\mathcal{P}(S_{\mathcal{F}})$.

ii) A *model class* is a structure $\langle \boldsymbol{\mathcal{M}}, \mathcal{L}, \mathrm{Ap} \rangle$, where

   a) $\mathcal{L}$ is a label universe,

   b) Ap is a set of atomic propositions, and

   c) $\boldsymbol{\mathcal{M}}$ is a class of models over $\mathcal{L}$ and Ap.

As for frames, a *rooted model* is a pair $\langle \sigma, \mathcal{M} \rangle$ with $\sigma \in S_{\mathcal{M}}$. We sometimes also consider *i-models*, for $i \in \{1, 2\}$, to be models equipped only with an i-valuation. Presently model classes are not very interesting entities. They turn out, however, to be very useful later when we introduce a parallel composition on models and thus need the label universes and atomic propositions to be in some way structured. For the time being we simply assume that we are working over some fixed model class.

We adopt some standard derived notions. First, $\Rightarrow$ denotes the reflexive, transitive closure of $\twoheadrightarrow$. Secondly the relation $\stackrel{\lambda}{\Rightarrow}$ is defined by $\stackrel{\lambda}{\Rightarrow} =_{\mathrm{def}} \Rightarrow \circ \stackrel{\lambda}{\rightarrow} \circ \Rightarrow$, where $\circ$ denotes relational composition. The relation $\stackrel{s}{\Rightarrow}$ for s a string of labels is derived from $\stackrel{\lambda}{\Rightarrow}$ in the standard way. Let further

$$\mathrm{init}(\sigma) = \{\lambda \in L \mid \exists \tau \in S.\ \sigma \stackrel{\lambda}{\rightarrow} \tau\}, \text{ and}$$

$$\mathrm{init}^{\Rightarrow}(\sigma) = \{\lambda \in L \mid \exists \tau \in S.\ \sigma \stackrel{\lambda}{\Rightarrow} \tau\}.$$

We finally introduce some computational notions related to sets $\Sigma \subseteq S$ of states similar to those adopted in chapter 3:

i) Σ can λ iff there are $\sigma \in \Sigma$ and $\tau \in S$ s.t. $\sigma \stackrel{\lambda}{\Rightarrow} \tau$.

ii) $\Sigma \Uparrow$ (Σ *divergent*) iff there is a $\sigma \in \Sigma$ and either an infinite $\twoheadrightarrow$-derivation from σ or for some $\tau \in S$, $\sigma \Rightarrow \tau$ and $\tau \uparrow$. Let $\Sigma \Downarrow$ (Σ *convergent*) iff not $\Sigma \Uparrow$.

iii) $\Sigma/\lambda = \{\tau \in S \mid \exists\sigma \in \Sigma.\ \sigma \stackrel{\lambda}{\Rightarrow} \tau\}$.

iv) $V_1{}^\dagger(\alpha) = \{\Sigma \subseteq S \mid \exists\sigma \in \Sigma.\ \sigma \in V_1(\alpha)\}$.

v) $V_2{}^\dagger(\alpha) = \{\Sigma \subseteq S \mid \forall\sigma \in \Sigma.\ \sigma \in V_2(\alpha)\}$.

Hence we obtain $\Sigma_1 \stackrel{\lambda}{\rightrightarrows} \Sigma_2$ iff Σ_1 can λ and $\Sigma_2 = \Sigma_1/\lambda$. In all cases we obtain corresponding notions on single states by specialisation as in the previous chapter, e.g. σ can λ iff $\{\sigma\}$ can λ.

If $\mathcal{M}$ is a model then its linearisation $\mathcal{M}^\dagger$ will contain some subset $\mathcal{S}$ of $\mathcal{P}(S_\mathcal{M})$ as its set of states, and transition relations $\stackrel{\lambda}{\rightrightarrows}$ plus the divergence predicate and extended valuations. What should the set $\mathcal{S}$ be? If we want to derive the weak preorders on elements from the weak preorders on linearised frames we must assume $\mathcal{S}$ to contain at least all singletons $\{\sigma\}$, for $\sigma \in S$. Moreover, $\mathcal{S}$ should clearly be closed under $\stackrel{\lambda}{\rightrightarrows}$. So these two requirements emerge as the minimal closure conditions for $\mathcal{S}$. We shall in fact assume closure under subset and finite unions also—this is due to the fact that we here and in the following chapter introduce operations on models such as a parallel composition, and wish linearisation to respect these operations up to weak equivalence. We could, on the other hand, allow $\mathcal{S}$ to be the full powerset of S, and indeed almost all results of this and the following chapter would go through should this assumption be made. It is, however, sometimes convenient to be able to restrict attention to sets that at least in some sense are "reachable"—theorem 4.6 provides an example of this. So we define the family of mappings lin_n, $n \geq 0$ inductively by

$$
\begin{aligned}
\mathrm{lin}_0(\mathcal{S}) &= \mathcal{S},\\
\mathrm{lin}_{n+1}(\mathcal{S}) &= \{\Sigma' \mid \Sigma' \neq \emptyset \text{ and } \exists\Sigma \in \mathrm{lin}_n(\mathcal{S}).\ \Sigma' \subseteq \Sigma\}\\
&\quad \cup \{\Sigma_1 \cup \Sigma_2 \mid \Sigma_1, \Sigma_2 \in \mathrm{lin}_n(\mathcal{S})\}\\
&\quad \cup \{\Sigma/\lambda \mid \Sigma \in \mathrm{lin}_n(\mathcal{S}).\ \Sigma \text{ can } \lambda\}
\end{aligned}
$$

and let $\mathrm{lin}(\mathcal{S}) = \bigcup_{n\geq 0} \mathrm{lin}_n(\mathcal{S})$. Then the *linearisation* of the model

$$\mathcal{M} = \langle\langle S, L, \twoheadrightarrow, \{\stackrel{\lambda}{\rightarrow}\}_{\lambda\in L}, \uparrow\rangle, V_1, V_2\rangle$$

is the model

$$\mathcal{M}^\dagger = \langle\langle \mathrm{lin}\{\{\sigma\} \mid \sigma \in S\}, L, \emptyset, \{\stackrel{\lambda}{\rightrightarrows}\}_{\lambda\in L}, \Uparrow\rangle, V_1{}^\dagger, V_2{}^\dagger\rangle,$$

where $\rightrightarrows$, $\Uparrow$, $V_1{}^\dagger$ and $V_2{}^\dagger$ are as given above. Clearly lin is a closure operator, and $\mathrm{lin}(\mathcal{S})$ is the least lin-closed set containing $\mathcal{S}$. Thus $S_{\mathcal{M}^\dagger}$ is the least set containing all singletons $\{\sigma\}$ for $\sigma \in S_{\mathcal{M}}$ which is closed under nonempty subsets, finite unions and the $\stackrel{\lambda}{\Rightarrow}$ for $\lambda \in L$. Clearly also the linearisation of a model is itself a model.

4.1.2 The weak orderings

We next introduce the "weak" orderings on rooted models. These are defined first on linearised models in very much the same spirit as the recursive characterisations of the may- and must-orderings of the previous chapter, theorem 3.3, and then inherited from linearisations to the models proper. For this purpose let R range over binary relations on rooted, linearised models $\langle \Sigma, \mathcal{M}^{\dagger} \rangle$. For ease of notation we usually let the models be understood from the context, and abbreviate the holding of $\langle \Sigma_1, {\mathcal{M}_1}^{\dagger} \rangle \; R \; \langle \Sigma_2, {\mathcal{M}_2}^{\dagger} \rangle$ by $\Sigma_1 R \, \Sigma_2$. Define the operators B_1, B_2, B on relations R by

i) $\Sigma_1(B_1(R))\Sigma_2$ iff

  a) for all $\alpha \in \mathrm{Ap}$, if $\Sigma_1 \in {V_1}^{\dagger}(\alpha)$ then $\Sigma_2 \in {V_1}^{\dagger}(\alpha)$, and

  b) for all $\lambda \in L_{\mathcal{M}_1}$, if Σ_1 can λ then ($\lambda \in L_{\mathcal{M}_2}$,) Σ_2 can λ and $(\Sigma_1/\lambda)R(\Sigma_2/\lambda)$.

ii) $\Sigma_1(B_2(R))\Sigma_2$ iff

  a) for all $\alpha \in \mathrm{Ap}$, if $\Sigma_1 \in {V_2}^{\dagger}(\alpha)$ then $\Sigma_2 \in {V_2}^{\dagger}(\alpha)$, and

  b) if $\Sigma_1 \Downarrow$ then $\Sigma_2 \Downarrow$ and for all $\lambda \in L_{\mathcal{M}_2}$, if Σ_2 can λ then ($\lambda \in L_{\mathcal{M}_1}$,) Σ_1 can λ and $(\Sigma_1/\lambda)R(\Sigma_2/\lambda)$.

iii) $\Sigma_1(B(R))\Sigma_2$ iff $\Sigma_1(B_1(R))\Sigma_2$ and $\Sigma_1(B_2(R))\Sigma_2$.

We then define the relations $\sqsubseteq_i^n$, $i \in \{1,2\}$, and $\sqsubseteq^n$ inductively by $\Sigma_1 \sqsubseteq_{(i)}^0 \Sigma_2$ for all Σ_1, Σ_2, and $\sqsubseteq_{(i)}^{n+1} = B_i(\sqsubseteq_{(i)}^n)$. As in chapter 3 the weak preorders are then obtained as the greatest fixed points of the operators B_1, B_2, B respectively.

Theorem 4.2 (The weak preorders). *The operators $B_{(i)}$, $i \in \{1,2\}$, preserves arbitrary intersections and hence possesses greatest fixed points $\sqsubseteq_{(i)}$, given by $\sqsubseteq_{(i)} = \bigcap_{n \geq 0} \sqsubseteq_{(i)}^n$.*

PROOF: Similar to theorem 3.3. □

We refer in general to $\sqsubseteq_1$ as the *1-weak preorder*, or the may-preorder, to $\sqsubseteq_2$ as the *2-weak preorder*, or the must-preorder, and to $\sqsubseteq$ as the *weak preorder*;

the *weak equivalences* $\simeq_{(i)}$ are the equivalences induced by the $\sqsubseteq_{(i)}$. The weak preorders and equivalences on rooted models for R one of the $\sqsubseteq_{(i)}$ or $\simeq_{(i)}$ are obtained by specialisation, i.e. $\langle\sigma_1, \mathcal{M}_1\rangle R \langle\sigma_2, \mathcal{M}_2\rangle$ iff $\langle\{\sigma_1\}, {\mathcal{M}_1}^\dagger\rangle R \langle\{\sigma_2\}, {\mathcal{M}_2}^\dagger\rangle$. The ambiguity involved is harmless as it is easy to see that for all Σ_1, Σ_2, $\Sigma_1 \sqsubseteq_{(i)} \Sigma_2$ according to the specialisation orders on linearised models iff $\Sigma_1 \sqsubseteq_{(i)} \Sigma_2$.

4.1.3 Basic properties

If we ignore atomic propositions the ordering $\sqsubseteq$ on linearised models coincide with a version of bisimulation taking divergence into account used by Milner [77] and studied by Abramsky [3] and Walker [114]. Notice also that the testing preorders of section 3.1.2 are easily rendered in the present setting by taking Ap to be finite, nonempty sets Λ of labels, $V_1(\Lambda)$ always to be the whole set $\mathrm{Pc}^\oplus$, $p \in V_2(\Lambda)$ iff p must Λ and by allowing deadlocked processes to diverge.

The weak preorders satisfy the following monotonicity properties:

Proposition 4.3

i) $\Sigma_1 \sqsubseteq_1 \Sigma_2$ *and* $\Sigma_1' \sqsubseteq_1 \Sigma_2$ *iff* $\Sigma_1 \cup \Sigma_1' \sqsubseteq_1 \Sigma_2$,

ii) $\Sigma_1 \sqsubseteq_2 \Sigma_2$ *and* $\Sigma_1 \sqsubseteq_2 \Sigma_2'$ *iff* $\Sigma_1 \sqsubseteq_2 \Sigma_2 \cup \Sigma_2'$,

iii) if $\Sigma_1 \sqsubseteq \Sigma_2$ *and* $\Sigma_1' \sqsubseteq \Sigma_2'$ *then* $\Sigma_1 \cup \Sigma_1' \sqsubseteq \Sigma_2 \cup \Sigma_2'$.

PROOF: Straightforward. □

We derive natural notions of embedding and isomorphism of models from the weak equivalences. Let i range over $\{1,2\}$. We say that a relation $R \subseteq S_{{\mathcal{M}_1}^\dagger} \times S_{{\mathcal{M}_2}^\dagger}$ is an *(i-)embedding* (of ${\mathcal{M}_1}^\dagger$ into ${\mathcal{M}_2}^\dagger$), if for each $\Sigma_1 \in S_{{\mathcal{M}_1}^\dagger}$ there is some $\Sigma_2 \in S_{{\mathcal{M}_2}^\dagger}$ s.t. $\Sigma_1 R\, \Sigma_2$, and whenever $\Sigma_1 R\, \Sigma_2$ then $\Sigma_1 \simeq_{(i)} \Sigma_2$. It is an *(i-)isomorphism* if in addition for every Σ_2 there is some Σ_1 s.t. $\Sigma_1 R\, \Sigma_2$; i.e. iff both R and R^{-1} are (i-)embeddings. We say $\mathcal{M}_1$ is *(i-)embedded in* $\mathcal{M}_2$, $\mathcal{M}_1 \precapprox_{(i)} \mathcal{M}_2$, iff there is some ($i$-)embedding $R \subseteq S_{{\mathcal{M}_1}^\dagger} \times S_{{\mathcal{M}_2}^\dagger}$, and finally $\mathcal{M}_1$ and $\mathcal{M}_2$ are *(i-)isomorphic*, $\mathcal{M}_1 \cong_{(i)} \mathcal{M}_2$, iff there is some ($i$-)isomorphism $R \subseteq S_{{\mathcal{M}_1}^\dagger} \times S_{{\mathcal{M}_2}^\dagger}$. The notions of embedding and isomorphism should be largely self-motivating. If $\mathcal{M}_1$ is embedded in $\mathcal{M}_2$ then every "reachable stateset" Σ_1 of $\mathcal{M}_1$ can—up to weak equivalence—be indistinguishably represented by some "reachable stateset" Σ_2 of $\mathcal{M}_2$, and if they are isomorphic, this goes the other way as well.

Under a couple of quite natural assumptions, the ordering $\sqsubseteq$ is obtained as the intersection of $\sqsubseteq_1$ and $\sqsubseteq_2$. Let $\mathcal{M}$ be a model.

i) $\mathcal{M}$ is *strong*, if for all $\sigma \in \mathcal{M}$ and $\alpha \in \mathrm{Ap}$, if $\sigma \in V_2(\alpha)$ then $\sigma \Downarrow$. A model class $\boldsymbol{\mathcal{M}}$ is *strong* if all its members are.

ii) $\mathcal{M}$ *preserves divergence*, if for all $\sigma_1, \sigma_2 \in \mathcal{M}$ and $\lambda \in L$, if $\sigma_1 \stackrel{\lambda}{\Rightarrow} \sigma_2$ and $\sigma_1 \Uparrow$ then $\sigma_2 \Uparrow$.

Proposition 4.4 *For all rooted models $\langle \sigma_1, \mathcal{M}_1 \rangle$, $\langle \sigma_2, \mathcal{M}_2 \rangle$, if $\mathcal{M}_1$ is strong and preserves divergence then $\sigma_1 \sqsubseteq \sigma_2$ iff $\sigma_1 \sqsubseteq_1 \sigma_2$ and $\sigma_1 \sqsubseteq_2 \sigma_2$,*

PROOF: The only-if direction is straightforward to check. For the if-direction it suffices to show that whenever $\Sigma_1 \sqsubseteq_1^n \Sigma_2$ and $\Sigma_1 \sqsubseteq_2^n \Sigma_2$ then $\Sigma_1 \sqsubseteq^n \Sigma_2$ for all n, assuming that $\mathcal{M}_1$ is strong and preserves divergence, by theorem 4.2. The base case is clear. So assume $\Sigma_1 \sqsubseteq_1^{n+1} \Sigma_2$ and $\Sigma_1 \sqsubseteq_2^{n+1} \Sigma_2$ and we show $\Sigma_1 B(\sqsubseteq^n)\Sigma_2$. Conditions i.a) and ii.a) of the definition of B are trivial. To check ii.b) assume that $\Sigma_1 \Downarrow$. Then $\Sigma_2 \Downarrow$ and whenever Σ_2 can λ then Σ_1 can λ and $\Sigma_1/\lambda \sqsubseteq_2^n \Sigma_2/\lambda$. But as Σ_1 can λ also $\Sigma_1/\lambda \sqsubseteq_1^n \Sigma_2/\lambda$, so indeed $\Sigma_1/\lambda \sqsubseteq^n \Sigma_2/\lambda$ by the induction hypothesis. For i.b) assume that Σ_1 can λ. Then Σ_2 can λ and $\Sigma_1/\lambda \sqsubseteq_1^n \Sigma_2/\lambda$. Now if $\Sigma_1 \Uparrow$ then $\Sigma_1/\lambda \Uparrow$ as $\mathcal{M}_1$ preserves divergence. But then $\Sigma_1/\lambda \sqsubseteq_2^n \Sigma_2/\lambda$ as $\mathcal{M}_1$ is strong, so indeed $\Sigma_1/\lambda \sqsubseteq^n \Sigma_2/\lambda$ by the induction hypothesis. If on the other hand $\Sigma_1 \Downarrow$ then $\Sigma_1/\lambda \sqsubseteq_2^n \Sigma_2/\lambda$ as well, and again $\Sigma_1/\lambda \sqsubseteq^n \Sigma_2/\lambda$, so $\Sigma_1 \sqsubseteq^{n+1} \Sigma_2$ and we are done. □

This result justifies our (and indeed others) focus of interest on $\sqsubseteq_1$ and $\sqsubseteq_2$. Certainly the assumption of "strongness" is satisfied in all the concrete interpretations of atomic propositions we consider later on. The assumption of divergence preservation is slightly more contentious. We show that up to $\simeq_1 \cap \simeq_2$ the assumption is harmless.

Proposition 4.5 *For any strong rooted model $\langle \sigma, \mathcal{M} \rangle$ there is a strong, divergence preserving, rooted model $\langle \sigma^*, \mathcal{M}^* \rangle$ s.t. $\sigma \simeq_1 \sigma^*$ and $\sigma \simeq_2 \sigma^*$.*

PROOF: Fix $\mathcal{M} = \langle \langle S, L, \twoheadrightarrow, \{\stackrel{\lambda}{\rightarrow}\}_{\lambda \in L}, \uparrow \rangle, V_1, V_2 \rangle$ and assume $\mathcal{M}$ is strong. We then define $\mathcal{M}^*$ by letting $S_{\mathcal{M}^*} = S \times \{0, 1\}$, $L_{\mathcal{M}^*} = L$, $\uparrow_{\mathcal{M}^*} = \emptyset$, $\langle \sigma, i \rangle \in V_{1,\mathcal{M}^*}(\alpha)$ iff $\sigma \in V_1(\alpha)$ and $\langle \sigma, i \rangle \in V_{2,\mathcal{M}^*}(\alpha)$ iff $\sigma \in V_2(\alpha)$ and $i = 0$, and the transition relations by

i) $\langle\sigma, i\rangle \twoheadrightarrow \langle\sigma', j\rangle$ iff either

a) $\sigma \twoheadrightarrow \sigma'$ and ($j = 0$ iff $i = 0$ and $\sigma \Downarrow$), or

b) $\sigma = \sigma'$ and $i = j = 1$,

ii) $\langle\sigma, i\rangle \xrightarrow{\lambda} \langle\sigma', j\rangle$ iff $\sigma \xrightarrow{\lambda} \sigma'$ and ($j = 0$ iff $i = 0$ and $\sigma \Downarrow$).

So we simply add a divergent copy of each state in $\mathcal{M}$. It is easy to check that indeed $\mathcal{M}^*$ is a strong and divergence-preserving model. Now each $\Sigma \in S_{\mathcal{M}^*\dagger}$ is represented by $\Sigma^* = \{\langle\sigma, 0\rangle \mid \sigma \in \Sigma\}$. Check next that for each $\Sigma \in S_{\mathcal{M}\dagger}$, Σ can λ iff Σ^* can λ, $\Sigma \Downarrow$ iff $\Sigma^* \Downarrow$, $\Sigma \in V_i(\alpha)$ iff $\Sigma^* \in V_i(\alpha)$, $i \in \{1, 2\}$, and further that whenever Σ can λ then $(\Sigma/\lambda)^* \simeq_1 \Sigma^*/\lambda$, and if also $\Sigma \Downarrow$ then $(\Sigma/\lambda)^* \simeq_2 \Sigma^*/\lambda$. But then the result follows as we have shown $(\cdot)^*$ to be a fixed point of both B_1 and B_2. □

Let us point out a little corollary of theorem 4.5. Say $\mathcal{M}$ is *image-finite*, if for all $\sigma \in S$ and $\lambda \in L$, $\{\sigma' \mid \sigma \xrightarrow{\lambda} \sigma'\}$ is finite.

Proposition 4.6 *For any strong, image-finite, rooted model $\langle\sigma, \mathcal{M}\rangle$ there is a rooted model $\langle\sigma', \mathcal{M}'\rangle$ s.t. $\sigma \simeq_1 \sigma'$ and $\sigma \simeq_2 \sigma'$ with the property that for all $\Sigma' \in S_{\mathcal{M}'\dagger}$, either Σ' is finite or $\Sigma' \Uparrow$.*

PROOF: Let $\langle\sigma, \mathcal{M}\rangle$ be strong and image-finite. By 4.5, $\sigma \simeq_i \sigma^*$ and $\mathcal{M}^*$ is strong and preserves divergence. Clearly by the proof of proposition 4.5, $\mathcal{M}^*$ may also be assumed to be image-finite. Let then $\Sigma \in S_{\mathcal{M}^*\dagger}$, and assume that Σ can λ. Now if $\Sigma \Uparrow$ then $\Sigma/\lambda \Uparrow$ and we are done, as $\mathcal{M}^*$ preserves divergence. On the other hand if Σ is finite, Σ can λ and $\Sigma/\lambda \Downarrow$ then Σ/λ is finite as well, and the proof is complete. □

4.1.4 Disjoint union

We introduce the semantical correlate of the internal sum operator which we relied so heavily on in the previous chapter. In the present setting this is captured by disjoint union of frames (c.f. [111]). With respect to the 1-weak preorder the disjoint union turns out to coincide with the sup, and with respect to the 2-weak preorder the inf—it is hence a very useful operation in the present context, which is why we introduce it at this point.

Let $\mathcal{M}_j = \langle\langle S_j, L_j, \twoheadrightarrow_j, \{\xrightarrow{\lambda}_j\}_{\lambda \in L_j}, \uparrow_j\rangle, V_{1,j}, V_{2,j}\rangle$, $j \in \{1,2\}$ be models. Then the *disjoint union* of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the model [1]

$$\mathcal{M}_1 \oplus \mathcal{M}_2 = \langle\langle S_1 \oplus S_2, L_1 \cup L_2, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda \in L_1 \cup L_2}, \uparrow\rangle, V_1, V_2\rangle,$$

where $S_1 \oplus S_2 = \{\langle\sigma, 1\rangle \mid \sigma \in S_1\} \cup \{\langle\sigma, 2\rangle \mid \sigma \in S_2\}$ is the disjoint union of sets S_1, S_2, and for all $k, l \in \{1,2\}$:

i) $\langle\sigma, k\rangle \twoheadrightarrow \langle\sigma', l\rangle$ iff $k = l$ and $\sigma \twoheadrightarrow_l \sigma'$,

ii) $\langle\sigma, k\rangle \xrightarrow{\lambda} \langle\sigma', l\rangle$ iff $k = l$ and $\sigma \xrightarrow{\lambda}_l \sigma'$,

iii) $\langle\sigma, k\rangle \uparrow$ iff $\sigma \uparrow_k$,

iv) $\langle\sigma, k\rangle \in V_1(\alpha)$ iff $\sigma \in V_{1,k}(\alpha)$,

v) $\langle\sigma, k\rangle \in V_2(\alpha)$ iff $\sigma \in V_{2,k}(\alpha)$.

It is easy to check that $\mathcal{M}_1 \oplus \mathcal{M}_2$ is well-defined—i.e. that the V_i, $i \in \{1,2\}$, of the definition are indeed 1- and 2-valuations respectively. We obtain the following basic properties of the disjoint union:

Proposition 4.7 *Let* $\Sigma_1 \in \mathcal{M}_1{}^\dagger$, $\Sigma_2 \in \mathcal{M}_2{}^\dagger$.

i) $\Sigma_1 \oplus \Sigma_2 \in (\mathcal{M}_1 \oplus \mathcal{M}_2)^\dagger$,

ii) $\Sigma_1 \oplus \Sigma_2$ can λ *iff* Σ_1 can λ *or* Σ_2 can λ,

iii) $(\Sigma_1 \oplus \Sigma_2)/\lambda = (\Sigma_1/\lambda) \oplus (\Sigma_2/\lambda)$,

iv) $(\Sigma_1 \oplus \Sigma_2) \Downarrow$ *iff* $\Sigma_1 \Downarrow$ *and* $\Sigma_2 \Downarrow$,

v) $\Sigma_1 \oplus \Sigma_2 \in V_1{}^\dagger(\alpha)$ *iff* $\Sigma_1 \in V_1{}^\dagger(\alpha)$ *or* $\Sigma_2 \in V_1{}^\dagger(\alpha)$,

vi) $\Sigma_1 \oplus \Sigma_2 \in V_2{}^\dagger(\alpha)$ *iff* $\Sigma_1 \in V_2{}^\dagger(\alpha)$ *and* $\Sigma_2 \in V_2{}^\dagger(\alpha)$.

PROOF: Straightforward. □

As in the previous chapter the $\oplus$ is the sup under $\sqsubseteq_1$ and the inf under $\sqsubseteq_2$ (up to isomorphism):

[1] We choose $\oplus$ here in order not to confuse with the CCS +-operator.

Proposition 4.8 *For all* $\mathcal{M}_1, \mathcal{M}_2$, $\Sigma_1 \in {\mathcal{M}_1}^\dagger$ *and* $\Sigma_2 \in {\mathcal{M}_2}^\dagger$,

i) $\Sigma_1 \sqsubseteq_1 \Sigma_2$ *iff* $\Sigma_1 \oplus \Sigma_2 \simeq_1 \Sigma_2$, *and*

ii) $\Sigma_1 \sqsubseteq_2 \Sigma_2$ *iff* $\Sigma_1 \oplus \Sigma_2 \simeq_2 \Sigma_1$.

Proof: Straightforward. □

We note further that $\oplus$ is well-behaved with respect to linearisation, and that $\oplus$ induces a semilattice structure on model classes—provided, of course, they are closed under this operation.

Proposition 4.9 *The following isomorphisms hold:*

i) $({\mathcal{M}_1}^\dagger \oplus {\mathcal{M}_2}^\dagger) \cong \mathcal{M}_1 \oplus \mathcal{M}_2$,

ii) $(\mathcal{M}_1 \oplus \mathcal{M}_2) \cong (\mathcal{M}_2 \oplus \mathcal{M}_1)$,

iii) $(\mathcal{M}_1 \oplus (\mathcal{M}_2 \oplus \mathcal{M}_3)) \cong ((\mathcal{M}_1 \oplus \mathcal{M}_2) \oplus \mathcal{M}_3)$,

iv) $(\mathcal{M} \oplus \mathcal{M}) \cong \mathcal{M}$.

Proof: Straightforward. □

4.2 Characterising the weak relations

The may- and must-orderings are not too transparent as they stand. In this section we investigate their characterisations in terms of tests and in terms of modal logics.

4.2.1 The testing characterisation

A non-recursive characterisation may be obtained using extremely simple notions of tests, generalising the "brooms" of De Nicola and Hennessy [30, 50] and failures of Brookes, Hoare and Roscoe [15]. Assume a label universe $\mathcal{L}$ and set of atomic proposition symbols Ap fixed. Then the set of *tests* $t \in \mathcal{T}$ is generated by the abstract syntax

$$t ::= 0 \mid \surd \mid \alpha \mid \lambda t,$$

where $\alpha \in \mathrm{Ap}$ and $\lambda \in L$. The *size*, $|\,t\,|$, of a test t is defined inductively by

$$|\,0\,| = |\surd| = 0,$$

$|\alpha| = 1,$

$|\lambda t| = |t| + 1.$

Let then $\mathcal{T}_n = \{t \in \mathcal{T} \mid |t| \leq n\}$. Intuitively tests have the power to explore the holding of atomic propositions along different paths in the model and the size of a test determines to which depth. We define, as in the previous chapter, the relations may and must between rooted models $\langle \sigma, \mathcal{M} \rangle$ and tests inductively by

i) σ m~~a~~y 0, σ may $\surd$,

σ may α iff $\sigma \in V_1(\alpha)$,

σ may λt iff ($\lambda \in L$,) σ can λ and for some σ' s.t. $\sigma \overset{\lambda}{\Rightarrow} \sigma'$, σ' may t,

ii) σ m~~u~~st 0, σ must $\surd$,

σ must α iff $\sigma \in V_2(\alpha)$,

σ must λt iff $\sigma \Downarrow$ and for all σ', if ($\lambda \in L$ and) $\sigma \overset{\lambda}{\Rightarrow} \sigma'$ then σ' must t.

Note that if $\sigma \Rightarrow \sigma'$ and σ' may t then σ may t as well, and similarly if $\sigma \Rightarrow \sigma'$ and σ must t then so σ' must t. It is not hard now to verify the following:

Theorem 4.10 (The testing characterisation). *For all rooted models $\langle \sigma_1, \mathcal{M}_1 \rangle$ and $\langle \sigma_2, \mathcal{M}_2 \rangle$,*

i) $\sigma_1 \sqsubseteq_1 \sigma_2$ iff for all $t \in \mathcal{T}$, if σ_1 may *t then σ_2* may *t, and*

ii) $\sigma_1 \sqsubseteq_2 \sigma_2$ iff for all $t \in \mathcal{T}$, if σ_1 must *t then σ_2* must *t.*

PROOF: It suffices to prove for all Σ_1, Σ_2 and $n \geq 0$ that

i) $\Sigma_1 \sqsubseteq_1^n \Sigma_2$ iff for all $t \in \mathcal{T}_n$, if Σ_1 may t then Σ_2 may t,

ii) $\Sigma_1 \sqsubseteq_2^n \Sigma_2$ iff for all $t \in \mathcal{T}_n$, if Σ_1 must t then Σ_2 must t,

where the may- and must-relations are extended to sets Σ in the obvious way. The proof is then straightforward. $\square$

4.2.2 The logical characterisation

We can generalise this characterisation of the preorders $\sqsubseteq_1$ and $\sqsubseteq_2$ to cover more general logics. We present logics that induce the preorders $\sqsubseteq_1$, $\sqsubseteq_2$ and $\sqsubseteq$ and their associated equivalences $\simeq_1$, $\simeq_2$ and $\simeq$ on models. These straightforward results can be viewed as specialisations of more general ones relating to intuitionistic modal logic, the bisimulation preorder along the lines of [106] to the present setting where models are linearised. Pursuing this, however, will take us too far astray and we stick instead to the simpler case at hand.

We start by introducing the language of modal formulas to be considered. The language of *ground formulas* (over $\mathcal{L}$ and Ap) $\phi \in$ GTFm is generated by the abstract syntax:

$$\phi ::= \alpha^1 \mid \alpha^2 \mid \top \mid \bot \mid \langle\lambda\rangle\phi \mid [\lambda]\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \neg\phi,$$

where $\lambda \in \mathcal{L}$ and $\alpha \in$ Ap. The *1- or may-fragment*, GTFm_1, is the sublanguage of GTFm whose only atomic proposition symbols are those of the form α^1 and whose only modalities are of the form $\langle\lambda\rangle$; similarly the *2- or must-fragment*, GTFm_2, is the sublanguage of GTFm containing only atomic propositions α^2 and modalities $[\lambda]$. For each sublanguage X of GTFm, the *positive fragment* X^+ of X is the sublanguage of X consisting of all negation-free formulas. As usual we assume that negation and the modal operators bind more strongly than $\wedge$ and $\vee$ and otherwise use parantheses to disambiguate. We define a (slightly nonstandard) modal depth of formulas. The *size*, or *modal depth*, $\mid \phi \mid$ of a $\phi \in$ GTFm is defined inductively by

$$|\top| = |\bot| = 0,$$

$$|\alpha^1| = |\alpha^2| = 1,$$

$$|\langle\lambda\rangle\phi| = |[\lambda]\phi| = |\phi| + 1,$$

$$|\phi \wedge \psi| = |\phi \vee \psi| = max(|\phi|, |\psi|), \text{ and}$$

$$|\neg\phi| = |\phi|,$$

and we let for each sublanguage X of GTFm and $n \geq 0$, $X_n = \{\phi \in X \mid |\phi| \leq n\}$.

We next define the relation $\models$ of *satisfaction* between rooted linearised models $\langle\Sigma, \mathcal{M}^\dagger\rangle$ and formulas $\phi \in$ GTFm. Instead of $\langle\Sigma, \mathcal{M}^\dagger\rangle \models \phi$ we sometimes write $\Sigma \models_{\mathcal{M}^\dagger} \phi$, or where $\mathcal{M}$ is understood from the context, write $\Sigma \models \phi$ instead. Now $\models$ is inductively defined by

$\Sigma \models \alpha^1$ iff $\Sigma \in {V_1}^\dagger(\alpha)$,

$\Sigma \models \alpha^2$ iff $\Sigma \in {V_2}^\dagger(\alpha)$,

$\Sigma \models \top$ for all Σ,

$\Sigma \not\models \bot$ for all Σ,

$\Sigma \models \langle\lambda\rangle\phi$ iff ($\lambda \in L_{\mathcal{M}}$,) Σ can λ and $\Sigma/\lambda \models \phi$,

$\Sigma \models [\lambda]\phi$ iff $\Sigma \Downarrow$ and if ($\lambda \in L_{\mathcal{M}}$ and) Σ can λ then $\Sigma/\lambda \models \phi$,

$\Sigma \models \phi \wedge \psi$ iff $\Sigma \models \phi$ and $\Sigma \models \psi$,

$\Sigma \models \phi \vee \psi$ iff $\Sigma \models \phi$ or $\Sigma \models \psi$,

$\Sigma \models \neg\phi$ iff $\Sigma \not\models \phi$.

Again satisfaction on single states is obtained by specialisation, i.e. $\langle\sigma, \mathcal{M}\rangle \models \phi$ iff $\langle\{\sigma\}, \mathcal{M}^\dagger\rangle \models \phi$.

Note the similarity between the modal operators as interpreted here and those we introduced in the conclusion of the preceding chapter. Here we revert to the "standard" satisfaction condition for the disjunction as compared with chapter 3. This is effectively forced upon us by the consideration of classical negation. It hardly needs emphasising that in proving properties for the full languages GTFm and GTFm_i we can take for instance $\vee$, $\top$ and $\bot$ to be derived operators. But it should be noted that $\langle\lambda\rangle$ and $[\lambda]$ are *not* dual.

The characteristic property for i-valuations extends to the whole of the positive i-fragments. For instance if $\sigma \twoheadrightarrow \sigma'$ and $\sigma' \models \phi$, $\phi \in \mathrm{GTFm}_1^+$ then $\sigma \models \phi$, and if $\Sigma \subseteq \Sigma'$ and $\Sigma \models \phi$, $\phi \in \mathrm{GTFm}_1^+$ then $\Sigma' \models \phi$, and correspondingly for the must-fragment. This is seen by an easy induction on the structure of formulas.

This logic gives us yet another set of characterisations of the weak orderings and equivalences. Let X be any sublanguage of GTFm. Say X *induces (the binary relation)* R on rooted models, if for all $\mathcal{M}_1, \mathcal{M}_2$, $\sigma_1 \in \mathcal{M}_1$, $\sigma_2 \in \mathcal{M}_2$, $\sigma_1 R \sigma_2$ iff for all $\phi \in X$, if $\sigma_1 \models \phi$ then $\sigma_2 \models \phi$. Evidently, if X induces R then R is a preorder, and if X is closed under negations then R is an equivalence. We obtain:

Theorem 4.11 (The logical characterisation, ground fragments). *For any model class* $\boldsymbol{\mathcal{M}}$ *and* $i \in \{1, 2\}$, $\mathrm{GTFm}_{(i)}^+$ *induces* $\sqsubseteq_{(i)}$ *and* $\mathrm{GTFm}_{(i)}$ *induces* $\simeq_{(i)}$.

PROOF: We prove only that GTFm induces $\simeq$. The proofs of the other cases are easily derived from this. The proof that whenever $\Sigma_1 \simeq \Sigma_2$ and $\Sigma_1 \models \phi$ then $\Sigma_2 \models \phi$ is an easy induction on the structure of ϕ. To complete the proof it suffices to show for all $n \geq 0$ that if for all $\phi \in \mathrm{GTFm}_n$, if $\Sigma_1 \models \phi$ then $\Sigma_2 \models \phi$, then also $\Sigma_1 \simeq^n \Sigma_2$, where we let $\Sigma_1 \simeq^n \Sigma_2$ iff $\Sigma_1 \sqsubseteq^n \Sigma_2$ and $\Sigma_2 \sqsubseteq^n \Sigma_1$. So assume that for all $\phi \in \mathrm{GTFm}_n$, $\Sigma_1 \models \phi$ implies $\Sigma_2 \models \phi$. Then also for all $\phi \in \mathrm{GTFm}_n$, $\Sigma_1 \models \phi$ iff $\Sigma_2 \models \phi$. We proceed by induction on n to show that then $\Sigma_1 \sqsubseteq^n \Sigma_2$—then by a symmetrical argument we obtain $\Sigma_1 \simeq^n \Sigma_2$. The base case is trivial, so let $n = n' + 1$. Clearly if $\Sigma \in {V_1}^{\dagger}(\alpha)$ ($\Sigma_1 \in {V_2}^{\dagger}(\alpha)$) also $\Sigma_2 \in {V_1}^{\dagger}(\alpha)$ ($\Sigma_2 \in {V_2}^{\dagger}(\alpha)$). Assume then that Σ_1 can λ. Then $\Sigma_1 \models \langle\lambda\rangle\top$ so also Σ_2 can λ. Now if Σ_1 can λ and $\Sigma_1/\lambda \models \phi \in \mathrm{GTFm}_{n'}$ then $\Sigma_1 \models \langle\lambda\rangle\phi$, so $\Sigma_2/\lambda \models \phi$ as well. Then by the induction hypothesis, $\Sigma_1/\lambda \sqsubseteq^{n'} \Sigma_2/\lambda$. Assume next that $\Sigma_1 \Downarrow$. Then $\Sigma_1 \models [\lambda]\top$ for all $\lambda \in \mathcal{L}$—which is nonempty, hence $\Sigma_2 \Downarrow$. Now if Σ_1 c~~a~~n λ then $\Sigma_1 \models [\lambda]\bot$, so Σ_2 c~~a~~n λ as well. We have already checked that if Σ_2 can λ and Σ_1 can λ then $\Sigma_1/\lambda \sqsubseteq^{n'} \Sigma_2/\lambda$ (in the must-case we need the $[\lambda]$ here). We have thus checked that $\Sigma_1 \sqsubseteq^n \Sigma_2$ and we are done. □

There is a close correspondence between the characterisation of the weak orders in terms of tests and in terms of the logic. We can translate tests $t \in \mathcal{T}$ into GTFm_i^+, $i \in \{1,2\}$ by the mappings $(\cdot)_1$ and $(\cdot)_2$ defined inductively by

i) $(0)_1 = (0)_2 = \bot$,

ii) $(\surd)_1 = (\surd)_2 = \top$,

iii) $(\alpha)_1 = \alpha^1$, $(\alpha)_2 = \alpha^2$, and

iv) $(\lambda t)_1 = \langle\lambda\rangle(t)_1$, $(\lambda t)_2 = [\lambda](t)_2$,

and it is easy to check that for all Σ and t, Σ may t iff $\Sigma \models (t)_1$ and Σ must t iff $\Sigma \models (t)_2$ thus obtaining theorem 4.10 as a corollary of (the proof of) theorem 4.11.

4.3 Examples: Testing, failures and refusal testing

We next turn to demonstrating how our general notions of models and weak preorders are capable of capturing important notions of process equivalence such as the testing equivalence of De Nicola and Hennessy [30], the failures and improved failures equivalences of Brookes, Hoare and Roscoe [15, 16], the readiness model of Olderog and Hoare [82] and the refusal testing equivalence of Phillips [86]. We emphasise yet again that the close connection between the testing and failures models has basically been etablished by De Nicola in his thesis [28], together with connections to a host of other, similar models—on this point our work essentially just fills in some details. Refusal testing on the other hand is a notion of equivalence strictly stronger than the notion of testing equivalence as studied by De Nicola and Hennessy, and it may thus be somewhat surprising that we can indeed capture refusal testing equivalence in the present framework. Actually we do not capture refusal testing in quite as direct a way as the other notions of equivalence—there we have to appeal to transformations of frames, essentially by adding explicit refusal transitions.

4.3.1 Testing equivalence

The notion of testing equivalence is based on the idea of equating systems according to whether or not one can find some experiment, or test, on the systems distinguishing them. This idea can be formalised in a number of ways. Here, following De Nicola (see [28], sec.1.6), the class of systems under consideration is frames [2], which we for the purpose of the present chapter assume to be over some fixed universe $\mathcal{L}$ of labels. The important points to be defined is to fix some class of *observers*, or tests, and to clarify what it means to run a test on a system and when such a run should be taken to be successful. As De Nicola we take observers to be frames themselves, equipped with an additional label to be used for reporting success. Other variations are possible: Abramsky [3] shows that by suitably increasing the expressive power of tests (to include for instance a capability of copying), observational equivalence [76, 77] can be rendered as a testing equivalence which is strictly finer than the one considered here.

So let the class, $\mathcal{O}$, of *observers* be the class of all frames over label-universe $\mathcal{L} \cup \{\omega\}$, where ω is some fixed element not in $\mathcal{L}$. The label ω is used by observers as a label for reporting the successful outcome of a test. Frames are tested by letting them run in parallel with some observer. Let $\mathcal{F}$ be a frame and $O \in \mathcal{O}$ an observer.

i) Define the binary relation $\rightarrow \subseteq (S_{\mathcal{F}} \times S_O) \times (S_{\mathcal{F}} \times S_O)$ as the least s.t.

[2] In contrast to De Nicola we do not assume image-finiteness in general.

a) if $\sigma, \sigma' \in S_{\mathcal{F}}$ and $\sigma \twoheadrightarrow \sigma'$ then for all $o \in S_O$, $\langle \sigma, o\rangle \rightarrow \langle \sigma', o\rangle$,

b) if $o, o' \in S_O$ and $o \twoheadrightarrow o'$ then for all $\sigma \in S_{\mathcal{F}}$, $\langle \sigma, o\rangle \rightarrow \langle \sigma, o'\rangle$

c) if $\sigma, \sigma' \in S_{\mathcal{F}}$, $o, o' \in S_O$, $\sigma \xrightarrow{\lambda} \sigma'$ and $o \xrightarrow{\lambda} o'$ then $\langle \sigma, o\rangle \rightarrow \langle \sigma', o'\rangle$.

ii) For each $\sigma \in S_{\mathcal{F}}$, $o \in S_O$ the set $\mathrm{comp}(\sigma, o)$ of *computations from* $\langle \sigma, o\rangle$ is the set of all maximal $\rightarrow$-derivations ρ from $\langle \sigma, o\rangle$.

iii) A computation $\rho \in \mathrm{comp}(\sigma, o)$ is said to be *successful*, if for some $i \geq 0$, ρ_i (the i'th configuration of ρ) is defined and if $\rho_i = \langle \sigma', o'\rangle$, say, then $\omega \in \mathrm{init}(o')$.

iv Extend the partiality predicate to pairs $\langle \sigma, o\rangle$ by $\langle \sigma, o\rangle \uparrow$ iff either $\sigma \uparrow$ or $o \uparrow$. Let $\langle \sigma, o\rangle \downarrow$ iff not $\langle \sigma, o\rangle \uparrow$.

v) A computation $\rho \in \mathrm{comp}(\sigma, o)$ is said to be *strongly successful*, if for some $i \geq 0$, ρ_i is defined and if $\rho_i = \langle \sigma', o'\rangle$, say, then $\omega \in \mathrm{init}(o')$ and for all j, $0 \leq j < i$, $\rho_j \downarrow$.

vi) $\sigma\ \mathrm{may}^{\mathrm{DH}}\ o$ iff there is some successful $\rho \in \mathrm{comp}(\sigma, o)$,

vii) $\sigma\ \mathrm{must}^{\mathrm{DH}}\ o$ iff for all $\rho \in \mathrm{comp}(\sigma, o)$, ρ is strongly successful.

Here we again let e.g. $\sigma\ \mathrm{may}^{\mathrm{DH}}\ o$ abbreviate $\langle \sigma, \mathcal{F}\rangle\ \mathrm{may}^{\mathrm{DH}}\ \langle o, O\rangle$. From the $\mathrm{may}^{\mathrm{DH}}$ and $\mathrm{must}^{\mathrm{DH}}$ relations the testing preorders and equivalences are derived as usual, by

$$\sigma \sqsubseteq_1^{\mathrm{DH}} \tau \text{ iff for all rooted observers } o \in O, \text{ if } \sigma\ \mathrm{may}^{\mathrm{DH}}\ o \text{ then } \tau\ \mathrm{may}^{\mathrm{DH}}\ o,$$

$$\sigma \sqsubseteq_2^{\mathrm{DH}} \tau \text{ iff for all rooted observers } o \in O, \text{ if } \sigma\ \mathrm{must}^{\mathrm{DH}}\ o \text{ then } \tau\ \mathrm{must}^{\mathrm{DH}}\ o,$$

and then $\sqsubseteq^{\mathrm{DH}} = \sqsubseteq_1^{\mathrm{DH}} \cap \sqsubseteq_2^{\mathrm{DH}}$. Let further $\simeq_i^{\mathrm{DH}}$, $\simeq^{\mathrm{DH}}$ denote the equivalences induced by the corresponding $\sqsubseteq_i^{\mathrm{DH}}$, $\sqsubseteq^{\mathrm{DH}}$. In particular the $\simeq^{\mathrm{DH}}$ is referred to as the *testing equivalence*.

In view of the way in which the rest of the orders and equivalences are derived from $\sqsubseteq_1^{\mathrm{DH}}$ and $\sqsubseteq_2^{\mathrm{DH}}$, we focus interest on the latter two. We show that with a particular choice of atomic propositions, the testing preorders $\sqsubseteq_i^{\mathrm{DH}}$ and the i-weak preorders $\sqsubseteq_i$ coincide. This result is essentially due to De Nicola [28] who proved a modified version of Kennaway's weak equivalence [57] to coincide with the equivalence $\simeq_2^{\mathrm{DH}}$ for strongly convergent processes—that is, processes for which all reachable states are convergent. Let $\mathrm{Ap}^{\mathrm{DH}} = \mathcal{P}(\mathcal{L})$ and for each frame $\mathcal{F}$ extend $\mathcal{F}$ to a model $\mathcal{F}^{\mathrm{DH}}$ by defining for each $\sigma \in \mathcal{F}$ and nonempty $\Lambda \subseteq \mathcal{L}$,

$\sigma \in V_1(\Lambda)$ iff for some $\lambda \in \Lambda$, σ can λ, and

$\sigma \in V_2(\Lambda)$ iff $\sigma \Downarrow$ and for all σ', if $\sigma \Rightarrow \sigma'$ then for some $\lambda \in \Lambda$, σ' can λ.

Tests $t \in \mathcal{T}$ have more or less obvious representations as observers. We define the mappings $(\cdot)_i^{\mathrm{DH}}$, $i \in \{1,2\}$, giving for each $t \in \mathcal{T}$ a rooted observer $\langle o, O\rangle$. Note that we need two mappings, as in contrast to the may-case, the must-representation of a test λt need the possibility of a "top-level" internal transition to a successful state. First for $(\cdot)_1^{\mathrm{DH}}$:

i) $(0)_1^{\mathrm{DH}} = \langle \emptyset, \langle \{\emptyset\}, \emptyset, \emptyset, \emptyset, \emptyset\rangle\rangle$,

ii) $(\surd)_1^{\mathrm{DH}} = \langle \emptyset, \langle \{\emptyset\}, \{\omega\}, \emptyset, \{\xrightarrow{\omega}\}, \emptyset\rangle\rangle$, where $\xrightarrow{\omega} = \{\langle \emptyset, \emptyset\rangle\}$,

iii) $(\Lambda)_1^{\mathrm{DH}} = \langle \emptyset, \langle \{\emptyset\} \cup \{\{\lambda\} \mid \lambda \in \Lambda\}, \Lambda \cup \{\omega\}, \emptyset, \{\xrightarrow{\lambda}\}_{\lambda \in \Lambda \cup \{\omega\}}, \emptyset\rangle\rangle$, where

   a) $\xrightarrow{\omega} = \{\langle \{\lambda\}, \{\lambda\}\rangle \mid \lambda \in \Lambda\}$, and

   b) for $\lambda \in \Lambda$, $\xrightarrow{\lambda} = \{\langle \emptyset, \{\lambda\}\rangle\}$,

iv) Let $(t)_1^{\mathrm{DH}} = \langle o, \langle S, L, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda \in L}, \uparrow\rangle\rangle$. Then
$(\lambda t)_1^{\mathrm{DH}} = \langle \emptyset, \langle S', L \cup \{\lambda\}, \twoheadrightarrow', \{\xrightarrow{\lambda'}{}'\}_{\lambda' \in L \cup \{\lambda\}}, \{\{o\} \mid o \uparrow\}\rangle\rangle$, where

   a) $S' = \{\{o\} \mid o \in S\} \cup \{\emptyset\}$,

   b) $\twoheadrightarrow' = \{\langle \{o_1\}, \{o_2\}\rangle \mid o_1 \twoheadrightarrow o_2\}$,

   c) for all λ' s.t. $\lambda \neq \lambda'$, $\xrightarrow{\lambda'}{}' = \{\langle \{o_1\}, \{o_2\}\rangle \mid o_1 \xrightarrow{\lambda'} o_2\}$,

   d) $\xrightarrow{\lambda}{}' = \{\langle \emptyset, \{o\}\rangle\} \cup \{\langle \{o_1\}, \{o_2\}\rangle \mid o_1 \xrightarrow{\lambda} o_2\}$.

For $(\cdot)_2^{\mathrm{DH}}$ take i)–iii) as above, and

iv) Let $(t)_2^{\mathrm{DH}} = \langle o, \langle S, L, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda \in L}, \uparrow\rangle\rangle$. Then
$(\lambda t)_2^{\mathrm{DH}} = \langle \emptyset, \langle S', L \cup \{\lambda, \omega\}, \twoheadrightarrow', \{\xrightarrow{\lambda'}{}'\}_{\lambda' \in L \cup \{\lambda, \omega\}}, \{\{\{o\}\} \mid o \uparrow\}\rangle\rangle$, where

   a) $S' = \{\emptyset\} \cup \{\{\emptyset\}\} \cup \{\{\{o\}\} \mid o \in S\}$,

   b) $\twoheadrightarrow' = \{\langle \{\{o_1\}\}, \{\{o_2\}\}\rangle \mid o_1 \twoheadrightarrow o_2\} \cup \{\langle \emptyset, \{\emptyset\}\rangle\}$,

   c) $\xrightarrow{\omega}{}' = \{\langle \{\emptyset\}, \{\emptyset\}\rangle\} \cup \{\langle \{\{o_1\}\}, \{\{o_2\}\}\rangle \mid o_1 \xrightarrow{\omega} o_2\}$,

   d) for all λ' s.t. $\lambda \neq \lambda'$ and $\lambda \neq \omega$, $\xrightarrow{\lambda'}{}' = \{\langle \{\{o_1\}\}, \{\{o_2\}\}\rangle \mid o_1 \xrightarrow{\lambda'} o_2\}$,

   e) $\xrightarrow{\lambda}{}' = \{\langle \emptyset, \{\{o\}\}\rangle\} \cup \{\langle \{\{o_1\}\}, \{\{o_2\}\}\rangle \mid o_1 \xrightarrow{\lambda} o_2\}$, This looks far more complicated than it actually is. Intuitively we can rephrase it in CCS-terms (see [76, 79]) as

$$(0)_1^{\mathrm{DH}} = 0,$$

$$(\surd)_1^{\mathrm{DH}} = \omega.0,$$

$$(\Lambda)_1^{\mathrm{DH}} = \textstyle\sum_{\lambda\in\Lambda} \lambda.\omega.0,$$

$$(\lambda t)_1^{\mathrm{DH}} = \lambda.(t)_1^{\mathrm{DH}},$$

and for $(\cdot)_2^{\mathrm{DH}}$ replacing the last clause by

$$(\lambda t)_2^{\mathrm{DH}} = \tau.\omega.0 + \lambda.(t)_2^{\mathrm{DH}}.$$

We verify first that these translations are indeed correct.

Proposition 4.12

i) σ may t *iff* σ may$^{\mathrm{DH}}$ $(t)_1^{\mathrm{DH}}$,

ii) σ must t *iff* σ must$^{\mathrm{DH}}$ $(t)_2^{\mathrm{DH}}$.

PROOF: A tedious, but straightforward induction on the structure of t. □

This result suffice to prove the containments $\sqsubseteq_i^{\mathrm{DH}} \subseteq \sqsubseteq_i$ and indeed $\sqsubseteq^{\mathrm{DH}} \subseteq \sqsubseteq$. To extend the former to equalities—and for strong and divergence preserving frames also the latter—we must show that tests are sufficiently expressive to separate any two frames separated by observers.

Lemma 4.13

i) if σ_1 may$^{\mathrm{DH}}$ o*,* σ_2 $\not\text{may}^{\mathrm{DH}}$ o *then for some* $t \in \mathcal{T}$*,* σ_1 may$^{\mathrm{DH}}$ $(t)_1^{\mathrm{DH}}$ *and* σ_2 $\not\text{may}^{\mathrm{DH}}$ $(t)_1^{\mathrm{DH}}$,

ii) if σ_1 must$^{\mathrm{DH}}$ o*,* σ_2 $\not\text{must}^{\mathrm{DH}}$ o *then for some* $t \in \mathcal{T}$*,* σ_1 must$^{\mathrm{DH}}$ $(t)_2^{\mathrm{DH}}$ *and* σ_2 $\not\text{must}^{\mathrm{DH}}$ $(t)_2^{\mathrm{DH}}$.

PROOF: Notice first that whenever $\rho = \langle\sigma_0, o_0\rangle \to \ldots \to \langle\sigma_n, o_n\rangle \to \ldots$ is a derivation from $\langle\sigma, o\rangle$ then there are corresponding derivations

$$\rho_\sigma = \sigma_0 \stackrel{(\lambda_0)}{\to} \ldots \stackrel{(\lambda_{n-1})}{\to} \sigma_n \stackrel{(\lambda_n)}{\to} \ldots,$$

and

$$\rho_o = o_0 \stackrel{(\lambda_0)}{\rightarrow} \ldots \stackrel{(\lambda_{n-1})}{\rightarrow} o_n \stackrel{(\lambda_n)}{\rightarrow} \ldots,$$

from σ and o respectively, where $\stackrel{(\lambda)}{\rightarrow}$ can be either the relation $\twoheadrightarrow$ or the relation $\stackrel{\lambda}{\rightarrow}$. Let $\mathrm{string}(\rho_\sigma) = \mathrm{string}(\rho_o)$ denote the—possibly infinite—string of labels occurring in ρ_σ/ρ_o.

i). Assume that σ_1 $\mathrm{may}^{\mathrm{DH}}$ o and σ_2 $\not\mathrm{may}^{\mathrm{DH}}$ o. Then there is a derivation $\rho \in \mathrm{comp}(\sigma_1, o)$ and an $n \geq 0$ s.t. ρ_n is defined and equal to, say, $\langle \sigma', o' \rangle$, and $\omega \in \mathrm{init}(o')$. Let $s = \mathrm{string}(\rho_{\sigma_1})$. Then σ_1 $\mathrm{may}^{\mathrm{DH}}$ $(s\surd)_1^{\mathrm{DH}}$ and σ_2 $\not\mathrm{may}^{\mathrm{DH}}$ $(s\surd)_1^{\mathrm{DH}}$.

ii). In proving ii) we have to be a little bit more careful. We first extend the $\mathrm{must}^{\mathrm{DH}}$ relation to sets Σ in the usual way, i.e. Σ $\mathrm{must}^{\mathrm{DH}}$ o iff for all $\sigma \in \Sigma$, σ $\mathrm{must}^{\mathrm{DH}}$ o. Next we extend the can-relation and successor map $\cdot/\cdot$ to strings $s \in \mathcal{L}^*$ by Σ can ε and Σ can λs iff Σ can λ and Σ/λ can s; and $\Sigma/\varepsilon = \Sigma$ and $\Sigma/(\lambda s) = (\Sigma/\lambda)/s$. Now assume that Σ_1 $\mathrm{must}^{\mathrm{DH}}$ o and $\sigma_2 \not\mathrm{must}^{\mathrm{DH}}$ o. Then either

a) there is a finite derivation $\rho = \langle \sigma'_0, o_0 \rangle \rightarrow \ldots \rightarrow \langle \sigma'_n, o_n \rangle$ from $\langle \sigma_2, o \rangle$ s.t. $\langle \sigma'_n, o_n \rangle \not\rightarrow$ and for all j, $0 \leq j \leq n$, $\langle \sigma'_j, o_j \rangle \downarrow$ and $\omega \notin \mathrm{init}(o_j)$, or

b) there is a finite derivation $\rho = \langle \sigma'_0, o_0 \rangle \rightarrow \ldots \rightarrow \langle \sigma'_n, o_n \rangle$ from $\langle \sigma_2, o \rangle$ s.t. $\langle \sigma'_n, o_n \rangle \uparrow$ and for all j, $0 \leq j \leq n$, $\omega \notin \mathrm{init}(o_j)$, or

c) there is an infinite derivation $\rho = \langle \sigma'_0, o_0 \rangle \rightarrow \ldots$ from $\langle \sigma_2, o \rangle$ s.t. for all $n \geq 0$, $\langle \sigma'_n, o_n \rangle \downarrow$ and $\omega \notin \mathrm{init}(o_n)$.

Note that if $\sigma_2 \not\mathrm{must}^{\mathrm{DH}}$ o then $\omega \notin \mathrm{init}(o)$ and hence, as Σ_1 $\mathrm{must}^{\mathrm{DH}}$ o, it must be the case that $\Sigma_1 \Downarrow$. Let now $\rho = \langle \sigma'_0, o_0 \rangle \twoheadrightarrow \ldots$ be some maximal derivation from $\langle \sigma_2, o \rangle$ which is not strongly successful, and let $s = \mathrm{string}(\rho_{\sigma_2})$. Assume first that s is infinite and that case b) fails to hold for any prefix of ρ—i.e. that for all $n \geq 0$, $\langle \sigma'_n, o_n \rangle \downarrow$. Then case c) holds for ρ. Now if there is some $\sigma_1 \in \Sigma_1$ s.t. there is an infinite derivation ρ_{σ_1} from σ_1 with $\mathrm{string}(\rho_{\sigma_1}) = s$ then $\Sigma_1 \not\mathrm{must}^{\mathrm{DH}}$ o, as we can construct a derivation ρ' from $\langle \sigma_1, o \rangle$ mimicing ρ and satisfying either b) or c). Then there must be some finite prefix $s_1\lambda$ of s s.t. Σ_1 can s_1 and $\Sigma_1/s_1 \not\mathrm{can}$ λ. Further, whenever Σ_1 can s'_1 for some prefix s'_1 of s_1 then $\Sigma_1/s'_1 \Downarrow$. Now we define, by induction on the length of s_1 a $t(s_1) \in \mathcal{T}$ s.t. $\sigma_2 \not\mathrm{must}^{\mathrm{DH}}$ $(t(s_1))_2^{\mathrm{DH}}$ and Σ_1 $\mathrm{must}^{\mathrm{DH}}$ $(t(s_1))_2^{\mathrm{DH}}$, by

1) $t(\varepsilon) = \lambda 0$,

2) $t(\lambda' s') = \lambda' t(s')$.

The proof of this is by an easy induction on the length of s_1. So we can assume that one of the cases a)–c) holds for ρ and if case c) holds then s is finite. We then proceed by a case-analysis.
Case c): As s is finite there must be some $n \geq 0$ s.t either $\sigma'_n \Uparrow$ or there is an infinite derivation $o_n \rightarrow o_{n+1} \rightarrow \ldots$ s.t. for no $m \geq n$ does $\omega \in \text{init}(o_m)$. Suppose the first subcase holds. Then we, as above, define by induction on the length of s a $t(s) \in \mathcal{T}$ s.t. $\Sigma_1 \text{ must}^{\text{DH}}\ (t(s))_2^{\text{DH}}$ and $\sigma_2\, \text{m}\!\!\not{u}\text{st}^{\text{DH}}\ (t(s))_2^{\text{DH}}$:

1) $t(\varepsilon) = \lambda\surd$, where $\lambda \in \mathcal{L}$ is arbitrary,

2) $t(\lambda' s') = \lambda' t(s')$.

To see this, assume $s = \varepsilon$. Clearly $\sigma_2\ \text{m}\!\!\not{u}\text{st}^{\text{DH}}\ (t(\varepsilon))_2^{\text{DH}}$ as $\sigma_2 \Uparrow$. But $\Sigma_1 \Downarrow$ as we noted, so whether Σ_1 can λ or not, $\Sigma_1 \text{ must}^{\text{DH}}\ (t(\varepsilon))_2^{\text{DH}}$. The induction step is straightforward. If the second subcase holds, then $\Sigma_1\ \text{c}\!\not{a}\text{n}\ s$ and the proof proceeds as in the case where s was infinite.
Case a): We proceed again by induction on the length of s. So assume that $s = \varepsilon$. Then $\sigma'_0 \Rightarrow \sigma'_n \not\twoheadrightarrow$ and $o_0 \Rightarrow o_n \not\twoheadrightarrow$. Then $\text{init}(\sigma'_n) \cap \text{init}(o_n) = \emptyset$ and for all j, $0 \leq j \leq n$, $\omega \notin \text{init}(o_j)$. Then, as $\Sigma_1 \text{ must}^{\text{DH}}\ o$, it must be the case that $\Sigma_1 \text{ must}^{\text{DH}}\ (\text{init}(o_n))_2^{\text{DH}}$, and we have $\sigma_2\, \text{m}\!\!\not{u}\text{st}^{\text{DH}}\ (\text{init}(o_n))_2^{\text{DH}}$. Assume then that $s = \lambda s'$. If $\Sigma_1\ \text{c}\!\not{a}\text{n}\ \lambda$ then $\Sigma_1 \text{ must}^{\text{DH}}\ (\lambda 0)_2^{\text{DH}}$ and $\sigma_2\ \text{m}\!\!\not{u}\text{st}^{\text{DH}}\ (\lambda 0)_2^{\text{DH}}$. If Σ_1 can λ let o_m, $0 \leq m \leq n$, be the immediate successor of the first λ-transition in ρ_o—i.e. m satisfies: λ_{m-1} is defined and equal to λ and whenever j satisfies $0 \leq j < m-1$ then λ_j is not defined—i.e. the j'th transition is unlabelled. Then $\Sigma_1/\lambda \text{ must}^{\text{DH}}\ o_m$ and $\sigma'_m\, \text{m}\!\!\not{u}\text{st}^{\text{DH}}\ o_m$ and then by the induction hypothesis we find some $t' \in \mathcal{T}$ s.t $\Sigma_1/\lambda \text{ must}^{\text{DH}}\ (t')_2^{\text{DH}}$. But then $\Sigma_1 \text{ must}^{\text{DH}}\ (\lambda t')_2^{\text{DH}}$ and $\sigma_2\, \text{m}\!\!\not{u}\text{st}^{\text{DH}}\ (\lambda t')_2^{\text{DH}}$, and we are done.
Case b): Again by induction on the length of s. If $s = \varepsilon$ then it must be the case that $\sigma_2 \Uparrow$. Then, as for case c), it will be the case that for all $\lambda \in \mathcal{L}$, $\Sigma_1 \text{ must}^{\text{DH}}\ (\lambda\surd)_2^{\text{DH}}$ and $\sigma_2\, \text{m}\!\!\not{u}\text{st}^{\text{DH}}\ (\lambda\surd)_2^{\text{DH}}$. The induction step is similar to the induction step of case b). We have thus completed the proof. □

Theorem 4.14 *For $i \in \{1, 2\}$,*

$$\langle \sigma_1, \mathcal{F}_1 \rangle \sqsubseteq_i^{\text{DH}} \langle \sigma_2, \mathcal{F}_2 \rangle \text{ iff } \langle \sigma_1, {\mathcal{F}_1}^{\text{DH}} \rangle \sqsubseteq_i \langle \sigma_2, {\mathcal{F}_2}^{\text{DH}} \rangle.$$

Proof: Use 4.12 and 4.13. □

We can in addition make the following observation: Let a frame $\mathcal{F}$ be *finitely branching*, if for all $\sigma \in \mathcal{F}$, $\{\lambda \mid \sigma \xrightarrow{\lambda} \sigma' \text{ for some } \sigma' \in \mathcal{F}\}$ is finite. Then by inspecting the proof of lemma 4.13 it can be seen that for finitely-branching frames [3] it suffices to take $\mathrm{Ap}^{\mathrm{DH}} = \{\Lambda \subseteq \mathcal{L} \mid \Lambda \text{ finite}\}$.

It should be emphasized again that there is nothing essentially new about theorem 4.14. The equivalence of the "De Nicola-Hennessy" preorders $\sqsubseteq_i^{\mathrm{DH}}$ and the i-weak preorders for $\mathrm{Ap}^{\mathrm{DH}}$ was shown in [28, 29]. There it is further shown that for strongly convergent systems, $\simeq_2^{\mathrm{DH}}$ coincides with a modified Kennaway's weak equivalence [57], failures equivalence [15] (which we return to below), and other equivalences (c.f. [26]).

4.3.2 Failures and readiness equivalence

The notion of failures equivalence, due to Brookes, Hoare and Roscoe [15], was proposed to serve as an operationally motivated denotational model for an abstract version of Hoare's CSP [56], known as TCSP. We adapt failures equivalence to the present setting and show again that it is captured by our notions of weak preorders with a particular choice of atomic propositions and their interpretations. Let

$$\mathrm{refusals}(\sigma) = \{\Lambda \subseteq \mathcal{L} \mid \exists \sigma'.\ \sigma \Rightarrow \sigma', \mathrm{init}^{\Rightarrow}(\sigma') \cap \Lambda = \emptyset\},$$

$$\mathrm{failures}(\sigma) = \{\langle s, \Lambda\rangle \mid \exists \sigma'.\ \sigma \stackrel{s}{\Rightarrow} \sigma', \Lambda \in \mathrm{refusals}(\sigma')\}, \text{ and}$$

$$\sigma_1 \sqsubseteq^{\mathrm{BHR}} \sigma_2 \text{ iff } \mathrm{failures}(\sigma_2) \subseteq \mathrm{failures}(\sigma_1).$$

As in the previous section we let $\mathrm{Ap}^{\mathrm{BHR}} = \mathcal{P}(\mathcal{L})$ and extend each frame $\mathcal{F}$ to a 1-model $\mathcal{F}^{\mathrm{BHR}}$ by defining for each $\sigma \in \mathcal{F}$ and $\Lambda \in \mathcal{L}$, $\sigma \in V_1(\Lambda)$ iff $\Lambda \in \mathrm{refusals}(\sigma)$. Clearly V_1 is a 1-valuation. We obtain

Theorem 4.15 $\langle \sigma_1, \mathcal{F}_1\rangle \sqsubseteq^{\mathrm{BHR}} \langle \sigma_2, \mathcal{F}_2\rangle$ *iff* $\langle \sigma_1, {\mathcal{F}_1}^{\mathrm{BHR}}\rangle \sqsubseteq_1 \langle \sigma_2, {\mathcal{F}_2}^{\mathrm{BHR}}\rangle$.

PROOF: We can regard every $\langle s, \Lambda\rangle \in \mathrm{failures}(\sigma)$ as a test in $\mathcal{T}$, and the result follows when we show that $\langle s, \Lambda\rangle \in \mathrm{failures}(\sigma)$ iff σ may $s\Lambda$—but this is a straightforward induction on the length of s. $\Box$

Failures equivalence may be critisized for not taking the possibility of divergence properly into account (c.f. [27, 96]). One may define an alternative version [16] of failures equivalence in the following way. Fix some distinguished symbol $\uparrow$. Let

[3] And *not* for image-finite ones as in [28].

i) $\sigma_1 \Uparrow s$ iff for some $\sigma_1' \in \mathcal{F}_1$ and prefix s' of s, $\sigma_1 \stackrel{s'}{\Rightarrow} \sigma_1'$ and $\sigma_1' \Uparrow$,

ii) $\text{failures}^{\uparrow}(\sigma_1) = \text{failures}(\sigma_1) \cup \{\langle s, \Lambda\rangle, \langle s, \uparrow\rangle \mid \sigma_1 \Uparrow s\}$,

iii) $\sigma_1 \sqsubseteq^{\text{BR}} \sigma_2$ iff $\text{failures}^{\uparrow}(\sigma_2) \subseteq \text{failures}^{\uparrow}(\sigma_1)$.

So this, intuitively, is a "catastrophic" view of divergence: If σ after s can diverge then after s anything might happen. This definition is an adaptation of the ideas of [16] to an operational setting; in [18] this and the model of [16] was shown to coincide for TCSP-processes. We capture $\sqsubseteq^{\text{BR}}$ in a very similar way to the examples we have already seen. We let (as usual) $\text{Ap}^{\text{BR}} = \mathcal{P}(\mathcal{L})$ and extend each frame $\mathcal{F}$ to a 2-model $\mathcal{F}^{\text{BR}}$ by defining for each $\sigma \in \mathcal{F}$, $\Lambda \subseteq \mathcal{L}$, $\sigma \in V_2(\Lambda)$ iff $\sigma \Downarrow$ and $\Lambda \notin \text{refusals}(\sigma)$. It is easy to see that this version of V_2 and the one for testing equivalence coincide.

Theorem 4.16

$$\langle \sigma_1, \mathcal{F}_1\rangle \sqsubseteq^{\text{BR}} \langle \sigma_2, \mathcal{F}_2\rangle \textit{ iff } \langle \sigma_1, {\mathcal{F}_1}^{\text{BR}}\rangle \sqsubseteq_2 \langle \sigma_2, {\mathcal{F}_2}^{\text{BR}}\rangle \textit{ iff } \langle \sigma_1, \mathcal{F}_1\rangle \sqsubseteq_2^{\text{DH}} \langle \sigma_2, \mathcal{F}_2\rangle.$$

PROOF: The last assertion is just 4.14 for $i = 2$ with the above observation in mind. For the first we extend first $\text{failures}^{\uparrow}$ to sets by $\text{failures}^{\uparrow}(\Sigma) = \bigcup_{\sigma\in\Sigma} \text{failures}^{\uparrow}(\sigma)$. So assume first that $\Sigma_1 \sqsubseteq^{\text{BR}} \Sigma_2$ and we proceed to show that for all $n \geq 0$, $\Sigma_1 \sqsubseteq_2^n \Sigma_2$—for then $\Sigma_1 \sqsubseteq_2 \Sigma_2$ by 4.2.ii). The base case is trivial. If $\Sigma_1 \in V_2(\Lambda)$ then $\langle \varepsilon, \uparrow\rangle, \langle \varepsilon, \Lambda\rangle \notin \text{failures}^{\uparrow}(\Sigma_1)$ so $\langle \varepsilon, \uparrow\rangle, \langle \varepsilon, \Lambda\rangle \notin \text{failures}^{\uparrow}(\Sigma_2)$ and hence $\Sigma_2 \in V_2(\Lambda)$. Assume next that $\Sigma_1 \Downarrow$ and Σ_2 can λ. Then $\langle \varepsilon, \uparrow\rangle \notin \text{failures}^{\uparrow}(\Sigma_1)$ so $\Sigma_2 \Downarrow$ as well, and $\langle \lambda, \emptyset\rangle \in \text{failures}^{\uparrow}(\Sigma_2)$ so Σ_1 can λ. Check then that $\text{failures}^{\uparrow}(\Sigma_2/\lambda) \subseteq \text{failures}^{\uparrow}(\Sigma_1/\lambda)$, so by the induction hypothesis $\Sigma_1/\lambda \sqsubseteq_2^n \Sigma_2/\lambda$ and thus $\Sigma_1 \sqsubseteq^{n+1} \Sigma_2$.

For the converse direction assume that $\Sigma_1 \sqsubseteq_2 \Sigma_2$ and assume that $\langle s, \Lambda\rangle \in \text{failures}^{\uparrow}(\Sigma_2)$. We proceed by induction on the length of s to show that then $\langle s, \Lambda\rangle \in \text{failures}^{\uparrow}(\Sigma_1)$. If $s = \varepsilon$ then either $\Sigma_2 \Uparrow$ or $\Sigma_2 \Downarrow$ and $\Sigma_2 \notin V_2(\Lambda)$. In the first case also $\Sigma_1 \Uparrow$ and in the second $\Sigma_1 \notin V_2(\Lambda)$, and in either case $\langle s, \Lambda\rangle \in \text{failures}^{\uparrow}(\Sigma_1)$. Let then $s = \lambda s'$. It can either be the case that $\Sigma_2 \Uparrow$ or if not that Σ_2 can λ and $\langle s', \Lambda\rangle \in \text{failures}^{\uparrow}(\Sigma_2)$. If $\Sigma_1 \Uparrow$ then $\langle s, \Lambda\rangle \in \text{failures}^{\uparrow}(\Sigma_1)$ so assume that $\Sigma_1 \Downarrow$. Then $\Sigma_2 \Downarrow$ as well, so Σ_2 can λ and then Σ_1 can λ and $\Sigma_1/\lambda \sqsubseteq_2 \Sigma_2/\lambda$. By the induction hypothesis, $\langle s', \Lambda\rangle \in \text{failures}^{\uparrow}(\Sigma_1/\lambda)$, and then $\langle s, \Lambda\rangle \in \text{failures}^{\uparrow}(\Sigma_1)$ as desired. The proof that $\langle s, \uparrow\rangle \in \text{failures}^{\uparrow}(\Sigma_2)$ implies $\langle s, \uparrow\rangle \in \text{failures}^{\uparrow}(\Sigma_1)$ is similar. □

It should be noted that in [15] refusal sets Λ are assumed to be finite. Of course 4.15 and 4.16 goes through smoothly with this assumption.

As the final close relative to the present equivalences we consider the readiness model of Olderog and Hoare [82]. This is a slight modification of the failures model based on "ready-sets" rather than refusal-sets, related to the linear-history model of Francez, Lehmann and Pnueli [38]. We assume again $\uparrow$ to be some distinguished symbol, and let

i) σ_1 stable iff $\sigma_1 \downarrow$ and for no σ_1' does $\sigma_1 \twoheadrightarrow \sigma_1'$,

ii) $\mathrm{traces}(\sigma_1) = \{s \mid \exists \sigma_1'. \sigma_1 \stackrel{s}{\Rightarrow} \sigma_1'\}$,

iii) $\mathrm{readies}(\sigma_1) = \{\langle s, \Lambda\rangle \mid \exists \sigma_1'. \sigma_1 \stackrel{s}{\Rightarrow} \sigma_1', \sigma_1'$ stable and $\Lambda = \mathrm{init}(\sigma_1')\}$,

iv) $\mathrm{observations}(\sigma_1) = \mathrm{traces}(\sigma_1) \cup \mathrm{readies}(\sigma_1) \cup \{s, \langle s, \uparrow\rangle, \langle s, \Lambda\rangle \mid \sigma_1 \Uparrow s\}$,

v) $\sigma_1 \sqsubseteq^{\mathrm{OH}} \sigma_2$ iff $\mathrm{observations}(\sigma_2) \subseteq \mathrm{observations}(\sigma_1)$.

Again it should be noted that this definition is actually an adaptation of the definitions of Olderog and Hoare to the present operationally based setting. We let $\mathrm{Ap}^{\mathrm{OH}} = \mathcal{P}(\mathcal{L})$ and extend any frame $\mathcal{F}$ to a 2-model $\mathcal{F}^{\mathrm{OH}}$ by defining $\sigma \in V_2(\Lambda)$ iff $\sigma \Downarrow$ and for all σ', if $\sigma \Rightarrow \sigma'$ and σ' stable then $\Lambda \neq \mathrm{init}(\sigma')$. Again V_2 is obviously a 2-valuation, so the extension is well-defined. We obtain

Theorem 4.17 *For each $\mathcal{F}_1, \mathcal{F}_2$, $\sigma_1 \in \mathcal{F}_1$, $\sigma_2 \in \mathcal{F}_2$,*

$$\langle \sigma_1, \mathcal{F}_1\rangle \sqsubseteq^{\mathrm{OH}} \langle \sigma_2, \mathcal{F}_2\rangle \quad \textit{iff} \quad \langle \sigma_1, \mathcal{F}_1^{\mathrm{OH}}\rangle \sqsubseteq_2 \langle \sigma_2, \mathcal{F}_2^{\mathrm{OH}}\rangle.$$

PROOF: As for 4.16. □

4.3.3 Refusal testing

As our last example we consider the extension of testing equivalence due to Phillips [86], where processes are in addition allowed to be tested for their abilities to reject actions. This is a genuine strengthening of testing equivalence, due to the fact that we can continue testing even after the potential refusal of some action has been registered [86]. This again means that the approach of the first examples where refusals and acceptances are captured by suitably instantiating the atomic propositions must be abandoned, and refusals turned into a dynamic notion on a par with "ordinary" transitions instead. The set of *refusal tests* $t \in \mathcal{T}^{\mathrm{P}}$ be given by the abstract syntax

$$t ::= \surd \mid 0 \mid \lambda t \mid \tilde{\lambda} t \mid \underline{\lambda} t \mid \underline{\tilde{\lambda}} t,$$

where $\lambda \in \mathcal{L}$, and define for each $\lambda \in \mathcal{L}$ the binary relation $\xrightarrow{\lambda}, \xrightarrow{\tilde{\lambda}} \subseteq \mathcal{T}^{\mathrm{P}} \times \mathcal{T}^{\mathrm{P}}$ as the least s.t.

i) $\lambda t \xrightarrow{\lambda} t$, $\lambda t \xrightarrow{\tilde{\lambda}} 0$,

ii) $\tilde{\lambda} t \xrightarrow{\lambda} 0$, $\tilde{\lambda} t \xrightarrow{\tilde{\lambda}} t$,

iii) $\underline{\lambda} t \xrightarrow{\lambda} \surd$, $\underline{\lambda} t \xrightarrow{\tilde{\lambda}} t$,

iv) $\underline{\tilde{\lambda}} t \xrightarrow{\lambda} t$, $\underline{\tilde{\lambda}} t \xrightarrow{\tilde{\lambda}} \surd$.

We define, as for testing equivalence, for each σ and $t \in \mathcal{T}^{\mathrm{P}}$ the set $\mathrm{comp}^{\mathrm{P}}(\sigma, t)$ of computations from $\langle \sigma, t \rangle$. First for each frame $\mathcal{F}$ let $\rightarrow \subseteq \mathcal{F} \times \mathcal{T}^{\mathrm{P}}$ be the least relation s.t. for all $\sigma, \sigma' \in \mathcal{F}$ and $t, t' \in \mathcal{T}^{\mathrm{P}}$,

i) if $\sigma \twoheadrightarrow \sigma'$ then for all $t \in \mathcal{T}^{\mathrm{P}}$, $\langle \sigma, t \rangle \rightarrow \langle \sigma', t \rangle$,

ii) if $\sigma \xrightarrow{\lambda} \sigma'$, $t \xrightarrow{\lambda} t'$ then $\langle \sigma, t \rangle \rightarrow \langle \sigma', t' \rangle$,

iii) if σ stable, σ c~~a~~n λ and $t \xrightarrow{\tilde{\lambda}} t'$ then $\langle \sigma, t \rangle \rightarrow \langle \sigma, t' \rangle$.

Then the set $\mathrm{comp}^{\mathrm{P}}(\sigma, t)$ of *refusal-* or *(P-)computations from* $\langle \sigma, t \rangle$ is the set of all maximal $\rightarrow$-derivations from $\langle \sigma, t \rangle$; say a $\rho \in \mathrm{comp}^{\mathrm{P}}(\sigma, t)$ is *P-successful* if for some $i \geq 0$, ρ_i is defined and has the form $\langle \sigma_i, \surd \rangle$; and say ρ is *(P-)strongly successful*, if for some $i \geq 0$, ρ_i is defined and has the form $\langle \sigma_i, \surd \rangle$ and for all j, $0 \leq j < i$, if $\rho_j = \langle \sigma_j, t_j \rangle$ then $\sigma_j \downarrow$. Then, as for testing equivalence, the relations $\mathrm{may}^{\mathrm{P}}$, $\mathrm{must}^{\mathrm{P}}$ and the corresponding preorders are defined by

σ $\mathrm{may}^{\mathrm{P}}$ t iff there is some successful $\rho \in \mathrm{comp}^{\mathrm{P}}(\sigma, t)$,

σ $\mathrm{must}^{\mathrm{P}}$ t iff for all $\rho \in \mathrm{comp}^{\mathrm{P}}(\sigma, t)$, ρ is strongly successful.

$\sigma_1 \sqsubseteq_1^{\mathrm{P}} \sigma_2$ iff for all $t \in \mathcal{T}^{\mathrm{P}}$, if σ_1 $\mathrm{may}^{\mathrm{P}}$ t then σ_2 $\mathrm{may}^{\mathrm{P}}$ t,

$\sigma_1 \sqsubseteq_2^{\mathrm{P}} \sigma_2$ iff for all $t \in \mathcal{T}^{\mathrm{P}}$, if σ_1 $\mathrm{must}^{\mathrm{P}}$ t then σ_2 $\mathrm{must}^{\mathrm{P}}$ t,

We capture the refusal testing orderings as the weak orderings on frames where refusals have been made explicit by adding them as genuine transitions. First, however, we recall a result from [86]. Let the sublanguage $\mathcal{T}_1^{\mathrm{P}}$ of $\mathcal{T}^{\mathrm{P}}$ be generated by

$$t ::= \surd \mid \lambda t \mid \tilde{\lambda} t,$$

and correspondingly the sublanguage $\mathcal{T}_2^{\mathrm{P}}$ of $\mathcal{T}^{\mathrm{P}}$ be generated by

$$t ::= 0 \mid \underline{\lambda} t \mid \underline{\tilde{\lambda}} t.$$

Lemma 4.18 ([86]). *Let $t \in \mathcal{T}^{\mathrm{P}}$. Then there are finite sets $T_1(t) \subseteq \mathcal{T}_1^{\mathrm{P}}$, $T_2(t) \subseteq \mathcal{T}_2^{\mathrm{P}}$ s.t. for any frame $\mathcal{F}$, $\sigma \in \mathcal{F}$,*

i) $\sigma\ \mathrm{may}^{\mathrm{P}}\ t$ iff for some $t' \in T_1(t)$, $\sigma\ \mathrm{may}^{\mathrm{P}}\ t'$, and

ii) $\sigma\ \mathrm{must}^{\mathrm{P}}\ t$ iff for all $t' \in T_2(t)$, $\sigma\ \mathrm{must}^{\mathrm{P}}\ t'$.

Proof: See [86]. □

By this lemma we can cut down the number of cases to consider substantially. We obtain the following inductive characterisation of the $\mathrm{may}^{\mathrm{P}}$ and $\mathrm{must}^{\mathrm{P}}$ relations:

Proposition 4.19 *For any $\mathcal{F}$, $\sigma \in \mathcal{F}$ and $t \in \mathcal{T}^{\mathrm{P}}$:*

i) $\sigma\ \mathrm{may}^{\mathrm{P}}\ \surd$,

$\sigma\ \mathrm{may}^{\mathrm{P}}\ \lambda t$ iff for some σ' s.t. $\sigma \overset{\lambda}{\Rightarrow} \sigma'$, $\sigma'\ \mathrm{may}^{\mathrm{P}}\ t$,

$\sigma\ \mathrm{may}^{\mathrm{P}}\ \tilde{\lambda} t$ iff for some σ' stable s.t. $\sigma \Rightarrow \sigma'$, $\sigma'\ \mathrm{c\not{a}n}\ \lambda$ and $\sigma'\ \mathrm{may}^{\mathrm{P}}\ t$.

ii) $\sigma\ \mathrm{m\not{u}st}^{\mathrm{P}}\ 0$,

$\sigma\ \mathrm{must}^{\mathrm{P}}\ \underline{\lambda} t$ iff $\sigma \Downarrow$ and for all σ' stable, if $\sigma \Rightarrow \sigma'$ and $\sigma'\ \mathrm{c\not{a}n}\ \lambda$ then $\sigma'\ \mathrm{must}^{\mathrm{P}}\ t$,

$\sigma\ \mathrm{must}^{\mathrm{P}}\ \underline{\tilde{\lambda}} t$ iff $\sigma \Downarrow$ and for all σ', if $\sigma \overset{\lambda}{\Rightarrow} \sigma'$ then $\sigma'\ \mathrm{must}^{\mathrm{P}}\ t$.

Proof: By the definitions. □

We capture refusal testing by letting $\mathrm{Ap}^{\mathrm{P}} = \emptyset$ and $\tilde{\mathcal{L}}$ denote a disjoint copy of $\mathcal{L}$, $\mathcal{L}$ and $\tilde{\mathcal{L}}$ connected via the bijection $\tilde{(\cdot)}$. Given a frame

$$\mathcal{F} = \langle S, L, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda \in L}, \uparrow\rangle$$

over $\mathcal{L}$ we define the model (or frame; with $\mathrm{Ap}^{\mathrm{P}} = \emptyset$ it makes no difference, of course):

$$\mathcal{F}^{\mathrm{P}} = \langle S, L \cup \tilde{L}, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda \in L \cup \tilde{L}}, \uparrow\rangle$$

over $\mathcal{L} \cup \tilde{\mathcal{L}}$ (and $\emptyset$), where $\tilde{L} = \{\tilde{\lambda} \mid \lambda \in L\}$ and $\sigma \xrightarrow{\lambda} \sigma'$ iff either $\lambda \in L$ and $\sigma \xrightarrow{\lambda} \sigma'$ in $\mathcal{F}$, or $\lambda \in \tilde{L}$, σ stable, $\sigma\ \mathrm{c\not{a}n}\ \tilde{\lambda}$ and $\sigma = \sigma'$. We prove the refusal testing preorders and the weak preorders over $\mathcal{L} \cup \tilde{\mathcal{L}}$ to coincide by appealing to the characterisation of the weak preorders of theorem 4.10. Define the translations $(\cdot)_1^{\mathrm{P}}$ and $(\cdot)_2^{\mathrm{P}}$ of tests over $\mathcal{L} \cup \tilde{\mathcal{L}}$ to refusal tests by

$(0)_1^{\mathrm{P}} = (0)_2^{\mathrm{P}} = 0,$

$(\surd)_1^{\mathrm{P}} = (\surd)_2^{\mathrm{P}} = \surd,$

$(\lambda t)_1^{\mathrm{P}} = \lambda(t)_1^{\mathrm{P}},\ (\lambda t)_2^{\mathrm{P}} = \underline{\tilde{\lambda}}(t)_2^{\mathrm{P}}.$

We obtain

Lemma 4.20 *For all t over $\mathcal{L} \cup \tilde{\mathcal{L}}$,*

i) $\langle \sigma, \mathcal{F}^{\mathrm{P}} \rangle$ may t *iff* $\langle \sigma, \mathcal{F} \rangle$ may$^{\mathrm{P}}$ $(t)_1^{\mathrm{P}}$,

ii) $\langle \sigma, \mathcal{F}^{\mathrm{P}} \rangle$ must t *iff* $\langle \sigma, \mathcal{F} \rangle$ must$^{\mathrm{P}}$ $(t)_2^{\mathrm{P}}$.

Proof: An easy induction on t. □

And now the ingredients are all in place:

Theorem 4.21 *For $i \in \{1, 2\}$, $\mathcal{F}_1$,$\mathcal{F}_2$ frames over $\mathcal{L}$, $\sigma_1 \in \mathcal{F}_1$, and $\sigma_2 \in \mathcal{F}_2$,*

$$\langle \sigma_1, \mathcal{F}_1 \rangle \sqsubseteq_i^{\mathrm{P}} \langle \sigma_2, \mathcal{F}_2 \rangle \text{ iff } \langle \sigma_1, \mathcal{F}_1{}^{\mathrm{P}} \rangle \sqsubseteq_i \langle \sigma_2, \mathcal{F}_2{}^{\mathrm{P}} \rangle.$$

Proof: The only-if direction follows by 4.20. For the if-directions, if $\sigma_1 \not\sqsubseteq_1^{\mathrm{P}} \sigma_2$ first there is a test $t \in \mathcal{T}^{\mathrm{P}}$ s.t. σ_1 may$^{\mathrm{P}}$ t and σ_2 m$\not{a}$y$^{\mathrm{P}}$ t, and then there is by 4.18.i) some $t_1 \in T_1(t)$ s.t. σ_1 may$^{\mathrm{P}}$ t_1 and σ_2 m$\not{a}$y$^{\mathrm{P}}$ t_1. But $t_1 = (t'_1)_1^{\mathrm{P}}$ for some $t'_1 \in \mathcal{T}$ over $\mathcal{L} \cup \tilde{\mathcal{L}}$ and then by 4.20.i) and 4.10 σ_1 may $(t'_1)_1^{\mathrm{P}}$ and σ_2 m$\not{a}$y $(t'_1)_1^{\mathrm{P}}$, whence $\langle \sigma_1, \mathcal{F}_1{}^{\mathrm{P}} \rangle \not\sqsubseteq_1 \langle \sigma_2, \mathcal{F}_2{}^{\mathrm{P}} \rangle$.
Similarly, for $i = 2$, if $\sigma_1 \not\sqsubseteq_2^{\mathrm{P}} \sigma_2$ we find some $t_2 \in \mathcal{T}_2^{\mathrm{P}}$ s.t. σ_1 must$^{\mathrm{P}}$ t_2, σ_2 m$\not{u}$st$^{\mathrm{P}}$ t_2 and then, as again $t_2 = (t'_2)_2^{\mathrm{P}}$ for some $t'_2 \in \mathcal{T}$ over $\mathcal{L} \cup \tilde{\mathcal{L}}$, the result obtains. □

4.4 Axiomatising the ground logics

Axiomatising the ground logics introduced in section 4.1 is a really straightforward matter—the only (slight) complication stemming from the necessity of taking divergence into account in the must-case. First let us make clear what the semantic notions we are interested in are.

Definition 4.22 For each strong model class $\boldsymbol{\mathcal{M}}$ define for $\phi, \psi \in$ GTFm,

i) Local consequence: ϕ $\boldsymbol{\mathcal{M}}$-*entails* ψ ($\phi \models_{\mathcal{M}} \psi$) iff for all $\mathcal{M} \in \boldsymbol{\mathcal{M}}$, $\Sigma \in S_{\mathrm{M}^\dagger}$, if $\langle \Sigma, \mathcal{M}^\dagger \rangle \models \phi$ then $\langle \Sigma, \mathcal{M}^\dagger \rangle \models \psi$. Further ϕ *entails* ψ ($\phi \models \psi$) iff for all model classes $\boldsymbol{\mathcal{M}}$, $\phi \models_{\mathcal{M}} \psi$.

ii) Validity: ϕ $\boldsymbol{\mathcal{M}}$-*valid* ($\models_{\mathcal{M}}\phi$) iff for all $\mathcal{M} \in \boldsymbol{\mathcal{M}}$, $\Sigma \in S_{\mathrm{M}^\dagger}$, $\langle \Sigma, \mathcal{M}^\dagger \rangle \models \phi$. Further ϕ *valid* ($\models \phi$) iff for all model classes $\boldsymbol{\mathcal{M}}$, $\models_{\mathcal{M}}\phi$.

iii) Consistency: ϕ $\boldsymbol{\mathcal{M}}$-*consistent* iff $\phi \not\models_{\mathcal{M}}\bot$ and ϕ *consistent* iff $\phi \not\models \bot$.

iv) Equivalence: ϕ and ψ are $\boldsymbol{\mathcal{M}}$-*equivalent* ($\phi \sim_{\mathcal{M}} \psi$) iff $\phi \models_{\mathcal{M}} \psi$ and $\psi \models_{\mathcal{M}} \phi$; and they are *equivalent* ($\phi \sim \psi$) iff $\phi \models \psi$ and $\psi \models \phi$.

4.4.1 Axiomatisation

We concentrate on the fragments GTFm_i^+ for $i \in \{1,2\}$. There is no difficulty in axiomatising the fuller fragments; we haven't, however, yet extended those to the compositional logics we shall consider later. Note also that we focus on *strong* model classes; these are more natural, we believe, but there is no difficulty in relaxing this requirement in the general work to follow. For $i \in \{1,2\}$ let $\mathbf{GL}^i$ be the logic over sequents $\phi \vdash_i \psi$ with $\phi, \psi \in \mathrm{GTFm}_i^+$ axiomatised by the following axioms and rules:

Axioms and rules common to $\vdash_1$ and $\vdash_2$:

1_i.	$\phi \vdash_i \phi$	(Reflexivity)
2_i.	$\phi \wedge \psi \vdash_i \phi$	($\wedge$-elim-1)
3_i.	$\phi \wedge \psi \vdash_i \psi$	($\wedge$-elim-2)
4_i.	$\phi \vdash_i \phi \vee \psi$	($\vee$-intro-1)
5_i.	$\psi \vdash_i \phi \vee \psi$	($\vee$-intro-2)
6_i.	From $\phi \vdash_i \psi$ and $\psi \vdash_i \gamma$ infer $\phi \vdash_i \gamma$	(Transitivity)
7_i.	From $\phi \vdash_i \psi$ and $\phi \vdash_i \gamma$ infer $\phi \vdash_i \psi \wedge \gamma$	($\wedge$-intro)
8_i.	From $\phi \vdash_i \gamma$ and $\psi \vdash_i \gamma$ infer $\phi \vee \psi \vdash_i \gamma$	($\vee$-elim)
9_i.	$\phi \wedge (\psi \vee \gamma) \vdash_i (\phi \wedge \psi) \vee \gamma$	(Distribution)
10_i.	$\bot \vdash_i \phi$	($\bot$-elim)
11_i.	$\phi \vdash_i \top$	($\top$-intro)

Axioms and rules for $\vdash_1$:

12_1.	$\langle\lambda\rangle\phi \wedge \langle\lambda\rangle\psi \vdash_1 \langle\lambda\rangle(\phi \wedge \psi)$	($\Diamond$-$\wedge$-preservation)
13_1.	$\langle\lambda\rangle(\phi \vee \psi) \vdash_1 \langle\lambda\rangle\phi \vee \langle\lambda\rangle\psi$	($\Diamond$-$\vee$-preservation)
14_1.	$\langle\lambda\rangle\bot \vdash_1 \bot$	($\Diamond$-strictness)
15_1.	From $\phi \vdash_1 \psi$ infer $\langle\lambda\rangle\phi \vdash_1 \langle\lambda\rangle\psi$	($\Diamond$-monotonicity)

Axioms and rules for $\vdash_2$:

12_2.	$[\lambda]\phi \wedge [\lambda]\psi \vdash_2 [\lambda](\phi \wedge \psi)$	($\Box$-$\wedge$-preservation)
13_2.	$[\lambda](\phi \vee \psi) \vdash_2 [\lambda]\phi \vee [\lambda]\psi$	($\Box$-$\vee$-preservation)

14_2.	$[\lambda]\phi \vdash_2 [\mu]\top$	($\Box$-convergence)
15_2.	From $\phi \vdash_2 \psi$ infer $[\lambda]\phi \vdash_2 [\lambda]\psi$	($\Box$-monotonicity)
16_2.	$\alpha^2 \vdash_2 [\lambda]\top$	(Strongness)

The axioms and rules are the expected ones: The first group expresses that formulas under $\wedge$, $\vee$ forms a distributive lattice with $\top$ unit and $\bot$ zero (for $\wedge$). The next groups captures the forwards linearity of our accessibility relation with 14_1 and 14_2 being the axioms that put $\langle\lambda\rangle$ and $[\lambda]$ apart. Soundness of the axiomatisations is readily obtained.

Theorem 4.23 (Soundness of $\mathbf{GL}^i$). *For all (strong) model classes $\boldsymbol{\mathcal{M}}$, $i \in \{1,2\}$, and $\phi, \psi \in \mathrm{GTFm}_i^+$, if $\phi \vdash_{\mathbf{GL}^i} \psi$ then $\phi \models_{\mathcal{M}} \psi$.*

PROOF: Show as usual the axioms to be valid entailments and the rules to preserve validity of entailments. $\Box$

4.4.2 Normal forms

Any of the standard modal logic techniques are likely to be adequate for proving completeness. The proof we present here is based on rewriting formulas into a normal form. These are needed in the following chapter, justifying the otherwise quite unnecessary amount of work having to be done. Note first the for the purpose of rewriting indispensable replacement property:

Proposition 4.24 *For $i \in \{1,2\}$, and any logic $\mathbf{L}$ extending $\mathbf{GL}^i$, $\vdash_{\mathbf{GL}^i}$ is a precongruence and $\equiv_{\mathbf{GL}^i}$ a congruence w.r.t. the connectives in GTFm_i^+.*

PROOF: Straightforward. $\Box$

We proceed to introduce normal forms and their orderings. For this purpose say a subset $A \subseteq L$ of a preordered set L is a *boundary*, if whenever $x \leq y$, $y \not\leq x$ and $x \in A$ then $y \notin A$.

Definition 4.25

i) The set $\mathrm{TRNF} \subseteq \mathrm{GTFm}_1^+$ of *trace normal forms*, or just *traces*, is the least set containing $\top$ and α^1 for all $\alpha \in \mathrm{Ap}$, and s.t. for all $\lambda \in \mathcal{L}$, $\langle\lambda\rangle\phi \in \mathrm{TRNF}$ whenever $\phi \in \mathrm{TRNF}$.

ii) $\leq$ is the least relation on TRNF s.t.

a) $\alpha^1 \leq \alpha^1$, for all $\alpha \in \mathrm{Ap}$,

b) $\phi \leq \top$, for all $\phi \in \mathrm{TRNF}$, and

c) for all $\lambda \in \mathcal{L}$, $\langle\lambda\rangle\phi \leq \langle\lambda\rangle\psi$ whenever $\phi \leq \psi$.

iii) The set PNF_1 of *1-prime normal forms* is given by

$$\mathrm{PNF}_1 = \{\bigwedge \Phi \mid \Phi \subseteq \mathrm{TRNF},\ \Phi \text{ a finite, nonempty boundary}\}.$$

iv) PNF_1 is ordered by for all finite, nonempty boundaries $\Phi, \Psi \subseteq \mathrm{TRNF}$ letting $\bigwedge \Phi \leq \bigwedge \Psi$ iff for all $\psi \in \Psi$ there is some $\phi \in \Phi$ s.t. $\phi \leq \psi$.

v) Given some $\lambda \in \mathcal{L}$ the set $(\lambda\text{-})\mathrm{PNF}_2$ of *2-prime normal forms* is the least s.t.

a) $\bot \in \lambda\text{-}\mathrm{PNF}_2$,

b) $(\bigwedge_{j\in J} \alpha_j^2) \wedge (\bigwedge_{\lambda\in\Lambda}[\lambda]\phi_\lambda) \in \lambda\text{-}\mathrm{PNF}_2$, if

1) J, Λ are finite sets,

2) for all $\lambda \in \Lambda$, $\phi_\lambda \in \lambda\text{-}\mathrm{PNF}_2$,

3) if $J \neq \emptyset$ then for all $\lambda \in \Lambda$, $\phi_\lambda \neq \top$, and

4) if $J = \emptyset$ and for some $\lambda \in \Lambda$, $\phi_\lambda = \top$ then $\Lambda = \{\lambda\}$.

vi) $\leq$ is the least relation on $(\lambda\text{-})\mathrm{PNF}_2$ s.t.

a) $\bot \leq \phi$ for all $\phi \in \lambda\text{-}\mathrm{PNF}_2$,

b) $(\bigwedge_{j\in J} \alpha_j^2) \wedge (\bigwedge_{\lambda\in\Lambda}[\lambda]\phi_\lambda) \leq (\bigwedge_{k\in K} \beta_k^2) \wedge (\bigwedge_{\lambda'\in\Lambda'}[\lambda']\psi_{\lambda'})$, if

1) $\{\beta_k^2 \mid k \in K\} \subseteq \{\alpha_j^2 \mid j \in J\}$,

2) if $\psi_{\lambda'} = \top$, $\lambda' \in \Lambda'$ then either $J \neq \emptyset$ or $\Lambda \neq \emptyset$,

3) if for all $\lambda' \in \Lambda'$, $\psi_{\lambda'} \neq \top$ then $\Lambda' \subseteq \Lambda$, and

4) for all $\lambda' \in \Lambda \cap \Lambda'$, $\phi_{\lambda'} \leq \psi_{\lambda'}$.

vii) For $i \in \{1, 2\}$ the set NF_i of *i-normal forms* is the set of all formulas $\bigvee \Phi$ where $\Phi \subseteq (\lambda\text{-})\mathrm{PNF}_i$ is a finite, and if $i = 2$ also nonempty boundary.

viii) NF_i is ordered by for all boundaries $\Phi, \Psi \subseteq \mathrm{PNF}_i$ letting $\bigvee \Phi \leq \bigvee \Psi$ iff for all $\phi \in \Phi$ there is a $\psi \in \Psi$ s.t. $\phi \leq \psi$.

For $\leq$ any of the orderings defined here, $\doteq$ denotes the induced equivalence—i.e. $\doteq = \leq \cap \geq$. The indexing by the label λ in the must-case only serves to pick out a canonical formula of the form $[\lambda]\top$, corresponding to the convergence predicate. We shall almost always leave this distinguished label understood from the context. The disheartening amount of detail in this definition is due to the fact that we are hunting *unique* normal forms. We first verify that the relations $\leq$ of definition 4.25 are indeed partial orderings and the $\doteq$ equivalences.

Proposition 4.26 *All the $\leq$ of definition 4.25 are partial orders.*

PROOF: The latter assertion, of course, is just a trivial consequence of the first. First we check that the $\leq$ are preorders. For the may-case this will follow from the preorder property of $\leq$ on TRNF, and similarly for the must-case it will follow from PNF_2.

TRNF. The preorder property of $\leq$ follows by a straightforward induction on the size of formulas; for reflexivity, for instance, $\top \leq \top$ by 4.25.ii.b), $\alpha^1 \leq \alpha^1$ by 4.25.ii.a), and $\langle\lambda\rangle\phi \leq \langle\lambda\rangle\phi$ by 4.25.ii.c), as $\phi \leq \phi$ by the induction hypothesis.

PNF_2. Reflexivity follow by an easy induction on the size of formulas. To prove transitivity, assume $\phi \leq \psi$, $\psi \leq \gamma$, with $\phi, \psi, \gamma \in \mathrm{PNF}_2$. If $\phi = \bot$ we are done. If $\phi = \top$ then $\psi = \top$—this is straightforward to check—and then similarly $\gamma = \top$, so $\phi \leq \gamma$ by reflexivity. So assume that

$$\phi = \bigwedge_{j \in J} \alpha_j^2 \wedge \bigwedge_{\lambda_1 \in \Lambda_1} [\lambda_1]\phi_{\lambda_1},$$

with either J or Λ_1 nonempty. We cannot have $\psi = \bot$, and if $\psi = \top$ again $\gamma = \top$ and $\phi \leq \gamma$ so assume that

$$\psi = \bigwedge_{k \in K} \beta_k^2 \wedge \bigwedge_{\lambda_2 \in \Lambda_2} [\lambda_2]\psi_{\lambda_2},$$

with either K or Λ_2 nonempty. Similarly we can assume that

$$\gamma = \bigwedge_{h \in H} \zeta_h^2 \wedge \bigwedge_{\lambda_3 \in \Lambda_3} [\lambda_3]\gamma_{\lambda_3},$$

with either H or Λ_3 nonempty. Clearly, for all $h \in H$ there is an $j \in J$ s.t. $\zeta_h^2 = \alpha_j^2$, so condition 4.25.vi.b.1) is satisfied. Next if $\gamma_{\lambda_3} = \top$ then we have already shown either J or Λ_1 to be nonempty, so also condition 4.25.vi.b.2) is satisfied. Condition vi.b.3) is straightforward, so assume that $\lambda \in \Lambda_1 \cap \Lambda_3$. If $\lambda \in \Lambda_2$ as well we are immediately done, so assume not. Then there must be some $\lambda_3 \in \Lambda_3$ s.t. $\gamma_{\lambda_3} = \top$ and then $\Lambda_3 = \{\lambda\}$ and then $\gamma_\lambda = \top$. But then $\phi_\lambda \leq \top$—this is easy to check—and we are done.

So we proceed to prove antisymmetry:

TRNF. Assume $\phi \leq \psi$ and $\psi \leq \phi$. If $\phi = \top$ then $\psi = \top$, if $\phi = \alpha^1$ then $\psi = \alpha^1$, and if $\phi = \langle\lambda\rangle\phi'$ then either $\psi = \top$—but then $\psi \not\leq \phi$—or $\psi = \langle\lambda\rangle\psi'$ and then $\phi' \doteq \psi'$. Then by the induction hypothesis, $\phi' = \psi'$ and so $\phi = \psi$.

PNF$_1$. Let Φ, Ψ be finite, nonempty boundaries and $\bigwedge\Phi \doteq \bigwedge\Psi$. Then for each $\phi \in \Phi$ there is a $\psi \in \Psi$ s.t. $\psi \leq \phi$ and then there is a $\phi' \in \Phi$ s.t. $\phi' \leq \psi$, thus $\phi' \leq \phi$. Now Φ is a boundary, so if $\phi \not\leq \phi'$ then $\phi \notin \Phi$—hence $\phi \doteq \phi'$. But then $\phi = \phi'$ by the previous case and then $\phi = \psi$. The converse direction is symmetrical.

NF$_1$. Antisymmetry is inherited from PNF$_1$ in a completely similar fashion.

PNF$_2$. Assume that $\phi \leq \psi \leq \phi$, $\phi, \psi \in$ PNF$_2$. If $\phi = \bot$ then $\psi = \bot$ as well, so let $\phi = \bigwedge_{j\in J}\alpha^2_j \wedge \bigwedge_{\lambda_1\in\Lambda_1}[\lambda_1]\phi_{\lambda_1}$ and $\psi = \bigwedge_{k\in K}\beta^2_k \wedge \bigwedge_{\lambda_2\in\Lambda_2}[\lambda_2]\psi_{\lambda_2}$. By condition 4.25.vi.b.1), $\{\alpha^2_j \mid j \in J\} = \{\beta^2_k \mid k \in K\}$. Let $\lambda \in \Lambda_1$. Either $\Lambda_1 = \{\lambda\}$, $\phi_\lambda = \top$ and $J = \emptyset$ or for all $\lambda' \in \Lambda_1$, $\phi_{\lambda'} \neq \top$ by vii.b.2-3). In the first case we obtain $\Lambda_2 = \{\lambda\}$ as well and $\phi_\lambda = \psi_\lambda$, and in the second, by vi.b.3), $\Lambda_1 = \Lambda_2$ and by vi.b.4) and the induction hypothesis that for all $\lambda \in \Lambda_1$, $\phi_\lambda = \psi_\lambda$; and we may thus conclude that $\phi = \psi$.

NF$_2$. Antisymmetry is inherited from PNF$_2$ as in the case for PNF$_1$. □

4.4.3 Provability and entailment on normal forms

We then wish to show that the orderings $\leq$ on normal forms coincide with both the provability relation $\vdash_{\mathbf{GL}^i}$ and the semantical relation of entailment, or consequence, First for the provability relation:

Proposition 4.27 *For all $i \in \{1, 2\}$, $X \in \{$(TR,)P,()$\}$ and $\phi, \psi \in X$NF$_{(i)}$, if $\phi \leq \psi$ then $\phi \vdash_{\mathbf{GL}^{(i)}} \psi$.*

PROOF: The proof is a long, but straightforward exercise in applying the axioms and proof rules. □

By soundness, theorem 4.23, we know that for all $\phi, \psi \in X$NF$_{(i)}$, if $\phi \vdash_{\mathbf{GL}^i} \psi$ then for each model class $\boldsymbol{\mathcal{M}}$, $\phi \models_{\mathcal{M}} \psi$. We proceed to show the converse to be true as well.

Lemma 4.28 *For $i \in \{1,2\}$, $X \in \{(\mathrm{TR},)\mathrm{P},()\}$ and $\phi \in X\mathrm{NF}_{(i)}$,*

i) ϕ is consistent iff $\phi \neq \bot$,

ii) ϕ valid iff $\phi = \top$.

Moreover, whenever $\phi \in \mathrm{PNF}_i$ is consistent then we can find a i-least rooted model Σ_ϕ s.t. $\Sigma_\phi \models \phi$ and whenever $\Sigma' \models \phi$ then $\Sigma_\phi \sqsubseteq_i \Sigma'$.

Proof: i). The if-direction is trivial. For the only if direction we proceed by cases, assuming $\phi \neq \bot$.

TRNF. $\top \not\models \bot$, for let Σ be without transitions and having no $\alpha \in \mathrm{Ap}$ holding for it—then Σ is least w.r.t. $\sqsubseteq_1$. Secondly, $\alpha^1 \not\models \bot$, for let Σ be without transitions and only α^1 holding, then $\Sigma \models \alpha^1$ and Σ is 1-least with that property. Thirdly, if $\phi \not\models \bot$ then $\langle\lambda\rangle\phi \not\models \bot$, for let $\Sigma \models \phi$. Then we easily build another model Σ' s.t. Σ' can λ and $\Sigma'/\lambda = \Sigma$, i.e. $\Sigma' \models \langle\lambda\rangle\phi$. Moreover Σ' may be constructed such that no atomic propositions α^1 holds for it and whenever Σ' can λ' then $\lambda' = \lambda$—then Σ' is 1-least satisfying ϕ.

PNF_1. If $\phi = \bigwedge\Phi$ for Φ a finite and nonempty boundary, then we use the previous case to build for each $\phi' \in \Phi$ a 1-least $\Sigma_{\phi'}$ s.t. $\Sigma_{\phi'} \models \phi'$. Then we can use the disjoint sum model construction introduced in section 4.1 and obtain $\sum_{\phi'\in\Phi}\Sigma_{\phi'} \models \phi'$ for each $\phi' \in \Phi$, by 4.8 and 4.11, thus $\sum_{\phi'\in\Phi}\Sigma_{\phi'} \models \phi$ as well; and by 4.8, $\sum_{\phi'\in\Phi}\Sigma_{\phi'}$ will be 1-least with that property.

NF_1. If $\phi \in \mathrm{NF}_1 \neq \bot$ then $\phi = \bigvee\Phi$ with $\Phi \subseteq \mathrm{PNF}_1$ a finite and nonempty boundary, and then ϕ is consistent.

PNF_2. Let $\phi \in \mathrm{PNF}_2$. If $\phi = \bot$ then we are done, so assume instead that

$$\phi = \bigwedge_{j\in J} \alpha_j^2 \wedge \bigwedge_{\lambda\in\Lambda} [\lambda]\phi_\lambda.$$

Let $\lambda \in \Lambda$. Now if $\phi_\lambda \neq \bot$ then by the induction hypothesis we can find a 2-least Σ_λ s.t. $\Sigma_\lambda \models \phi_\lambda$. Build now a new rooted model Σ, s.t. $\Sigma \Downarrow$ iff either J or Λ are nonempty, $\Sigma \models \alpha_j^2$ for all $j \in J$ and such that if $\lambda \notin \Lambda$ then Σ can λ and Σ/λ is a divergent model, and if $\lambda \in \Lambda$ and ϕ_λ consistent then Σ can λ and $\Sigma/\lambda = \Sigma_\lambda$. Then $\Sigma \models \phi$, and moreover it is easy to check that Σ is 2-least with that property.

NF_2. If $\phi \in \mathrm{NF}_2$ and $\phi \neq \bot$ then $\phi = \bigvee\Phi$ with $\Phi \subseteq \mathrm{PNF}_2$ a finite and nonempty boundary. But then $\bot \notin \Phi$ so Φ contains some consistent ϕ'—and we are done.

ii) Similar to i). □

We then complete the circle by proving that the relation of entailment is contained in the partial orders on normal forms.

Proposition 4.29 *For $i \in \{1,2\}$, $X \in \{(\mathrm{TR},)\mathrm{P},()\}$ and $\phi,\psi \in X\mathrm{NF}_{(i)}$, if $\phi \models \psi$ then $\phi \leq \psi$.*

Proof: Assume that $\phi,\psi \in X\mathrm{NF}_{(i)}$ and that $\phi \not\leq \psi$. Note that for $\phi \in \mathrm{PNF}_i$, $\phi \models \psi$ iff $\Sigma_\phi \models \psi$. The only-if direction is clear. For the if-direction, if $\Sigma_\phi \models \psi$ and $\Sigma' \models \phi$ then $\Sigma \sqsubseteq_i \Sigma'$, so $\Sigma' \models \psi$ as well, by 4.11.
TRNF. We cannot have $\psi = \top$, so assume first that $\psi = \alpha^1$. Then $\phi \neq \alpha^1$, and $\Sigma_\phi \not\models \alpha^1$ (note that all $\phi \in \mathrm{TRNF}$ are consistent by lemma 4.28). Assume instead that $\psi = \langle\lambda\rangle\psi'$ for some $\psi' \in \mathrm{TRNF}$. Either $\phi = \top$, and then any model Σ s.t. Σ $\cancel{\mathrm{can}}$ λ will furnish a counterexample for $\phi \models \psi$. Next if $\phi = \alpha^1$ then any Σ s.t. $\Sigma \models \alpha^1$ and Σ $\cancel{\mathrm{can}}$ λ will do, so finally assume that $\phi = \langle\mu\rangle\phi'$. If $\lambda \neq \mu$ then fix a Σ s.t. Σ can μ and Σ $\cancel{\mathrm{can}}$ λ, and if $\lambda = \mu$ then by the induction hypothesis we find some Σ' s.t $\Sigma' \models \phi'$ and $\Sigma' \not\models \psi'$ and then let Σ have the property that Σ can λ and $\Sigma/\lambda = \Sigma'$. Then $\Sigma \models \phi$ and $\Sigma \not\models \psi$.
PNF_1. We have $\phi = \bigwedge\Phi$ and $\psi = \bigwedge\Psi$, for $\Phi,\Psi \subseteq \mathrm{TRNF}$ finite and nonempty boundaries. Now if $\phi \not\leq \psi$ then we find a $\psi' \in \Psi$ s.t. for all $\phi' \in \Phi$, $\phi' \not\leq \psi'$, hence by the previous case, $\phi' \not\models \psi'$. Then for each $\phi' \in \Phi$ $\Sigma_{\phi'} \models \phi'$ and $\Sigma_{\phi'} \not\models \psi'$. Then $\boldsymbol{\Sigma}_{\phi'\in\Phi}\Sigma_{\phi'} \models \phi$ and $\boldsymbol{\Sigma}_{\phi'\in\Phi}\Sigma_{\phi'} \not\models \psi'$ by 4.8 and 4.11, thus $\phi \not\models \psi$.
NF_1. We have $\phi = \bigvee\Phi$ and $\psi = \bigvee\Psi$, $\Phi,\Psi \subseteq \mathrm{PNF}_1$, and for some $\phi' \in \Phi$, for all $\psi' \in \Psi$, $\phi' \not\leq \psi'$. By the previous case for all $\psi' \in \Psi$, $\phi' \not\models \psi'$. Then ϕ' is consistent and $\Sigma_{\phi'} \models \phi$ and for each $\psi' \in \Psi$, $\Sigma_{\phi'} \not\models \psi'$, thus $\Sigma_{\phi'} \not\models \psi$.
PNF_2. We may assume that $\phi = \bigwedge_{j\in J}\alpha^2_j \wedge \bigwedge_{\lambda_1\in\Lambda_1}[\lambda_1]\phi_{\lambda_1}$ and $\psi = \bigwedge_{k\in K}\beta^2_k \wedge \bigwedge_{\lambda_2\in\Lambda_2}[\lambda_2]\psi_{\lambda_2}$. If $\phi \models \psi$ then—as we saw—$\Sigma_\phi \models \psi$. So for each $k \in K$ it must be the case that $\phi \models \beta^2_k$, but this can only be the case if for some $j \in J$, $\alpha^2_j = \beta^2_k$. So condition 4.25.vi.b.1) is satisfied. Next, if $\psi_{\lambda_2} = \top$ and both J and Λ_1 are empty then $\Sigma_\phi \Uparrow$ and then $\Sigma_\phi \not\models \psi$, so condition vi.b.2) is satisfied as well. Assume then that for all $\lambda_2 \in \Lambda_2$, $\psi_{\lambda_2} \neq \top$, and assume that for some $\lambda_2 \in \Lambda_2$, $\lambda_2 \notin \Lambda_1$. We must have $\Sigma_\phi \Downarrow$, and moreover Σ_ϕ can λ_2 and $\Sigma_\phi/\lambda_2 \Uparrow$, as otherwise Σ_{λ_2} would not be 2-least satisfying ϕ. But if $\psi_{\lambda_2} \neq \top$ then by lemma 4.28 there is some Σ' s.t. $\Sigma' \not\models \psi_{\lambda_2}$ and then in particular $\Sigma_{\lambda_2}/\lambda_2 \not\models \psi_{\lambda_2}$—a contradiction. So it must be the case that $\lambda_2 \in \Lambda_1$, and condition vi.b.3) is satisfied. Finally the verification of vi.b.4) is straightforward.

NF_2. Then $\phi = \bigvee \Phi$ and $\psi = \bigvee \Psi$, with $\Phi, \Psi \subseteq \mathrm{PNF}_2$ finite and nonempty boundaries. Let then $\phi' \in \Phi$. If $\phi' = \bot$ then $\bot \leq \psi$ and we are done. Otherwise $\Sigma_{\phi'} \models \psi$, and then for some $\psi' \in \Psi$, $\Sigma_{\phi'} \models \psi'$ and then we may conclude that $\phi' \models \psi'$. But then by the previous case $\phi \leq \psi$, and we are done. □

4.4.4 Completeness

We have thus completed the characterisation of the provability relations on normal forms and it remains to be checked that each formula can be rewritten using the axioms and rules into an equivalent normal form.

Lemma 4.30 *For $i \in \{1, 2\}$ and any $\phi \in \mathrm{GTFm}_i^+$ there is a $\phi' \in \mathrm{NF}_i$ s.t $\phi \equiv_{\mathbf{GL}^i} \phi'$, and moreover the size of ϕ' does not exceed the size of ϕ.*

PROOF: For both $i = 1$ and $i = 2$ we proceed by induction on the structure of ϕ.

First for $i = 1$. The cases for $\phi = \bot$, $\phi = \top$ and $\phi = \alpha^1$ are trivial.

$\phi = \phi_1 \wedge \phi_2$. By the induction hypothesis there are $\phi_1', \phi_2' \in \mathrm{NF}_1$ s.t. $\phi_1 \equiv_{\mathbf{GL}^1} \phi_1'$, $\phi_2 \equiv_{\mathbf{GL}^2} \phi_2'$ with sizes not exceeding those of ϕ_1, ϕ_2, and then—by 4.24 (which we use so often here that we omit references to it from now on)—$\phi \equiv_{\mathbf{GL}^1} \phi_1' \wedge \phi_2'$. Let $\phi_1' = \bigvee \Phi$ and $\phi_2' = \bigvee \Psi$ with Φ, Ψ finite, nonempty subsets of PNF_1, and then by distribution of $\wedge$ over $\vee$ we obtain $\phi \equiv_{\mathbf{GL}^1} \bigvee \Theta$, where $\Theta = \{\phi_1 \wedge \psi_1 \mid \phi_1 \in \Phi, \psi_1 \in \Psi\}$. Note that the size of $\bigvee \Theta$ does not exceed that of ϕ. If one of Φ or Ψ are empty we are done as then $\bigvee \Theta \in \mathrm{NF}_1$, so assume not. Now each $\phi_1 \in \Phi$, $\psi_1 \in \Psi$ are in PNF_1, so $\phi_1 = \bigwedge \Phi_1$, $\psi_1 = \bigwedge \Psi_1$ for Φ_1, Ψ_1 a finite, nonempty boundary of TRNF. Then $\phi_1 \wedge \psi_1$ is in PNF_1 except it might not be a boundary. So let $\gamma_1, \gamma_2 \in \Phi_1 \cup \Psi_1$, $\gamma_1 \leq \gamma_2$ and $\gamma_2 \not\leq \gamma_1$. By proposition 4.27 $\gamma_1 \vdash_{\mathbf{GL}^1} \gamma_2$ and then we obtain

$$\bigwedge((\Phi_1 \cup \Psi_1) \setminus \{\gamma_2\}) \equiv_{\mathbf{GL}^1} \bigwedge(\Phi_1 \cup \Psi_1).$$

By induction on the size of the set $\Phi_1 \cup \Psi_1$ all such γ_2 can be eliminated to produce a set $\Theta_1 \subseteq \Phi_1 \cup \Psi_1$ which is a boundary and $\bigwedge \Theta_1 \equiv_{\mathbf{GL}^1} \bigwedge(\Phi_1 \cup \Psi_1)$. But then

$$\phi \equiv_{\mathbf{GL}^1} \bigvee_{\phi_1 \in \Phi, \phi_2 \in \Psi} (\bigwedge \Theta_{\phi_1 \wedge \phi_2}),$$

where $\Theta_{\phi_1 \wedge \phi_2}$ is the boundary Θ_1 produced from $\Phi_1 \cup \Psi_1$ in this way. But this might again not be in NF_1 as there might be ϕ_1, $\phi_2 \in \Phi$, ψ_1, $\psi_2 \in \Psi$ s.t.

$\bigwedge\Theta_{\phi_1\wedge\psi_1} \leq \bigwedge\Theta_{\phi_2\wedge\psi_1}$ but not $\bigwedge\Theta_{\phi_2\wedge\psi_1} \leq \bigwedge\Theta_{\phi_1\wedge\psi_1}$. But this is rectified by applying the above procedure "in reverse"—this is justified by the $\vee$-intro and -elim rules using proposition 4.27, thus obtaining a $\phi' \in \mathrm{NF}_1$ s.t. $\phi \equiv_{\mathbf{GL}^1} \phi'$, and moreover the size of ϕ' does not exceed that of ϕ.

$\phi = \phi_1 \vee \phi_2$. Here we just need to apply the procedure of the latter part of the previous case and the induction hypothesis to obtain a $\phi' \in \mathrm{NF}_1$ s.t. $\phi \equiv_{\mathbf{GL}^1} \phi'$ and $|\phi'| \leq |\phi|$.

$\phi = \langle\lambda\rangle\phi_1$. By the induction hypothesis we find a $\phi_1' \in \mathrm{NF}_1$ s.t. $\phi_1 \equiv_{\mathbf{GL}^1} \phi_1'$. If $\phi_1' = \perp$ then by $\diamond$-strictness we obtain $\phi \equiv_{\mathbf{GL}^1} \perp$. Otherwise we just need to apply the preservation of $\wedge$ and $\vee$ under $\langle\lambda\rangle$ which follows from $\diamond$-$\wedge$-preservation, $\diamond$-$\vee$-preservation and $\diamond$-monotonicity. It is not hard to see that this process preserves the boundary property, and that it does not increase size.

Then for $i = 2$. Again the cases for $\phi = \perp$, $\phi = \top$ and $\phi = \alpha^2$ are trivial.

$\phi = \phi_1 \wedge \phi_2$. We find by the induction hypothesis finite and nonempty boundaries $\Phi_1, \Phi_2 \subseteq \mathrm{PNF}_2$ s.t $\phi_1 \equiv_{\mathbf{GL}^2} \bigvee\Phi_1$ and $\phi_2 \equiv_{\mathbf{GL}^2} \bigvee\Phi_2$, thus $\phi \equiv_{\mathbf{GL}^2} (\bigvee\Phi_1)\wedge(\bigvee\Phi_2)$ and then by distribution of $\wedge$ over $\vee$ $\phi \equiv_{\mathbf{GL}^2} \bigvee\Phi$, where $\Phi = \{\psi_1 \wedge \psi_2 \mid \psi_1 \in \Phi_1, \psi_2 \in \Phi_2\}$. Let then $\psi_1 \in \Phi_1$, $\psi_2 \in \Phi_2$ and we rewrite $\psi_1 \wedge \psi_2$ into a normal form in NF_2. If either $\psi_1 = \perp$ or $\psi_2 = \perp$ then $\psi_1 \wedge \psi_2 \equiv_{\mathbf{GL}^2} \perp \in \mathrm{PNF}_2$ so let instead for $k \in \{1,2\}$

$$\psi_k = \bigwedge_{j\in J_k} \alpha^2_{j,k} \wedge \bigwedge_{\lambda\in\Lambda_k} [\lambda]\psi_{k,\lambda}$$

By the semilattice properties of $\wedge$ we obtain

$$\psi_1 \wedge \psi_2 \equiv_{\mathbf{GL}^2} \bigwedge_{j\in J_1\oplus J_2} \beta^2_j \wedge \bigwedge_{\lambda\in\Lambda_1\cup\Lambda_2} [\lambda]\psi_{3,\lambda},$$

where for $j = \mathrm{in}_1(j')$, $\beta^2_j = \alpha^2_{j',1}$, for $j = \mathrm{in}_2(j')$, $\beta^2_j = \alpha^2_{j',2}$; for $\lambda \in \Lambda_1 \setminus \Lambda_2$, $\psi_{3,\lambda} = \psi_{1,\lambda}$, for $\lambda \in \Lambda_2 \setminus \Lambda_1$, $\psi_{3,\lambda} = \psi_{2,\lambda}$, and for $\lambda \in \Lambda_1 \cap \Lambda_2$, $\psi_{3,\lambda} = \psi_{1,\lambda} \wedge \psi_{2,\lambda}$. By the inner induction hypothesis (on the size of ϕ) we find for each $\lambda \in \Lambda_1 \cup \Lambda_2$ a $\gamma_\lambda \in \mathrm{PNF}_2$ s.t. $\psi_{3,\lambda} \equiv_{\mathbf{GL}^2} \gamma_\lambda$ and then

$$\psi_1 \wedge \psi_2 \equiv_{\mathbf{GL}^2} \bigwedge_{j\in J_1\oplus J_2} \beta^2_j \wedge \bigwedge_{\lambda\in\Lambda_1\cup\Lambda_2} [\lambda]\gamma_\lambda = \gamma.$$

Hence γ satisfies condition v.b.1) of definition 4.25. Suppose now that $J_1 \oplus J_2 \neq \emptyset$. Let $\Lambda_3' = \{\lambda \in \Lambda_1 \cup \Lambda_2 \mid \gamma_\lambda \neq \top\}$ and by $\wedge$-elim and the "strongness"-axiom we obtain

$$\psi_1 \wedge \psi_2 \equiv_{\mathbf{GL}^2} \bigwedge_{j\in J_1\oplus J_2} \beta^2_j \wedge \bigwedge_{\lambda\in\Lambda_3'} [\lambda]\gamma_\lambda = \gamma_1$$

and $\gamma_1 \in \mathrm{PNF}_2$. If instead $J_1 \oplus J_2 = \emptyset$, suppose there is some $\lambda \in \Lambda_1 \cup \Lambda_2$ s.t. $\gamma_\lambda \neq \top$. Let then Λ_3' be as above and by $\wedge$-elim and the $\Box$-convergence axiom we obtain

$$\psi_1 \wedge \psi_2 \equiv_{\mathbf{GL}^2} \bigwedge_{\lambda \in \Lambda_3'} [\lambda]\gamma_\lambda = \gamma_2$$

and $\gamma_2 \in \mathrm{PNF}_2$. Finally, if $J_1 \oplus J_2 = \emptyset$ and for all $\lambda \in \Lambda_3'$, $\gamma_\lambda = \top$ then by $\wedge$-elim and the $\Box$-convergence axiom we obtain $\psi_1 \wedge \psi_2 \equiv_{\mathbf{GL}^2} [\lambda]\top$ where λ is the distinguished label picked out in the definition of PNF_2.
Let then $\mathrm{nf}(\psi_1 \wedge \psi_2)$ denote the $\gamma' \in \mathrm{PNF}_2$ s.t. $\psi_1 \wedge \psi_2 \equiv_{\mathbf{GL}^2} \gamma'$ obtained by this procedure. Then

$$\phi \equiv_{\mathbf{GL}^2} \bigvee\{\mathrm{nf}(\psi_1 \wedge \psi_2) \mid \psi_1 \in \Phi_1, \psi_2 \in \Phi_2\} = \phi'.$$

Now ϕ' might not be a boundary, but then exactly as in the corresponding case for $i = 1$, ϕ' may be rewritten into a disjunction $\bigvee \Phi'$ with Φ' a boundary and s.t. $\phi \equiv_{\mathbf{GL}^2} \bigvee \Phi' \in \mathrm{NF}_2$. It is not hard to check that $|\bigvee \Phi'| \leq |\phi|$.
$\phi = \phi_1 \vee \phi_2$. By the induction hypothesis we find boundaries $\Phi_1, \Phi_2 \subseteq \mathrm{PNF}_2$ s.t. $\phi_1 \equiv_{\mathbf{GL}^2} \bigvee \Phi_1$, $\phi_2 \equiv_{\mathbf{GL}^2} \Phi_2$ and then

$$\phi \equiv_{\mathbf{GL}^2} (\bigvee \Phi_1) \vee (\bigvee \Phi_@) \equiv_{\mathbf{GL}^2} \bigvee(\Phi_1 \cup \Phi_2);$$

and then it just remains—as we have done several times now—to cut down $\Phi_1 \cup \Phi_2$ to a boundary $\Phi' \subseteq \Phi_1 \cup \Phi_2$ and then $\phi \equiv_{\mathbf{GL}^2} \bigvee \Phi' \in \mathrm{NF}_2$.
$\phi = [\lambda]\phi_1$. We find a boundary $\Phi \subseteq \mathrm{PNF}_2$ s.t. $\bigvee \Phi \in \mathrm{NF}_2$ and $\phi \equiv_{\mathbf{GL}^2} [\lambda](\bigvee \Phi)$. Then by $\Box$-$\vee$-preservation and $\Box$-monotonicity,

$$\phi \equiv_{\mathbf{GL}^2} \bigvee\{[\lambda]\phi_1' \mid \phi_1' \in \Phi\}.$$

Now whenever $\phi_1' \in \Phi$ is different from $\top$ then $[\lambda]\phi_1' \in \mathrm{PNF}_2$. Otherwise we apply the $\Box$-convergence axiom to replace $[\lambda]\top$ by $[\lambda']\top$, where λ' is the distinguished label picked out in the definition of 2-normal forms. Then it just remains to cut down the resulting set to a boundary to obtain a $\phi' \in \mathrm{NF}_2$ s.t. $\phi \equiv_{\mathbf{GL}^2} \phi'$, and the proof is completed. $\Box$

Now all the ingredients for the completeness theorem are in place:

Theorem 4.31 (Completeness of $\mathbf{GL}^i$). *For all $i \in \{1,2\}$ and $\phi, \psi \in \mathrm{GTFm}_i^+$, $\phi \vdash_{\mathbf{GL}^i} \psi$ iff $\phi \models \psi$.*

PROOF: The only-if direction follow by the soundness 4.23. For the if-direction assume that $\phi \models \psi$. By 4.30 we find $\phi', \psi' \in \mathrm{NF}_i$ s.t. $\phi \equiv_{\mathbf{GL}^i} \phi'$, $\psi \equiv_{\mathbf{GL}^i} \psi'$ and then $\phi \sim \phi'$ and $\psi \sim \psi'$ by soundness. Then $\phi' \models \psi'$ and then by 4.29, $\phi' \leq \psi'$ whence $\phi' \vdash_{\mathbf{GL}^i} \psi'$ by 4.27. But then $\phi \vdash_{\mathbf{GL}^i} \psi$ and we are done. □

This proof is probably far more complicated than need be. There is little doubt that an ordinary canonical model construction could be made to work here—all that needs be taken into account is the divergence predicate and the strong interpretation of must- atomic propositions. However the present proof is far more adaptable to the completeness for the full logics—for those we know only techniques based on rewriting (sadly so) and hence the proof has laid out a lot of the groundwork needed for the following chapter.

A finite model property for our logics $\mathbf{GL}^i$, $i \in \{1, 2\}$ is very easy: The least rooted model $\Sigma_\top$ will fail to satisfy any non-valid formula and may be assumed to be finite. Conversely if ϕ is consistent then to find a finite model satisfying ϕ just rewrite ϕ into a normal form $\bigvee \Phi$ with $\Phi \subseteq \mathrm{PNF}_i$, and pick any $\phi' \in \Phi$. As Φ is a boundary, ϕ' is consistent, and then $\Sigma_{\phi'}$ is defined and satisfies ϕ. It is not hard to see that $\Sigma_{\phi'}$ can be constructed such that it is finite. Follow the construction of $\Sigma_{\phi'}$ in the proof of proposition 4.28. For $i = 1$ finiteness is straightforward. For $i = 2$ note that for every state σ even if it may be the case that Σ can λ for an infinite number of λ, only a finite number of sets Σ/λ need be distinct from $\Sigma_\top$. Moreover, we can make sure that the linearisation of the model $\Sigma_{\phi'}$ contains only singleton sets, and then the proof that $\Sigma_{\phi'}$ is finite may proceed by induction on the size of ϕ'.

4.5 Concluding remarks

We have introduced the notions of weak preorders and equivalences on modal models, based on the idea of applying bisimulation (or more generally zig-zag relation) like orderings to models that have been suitably transformed, or linearised. This involves in particular a shift of emphasis from states to sets of states and from the transition relations to partial functional successor relations. The weak preorders and equivalences generalises a variety of well-known process-oriented equivalences such as testing and failures equivalence, an observation basically due to De Nicola [28, 29].

The weak preorders have logical characterisations in terms of (fragments of) a normal modal logic with indexed forwards linear modalities, one variety of which takes the possibility of divergence into account where the other does not. We presented, as the main result of the chapter, sound and complete axiomatisations of the logics obtained.

Our approach serve to emphasise the importance in the theory of concurrent processes of general modal models as opposed to just frames, or transition systems; and their companions, the zig-zag relations (c.f. [107, 111]) generalising the bisimulation relations on frames. The huge loss of information resulting from the linearisation of frames is compensated for by the admission of even very restricted atomic propositions, and it is interesting that such apparently very different notions of equivalence as bisimulation and testing, refusal testing and failures can be captured in the common framework of zig-zag relations.

We finally point out some possible directions for future work. First of all, axiomatisations for fragments other than GTFm_1 and GTFm_2 should be developed. Secondly it is very likely that most of the results presented here can be obtained as special cases of results pertaining to intuitionistic modal logics along the line of Stirling [106], due to the zig-zag relation-like nature of the weak preorders and equivalences on linearised frames. These connections should be investigated.

Chapter 5

Concurrency and compositionality

In this chapter we consider model classes closed under a parallel composition and use such classes to interpret relevant extensions of the ground logics introduced in chapter 4. We give, as the main result of the chapter, for each suitably closed class of models sound and complete axiomatisations of the extended logics, relative to complete axiomatisations of their ground fragments.

We start by introducing the notion of concurrent model class and investigate its basic properties. In section 2 the ground logics introduced in chapter 4 are extended by a relevant implication in the style of the introduction. The completeness proof proceeds, as the proof of 3.35, by rewriting, and we hence need strong decomposition properties for implications. We identify subclasses of prime and coprime formulas for this purpose. To be able to eliminate implications completely we need principally the existence of least models satisfying consistent formulas. In section 3 we introduce a suitable such notion of completeness and prove the final decomposition lemmas needed. We thus proceed, in section 4, to present axiomatisations of the full logics relative to sound and complete axiomatisations of their ground fragments, and prove their soundness, and in section 5 we proceed to relative completeness.

We then, in section 6, turn to the concrete example of testing models, and show this model class indeed to be complete, and finally, in section 7, we show how unrelativised complete axiomatisations are obtained from the earlier, more general results.

5.1 Adding structure: Concurrent model classes

We introduce an operationally motivated parallel composition on models, intended to capture in general terms a variety of notions of both synchronous and asynchronous parallel composition such as those known from CCS and SCCS.

5.1.1 Basic definitions

We define the parallel composition $\mathcal{M}_1 \parallel \mathcal{M}_2$ of models $\mathcal{M}_1$ and $\mathcal{M}_2$. The issues are how to define the transitions and the valuations of $\mathcal{M}_1 \parallel \mathcal{M}_2$ in terms of those of $\mathcal{M}_1$ and $\mathcal{M}_2$. Concerning the transitions we assume—as in CCS, SCCS and the example of the chapter 3—that the label universe $\mathcal{L}$ comes equipped with a (potentially partial) operation of label multiplication to account for the simultaneous occurrence of actions. We refer to such a structure as a *synchronisation algebra*. In this setting asynchrony can be obtained by assuming a special "idling" label, as for instance in [116]. Note, however, that in the present setting this also amounts to the assumption that any transition arising as the simultaneous occurrence of labelled—and hence observable—transitions is itself labelled and hence observable. It is an important point for future investigation to allow this to be relaxed.

Concerning the valuations we assume that there is some compositional way of deducing the holding of atomic propositions for parallel composed states. That is, we assume that with the set Ap comes *decomposition maps*

$$\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 : \mathrm{Ap} \to \mathcal{P}(\mathrm{GTFm} \times \mathrm{GTFm})$$

with the property, intuitively, that a parallel composition will have the (e.g.) may-property α iff for some $\phi_1, \phi_2 \in \mathrm{GTFm}$ s.t. $\langle \phi_1, \phi_2 \rangle \in \boldsymbol{\delta}_1(\alpha)$ one component will satisfy ϕ_1 and one will satisfy ϕ_2. Or in other words that for each $\alpha \in \mathrm{Ap}$ and $i \in \{1, 2\}$ there is some sound and complete set of proof rules of the form

$$\frac{\sigma_1 \models \phi_1 \quad \sigma_2 \models \phi_2}{\sigma_1 \times \sigma_2 \models \alpha^i}$$

where $\times$ denotes parallel composition. In this connection it seems reasonable moreover to assume that atomic may-propositions can be deduced in terms of positive may-formulas only—these, after all were seen to be sufficiently expressive to induce the may-preorder on the frames themselves—and similarly for the must-case. That is, that for $i \in \{1, 2\}$, $\boldsymbol{\delta}_i$ has the functionality

$$\boldsymbol{\delta}_i : \mathrm{Ap} \to \mathcal{P}(\mathrm{GTFm}_i^+ \times \mathrm{GTFm}_i^+).$$

We moreover, for technical reasons, assume that atomic propositions are always decomposed into formulas of equal size, i.e. such that whenever $\langle \phi_1, \phi_2 \rangle \in \boldsymbol{\delta}_i(\alpha)$, $i \in \{1,2\}$, then $|\phi_1| = |\phi_2|$. We need this assumption in the full completeness proof for assuring that the size of formulas does not increase under reduction.

Definition 5.1 A *basic concurrent model class* is a structure

$$\langle \boldsymbol{\mathcal{M}}, \mathcal{L}, \cdot, \mathrm{Ap}, \boldsymbol{\delta}_1, \boldsymbol{\delta}_2 \rangle,$$

where

i) $\boldsymbol{\mathcal{M}}$ is a class of models over $\mathcal{L}$ and Ap,

ii) $\langle \mathcal{L}, \cdot \rangle$ is a synchronisation algebra.

iii) For $i \in \{1,2\}$, $\boldsymbol{\delta}_i : \mathrm{Ap} \rightarrow \mathcal{P}(\mathrm{GTFm}_i^+ \times \mathrm{GTFm}_i^+)$ are decomposition maps s.t. whenever $\langle \phi_1, \phi_2 \rangle \in \boldsymbol{\delta}_i(p)$ then $|\phi_1| = |\phi_2|$.

Multiplication of labels $\lambda_1, \lambda_2 \in \mathcal{L}$ is denoted by juxtaposition and the fact that $\lambda_1\lambda_2$ is defined is denoted by $\lambda_1\lambda_2\downarrow$. We sometimes use "Kleene equality", $\simeq$, on label expressions, defined by $\lambda \simeq \mu$ iff ($\lambda\downarrow$ iff $\mu\downarrow$), and if $\lambda\downarrow$ then $\lambda = \mu$. Parallel composition of models is now defined in the following way. For $j \in \{1,2\}$ let

$$\mathcal{M}_j = \langle \langle S_j, L_j, \twoheadrightarrow_j, \{\xrightarrow{\lambda}_j\}_{\lambda \in L_j}, \uparrow_j \rangle, V_{1,j}, V_{2,j} \rangle$$

be models in some basic concurrent model class $\boldsymbol{\mathcal{M}}$. Then the *parallel composition*, or "product", of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the model

$$\mathcal{M}_1 \parallel \mathcal{M}_2 = \langle \langle S_1 \times S_2, L_1 \cdot L_2, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda \in L}, \uparrow \rangle, V_1, V_2 \rangle,$$

where

i) $L_1 \cdot L_2 =_{\mathrm{def}} \{\lambda_1\lambda_1 \mid \lambda_1 \in L_1, \lambda_2 \in L_2, \lambda_1\lambda_2\downarrow\}$,

ii) $\langle \sigma_1, \sigma_2 \rangle \twoheadrightarrow \langle \sigma_1', \sigma_2' \rangle$ iff either

  a) $\sigma_1 \twoheadrightarrow_1 \sigma_1'$ and $\sigma_2 = \sigma_2'$, or

  b) $\sigma_2 \twoheadrightarrow_2 \sigma_2'$ and $\sigma_1 = \sigma_1'$.

iii) for all $\lambda \in L_1 \cdot L_2$, $\langle \sigma_1, \sigma_2 \rangle \xrightarrow{\lambda} \langle \sigma_1', \sigma_2' \rangle$ iff there are $\lambda_1 \in L_1$, $\lambda_2 \in L_2$ s.t. $\lambda_1\lambda_2 \simeq \lambda$, $\sigma_1 \xrightarrow{\lambda_1}_1 \sigma_1'$ and $\sigma_2 \xrightarrow{\lambda_2}_2 \sigma_2'$.

iv) $\langle\sigma_1, \sigma_2\rangle \uparrow$ iff $\sigma_1 \uparrow$ or $\sigma_2 \uparrow$.

v) $\langle\sigma_1, \sigma_2\rangle \in V_i(\alpha)$ iff there are $\phi_1, \phi_2 \in \mathrm{GTFm}_i^+$ s.t. $\langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_i(\alpha)$, $\sigma_1 \models \phi_1$ and $\sigma_2 \models \phi_2$, for $i \in \{1, 2\}$.

Note that parallel composition is well-defined—i.e. that the valuations V_i of v) are indeed i-valuations, for $i \in \{1, 2\}$. We could, in ii), allow both σ_1 and σ_2 to proceed in parallel as well without this making any difference in terms of the weak orderings.

5.1.2 Uniformity

Surprisingly, perhaps, the parallel composition turns out not to be very well-behaved in general when linearising—the must-ordering, in particular, fails in general to be preserved under $\|$. A simple example suffices to illustrate the problem:

Example 5.2 Let $\mathrm{Ap} = \{\alpha_a, \alpha_b, \alpha_c, \alpha_d\}$ and $\boldsymbol{\delta}_2(\alpha_d) = \{\langle\alpha_a, \alpha_c\rangle, \langle\alpha_b, \alpha_c\rangle\}$. Let $\mathcal{M}_a$, $\mathcal{M}_b$, $\mathcal{M}_c$ be models without transitions and all states convergent. Let $S_{\mathcal{M}_a} = \{\sigma_a, \sigma_b\}$ with $V_{2,\mathcal{M}_a}(\alpha_a) = \{\sigma_a\}$ and $V_{2,\mathcal{M}_a}(\alpha_b) = \{\sigma_b\}$, $S_{\mathcal{M}_b} = \{\sigma\}$, and $S_{\mathcal{M}_c} = \{\sigma_c\}$ with $V_{2,\mathcal{M}_c}(\alpha_c) = \{\sigma_c\}$, and for all other $j \in \{a, b, c, d\}$ and $k \in \{a, b, c\}$, let $V_{2,\mathcal{M}_k}(\alpha_j) = \emptyset$. Then $\{\sigma_a, \sigma_b\} \sqsubseteq_2 \{\sigma\}$ but

$$\{\sigma_a, \sigma_b\} \times \{\sigma_c\} \in V_{2,\mathcal{M}_a\|\mathcal{M}_c}(\alpha_d)$$

and

$$\{\sigma\} \times \{\sigma_c\} \notin V_{2,\mathcal{M}_b\|\mathcal{M}_c}(\alpha_d).$$

This is overcome by requiring must-decomposition to be uniform w.r.t the formation of sets. A basic concurrent model class $\boldsymbol{\mathcal{M}}$ is called *uniform*, if whenever $\Sigma_1 \times \Sigma_2 \in V_{2,(\mathcal{M}_1\|\mathcal{M}_2)^\dagger}(\alpha)$ for $\mathcal{M}_1, \mathcal{M}_2 \in \boldsymbol{\mathcal{M}}$ then for some $\phi_1, \phi_2 \in \mathrm{GTFm}_2^+$, $\langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_2(\alpha)$, $\Sigma_1 \models \phi_1$ and $\Sigma_2 \models \phi_2$. All the concrete instances of model classes considered in section 4.3 satisfy this uniformity requirement (provided the decomposition maps exist), because sets are expressible internally in them using some kind of internal sum construction. We note some basic properties of linearised parallel compositions in uniform model classes:

Proposition 5.3 *For all $\mathcal{M}_1, \mathcal{M}_2 \in \boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{M}}$ a uniform basic concurrent model class, and $\Sigma_1 \in {\mathcal{M}_1}^\dagger$, $\Sigma_2 \in {\mathcal{M}_2}^\dagger$,* *i)* $\Sigma_1 \times \Sigma_2 \in (\mathcal{M}_1 \parallel \mathcal{M}_2)^\dagger$,

ii) $(\Sigma_1 \times \Sigma_2)$ can λ *iff for some* $\lambda_1 \in L_{\mathcal{M}_1}$, $\lambda_2 \in L_{\mathcal{M}_2}$ *s.t.* $\lambda_1\lambda_2 \simeq \lambda$, Σ_1 can λ_1 *and* Σ_2 can λ_2,

iii) $(\Sigma_1 \times \Sigma_2)/\lambda = \bigcup_{\lambda_1,\lambda_2 \text{ s.t. } \lambda_1\lambda_2\simeq\lambda}(\Sigma_1/\lambda_1) \times (\Sigma_2/\lambda_2)$,

iv) $(\Sigma_1 \times \Sigma_2)\Downarrow$ *iff* $\Sigma_1 \Downarrow$ *and* $\Sigma_2 \Downarrow$,

v) $\Sigma_1 \times \Sigma_2 \in V_i(\alpha)$ *iff for some* $\phi_1, \phi_2 \in \mathrm{GTFm}_i^+$, $\langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_i(\alpha)$, $\Sigma_1 \models \phi_1$, $\Sigma_2 \models \phi_2$.

PROOF: Straightforward. □

The following proposition is important as it gives, up to isomorphism, a unique way of linearising parallel compositions.

Proposition 5.4 *For any uniform basic concurrent model class* $\boldsymbol{\mathcal{M}}$, *and* $\mathcal{M}_1$, $\mathcal{M}_2 \in \boldsymbol{\mathcal{M}}$, ${\mathcal{M}_1}^\dagger \parallel {\mathcal{M}_2}^\dagger \cong \mathcal{M}_1 \parallel \mathcal{M}_2$.

PROOF: Note first that ${\mathcal{M}_1}^\dagger \parallel {\mathcal{M}_2}^\dagger$ is indeed well-defined. The desired isomorphism takes any set $\underline{\Sigma} \in ({\mathcal{M}_1}^\dagger \parallel {\mathcal{M}_2}^\dagger)^\dagger$ into the set

$$f(\underline{\Sigma}) = \bigcup\{\Sigma_1 \times \Sigma_2 \mid \langle\Sigma_1, \Sigma_2\rangle \in \underline{\Sigma}\}.$$

It is not hard to check that f is indeed a surjection—indeed lin was defined specifically for this to hold.
Let now $\underline{\Sigma}$ range over $({\mathcal{M}_1}^\dagger \parallel {\mathcal{M}_2}^\dagger)^\dagger$. First $\underline{\Sigma}$ can λ iff for some $\Sigma_1 \in {\mathcal{M}_1}^\dagger$, $\Sigma_2 \in {\mathcal{M}_2}^\dagger$, λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$, Σ_1 can λ_1 and Σ_2 can λ_2 iff for some $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$, σ_1 can λ_1 and σ_2 can λ_2 iff $f(\underline{\Sigma})$ can λ.
Suppose then that $\underline{\Sigma}$ can λ. We calculate

$$\begin{aligned}
f(\underline{\Sigma}/\lambda) &= f\{\langle\Sigma_1', \Sigma_2'\rangle \mid \exists\Sigma_1, \Sigma_2 \text{ s.t. } \langle\Sigma_1, \Sigma_2\rangle \in \underline{\Sigma}, \langle\Sigma_1, \Sigma_2\rangle \xrightarrow{\lambda} \langle\Sigma_1', \Sigma_2'\rangle\} \\
&= \bigcup\{\Sigma_1/\lambda_1 \times \Sigma_2/\lambda_2 \mid \langle\Sigma_1, \Sigma_2\rangle \in \underline{\Sigma}, \lambda \simeq \lambda_1\lambda_2, \Sigma_1 \text{ can } \lambda_1, \Sigma_2 \text{ can } \lambda_2\} \\
&= \bigcup\{(\Sigma_1 \times \Sigma_2)/\lambda \mid \langle\Sigma_1, \Sigma_2\rangle \in \underline{\Sigma}, \Sigma_1 \times \Sigma_2 \text{ can } \lambda\} \\
&= (f(\underline{\Sigma}))/\lambda.
\end{aligned}$$

The rest of the proof is an easy check that $\underline{\Sigma}\Downarrow$ iff $f(\underline{\Sigma})\Downarrow$ and that $\underline{\Sigma} \in V_i(\alpha)$ iff $f(\underline{\Sigma}) \in V_i(\alpha)$, $i \in \{1, 2\}$. □

It is not hard to generalise example 5.2 above to prove a converse to proposition 5.4—i.e. if the proposition holds for all $\mathcal{M}_1, \mathcal{M}_2 \in \boldsymbol{\mathcal{M}}$ then $\boldsymbol{\mathcal{M}}$ is uniform. Note next that we have indeed overcome the problems of example 5.2:

Proposition 5.5 *For any uniform basic concurrent model class, the orderings* $\sqsubseteq_i$, $i \in \{1,2\}$, *and* $\sqsubseteq$ *are preserved under* $\|$.

PROOF: Use 5.3, 4.3 and the uniformity property. □

Finally we obtain preservation of $\oplus$ under $\|$ in both arguments:

Proposition 5.6 *For any uniform, concurrent model class* $\boldsymbol{\mathcal{M}}$ *and* $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3 \in \boldsymbol{\mathcal{M}}$,

i) $\mathcal{M}_1 \parallel (\mathcal{M}_2 \oplus \mathcal{M}_3) \overset{\sqsubset}{\sim} ((\mathcal{M}_1 \parallel \mathcal{M}_2) \oplus (\mathcal{M}_1 \parallel \mathcal{M}_3))$, *and*

ii) $(\mathcal{M}_1 \oplus \mathcal{M}_2) \parallel \mathcal{M}_3 \overset{\sqsubset}{\sim} (\mathcal{M}_1 \parallel \mathcal{M}_3) \oplus (\mathcal{M}_2 \parallel \mathcal{M}_3)$.

PROOF: ii) is proved by an argument symmetrical to i). We show that the relation R determined by

$$\Sigma_1 \times (\Sigma_2 \oplus \Sigma_3) R ((\Sigma_1 \times \Sigma_2) \oplus (\Sigma_1 \times \Sigma_2))$$

is an embedding, i.e. that

$$\Sigma_1 \times (\Sigma_2 \oplus \Sigma_3) \simeq^n (\Sigma_1 \times \Sigma_2) \oplus (\Sigma_1 \times \Sigma_2)$$

for all $n \geq 0$. This is shown—as usual—by induction on n. The base case is trivial, so let $n = n'+1$. Now $\Sigma_1 \times (\Sigma_2 \oplus \Sigma_3)$ can λ iff for some λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$, Σ_1 can λ_1 and either Σ_2 can λ_2 or Σ_3 can λ_2 iff $(\Sigma_1 \times \Sigma_2) \oplus (\Sigma_1 \times \Sigma_3)$ can λ. Then

$$\begin{array}{rcl}
(\Sigma_1 \times (\Sigma_2 \oplus \Sigma_3))/\lambda & = & \bigcup_{\lambda_1,\lambda_2.\lambda\simeq\lambda_1\lambda_2} (\Sigma_1/\lambda_1) \times ((\Sigma_2 \oplus \Sigma_3)/\lambda_2) \\
& = & \bigcup_{\lambda_1,\lambda_2.\lambda\simeq\lambda_1\lambda_2} (\Sigma_1/\lambda_1) \times ((\Sigma_2/\lambda_2) \oplus (\Sigma_3/\lambda_2)) \\
& \simeq^{n'} & \bigcup_{\lambda_1,\lambda_2.\lambda\simeq\lambda_1\lambda_2} ((\Sigma_1/\lambda_1) \times (\Sigma_2/\lambda_2)) \oplus \\
& & ((\Sigma_1/\lambda_1) \times (\Sigma_3/\lambda_2)) \\
& & \text{(by the induction hypothesis)} \\
& = & (\bigcup_{\lambda_1,\lambda_2.\lambda\simeq\lambda_1\lambda_2} (\Sigma_1/\lambda_1) \times (\Sigma_2/\lambda_2)) \oplus \\
& & (\bigcup_{\lambda_1,\lambda_2.\lambda\simeq\lambda_1\lambda_2} (\Sigma_1/\lambda_1) \times (\Sigma_3/\lambda_2)) \\
& = & ((\Sigma_1 \times \Sigma_2) \oplus (\Sigma_1 \times \Sigma_3))/\lambda.
\end{array}$$

Here we have implicitly made a harmless extension of $\oplus$ to the case where one of the summands is empty. Next $\Sigma_1 \times (\Sigma_2 \oplus \Sigma_3) \Downarrow$ iff $(\Sigma_1 \times \Sigma_2) \oplus (\Sigma_1 \times \Sigma_3) \Downarrow$, and for the valuations, $\Sigma_1 \times (\Sigma_2 \oplus \Sigma_3) \in V_1(\alpha)$ iff for some $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2 \oplus \Sigma_3$, $\langle \phi_1, \phi_2 \rangle \in \boldsymbol{\delta}_1(\alpha)$, $\sigma_1 \models \phi_1$ and $\sigma_2 \models \phi_2$ iff $(\Sigma_1 \times \Sigma_2) \oplus (\Sigma_1 \times \Sigma_3) \in V_1(\alpha)$. The corresponding result for V_2 is similar, and the proof is thus completed. □

Note that isomorphism fails to hold in general. In case i), for instance, there may well be $\Sigma_1, \Sigma_1' \in \mathcal{M}_1{}^\dagger$, $\Sigma_2 \in \mathcal{M}_2{}^\dagger$ and $\Sigma_3 \in \mathcal{M}_3{}^\dagger$ s.t. $(\Sigma_1 \times \Sigma_2) \oplus (\Sigma_1' \times \Sigma_3)$ is not weakly equivalent to any state in $(\mathcal{M}_1 \parallel (\mathcal{M}_2 \oplus \mathcal{M}_3))^\dagger$.

5.2 The full logics

Having thus set up our basic notion of model class structured with a parallel composition, we turn to enriching the base modal logic with a relevant implication to provide a compositional account of this operation in the same way as in chapter 3. Notice that, as the label universes and atomic proposition symbols are fixed on the level of concurrent model classes, formulas are really parametrized on those. So for each model class $\boldsymbol{\mathcal{M}}$ and $i \in \{1, 2\}$, $\mathrm{TFm}_{(i)}^{(+)}$ denotes the extension of $\mathrm{GTFm}_{(i)}^{(+)}$ by the binary connective $\to$ (which is assumed to be of smallest binding power). The notion of size, or modal depth, $|\cdot|$, is extended to formulas $\phi \in \mathrm{TFm}$ by $|\, \phi \to \psi \,| = \max(|\, \phi \,|, |\, \psi \,|)$. As in the previous chapter we are (almost) only concerned with the fragments TFm_1^+ and TFm_2^+. Formulas are interpreted over uniform concurrent model classes that are closed under parallel composition and disjoint union; such a class is called a *u.p.d.-model class*. For $\boldsymbol{\mathcal{M}}$ a u.p.d.-model class satisfaction is extended to TFm by:

$$\langle \Sigma, \mathcal{M}^\dagger \rangle \models \phi \to \psi \quad \text{iff} \quad \text{for all } \mathcal{M}' \in \boldsymbol{\mathcal{M}},\ \Sigma' \in S_{\mathcal{M}^\dagger}, \text{ if } \langle \Sigma', \mathcal{M}'^\dagger \rangle \models \phi$$
$$\text{then } \langle \Sigma \times \Sigma', (\mathcal{M} \parallel \mathcal{M}')^\dagger \rangle \models \psi.$$

Notice that we quantify over models as well as states—in contrast to chapter 3. Here it is model classes rather than the individual models that are structured. Note first that the logical characterisation of the weak preorders is maintained in the full languages:

Theorem 5.7 (The logical characterisation, full fragments). *For any u.p.d.-model class $\boldsymbol{\mathcal{M}}$ and $i \in \{1, 2\}$, $\mathrm{TFm}_{(i)}^+$ induces $\sqsubseteq_{(i)}$, $\mathrm{TFm}_{(i)}$ induces $\simeq_{(i)}$.*

PROOF: One half follows directly from theorem 4.11. The other is an easy induction on the structure of formulas, using 5.5. □

A little corollary of 5.7 is that for u.p.d.-model classes, whenever $\Sigma_1 \subseteq \Sigma_2$ and $\Sigma_1 \models \phi \in \mathrm{TFm}_1^+$ then $\Sigma_2 \models \phi$; and similarly whenever $\Sigma_1 \subseteq \Sigma_2$ and

$\Sigma_2 \models \phi \in \mathrm{TFm}_2^+$ then $\Sigma_1 \models \phi$. This is seen by 5.7 and 4.3.

5.2.1 Prime and coprime formulas

The semantical object of prime interest is the relation of local consequence, $\models_{\mathcal{M}}$. Evidently we cannot hope for a sound and complete axiomatisation of $\models_{\mathcal{M}}$ for arbitrary $\boldsymbol{\mathcal{M}}$. What we rather do is to look for properties of implications relative to $\models_{\mathcal{M}}$ restricted to ground formulas. The basic idea in axiomatising the local consequence relation is analogous to the rewriting based completeness proof for the logics $\mathbf{PL}^{i,D}$ of chapter 3. That is, given an implication $\phi \to \psi$ we look for ways in which $\phi \to \psi$ can be rewritten, or decomposed, into equivalent, but "simpler", formulas. An important ingredient is identifying sufficiently large collections of formulas that allow disjunctions and conjunctions in certain contexts to be "folded out" of implications. These are the prime and coprime formulas.

Let a u.p.d.-model class $\boldsymbol{\mathcal{M}}$ be given, and let $i \in \{1, 2\}$.

i) A $\phi \in \mathrm{TFm}_i^+$ is $\to$-*prime*, if for all $\psi, \gamma \in \mathrm{TFm}_i^+$,

$$\phi \to (\psi \vee \gamma) \models_{\mathcal{M}} (\phi \to \psi) \vee (\phi \to \gamma).$$

ii) A $\phi \in \mathrm{TFm}_i^+$ is $\to$-*coprime*, if for all $\psi, \gamma \in \mathrm{TFm}_i^+$,

$$(\psi \wedge \gamma) \to \phi \models_{\mathcal{M}} (\psi \to \phi) \vee (\gamma \to \phi).$$

Similarly a ϕ is *prime* if whenever $\phi \models_{\mathcal{M}} \psi \vee \gamma$ then either $\phi \models_{\mathcal{M}} \psi$ or $\phi \models_{\mathcal{M}} \gamma$, and ϕ is *coprime* if whenever $\psi \wedge \gamma \models_{\mathcal{M}} \phi$ then either $\psi \models_{\mathcal{M}} \phi$ or $\gamma \models_{\mathcal{M}} \phi$. Note that, referring back to chapter 2, $\to$-primeness is the property needed to ensure that the product of theories preserve primeness; that is, if each formula can be rewritten into an equivalent one which is the disjunction of $\to$-primes then the product $\nabla_1 \cdot \nabla_2$ of any two prime theories is prime itself.

A sufficiently large number of $\to$-(co-)primes can be characterised syntactically. Given a u.p.d.-model class the set $\mathrm{SPR}_2 \subseteq \mathrm{TFm}_2^+$ of 2- *syntactic* $\to$-*primes* is the least s.t.

i) $\top, \bot \in \mathrm{SPR}_2$,

ii) $\alpha^2 \in \mathrm{SPR}_2$, for all $\alpha \in \mathrm{Ap}$,

iii) $\phi \in \mathrm{SPR}_2$ only if $[\lambda]\phi \in \mathrm{SPR}_2$, for all $\lambda \in \mathcal{L}$,

iv) $\phi, \psi \in \mathrm{SPR}_2$ only if $\phi \wedge \psi \in \mathrm{SPR}_2$,

v) $\psi \in \mathrm{SPR}_2$ only if $\phi \rightarrow \psi \in \mathrm{SPR}_2$, for all $\phi \in \mathrm{TFm}_2^+$.

The set $\mathrm{SCPR}_1 \subseteq \mathrm{TFm}_1^+$ of 1- *syntactic* $\rightarrow$- *coprimes* is the least s.t.

i) $\top, \bot \in \mathrm{SCPR}_1$,

ii) $\alpha^1 \in \mathrm{SCPR}_1$, for all $\alpha \in \mathrm{Ap}$,

iii) $\phi \in \mathrm{SCPR}_1$ only if $\langle\lambda\rangle\phi \in \mathrm{SCPR}_1$, for all $\lambda \in \mathcal{L}$,

iv) $\phi, \psi \in \mathrm{SCPR}_1$ only if $\phi \vee \psi \in \mathrm{SCPR}_1$.

Let further $\mathrm{SCPR}_2 = \emptyset$. We define SCPR_2 here only because it helps for a systematic exposition—in contrast to the may-case there are in the must-case hardly any other $\rightarrow$-coprimes than $\top$ and $\bot$. The set SPR_1 is defined later. Note that implications are not admissible in SCPR_1. In proving that the syntactic $\rightarrow$-(co-)primes are indeed what they claim to be we arrive at the essential reason why disjoint union was introduced in addition to parallel composition. In fact for the syntactic $\rightarrow$-(co-)primes disjoint union acts exactly as internal choice did in chapter 3 for all formulas, due to the nonstandard interpretation of the disjunction adopted there:

Proposition 5.8 *For any u.p.d.-model class,*

i) for all $\phi \in \mathrm{SCPR}_1$, $\Sigma_1 \oplus \Sigma_2 \models \phi$ *iff* $\Sigma_1 \models \phi$ *or* $\Sigma_2 \models \phi$,

ii) for all $\phi \in \mathrm{SPR}_2$, $\Sigma_1 \oplus \Sigma_2 \models \phi$ *iff* $\Sigma_1 \models \phi$ *and* $\Sigma_2 \models \phi$.

Proof: i) The if-direction follows from 4.8. For the only-if direction we proceed by induction on the structure of ϕ. The cases for $\phi = \top$ and $\phi = \bot$ are trivial.
$\phi = \alpha^1$. Use 4.7.v).
$\phi = \langle\lambda\rangle\phi'$. If $\Sigma_1 \oplus \Sigma_2 \models \langle\lambda\rangle\phi'$ then $\Sigma_1 \oplus \Sigma_2$ can λ—hence Σ_1 can λ or Σ_2 can λ by 4.7.ii). Moreover by 4.7.iii), $(\Sigma_1 \oplus \Sigma_2)/\lambda = (\Sigma_1/\lambda) \oplus (\Sigma_2/\lambda) \models \phi'$, hence $\Sigma_1/\lambda \models \phi'$ or $\Sigma_2/\lambda \models \phi'$ by the induction hypothesis. Moreover one of the Σ_j/λ, $j \in \{1,2\}$, are nonempty and then $\Sigma_j \models \langle\lambda\rangle\phi'$.
$\phi = \phi_1 \vee \phi_2$. Immediate by the induction hypothesis.
ii) Here the only-if direction follow from 4.8. For the if-direction we proceed by induction as in i). The cases for $\phi = \top$, $\phi = \bot$ are trivial.
$\phi = \alpha^2$. If $\Sigma_1, \Sigma_2 \models \alpha^2$ then $\Sigma_1 \oplus \Sigma_2 \models \alpha^2$ by 4.7.vi).

$\phi = [\lambda]\phi'$. If $\Sigma_1, \Sigma_2 \models [\lambda]\phi'$ then $\Sigma_1 \Downarrow$, $\Sigma_2 \Downarrow$—so $\Sigma_1 \oplus \Sigma_2 \Downarrow$—and if Σ_i can λ then $\Sigma_i/\lambda \models \phi'$, $i \in \{1,2\}$. Now if $\Sigma_1 \oplus \Sigma_2$ can λ then $(\Sigma_1 \oplus \Sigma_2)/\lambda = \Sigma_1/\lambda \oplus \Sigma_2/\lambda \models \phi'$ by the induction hypothesis, so $\Sigma_1 \oplus \Sigma_2 \models [\lambda]\phi'$.
$\phi = \phi_1 \wedge \phi_2$. Immediate by the induction hypothesis.
$\phi = \phi_1 \rightarrow \phi_2$. Let $\Sigma_1, \Sigma_2 \models \phi_1 \rightarrow \phi_2$ and $\Sigma \models \phi_1$. Then $\Sigma_1 \times \Sigma$, $\Sigma_2 \times \Sigma \models \phi_2$ and then by the induction hypothesis, $(\Sigma_1 \times \Sigma) \oplus (\Sigma_2 \times \Sigma) \models \phi_2$ as well, but

$$(\Sigma_1 \times \Sigma) \oplus (\Sigma_2 \times \Sigma) \simeq (\Sigma_1 \oplus \Sigma_2) \times \Sigma$$

by (the proof of) 5.6, so $(\Sigma_1 \oplus \Sigma_2) \times \Sigma \models \phi_2$ by 5.7, thus $\Sigma_1 \oplus \Sigma_2 \models \phi_1 \rightarrow \phi_2$. □

Note that the proof of i) will not go through if implications are admitted in SCPR_1. We check that the syntactic $\rightarrow$-(co-)primes are indeed what they claim to be.

Proposition 5.9 *For any u.p.d.-model class,*

i) Any $\phi \in \mathrm{SCPR}_1$ is $\rightarrow$-coprime and coprime,

ii) Any $\phi \in \mathrm{SPR}_2$ is $\rightarrow$-prime and prime.

PROOF: i) Let $\phi \in \mathrm{SCPR}_1$ and assume $\Sigma \models \psi \wedge \gamma \rightarrow \phi$, and moreover, for a contradiction, that $\Sigma \not\models \psi \rightarrow \phi$ and $\Sigma \not\models \gamma \rightarrow \phi$. Then there are Σ_1, Σ_2 s.t. $\Sigma_1 \models \psi$, $\Sigma_2 \models \gamma$, $\Sigma \times \Sigma_1 \not\models \phi$ and $\Sigma \times \Sigma_2 \not\models \phi$. Then, by 5.8.i), $(\Sigma \times \Sigma_1) \oplus (\Sigma \times \Sigma_2) \not\models \phi$, but $(\Sigma \times \Sigma_1) \oplus (\Sigma \times \Sigma_2) \simeq_1 \Sigma \times (\Sigma_1 \oplus \Sigma_2)$ by 5.6 and hence $\Sigma \times (\Sigma_1 \oplus \Sigma_2) \not\models \phi$ by 5.7. But $\Sigma_1 \oplus \Sigma_2 \models \psi \wedge \gamma$ by 4.8 and 5.7, so $\Sigma \not\models \psi \wedge \gamma \rightarrow \phi$—a contradiction. The proof for coprimeness is similar.
ii) Assume $\phi \in \mathrm{SPR}_2$, $\Sigma \models \phi \rightarrow \psi \vee \gamma$, and assume for a contradiction that $\Sigma \not\models \phi \rightarrow \psi$ and $\Sigma \not\models \phi \rightarrow \gamma$. Then there are Σ_1, Σ_2 s.t. $\Sigma_1, \Sigma_2 \models \phi$, $\Sigma \times \Sigma_1 \not\models \psi$ and $\Sigma \times \Sigma_2 \not\models \gamma$. By 5.8.ii), $\Sigma_1 \oplus \Sigma_2 \models \phi$ and hence $\Sigma \times (\Sigma_1 \oplus \Sigma_2) \models \psi \vee \gamma$. But $\Sigma \times (\Sigma_1 \oplus \Sigma_2) \simeq_2 (\Sigma \times \Sigma_1) \oplus (\Sigma \times \Sigma_2)$ by 5.6, so also $(\Sigma \times \Sigma_1) \oplus (\Sigma \times \Sigma_2) \models \psi \vee \gamma$ by 5.7. But then either $(\Sigma \times \Sigma_1) \oplus (\Sigma \times \Sigma_2) \models \psi$ or $(\Sigma \times \Sigma_1) \oplus (\Sigma \times \Sigma_2) \models \gamma$ and then by 4.8 and 5.7 once more, either $\Sigma \times \Sigma_1 \models \psi$ or $\Sigma \times \Sigma_2 \models \gamma$—a contradiction. Again the proof of primeness is similar. □

Note that in particular $\mathrm{TRNF} \subseteq \mathrm{SCPR}_1$ and $\mathrm{PNF}_2 \subseteq \mathrm{SPR}_2$, so we have proved traces to be ($\rightarrow$-) coprime and 2-prime normal forms to be ($\rightarrow$-)prime. A little corollary of 5.9 is that validity in a u.p.d-model class for TFm_2^+ has the disjunction property—i.e. if $\models_{\mathcal{M}} \phi \vee \psi$ then $\models_{\mathcal{M}} \phi$ or $\models_{\mathcal{M}} \psi$. This is because $\top \in \mathrm{SPR}_2$.

5.3 Complete model classes

A large number of occurrences of disjunctions and conjunctions inside implications can be "eliminated", or rather, pushed out now, using either generally valid logical principles, or the syntactic characterisations of proposition 5.9. Moreover, by 4.30 any ground formula can be rewritten into an equivalent one which is a disjunction of "primes" (we haven't in fact yet justified this terminology for the may-case, remember, although it is not too hard to see that for $\models$—but not for $\models_{\mathcal{M}}$—the 1-prime normal forms are indeed prime).

Fixing $\mathcal{L}$ and Ap let now an i-prime normal form ϕ be given. If $\phi \neq \bot$ then it is an easy matter, as in the proof of lemma 4.28.i) to construct some model $\mathcal{M}$ and $\Sigma \in \mathcal{M}^\dagger$ s.t. $\Sigma \models \phi$, and moreover, $\langle \Sigma, \mathcal{M} \rangle$ may be constructed so that it is i-least among those satisfying ϕ. What was needed for this was in essence just a notion of prefixing as in CCS and—most crucially—a complete freedom to generate the appropriate valuations, using the implicit assumption that the dynamics—i.e. transitions—of models and their atomic properties do not interact. Such an assumption, however, is evidently totally unrealistic; in the above examples there is such an interaction, and freedom to generate arbitrary valuations cannot be assumed. However, the power of atomic propositions to express the dynamic properties of models, and this is an important point, for all the concrete examples we have seen is strictly limited to the immediate transition capabilities of the state under consideration and does not in any way depend upon successive reachable states. For this reason it will for the examples we have considered (modulo a remark to be made later) suffice to have available a least model and a notion of guarded choice generalising prefixing in addition to disjoint union, in order to ensure that for any ϕ in i-prime normal form which is *consistent*, such a construction of an i-least model $\langle \Sigma, \mathcal{M}^\dagger \rangle$ satisfying ϕ can be carried out. Moreover, this construction can be done uniformly in ϕ because all information concerning the properties of successors of Σ must be present explicitly in ϕ as formulas of the form $\langle \lambda \rangle \phi_\lambda / [\lambda] \phi_\lambda$.

This property—that i-least models satisfying consistent i-prime normal forms exist and are uniform in the structure of that formula—is the final property of model classes needed in order to allow implications to be completely eliminated, and thus complete axiomatisations in the spirit of the completeness proofs of the $\mathbf{PL}^{i,D}$ of the previous chapter to be given.

There is, however, one more remark we need to give. For note that this property entails the primeness of consistent atomic propositions w.r.t. $\models_{\mathcal{M}}$. This is not hard to see. For each α^i is in i-prime normal form and if it is consistent then Σ_{α^i}—the i-least model satisfying α^i—is defined and $\Sigma_{\alpha^i} \models \alpha^i$. Let then $\alpha^i \models_{\mathcal{M}} \phi \vee \psi$. Then $\Sigma_{\alpha^i} \models \phi$ or $\Sigma_{\alpha^i} \models \psi$ and then $\alpha^i \models_{\mathcal{M}} \phi$ or $\alpha^i \models_{\mathcal{M}} \psi$, and α^i is indeed prime. This is okay for $i = 2$, of course, for we have already seen (proposition 5.9) that in this case atomic propositions are prime. Not so, however, for the may-case. There may well be model-classes $\boldsymbol{\mathcal{M}}$ (for instance the testing models of 4.3.1) where this fails, even though it does in fact hold for the class of all models (over given $\mathcal{L}$ and Ap). It is, however, the case for the above examples that it is possible to pick out for each given $\boldsymbol{\mathcal{M}}$ a subset Ap_1 of Ap which consists of all the prime atomic propositions; and moreover Ap_1 suffice to generate, up to 1-equivalence, the whole set Ap in the sense that each $\alpha \in \mathrm{Ap}$ will be 1-equivalent to a finite disjunction of elements of Ap_1.

The notion of 1-normal form is thus relativised to such sets Ap_1. Given any subset $\mathrm{Ap}_1 \subseteq \mathrm{Ap}$, the sets $\mathrm{TRNF}_{\mathrm{Ap}_1}$, $\mathrm{PNF}_{1,\mathrm{Ap}_1}$ and $\mathrm{NF}_{1,\mathrm{Ap}_1}$ denote the subsets of TRNF, PNF_1 and NF_1 respectively that contains only atomic propositions α^1 s.t. $\alpha \in \mathrm{Ap}_1$. For any model class $\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{M}}$ is said to be 1-*finitely generated* (1-f.g.) by Ap_1 if for any $\alpha \in \mathrm{Ap}$ there is a finite set $A \subseteq \mathrm{Ap}_1$ s.t. $\alpha^1 \sim_{\mathcal{M}} \bigvee\{\beta^1 \mid \beta \in A\}$. The indexing of TRNF etc. by Ap_1 is usually omitted, where the set Ap_1 is understood from the context.

Let us then make precise what is meant by uniformity in the present context. For this purpose prime normal forms are equipped with an operation $\cdot/\lambda$ of successor. For TRNF let

i) $\top/\lambda = \top$,

ii) $\alpha^1/\lambda = \top$,

iii) $\langle\lambda_1\rangle\phi/\lambda_2 = \begin{cases} \top & \text{, if } \lambda_1 \neq \lambda_2 \\ \phi & \text{, otherwise.} \end{cases}$

For $\phi = \bigwedge \Phi \in \mathrm{PNF}_1$, $\Phi \subseteq \mathrm{TRNF}$, let $\phi/\lambda = \bigwedge\{\phi'/\lambda \mid \phi' \in \Phi\}$, and for PNF_2 let

i) $\bot/\lambda = \bot$,

ii) $(\bigwedge_{j\in J} \alpha_j^2 \wedge \bigwedge_{\lambda_1\in\Lambda}[\lambda_1]\phi_{\lambda_1})/\lambda_2 = \begin{cases} \top & \text{, if } \lambda_2 \notin \mathrm{PNF}_2 \\ \phi_{\lambda_2} & \text{, otherwise.} \end{cases}$

Clearly these operations are well-defined. The desired property of model classes is the following one:

Definition 5.10 For $i \in \{1,2\}$, an *i-complete model class* is a u.p.d.-model class $\mathcal{M}$ s.t. if $i = 1$ then $\mathcal{M}$ is 1-finitely generated by some Ap_1, and s.t. for all consistent $\phi \in \mathrm{PNF}_{i(,\mathrm{Ap}_1)}$ there is a rooted model $\langle \sigma_\phi, \mathcal{M}_\phi \rangle$ in $\mathcal{M}$ s.t.

i) $\sigma_\phi \models \phi$,

ii) whenever $\langle \sigma', \mathcal{M}' \rangle$ is in $\mathcal{M}$ and $\sigma' \models \phi$ then $\sigma_\phi \sqsubseteq_i \sigma'$,

iii) (Uniformity 1) if $i = 2$ and $\sigma_\phi \Downarrow$ then for all $\lambda \in \mathcal{L}$, σ_ϕ can λ iff ϕ/λ is $\mathcal{M}$-consistent, and

iv) (Uniformity 2) for all $\lambda \in \mathcal{L}$, if σ_ϕ can λ then $\sigma_\phi/\lambda \simeq_i \sigma_{\phi/\lambda}$.

Let $\Sigma_\phi = \{\sigma_\phi\}$, and call a model class *complete* if it is both 1- and 2-complete. This is a very strong property of model classes. We later demonstrate that the class of "testing models" form an example. Note that the only-if direction of 5.10.iii) is a trivial consequence of i).

5.3.1 Primes revisited

Completeness has a number of immediate consequences; in particular we obtain the lacking set of 1-syntactic $\rightarrow$-primes. Note first the following corollary:

Proposition 5.11 *Let $i \in \{1,2\}$ and $\mathcal{M}$ be i-complete. Then for every $\mathcal{M}$-consistent $\phi \in \mathrm{PNF}_i$, and any $\psi \in \mathrm{TFm}_i^+$, $\phi \models_{\mathcal{M}} \psi$ iff $\Sigma_\phi \models \psi$.*

Proof: If $\phi \models_{\mathcal{M}} \psi$ then—as $\Sigma_\phi \models \phi$ by 5.10.i)—$\Sigma_\phi \models \psi$. Conversely if $\Sigma_\phi \models \psi$ and $\Sigma \models \phi$ then $\Sigma_\phi \sqsubseteq_i \Sigma$ by 5.10.ii) and then $\Sigma \models \psi$ by 5.7. □

The strength of least members w.r.t. implications should be obvious—from this the $\rightarrow$-primeness of PNF_1 in particular follow:

Proposition 5.12 *Let $i \in \{1,2\}$ and $\mathcal{M}$ be i-complete. Then for every $\mathcal{M}$-consistent $\phi \in \mathrm{PNF}_i$, and any $\psi \in \mathrm{TFm}_i^+$,*

i) for all $\langle \Sigma, \mathcal{M}^\dagger \rangle$ in $\mathcal{M}$, $\Sigma \models \phi \rightarrow \psi$ iff $\Sigma \times \Sigma_\phi \models \psi$,

ii) ϕ is prime and $\rightarrow$-prime.

PROOF: i) If $\Sigma \models \phi \to \psi$ then $\Sigma \times \Sigma_\phi \models \psi$ as $\Sigma_\phi \models \phi$. If $\Sigma \times \Sigma_\phi \models \psi$ and $\Sigma' \models \phi$ then $\Sigma_\phi \sqsubseteq_i \Sigma'$ so $\Sigma \times \Sigma' \models \psi$ by 5.5 and then $\Sigma \models \phi \to \psi$.
ii) If $\Sigma \models \phi \to \psi \vee \gamma$ then $\Sigma \times \Sigma_\phi \models \psi$ or $\Sigma \times \Sigma_\phi \models \gamma$ as $\Sigma_\phi \models \phi$, and then $\Sigma \models \phi \to \psi$ or $\Sigma \models \phi \to \gamma$ by i). Primeness is similar (we have of course already proved primeness for $i = 2$). $\Box$

Thus for both $i = 1$ and $i = 2$, i-prime normal forms are indeed prime. On that basis the may-correlate of SPR_2 can be introduced. Let $\mathcal{M}$ be 1-complete and 1-f.g. by Ap_1. Then the set $\mathrm{SPR}_1 \subseteq \mathrm{TFm}_1^+$ of (1-) *syntactic $\to$-primes* is the least s.t.

i) $\top \in \mathrm{SPR}_1$,

ii) $\alpha \in \mathrm{Ap}_1$ only if $\alpha^1 \in \mathrm{SPR}_1$

iii) $\phi \in \mathrm{SPR}_1$ only if $\langle\lambda\rangle\phi \in \mathrm{SPR}_1$, for all $\lambda \in \mathcal{L}$,

iv) $\phi, \psi \in \mathrm{SPR}_1$ only if $\phi \wedge \psi \in \mathrm{SPR}_1$.

Proposition 5.13 *For any* 1*-complete model class $\mathcal{M}$, and any $\phi \in \mathrm{SPR}_1$, ϕ is $\to$-prime and prime.*

PROOF: Note just that for any $\phi \in \mathrm{SPR}_1$, the $\phi' \in \mathrm{NF}_1$ s.t. $\phi \equiv_{\mathbf{GL}^1} \phi'$ constructed using the rewriting procedure of 4.30 is in PNF_1, and then by 5.12.ii), ϕ' is prime. But then so is ϕ. $\Box$

5.3.2 Further decomposition properties

To complete the argument that for i-complete model classes implications are eliminable which we show in the next section, it is necessary to consider implications of the form $\phi \to \psi$, where ϕ is in prime normal form and ψ has either of the forms $\psi = \alpha^i$ or $\psi = \langle\lambda\rangle\psi'/[\lambda]\psi'$. First the case where ψ is an atomic proposition:

Lemma 5.14 *For $i \in \{1, 2\}$, $\mathcal{M}$ i-complete and $\phi \in \mathrm{PNF}_i$ $\mathcal{M}$-consistent, $\Sigma \models \phi \to \alpha^i$ iff for some $\phi_1, \phi_2 \in \mathrm{TFm}_i^+$, $\langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_i(\alpha)$, $\phi \models_{\mathcal{M}} \phi_2$, and $\Sigma \models \phi_1$.*

PROOF: Assume that $\Sigma \models \phi \to \alpha^i$, $\phi \in \mathrm{PNF}_i$ consistent. Then $\Sigma \times \Sigma_\phi \models \alpha^i$ and then there are $\phi_1, \phi_2 \in \mathrm{TFm}_i^+$ s.t. $\langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_i(\alpha)$, $\Sigma \models \phi_1$ and $\Sigma_\phi \models \phi_2$, i.e. $\phi \models_{\mathcal{M}} \phi_2$. Conversely if $\phi \models_{\mathcal{M}} \phi_2$, $\langle \phi_1, \phi_2 \rangle \in \boldsymbol{\delta}_i(\alpha)$ and $\Sigma \models \phi_1$ then if $\Sigma' \models \phi$ also $\Sigma' \models \phi_2$ and then $\Sigma \times \Sigma' \models \alpha^i$—so $\Sigma \models \phi \to \alpha^i$. □

Next the case where the outermost connective of ψ is a modal operator. Here the uniformity requirement is essential. We take first the case for $i = 2$.

Lemma 5.15 *For $\mathcal{M}$ 2-complete and $\phi \in \mathrm{PNF}_2$ $\mathcal{M}$-consistent, $\Sigma \models \phi \to [\lambda]\psi$ iff $\Sigma \Downarrow$ and $\phi \models [\lambda']\top$ for some λ', and for all λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$, $\Sigma \models [\lambda_1]((\phi/\lambda_2) \to \psi)$.*

Proof: For the only-if direction, if ϕ is consistent then ϕ has the form

$$\phi = \bigwedge_{j \in J} \alpha_j^2 \wedge \bigwedge_{\lambda \in \Lambda} [\lambda]\phi_\lambda$$

and Σ_ϕ is defined. Then $\Sigma \times \Sigma_\phi \models [\lambda]\psi$ so $\Sigma \Downarrow$ and $\Sigma_\phi \Downarrow$. But then $\Sigma_\phi \models [\lambda']\top$ for all $\lambda' \in \mathcal{L}$ and then $\phi \models [\lambda']\top$ by 5.11. Let then $\lambda \simeq \lambda_1\lambda_2$ and assume that Σ can λ_1-if not we are done. Assume first that Σ_ϕ $\cancel{\text{can}}$ λ_2. Then by definition 5.10.iii) ϕ/λ_2 is not consistent—but then $\Sigma/\lambda_1 \models (\phi/\lambda_2) \to \psi$ and we are done. Assume instead that Σ_ϕ can λ_2 and let $\Sigma' \models \phi/\lambda_2$, and we must show $(\Sigma/\lambda_1) \times \Sigma' \models \psi$. Now $\phi/\lambda_2 \in \mathrm{PNF}_2$ so by 5.10.iii), ϕ/λ_2 is consistent and then Σ_{ϕ/λ_2} is defined and $\Sigma_{\phi/\lambda_2} \simeq_2 \Sigma_\phi/\lambda_2$. Moreover $\Sigma_{\phi/\lambda_2} \sqsubseteq_2 \Sigma'$, thus $(\Sigma_\phi)/\lambda_2 \sqsubseteq_2 \Sigma'$. Now $\Sigma \times \Sigma_\phi \models [\lambda]\psi$, and as Σ can λ_1, Σ_ϕ can λ_2, $\Sigma \times \Sigma_\phi$ can λ and $\Sigma \times \Sigma_\phi/\lambda \models \psi$. Now

$$(\Sigma/\lambda_1) \times (\Sigma_\phi/\lambda_2) \subseteq (\Sigma \times \Sigma_\phi)/\lambda$$

so also $(\Sigma/\lambda_1) \times (\Sigma_\phi/\lambda_2) \models \psi$. But then by 5.5 and 5.7 $(\Sigma/\lambda_1) \times \Sigma' \models \psi$ and we are done.

For the if-direction assume instead that $\Sigma \Downarrow$ and $\phi \models [\lambda']\top$ for some $\lambda' \in \mathcal{L}$, and that whenever $\lambda \simeq \lambda_1\lambda_2$ then $\Sigma \models [\lambda_1]((\phi/\lambda_2) \to \psi)$. Let $\Sigma' \models \phi$ and we show $\Sigma \times \Sigma' \models [\lambda]\psi$. As ϕ consistent, $\Sigma_\phi \sqsubseteq_2 \Sigma'$ so it suffices to show $\Sigma \times \Sigma_\phi \models [\lambda]\psi$ by 5.5. Now $\Sigma \Downarrow$ and $\Sigma_\phi \Downarrow$ as $\phi \models [\lambda']\top$. If $\Sigma \times \Sigma_\phi$ $\cancel{\text{can}}$ λ then we are done so assume not. By 5.3.iii),

$$(\Sigma \times \Sigma_\phi)/\lambda = \bigcup_{\lambda_1, \lambda_2 . \lambda \simeq \lambda_1\lambda_2} (\Sigma/\lambda_1) \times \Sigma_\phi/\lambda_2),$$

and hence it suffices to show for each λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$, Σ can λ_1 and Σ_ϕ can λ_2 that $(\Sigma/\lambda_1) \times (\Sigma_\phi/\lambda_2) \models \psi$. Now by the assumption, $\Sigma/\lambda_1 \models (\phi/\lambda_2) \to \psi$, and $(\Sigma_\phi)/\lambda_2 \simeq_2 \Sigma_{\phi/\lambda_2}$ by definition 5.10.iv) so by 5.7, $(\Sigma_\phi)/\lambda_2 \models \phi/\lambda_2$—but then

$$(\Sigma/\lambda_1) \times (\Sigma_\phi/\lambda_2) \models \psi$$

and we are done. □

We proceed to prove a similar result for the may-case.

Lemma 5.16 *For any 1-complete model class $\boldsymbol{\mathcal{M}}$, $\mathcal{M} \in \boldsymbol{\mathcal{M}}$, $\Sigma \in \mathcal{M}^\dagger$, $\lambda \in \mathcal{L}$, any consistent $\phi \in \mathrm{PNF}_1$, and any $\psi \in \mathrm{SCPR}_1$, $\Sigma \models \phi \to \langle\lambda\rangle\psi$ iff for some λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$, $\phi \models_{\mathcal{M}} \langle\lambda_2\rangle\top$ and $\Sigma \models \langle\lambda_1\rangle((\phi/\lambda_2) \to \psi)$.*

Proof: For the only-if direction assume that $\Sigma \models \phi \to \langle\lambda\rangle\psi$, $\phi \in \mathrm{PNF}_1$ is consistent and $\psi \in \mathrm{SCPR}_1$. Then Σ_ϕ is defined and $\Sigma \times \Sigma_\phi \models \langle\lambda\rangle\psi$. Then $\Sigma \times \Sigma_\phi$ can λ and $(\Sigma \times \Sigma_\phi)/\lambda \models \psi$. Note here that for any Σ' and $\phi' \in \mathrm{SCPR}_1$, $\Sigma' \models \phi'$ iff for some $\sigma \in \Sigma'$, $\sigma \models \phi$. This is easily seen by induction on the structure of ϕ'. But then there is some pair $\langle\sigma', \sigma'_\phi\rangle \in (\Sigma\times\Sigma_\phi)/\lambda$ s.t. $\langle\sigma', \sigma'_\phi\rangle \models \psi$ and then there is some pair $\langle\sigma, \sigma_\phi\rangle \in \Sigma \times \Sigma_\phi$ and some λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$ and $\sigma \overset{\lambda_1}{\to} \sigma'$, $\sigma_\phi \overset{\lambda_2}{\to} \sigma'_\phi$. Then Σ_ϕ can λ_2 so $\phi \models \langle\lambda_2\rangle\top$ and Σ can λ_1. To prove $\Sigma/\lambda_1 \models (\phi/\lambda_2) \to \psi$ is suffices to show that $(\Sigma/\lambda_1) \times \Sigma_{\phi/\lambda_2} \models \psi$. But $\Sigma_{\phi/\lambda_2} \simeq_1 \Sigma_\phi/\lambda_2$ by 5.10.iv) so it suffices to show $(\Sigma/\lambda_1) \times (\Sigma_\phi/\lambda_2) \models \psi$. But $\sigma' \in \Sigma/\lambda_1$ and $\sigma'_\phi \in \Sigma_\phi/\lambda_2$ and then we are done as $\langle\sigma', \sigma'_\phi\rangle \models \psi$.
For the if-direction assume that $\phi \models \langle\lambda_2\rangle\top$ and $\Sigma \models \langle\lambda_1\rangle((\phi/\lambda_2) \to \psi)$ and that $\lambda \simeq \lambda_1\lambda_2$. Then $\Sigma \times \Sigma_\phi$ can λ and

$$(\Sigma/\lambda_1) \times (\Sigma_{\phi/\lambda_2}) \simeq_1 (\Sigma/\lambda_1) \times (\Sigma_\phi/\lambda_2) \models \psi.$$

But then also $(\Sigma \times \Sigma_\phi)/\lambda \models \psi$—hence $\Sigma \times \Sigma_\phi \models \langle\lambda\rangle\psi$. Thus $\Sigma \models \phi \to \langle\lambda\rangle\psi$ by 5.12.i) and we are done. □

5.4 Axiomatisation

All the ingredients are now in place for presenting our main result of this chapter, namely a sound and relatively complete axiomatisation of $\models_{\mathcal{M}}$. In order to derive suitable axioms from the decomposition lemmas 5.14, 5.16 and 5.15 it is first of all necessary to ensure that only finite conjunctions and disjunctions are needed.

i) A synchronisation algebra $\langle\mathcal{L}, \cdot\rangle$ is said to have the *finite factorisation property*, if for all $\lambda \in \mathcal{L}$, $\{\langle\lambda_1, \lambda_2\rangle \mid \lambda \simeq \lambda_1\lambda_2\}$ is a finite set. A concurrent model class $\boldsymbol{\mathcal{M}}$ is *finitely factorising* if its synchronisation algebra has this property.

ii) A model class $\boldsymbol{\mathcal{M}}$ is *sort-finite*, if its label universe is finite.

iii) A concurrent model class $\mathcal{M}$ has the *finite decomposition property*, if for all $\alpha \in \mathrm{Ap}$ and $i \in \{1,2\}$, $\boldsymbol{\delta}_i(\alpha)$ is a finite set.

iv) A concurrent model class is *finitely based* if it is finitely factorising and it has the finite decomposition property.

Fix now a finitely based, complete model class $\mathcal{M}$. Define the consistency predicate con by conϕ iff $\phi \not\models_{\mathcal{M}} \bot$. For $i \in \{1,2\}$ the logic $\mathbf{FL}^i(\mathcal{M})$ (or just $\mathbf{FL}^i$ when $\mathcal{M}$ is understood from the context) over TFm_i^+ is axiomatised by the following axiom and rule schemas:

Axioms derived from $\models_{\mathcal{M}}$:

$1{\rightarrow_i}$: $\phi \vdash_i \psi$ whenever $\phi \models_{\mathcal{M}} \psi$ and $\phi, \psi \in \mathrm{GTFm}_i^+$,

$2{\rightarrow_i}$: $(\phi \rightarrow \bot) \vdash_i \bot$ whenever $\mathrm{con}(\phi)$ and $\phi \in \mathrm{GTFm}_i^+$.

All **G** axioms and rules. Let in particular

$3{\rightarrow_i}$: $(\phi \rightarrow \psi) \wedge (\phi \rightarrow \gamma) \vdash_i \phi \rightarrow (\psi \wedge \gamma)$ ($\rightarrow$-$\wedge$-intro; A6 of **G**)

$4{\rightarrow_i}$: $(\phi \rightarrow \gamma) \wedge (\psi \rightarrow \gamma) \vdash_i (\phi \vee \psi) \rightarrow \gamma$ ($\rightarrow$-$\vee$-elim; A7 of **G**)

$5{\rightarrow_i}$: From $\psi \vdash_i \gamma$ infer $\phi \rightarrow \psi \vdash_i \phi \rightarrow \gamma$ (Covariance, R4 of **G**)

$6{\rightarrow_i}$: From $\psi \vdash_i \phi$ infer $\phi \rightarrow \gamma \vdash_i \psi \rightarrow \gamma$ (Contravariance, R5 of **G**)

The $\top/\bot$-related axioms:

$7{\rightarrow_i}$: $\top \vdash_i \phi \rightarrow \top$ ($\rightarrow$-$\top$-intro)

$8{\rightarrow_i}$: $\top \vdash_i \bot \rightarrow \phi$ ($\rightarrow$-$\bot$-elim)

$9{\rightarrow_i}$: $\phi \rightarrow (\psi \rightarrow \bot) \vdash_i \psi \rightarrow (\phi \rightarrow \bot)$ ($\bot$-permutation)

$10{\rightarrow_i}$: $\phi \rightarrow (\phi \rightarrow \bot) \vdash_i \phi \rightarrow \bot$ ($\bot$-contraction)

The primeness/coprimeness based axioms:

$11{\rightarrow_i}$: $\phi \rightarrow (\psi \vee \gamma) \vdash_i (\phi \rightarrow \psi) \vee (\phi \rightarrow \gamma)$ whenever $\phi \in \mathrm{SPR}_i$

$12{\rightarrow_i}$: $(\phi \wedge \psi) \rightarrow \gamma \vdash_i (\phi \rightarrow \gamma) \vee (\psi \rightarrow \gamma)$ whenever $\gamma \in \mathrm{SCPR}_i$

Axioms derived from lemma 5.14:

$13{\rightarrow_i}$: $\phi \vdash_i \psi \rightarrow \alpha^i$ whenever $\langle \phi, \psi \rangle \in \boldsymbol{\delta}_i(\alpha)$

$14{\rightarrow_i}$: $\phi \rightarrow \alpha^i \vdash_i \bigvee\{\phi_1 \mid \exists \phi_2 . \phi \models_{\mathcal{M}} \phi_2, \langle \phi_1, \phi_2 \rangle \in \boldsymbol{\delta}_i(\alpha)\}$

whenever $\phi \in \mathrm{PNF}_i$, conϕ

Axioms peculiar to $i = 1$:

$15{\rightarrow_1}$: $\langle \lambda_1 \rangle (\phi \rightarrow \psi) \vdash_1 (\langle \lambda_2 \rangle \phi) \rightarrow (\langle \lambda_1 \lambda_2 \rangle \psi)$ whenever $\lambda_1 \lambda_2 \downarrow$

$16{\rightarrow_1}$: $\phi \rightarrow (\langle \lambda \rangle \psi) \vdash_1 \bigvee\{\langle \lambda_1 \rangle ((\phi / \lambda_2) \rightarrow \psi) \mid \lambda_1 \lambda_2 \simeq \lambda, \phi \models_{\mathcal{M}} \langle \lambda_2 \rangle \top\}$

whenever $\phi \in \mathrm{PNF}_1$, conϕ and $\psi \in \mathrm{SCPR}_1$

Axioms and rules peculiar to $i = 2$:

$15{\rightarrow_2}$: $[\mu] \top \wedge (\phi \rightarrow [\lambda] \psi) \vdash_2 [\lambda_1]((\phi / \lambda_2) \rightarrow \psi)$

whenever $\phi \in \mathrm{PNF}_2$ and $\lambda_1\lambda_2 \simeq \lambda$

$16{\to_2}$: From $\phi \vdash_2 [\mu]\top$ infer

$$[\mu] \wedge (\bigwedge\{[\lambda_1]((\phi/\lambda_2) \to \psi) \mid \lambda \simeq \lambda_1\lambda_2\}) \vdash_2 \phi \to [\lambda]\psi$$

whenever $\phi \in \mathrm{PNF}_2$

$17{\to_2}$: From $\psi \vdash_i [\lambda]\top$ infer $\phi \to \psi \vdash_i [\lambda]\top$

whenever $\mathrm{con}(\phi)$ and $\phi \in \mathrm{GTFm}_i^+$

The role of most of these axioms and rules should be reasonably clear. The last part of the axiomatisation is concerned with the decomposition properties of implications investigated in the last two sections. Axiom $2{\to_i}$ internalises the consistency predicate on the ground fragments. The role of the $\top/\bot$-related axioms is a bit curious. They are sound, seems to be independent and suffice for the completeness proof, but there are others such as corresponding instances of transitivity, and the combinators S and K that are valid, but does not seem provable from the present axioms *as axiom schemas*, that is. We return to this issue in the conclusion. Here we proceed to prove soundness.

Theorem 5.17 (Soundness of $\mathbf{FL}^i$). *Let $i \in \{1,2\}$ and $\boldsymbol{\mathcal{M}}$ be any finitely based, complete model class. Then for all $\phi, \psi \in \mathrm{TFm}_i^+$, if $\phi \vdash_{\mathbf{FL}^i} \psi$ then $\phi \models_{\mathcal{M}} \psi$.*

PROOF: The proof of soundness for the $\mathbf{GL}^i$ axioms and rules transfers directly to the present setting. The soundness of the remaining axioms is in most cases immediate. For the rest:

$2 \to_i$. If $\Sigma \models \phi \to \bot$ then there can be no Σ' in $\boldsymbol{\mathcal{M}}$ s.t. $\Sigma' \models \phi$—but this contradicts the consistency of ϕ.

$11 \to_i$–$12 \to_i$. Use 5.9 and 5.13.

$14 \to_i$. Note first that if $\phi \in \mathrm{PNF}_i$ and $\langle \phi_1, \phi_2 \rangle \in \boldsymbol{\delta}_i(\alpha)$ then $\phi, \phi_2 \in \mathrm{GTFm}_i^+$, so the axiom is indeed admissible, in view of the finite decomposition property of $\boldsymbol{\mathcal{M}}$. Then the result follows from 5.14.

$15 \to_1$. Suppose that $\Sigma_1 \models \langle\lambda_1\rangle(\phi \to \psi)$ and $\Sigma_2 \models \langle\lambda_2\rangle\phi$, and $\lambda_1\lambda_2 \downarrow$. Then $\Sigma_1 \times \Sigma_2$ can $\lambda_1\lambda_2$. Moreover $(\Sigma_1/\lambda_1) \times (\Sigma_2/\lambda_2) \models \psi$, and

$$(\Sigma_1/\lambda_1) \times (\Sigma_2/\lambda_2) \subseteq (\Sigma_1 \times \Sigma_2)/(\lambda_1\lambda_2),$$

by 5.3, so also $(\Sigma_1 \times \Sigma_2)/(\lambda_1\lambda_2) \models \psi$.

$16 \to_1$. This is just a restatement of 5.16.

$15 \to_2$. Note that if Σ satisfies the antecedent then $\Sigma \Downarrow$. Hence if ϕ is inconsistent, the axiom holds—if ϕ is consistent, the result obtains from 5.15.

16 $\rightarrow_2$. If ϕ is in fact consistent the axiom is just a restatement of 5.15. If not soundness is immediate.
17 $\rightarrow_2$. If ϕ is consistent and $\psi \rightarrow [\mu]\top$ then let $\Sigma \models \phi \rightarrow \psi$. If $\Sigma \Downarrow$ we are done. If not, pick any Σ' s.t. $\Sigma' \models \phi$. Then $\Sigma \times \Sigma' \models \psi$. But then $\Sigma \times \Sigma' \Downarrow$—a contradiction. $\Box$

5.5 Relative completeness

Going on to completeness note the indispensable replacement property of $\equiv_{\mathbf{FL}^i}$:

Proposition 5.18 *For $i \in \{1, 2\}$, $\equiv_{\mathbf{FL}^i}$ is a congruence.*

PROOF: Just extend 4.24 by handling implications using the co- and contravariance rules 5 $\rightarrow_i$ and 6 $\rightarrow_i$. $\Box$

Completeness is proved by extending lemma 4.30 to the full languages. Note the following extension of lemma 4.28: If $\mathcal{M}$ is finitely based and complete and $\phi \in \mathrm{GTFm}_i^+$ then ϕ is consistent iff $\phi \rightarrow \bot \vdash_{\mathbf{FL}^i} \bot$. This follows directly from axiom 2 $\rightarrow_i$ together with the soundness theorem 5.17. We can then proceed to the normal form lemma proper:

Lemma 5.19 *For any finitely based, complete model class, any $i \in \{1, 2\}$, and any $\phi \in \mathrm{TFm}_i^+$ there is some $\phi' \in \mathrm{NF}_i$ s.t. $\phi \equiv_{\mathbf{FL}^i} \phi'$, and moreover the size of ϕ' does not exceed that of ϕ.*

PROOF: We proceed again by induction on the structure of ϕ. The cases for ϕ not an implication follow exactly the corresponding cases in the proof of lemma 4.30.
So assume that $\phi = \phi_1 \rightarrow \phi_2$, and we proceed by an inner induction on the size of ϕ. By the outer induction hypothesis we can find boundaries $\Phi, \Psi \subseteq \mathrm{PNF}_i$ s.t. $\phi_1 \equiv_{\mathbf{FL}^i} \bigvee \Phi$, $\phi_2 \equiv_{\mathbf{FL}^i} \bigvee \Psi$ and then by 5.18, $\phi \equiv_{\mathbf{FL}^i} (\bigvee \Phi) \rightarrow (\bigvee \Psi)$.
Moreover, the size of $(\bigvee \Phi) \rightarrow (\bigvee \Psi)$ does not exceed that of ϕ. If $\bigvee \Phi$ is inconsistent then $\phi \equiv_{\mathbf{FL}^i} \bot \rightarrow (\bigvee \Psi)$, by 1$\rightarrow_i$, 10_i and 6$\rightarrow_i$, and then by 8 $\rightarrow_i$ and 11_i, $\bot \rightarrow (\bigvee \Psi) \equiv_{\mathbf{FL}^i} \top$, so $\phi \equiv_{\mathbf{FL}^i} \top$. So assume instead that $\bigvee \Phi$ is consistent. Suppose that $\bigvee \Psi$ is inconsistent. Then $\bigvee \Psi \equiv_{\mathbf{FL}^i} \bot$ and then by 5 $\rightarrow_i$ we obtain $\phi \equiv_{\mathbf{FL}^i} (\bigvee \Phi) \rightarrow \bot$. But then by 4.28, 10_i and 2 $\rightarrow_i$ we get $\phi \equiv_{\mathbf{FL}^i} \bot$, so assume instead that also $\bigvee \Psi$ is consistent. By 4_i, 5_i, 8_i and 10_i we can assume that then

all $\phi' \in \Phi$ as well as all $\psi' \in \Psi$ are consistent. By 4 $\rightarrow_i$ and 6 $\rightarrow_i$ (and standard reasoning) we obtain

$$\phi \equiv_{\mathbf{FL}^i} \bigwedge\{\phi' \rightarrow (\bigvee \Psi) \mid \phi' \in \Phi\}.$$

Now, as for each $\phi' \in \Phi$, $\phi' \in \mathrm{SPR}_i$, by 11 $\rightarrow_i$, 5 $\rightarrow_i$ (and standard reasoning),

$$\phi \equiv_{\mathbf{FL}^i} \bigwedge\{\bigvee\{\phi' \rightarrow \psi' \mid \psi' \in \Psi\} \mid \phi' \in \Phi\}.$$

Note that for each $\phi' \in \Phi$, $\psi' \in \Psi$, $|\phi' \rightarrow \psi'| \leq |\phi|$. Let now $\phi' \in \Phi$ and $\psi' \in \Psi$ and we proceed to rewrite $\phi' \rightarrow \psi'$ into normal form.
Here we split the proof into two and consider first the case for $i = 1$:
$i = 1$.
Here $\phi' = \bigwedge \Phi'$ and $\psi' = \bigwedge \Psi'$ for $\Phi', \Psi' \subseteq$ TRNF boundaries—and hence nonempty. Now

$$\phi' \rightarrow \psi' \equiv_{\mathbf{FL}^1} \bigwedge\{(\phi' \rightarrow \psi'') \mid \psi'' \in \Psi'\}$$

by 3 $\rightarrow_1$ and 5 $\rightarrow_1$. Note that for each $\psi'' \in \Psi'$, $\psi'' \in \mathrm{SCPR}_1$, so by 12 $\rightarrow_1$,

$$\phi' \rightarrow \psi' \equiv_{\mathbf{FL}^1} \bigwedge\{\bigvee\{\phi'' \rightarrow \psi'' \mid \phi'' \in \Phi'\} \mid \psi'' \in \Psi'\}.$$

Let then $\phi'' \in \Phi'$, $\psi'' \in \Psi'$ and we proceed to rewrite $\phi'' \rightarrow \psi''$ into normal form, noting again that $|\phi''| \leq |\phi|$.
If ψ'' is valid then by 1 $\rightarrow_1$, 5 $\rightarrow_1$ and 7 $\rightarrow_i$, $\phi'' \rightarrow \psi'' \equiv_{\mathbf{FL}^1} \top$, so assume instead that ψ'' is not valid. Now ψ'' is of one of the forms $\psi'' = \alpha^1$, or $\psi'' = \langle\lambda\rangle\psi'''$. Assume the first. Then, as $\phi'' \in \mathrm{PNF}_1$ and $\mathrm{con}\phi''$, by 14 $\rightarrow_1$ we obtain

$$\phi'' \rightarrow \alpha^1 \vdash_{\mathbf{FL}^1} \bigvee\{\phi_1 \mid \exists\phi_2.\ \phi'' \models_{\mathcal{M}} \phi_2,\ \langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_1(\alpha)\}$$

and conversely, if $\langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_1(\alpha)$ then $\phi_1 \vdash_1 \phi_2 \rightarrow \alpha^1$ by 13 $\rightarrow_i$, and if in addition $\phi'' \models_{\mathcal{M}} \phi_2$ then by 6 $\rightarrow_i$ also $\phi_1 \vdash_1 \phi'' \rightarrow \alpha^1$, so by standard reasoning,

$$\phi'' \rightarrow \alpha^1 \equiv_{\mathbf{FL}^1} \bigvee\{\phi_1 \mid \exists\phi_2.\ \phi'' \models_{\mathcal{M}} \phi_2,\ \langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_1(\alpha)\}$$

Moreover, as we assumed that whenever $\langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_i(\alpha)$ then $|\phi_1| = |\phi_2|$, we obtain $|\phi'''| \leq |\phi|$. So assume instead that $\psi'' = \langle\lambda\rangle\psi'''$. Again, as $\phi'' \in \mathrm{PNF}_1$, $\mathrm{con}\phi''$, and $\psi''' \in \mathrm{SCPR}_1$ we can apply axiom 16 $\rightarrow_i$ and obtain

$$\phi'' \rightarrow \langle\lambda\rangle\psi''' \vdash_{\mathbf{FL}^1} \bigvee\{\langle\lambda_1\rangle((\phi''/\lambda_2) \rightarrow \psi''') \mid \lambda_1\lambda_2 \simeq \lambda,\ \phi'' \models_{\mathcal{M}} \langle\lambda_2\rangle\top\}$$

and conversely, if $\lambda_1\lambda_2 \simeq \lambda$ then

$$\langle\lambda_1\rangle((\phi''/\lambda_2) \rightarrow \psi''') \vdash_{\mathbf{FL}^1} (\langle\lambda_2\rangle(\phi''/\lambda_2)) \rightarrow (\langle\lambda\rangle\psi'''),$$

and if in addition $\phi'' \models_{\mathcal{M}} \langle\lambda_2\rangle\top$ then $\phi'' \models_{\mathcal{M}} \langle\lambda_2\rangle(\phi''/\lambda_2)$ (this is easy to see), thus

$$\langle\lambda_1\rangle((\phi''/\lambda_2) \to \psi''') \vdash_{\mathbf{FL}^1} \phi'' \to (\langle\lambda\rangle\psi'''),$$

by 6 $\to_1$ and then by standard reasoning we obtain

$$\phi'' \to \langle\lambda\rangle\psi''' \equiv_{\mathbf{FL}^1} \bigvee\{\langle\lambda_1\rangle((\phi''/\lambda_2) \to \psi''') \mid \lambda_1\lambda_2 \simeq \lambda,\ \phi'' \models_{\mathcal{M}} \langle\lambda_2\rangle\top\}$$

Now for each such λ_2 the size of $(\phi''/\lambda_2) \to \psi'''$ is strictly smaller than that of ϕ, so by the inner induction hypothesis we can find a $\gamma(\phi'', \lambda_2, \psi''') \in \mathrm{NF}_1$ s.t.

$$(\phi''/\lambda_2) \to \psi''' \equiv_{\mathbf{FL}^1} \gamma.$$

Then

$$\phi'' \to \langle\lambda\rangle\psi''' \equiv_{\mathbf{FL}^1} \bigvee\{\langle\lambda_1\rangle\gamma(\phi'', \lambda_2, \psi''') \mid \lambda_1\lambda_2 \simeq \lambda,\ \phi'' \models_{\mathcal{M}} \langle\lambda_2\rangle\top\}$$

Now, whatever form ψ'' takes we have completed a rewriting of each formula $\phi'' \to \psi''$ with $\phi'' \in \Phi'$ and $\psi'' \in \Psi'$ to a formula $\gamma' \in \mathrm{GTFm}_1^+$ s.t. $\phi'' \to \psi'' \equiv_{\mathbf{FL}^1} \gamma'$, and this procedure can be applied for each ϕ', ψ' as above, thus giving a rewriting of $\phi \to \psi$ into an $\mathbf{FL}^1$ equivalent formula in GTFm_1^+. This formula can then be rewritten into a $\mathbf{GL}^1$-equivalent formula in NF_1 by 4.30, thus completing the proof for case $i = 1$.
$i = 2$.
We assumed $\bigvee\Phi$ and $\bigvee\Psi$ to be consistent, so as Φ, Ψ are boundaries, both ϕ' and ψ' must be consistent as well. Hence ϕ' and ψ' must have the forms

$$\phi' = \bigwedge_{j\in J} \zeta_j^2 \wedge \bigwedge_{\lambda_2\in\Lambda_2} [\lambda_2]\phi_{\lambda_2}, \text{ and}$$

$$\psi' = \bigwedge_{k\in K} \alpha_k^2 \wedge \bigwedge \lambda \in \Lambda[\lambda]\psi_\lambda.$$

By 3 $\to_2$, 5 $\to_2$ and standard reasoning,

$$\phi' \to \psi' \equiv_{\mathbf{FL}^2} \bigwedge_{k\in K}(\phi' \to \alpha_k^2) \wedge \bigwedge_{\lambda\in\Lambda}(\phi' \to [\lambda]\psi_\lambda),$$

noting that if K and Λ are both empty then $\phi' \to \psi' \equiv_{\mathbf{FL}^2} \top$, so that we can assume either K or Λ to be inhabited. We then proceed by rewriting first each $\phi' \to \alpha_k^2$ and next each $\phi' \to [\lambda]\psi_\lambda$ into normal form.
Let then $k \in K$. Then by 14 $\to_2$,

$$\phi' \to \alpha_k^2 \vdash_{\mathbf{FL}^2} \bigvee\{\phi_1 \mid \exists\phi_2.\ \phi' \models_{\mathcal{M}} \phi_2,\ \langle\phi_1, \phi_2\rangle \in \boldsymbol{\delta}_2(\alpha_k^2)\}$$

Conversely let $\langle\phi_1,\phi_2\rangle \in \boldsymbol{\delta}_2(\alpha_k^2)$. Then by 13 $\rightarrow_2$, $\phi_1 \vdash_{\mathbf{FL}^2} \phi_2 \rightarrow \alpha_k^2$, and if in addition, $\phi' \models_{\mathcal{M}} \phi_2$ then by 1 $\rightarrow_2$ and 6 $\rightarrow_2$ also $\phi_1 \vdash_{\mathbf{FL}^2} \phi' \rightarrow \alpha_k^2$ and we have consequently shown $\phi' \rightarrow \alpha_k^2 \equiv_{\mathbf{FL}^2} \gamma$. Again we note that $|\gamma| \leq |\phi|$. Now $\gamma \in \mathrm{GTFm}_2^+$ so by 4.30, γ can be rewritten into a $\gamma_k \in \mathrm{NF}_2$ s.t. $\phi' \rightarrow \alpha_k^2 \equiv_{\mathbf{FL}^2} \gamma_k$. So let instead $\lambda \in \Lambda$. Now $\phi' \vdash_{\mathbf{FL}^2} [\mu_1]\top$ by 14_2 and 16_2 and standard reasoning and $[\lambda]\psi_\lambda \vdash_{\mathbf{FL}^2} [\mu_2]\top$ by 14_2 so then by 17 $\rightarrow_2$, as ϕ' is consistent,

$$\phi' \rightarrow [\lambda]\psi_\lambda \vdash_{\mathbf{FL}^2} [\mu]\top,$$

and thus by standard reasoning,

$$\phi' \rightarrow [\lambda]\psi_\lambda \vdash_{\mathbf{FL}^2} [\mu]\top \wedge (\phi' \rightarrow [\lambda]\psi_\lambda)$$

whence

$$\phi' \rightarrow [\lambda]\psi_\lambda \vdash_{\mathbf{FL}^2} [\lambda_1]((\phi'/\lambda_2) \rightarrow \psi_\lambda)$$

whenever $\lambda_1\lambda_2 \simeq \lambda$, by 15 $\rightarrow_2$, so

$$\phi' \rightarrow [\lambda]\psi_\lambda \vdash_{\mathbf{FL}^2} \bigwedge\{[\lambda_1]((\phi'/\lambda_2) \rightarrow \psi_\lambda) \mid \lambda_1\lambda_2 \simeq \lambda\} = \gamma.$$

If now there are no λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$ then

$$\phi' \rightarrow [\lambda]\psi_\lambda \equiv_{\mathbf{FL}^2} [\mu]\top,$$

for we have already seen $\vdash_{\mathbf{FL}^2}$ to hold, and the converse direction follows by 16 $\rightarrow_2$, as we have seen already that $\phi' \vdash_{\mathbf{FL}^2} [\mu]\top$. Also $|[\mu]\top| \leq |\phi|$. So assume instead that there indeed are λ_1, λ_2 s.t. $\lambda \simeq \lambda_1\lambda_2$. Then $\gamma \vdash_{\mathbf{FL}^2} [\mu]\top$ and thus $\gamma \vdash_{\mathbf{FL}^2} [\mu]\top \wedge \gamma$. Moreover, as we have already seen, $\phi' \vdash_{\mathbf{FL}^2} [\mu_1]\top$ and then by 16 $\rightarrow_2$, $\gamma \vdash_{\mathbf{FL}^2} \phi' \rightarrow [\lambda]\psi_\lambda$ whence we obtain $\phi' \rightarrow [\lambda]\psi_\lambda \equiv_{\mathbf{FL}^2} \gamma$. Now for each λ_2 s.t. for some λ_1, $\lambda_1\lambda_2 \simeq \lambda$, $(\phi'/\lambda_2) \rightarrow \psi_\lambda$ is of a size strictly smaller than that of ϕ and thus by the inner induction hypothesis we can find some $\gamma_{\lambda_2} \in \mathrm{NF}_2$ s.t.

$$(\phi'/\lambda_2) \rightarrow \psi_\lambda \equiv_{\mathbf{FL}^2} \gamma_{\lambda_2}$$

and we obtain

$$\phi' \rightarrow [\lambda]\psi_\lambda \equiv_{\mathbf{FL}^2} \bigwedge\{[\lambda_2]\gamma_{\lambda_2} \mid \lambda \simeq \lambda_1\lambda_2\} = \gamma',$$

and $|\gamma'| \leq |\phi|$. Now $\gamma' \in \mathrm{GTFm}_2^+$ so by 4.30, γ' can be rewritten into a $\gamma'_\lambda \in \mathrm{NF}_2$ s.t. $\phi' \rightarrow [\lambda]\psi_\lambda \equiv_{\mathbf{FL}^2} \gamma'_\lambda$. Then

$$\phi' \rightarrow \psi' \equiv_{\mathbf{FL}^2} (\bigwedge_{j\in J} \gamma_j) \wedge (\bigwedge_{\lambda\in\Lambda} \gamma'_\lambda)$$

and thus by applying this procedure for each ϕ', ψ' we can rewrite, by 4.30, ϕ into a normal form $\gamma'' \in \mathrm{NF}_2$ s.t. $\phi \equiv_{\mathbf{FL}^2} \gamma''$ and the proof is complete. □

With this lemma in place, the (relative) completeness of $\mathbf{FL}^i$, $i \in \{1,2\}$ is obtained as a corollary.

Corollary 5.20 (Relative completeness of $\mathbf{FL}^i$). *For any finitely-based, complete model class $\boldsymbol{\mathcal{M}}$, any $i \in \{1,2\}$, $\phi, \psi \in \mathrm{TFm}_i^+$, $\phi \vdash_{\mathbf{FL}^i} \psi$ iff $\phi \models_{\mathcal{M}} \psi$.*

Proof: As the proof of the $\mathbf{GL}^i$-completeness theorem, corollary 4.31, using lemma 5.19 in place of 4.30. □

5.6 Testing models

We apply the general results of the preceding sections to the concrete example of "testing models"—that is, models obtained from frames by adding to them atomic propositions of the form $\Lambda \subseteq \mathcal{L}$ with the interpretations of section 4.3.1. This, in particular, entails turning the class of all testing models into a complete model class. This will immediately provide the relatively complete axiomatisations of $\mathbf{FL}^i$ of the previous section for the class of all finitely-based testing models. Moreover, we can quite easily provide complete axiomatisations of the ground fragments $\mathbf{GL}^i$, as well as syntactically characterise the con predicate, thus obtaining directly unrelativized complete axiomatisations of the $\mathbf{FL}^i$.

Definition 5.21 Given a synchronisation algebra $\langle \mathcal{L}, \cdot \rangle$ let $\boldsymbol{\mathcal{M}}^T$ denote the class of all models $\mathcal{F}^{\mathrm{DH}}$ for $\mathcal{F}$ a frame over $\mathcal{L}$ and Ap. Let further, for uniformity of notation, $\mathrm{Ap}^T = \mathrm{Ap}^{\mathrm{DH}}$.

The first hurdle to overcome is to turn $\boldsymbol{\mathcal{M}}^T$ into a basic concurrent model class by giving decomposition map $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$—the parallel composition of frames is obtained as a straightforward specialisation of that of models. For frames $\mathcal{F}_1, \mathcal{F}_2$ let their parallel composition $\mathcal{F}_1 \parallel \mathcal{F}_2$ be defined as the parallel composition of models ignoring valuations.

5.6.1 Decomposing atomic propositions

Decomposing the may-sets is a trivial matter: to check $\langle \sigma_1, \sigma_2 \rangle \in V_{1,(\mathcal{F}_1 \parallel \mathcal{F}_2)^{\mathrm{DH}}}(\Lambda)$ find some $\lambda_1, \lambda_2 \in \mathcal{L}$ s.t. $\lambda \simeq \lambda_1\lambda_2$, $\sigma_1 \in V_{1,\mathcal{F}_1^{\mathrm{DH}}}(\{\lambda_1\})$ and $\sigma_2 \in V_{1,\mathcal{F}_2^{\mathrm{DH}}}(\{\lambda_2\})$.

Decomposing the must-sets is far less trivial. One might as an initial attempt think that a decomposition in terms of just atomic propositions might be possible. That this is not so is not too hard to see, however.

Let the frames $\mathcal{F}_1$ and $\mathcal{F}_2$ be given in the following way: Assume a synchronisation algebra $\mathcal{L}$ containing the labels $\lambda_{j,k}$ for $j \in \{1,2\}$ and $1 \leq k \leq 4$, all $\lambda_{j,k}$ asssumed to be distinct; and assume further that

$$\lambda_{1,1}\lambda_{2,1} = \lambda_{1,2}\lambda_{2,3} = \lambda_{1,3}\lambda_{2,2} = \lambda_{1,4}\lambda_{2,4}$$

and that any other product $\lambda_{1,k_1}\lambda_{2,k_2}$ is either undefined or distinct from $\lambda_{1,1}\lambda_{2,1}$. The usual CCS or SCCS label structures provide examples of synchronisation algebras satisfying these properties, as long as they contain at least 9 distinct labels. Let each $\mathcal{F}_j$, $j \in \{1,2\}$, contain 7 states, $\sigma_{j,k}$, $1 \leq k \leq 7$, and let the root states $\sigma_j = \sigma_{j,1}$. Let further

$$\begin{aligned}
\twoheadrightarrow_j &= \{\langle\sigma_{j,1},\sigma_{j,2}\rangle, \langle\sigma_{j,1},\sigma_{j,3}\rangle\}, \\
\stackrel{\lambda_{j,1}}{\rightarrow} &= \{\langle\sigma_{j,2},\sigma_{j,4}\rangle\}, \\
\stackrel{\lambda_{j,2}}{\rightarrow} &= \{\langle\sigma_{j,2},\sigma_{j,5}\rangle\}, \\
\stackrel{\lambda_{j,3}}{\rightarrow} &= \{\langle\sigma_{j,3},\sigma_{j,6}\rangle\}, \\
\stackrel{\lambda_{j,4}}{\rightarrow} &= \{\langle\sigma_{j,3},\sigma_{j,7}\rangle\}.
\end{aligned}$$

Then $\sigma_j \in V_2(\Lambda)$ iff Λ contains at least one of $\lambda_{j,1}$ or $\lambda_{j,2}$ and at least one of $\lambda_{j,3}$ or $\lambda_{j,4}$. Furthermore $\langle\sigma_1,\sigma_2\rangle \in V_2(\{\lambda_{1,1}\lambda_{2,1}\})$.

Suppose a decomposition map $\boldsymbol{\delta}_2$ exists which is *sound and complete*—i.e. whenever $\langle\Lambda_1,\Lambda_2\rangle \in \boldsymbol{\delta}_2(\Lambda)$, $\sigma_1 \in V_2(\Lambda_1)$ and $\sigma_2 \in V_2(\Lambda_2)$ then $\langle\sigma_1,\sigma_2\rangle \in V_2(\Lambda)$; and whenever $\langle\sigma_1,\sigma_2\rangle \in V_2(\Lambda)$ then Λ_1, Λ_2 s.t. $\langle\Lambda_1,\Lambda_2\rangle \in \boldsymbol{\delta}_2(\Lambda)$, $\sigma_1 \in V_2(\Lambda_1)$ and $\sigma_2 \in V_2(\Lambda_2)$ can be found. Let now $\langle\Lambda_1,\Lambda_2\rangle \in \boldsymbol{\delta}_2(\{\lambda_{1,1}\lambda_{2,1}\})$. Then $\sigma_1 \in V_2(\Lambda_1)$ and $\sigma_2 \in V_2(\Lambda_2)$ and then for $j \in \{1,2\}$, Λ_j contains at least one of $\lambda_{j,1}$ or $\lambda_{j,2}$ and at least one of $\lambda_{j,3}$ or $\lambda_{j,4}$. Without loss of generality assume that $\{\lambda_{1,1},\lambda_{1,3}\} \subseteq \Lambda_1$ and $\{\lambda_{2,1},\lambda_{2,3}\} \subseteq \Lambda_1$. Consider then the frames $\mathcal{F}'_j$, $j \in \{1,2\}$ obtained from $\mathcal{F}_j$ by removing the states $\sigma_{j,5}$ and $\sigma_{j,7}$ (and then of course the transitions leading to them as well); and let σ'_j denote the root state of $\mathcal{F}_j$. Then $\sigma'_j \in V_2(\Lambda_j)$ for $j \in \{1,2\}$ but

$$\langle\sigma'_1,\sigma'_2\rangle \notin V_2(\{\lambda_{1,1}\lambda_{2,1}\}),$$

hence V_2 is unsound; a contradiction. So our decompositions will have to be more subtle than that. It turns out that one can, in fact, decompose instead in terms of conjunctions of atomic propositions. For this purpose we introduce the following notions:

i) A relation $R \subseteq \mathcal{L} \times \mathcal{L}$ is a *decomposition* of Λ, if R is nonempty and for all $\lambda_1, \lambda_2 \in \mathcal{L}$, if $\lambda_1 R \lambda_2$ then $\lambda_1\lambda_2 \downarrow$ and $\lambda_1\lambda_2 \in \Lambda$.

ii) A pair $\langle \Lambda_1, \Lambda_2 \rangle$ is a *covering* of $R \subseteq \mathcal{L} \times \mathcal{L}$, if whenever $\lambda_1 R \lambda_2$ then either $\lambda_1 \in \Lambda_1$ or $\lambda_2 \in \Lambda_2$.

iii) A covering $\langle \Lambda_1, \Lambda_2 \rangle$ is *exact*, if for all $\lambda_1 \in \Lambda_1$ there is some $\lambda_2 \in \Lambda_2$ s.t. $\lambda_1 R \lambda_2$ and for all $\lambda_2 \in \Lambda_2$ there is some $\lambda_1 \in \Lambda_1$ s.t. $\lambda_1 R \lambda_2$.

We then arrive at the following:

Lemma 5.22 *For all rooted frames $\langle \sigma_j, \mathcal{F}_j \rangle$, $j \in \{1,2\}$, $\langle \sigma_1, \sigma_2 \rangle \in V_2(\Lambda)$ iff $\sigma_1 \Downarrow$, $\sigma_2 \Downarrow$ and there is some decomposition R of Λ s.t. for any exact covering $\langle \Lambda_1, \Lambda_2 \rangle$ of R, either $\sigma_1 \in V_2(\Lambda_1)$ or $\sigma_2 \in V_2(\Lambda_2)$.*

Proof: For the only-if direction assume that $\langle \sigma_1, \sigma_2 \rangle$ must Λ. Then $\sigma_1 \Downarrow$ and $\sigma_2 \Downarrow$. Let $\lambda_1 R \lambda_2$ iff σ_1 can λ_1, σ_2 can λ_2, $\lambda_1\lambda_2 \downarrow$ and $\lambda_1\lambda_2 \in \Lambda$. Then R is a decomposition of Λ. Let $\langle \Lambda_1, \Lambda_2 \rangle$ be any covering of R (exact or not) and suppose for a contradiction that σ_1 $\mathrm{m}\not\mathrm{u}\mathrm{st}$ Λ_1 and σ_2 $\mathrm{m}\not\mathrm{u}\mathrm{st}$ Λ_2. We cannot of course have $\sigma_1 \Uparrow$ nor $\sigma_2 \Uparrow$ so there must be some σ_1', σ_2' s.t. $\sigma_1 \Rightarrow \sigma_1'$, $\sigma_2 \Rightarrow \sigma_2'$ and for all $j \in \{1,2\}$ s.t. $\lambda_j \in \Lambda_j$, σ_j' $\mathrm{c}\not\mathrm{a}\mathrm{n}$ λ_j. On the other hand, as $\langle \sigma_1, \sigma_2 \rangle$ must Λ we know that as $\langle \sigma_1, \sigma_2 \rangle \Rightarrow \langle \sigma_1', \sigma_2' \rangle$, $\langle \sigma_1', \sigma_2' \rangle$ can λ for some $\lambda \in \Lambda$ and then for some λ_1, λ_2 s.t. $\lambda_1 R \lambda_2$, σ_1' can λ_1 and σ_2' can λ_2. But as $\langle \Lambda_1, \Lambda_2 \rangle$ covers R either $\lambda_1 \in \Lambda_1$ or $\lambda_2 \in \Lambda_2$—a contradiction.
For the if-direction assume that $\langle \sigma_1, \sigma_2 \rangle$ $\mathrm{m}\not\mathrm{u}\mathrm{st}$ Λ. If $\langle \sigma_1, \sigma_2 \rangle \Uparrow$ then $\sigma_1 \Uparrow$ or $\sigma_2 \Uparrow$, so assume instead that $\langle \sigma_1, \sigma_2 \rangle \Downarrow$. Then we find σ_1', σ_2' s.t. $\sigma_1 \Rightarrow \sigma_1'$, $\sigma_2 \Rightarrow \sigma_2'$ and for all $\lambda \in \Lambda$, $\langle \sigma_1', \sigma_2' \rangle$ $\mathrm{c}\not\mathrm{a}\mathrm{n}$ λ—i.e. for all λ_1, λ_2 s.t. σ_1' can λ_1, σ_2' can λ_2 either $\lambda_1\lambda_2 \uparrow$ or else $\lambda_1\lambda_2 \notin \Lambda$. Let a decomposition R of Λ be given, and we define a covering $\langle \Lambda_1, \Lambda_2 \rangle$ of R by letting

$$\Lambda_1 = \{\lambda_1 \mid \exists \lambda_2 . \lambda_1\lambda_2 \downarrow, \lambda_1\lambda_2 \in \Lambda, \sigma_1' \ \mathrm{c}\not\mathrm{a}\mathrm{n}\ \lambda_1, \exists \lambda_2 . \lambda_1 R \lambda_2\},$$

and similarly $\Lambda_2 = \{\lambda_2 \mid \exists \lambda_1 . \lambda_1\lambda_2 \downarrow, \lambda_1\lambda_2 \in \Lambda, \sigma_2' \ \mathrm{c}\not\mathrm{a}\mathrm{n}\ \lambda_2, \exists \lambda_1 . \lambda_1 R \lambda_2\}$. Now $\langle \Lambda_1, \Lambda_2 \rangle$ is indeed an exact covering of R, for let $\lambda_1 R \lambda_2$. Then $\lambda_1\lambda_2 \downarrow$ and $\lambda_1\lambda_2 \in \Lambda$ and then either σ_1' $\mathrm{c}\not\mathrm{a}\mathrm{n}$ λ_1 or σ_2' $\mathrm{c}\not\mathrm{a}\mathrm{n}$ λ_2 so either $\lambda_1 \in \Lambda_1$ or $\lambda_2 \in \Lambda_2$; and $\langle \Lambda_1, \Lambda_2 \rangle$ is exact by its construction. But certainly σ_1 $\mathrm{m}\not\mathrm{u}\mathrm{st}$ Λ_1 and σ_2 $\mathrm{m}\not\mathrm{u}\mathrm{st}$ Λ_2, and we are done. □

The decomposition maps $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are then defined by

i) $\boldsymbol{\delta}_1(\Lambda) = \{\langle\{\lambda_1\},\{\lambda_2\}\rangle \mid \lambda_1\lambda_2\downarrow, \lambda_1\lambda_2 \in \Lambda\}$,

ii) $\boldsymbol{\delta}_2(\Lambda) = \{\langle \bigwedge \boldsymbol{\Lambda}_1, \bigwedge \boldsymbol{\Lambda}_2\rangle \mid$

a) $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2 \subseteq \mathrm{Ap}^T$, $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2$ finite and nonempty,

b) there is some decomposition R of Λ s.t. for all exact coverings $\langle\Lambda_1,\Lambda_2\rangle$ of R, either $\Lambda_1 \in \boldsymbol{\Lambda}_1$ or $\Lambda_2 \in \boldsymbol{\Lambda}_2$ $\}$.

Theorem 5.23

i) $\boldsymbol{\delta}_1$ *is sound and complete (in the above sense),*

ii) If $\mathcal{M}^T$ *is sort-finite then* $\boldsymbol{\delta}_2$ *is sound and complete.*

PROOF: i) is trivial.
ii) Suppose that $\langle\sigma_1,\sigma_2\rangle$ must Λ. By lemma 5.22 we find some decomposition R of Λ s.t. for any exact covering $\langle\Lambda_1,\Lambda_2\rangle$ of R either σ_1 must Λ_1 or σ_2 must Λ_2. Clearly, R is a finite set. Define now

$$\begin{aligned}\boldsymbol{\Lambda}_1 &= \{\Lambda_1 \in \mathrm{Ap}^T \mid \sigma_1 \text{ must } \Lambda_1 \text{ and for some } \Lambda_2,\\ &\qquad \langle\Lambda_1,\Lambda_2\rangle \text{ is an exact covering of } R\},\end{aligned}$$

and $\boldsymbol{\Lambda}_2$ symmetrically. Clearly, if $\Lambda_1 \in \boldsymbol{\Lambda}_1$ then $\Lambda_1 \subseteq \{\lambda_1 \mid \exists\lambda_2.\lambda_1\lambda_2\downarrow, \lambda_1\lambda_2 \in \Lambda\}$ and similarly for $\boldsymbol{\Lambda}_2$. Moreover, $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are nonempty as can be seen from the proof of lemma 5.22; and finally $\sigma_1 \models \bigwedge\boldsymbol{\Lambda}_1$, $\sigma_2 \models \bigwedge\boldsymbol{\Lambda}_2$. The converse direction, soundness, i.e. that whenever $\sigma_1 \models \bigwedge\boldsymbol{\Lambda}_1$ and $\sigma_2 \models \bigwedge\boldsymbol{\Lambda}_2$ for $\langle\bigwedge\boldsymbol{\Lambda}_1,\bigwedge\boldsymbol{\Lambda}_2\rangle \in \boldsymbol{\delta}_2(\Lambda)$ then $\langle\sigma_1,\sigma_2\rangle \models \Lambda^2$ follows directly from lemma 5.22. □

5.6.2 Completeness of $\mathcal{M}^T$

Thus the appropriate model class is finally obtained:

Theorem 5.24 *For any sort-finite* $\mathcal{M}^T$, $\mathcal{M}^T$ *is a finitely-based u.p.d.-model class with* $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_2$ *as given above. Moreover, for each* ${\mathcal{F}_1}^{\mathrm{DH}}$, ${\mathcal{F}_2}^{\mathrm{DH}} \in \mathcal{M}^T$,

$${\mathcal{F}_1}^{\mathrm{DH}} \parallel {\mathcal{F}_2}^{\mathrm{DH}} = (\mathcal{F}_1 \parallel \mathcal{F}_2)^{\mathrm{DH}}.$$

PROOF: The last assertion follows from 5.23 and shows that $\mathcal{M}^T$ is closed under parallel composition. For disjoint union we just note that for each ${\mathcal{F}_1}^{\mathrm{DH}}$, ${\mathcal{F}_2}^{\mathrm{DH}} \in \mathcal{M}^T$,

$${\mathcal{F}_1}^{\mathrm{DH}} \oplus {\mathcal{F}_2}^{\mathrm{DH}} = (\mathcal{F}_1 \oplus \mathcal{F}_2)^{\mathrm{DH}}$$

for the $\oplus$ on frames being the obvious specialisation of $\oplus$ on models. Finally for uniformity it suffices to note that for any frame $\mathcal{F}$ and $\Sigma \in \mathcal{F}^\dagger$ we find another frame $\mathcal{F}'$ and $\sigma' \in \mathcal{F}'$ s.t. $\{\sigma'\} \simeq \Sigma$—$\mathcal{F}'$ is constructed from $\mathcal{F}$ by adding Σ as a new state with the transitions of $\mathcal{F}$ and in addition the transitions $\Sigma \twoheadrightarrow \sigma$ whenever $\sigma \in \Sigma$ and then taking $\sigma' = \Sigma$. □

It thus remains to be checked that $\boldsymbol{\mathcal{M}}^T$ is complete. To do this $\boldsymbol{\mathcal{M}}^T$ is equipped with some additional structure, introducing a couple of constants Ω and 0 and a notion of guarded choice. Let $\Omega = \langle\{\emptyset\}, \emptyset, \emptyset, \emptyset, \{\emptyset\}\rangle$ and $0 = \langle\{\emptyset\}, \emptyset, \emptyset, \emptyset, \emptyset\rangle$. As Ω and 0 contains only the state $\emptyset$, Ω and 0 are identified with their corresponding rooted frames and with the (rooted) testing models Ω^{DH} and 0^{DH} respectively. The frame Ω is the diverging frame, least w.r.t. both $\sqsubseteq_1$ and $\sqsubseteq_2$ and 0 is the convergent, but inactive frame. Thus for all rooted models $\langle\sigma, \mathcal{M}\rangle$ in $\boldsymbol{\mathcal{M}}^T$, $\Omega \sqsubseteq \sigma$ and $0 \simeq_1 \Omega$. For the guarded choice let $\Lambda \in \mathrm{Ap}^T$ and for each $\lambda \in \Lambda$, $f(\lambda)$ be the rooted frame

$$f(\lambda) = \langle\sigma_\lambda, \langle S_\lambda, L_\lambda, \twoheadrightarrow_\lambda, \{\xrightarrow{\lambda'}_\lambda\}_{\lambda' \in L_\lambda}, \uparrow_\lambda\rangle\rangle.$$

Then $\Lambda(f)$ is the rooted frame

$$\Lambda(f) = \langle \mathrm{in}_2(\emptyset), \langle(\textstyle\sum_{\lambda\in\Lambda} S_\lambda) \oplus \{\emptyset\}, (\bigcup_{\lambda\in\Lambda} L_\lambda) \cup \Lambda, \twoheadrightarrow, \{\xrightarrow{\lambda}\}_{\lambda\in L}, \uparrow\rangle\rangle,$$

where (letting $\sum$ denote n-ary disjoint union of sets and in_n the n'th injection, $n \geq 1$)

i) For all $\sigma_1, \sigma_2 \in S$, $\sigma_1 \twoheadrightarrow \sigma_2$ iff for some $\lambda \in \Lambda$, $\sigma_1', \sigma_2' \in S_\lambda$, $\sigma_1' \twoheadrightarrow_\lambda \sigma_2'$, $\sigma_1 = \mathrm{in}_1(\mathrm{in}_\lambda(\sigma_1'))$ and $\sigma_2 = \mathrm{in}_1(\mathrm{in}_\lambda(\sigma_2'))$.

ii) For all $\sigma_1, \sigma_2 \in S$, $\lambda' \in L$, $\sigma_1 \xrightarrow{\lambda'} \sigma_2$ iff either

  a) for some $\lambda \in \Lambda$, $\sigma_1', \sigma_2' \in S_\lambda$, $\sigma_1 = \mathrm{in}_1(\mathrm{in}_\lambda(\sigma_1'))$, $\sigma_2 = \mathrm{in}_1(\mathrm{in}_\lambda(\sigma_2'))$ and $\sigma_1' \xrightarrow{\lambda'}_\lambda \sigma_2'$, or

  b) $\lambda' \in \Lambda$, $\sigma_1 = \mathrm{in}_2(\emptyset)$ and $\sigma_2 = \mathrm{in}_1(\mathrm{in}_{\lambda'}(\sigma_{\lambda'}))$.

iii) For all $\sigma \in S$, $\sigma \uparrow$ iff for some $\lambda \in \Lambda$, $\sigma' \in S_\lambda$, $\sigma = \mathrm{in}_1(\mathrm{in}_\lambda(\sigma'))$ and $\sigma' \uparrow_\lambda$.

This definition, it has to be said, looks rather more complicated than it actually is, due to the symbol manipulation involved. If Λ is a singleton $\{\lambda\}$ and $f(\lambda) = \langle\sigma, \mathcal{F}\rangle$ $\Lambda(f)$ is sometimes abbreviated by $\lambda(\sigma, \mathcal{F})$ or if $\mathcal{F}$ is understood from the context just $\lambda(\sigma)$, corresponding to the prefixing of CCS and SCCS.

Proposition 5.25 *For any $\Lambda \in \mathrm{Ap}^T$ and mapping f of Λ into rooted frames:*

i) $\Lambda(f)$ can λ *iff* $\lambda \in \Lambda$,

ii) $\Lambda(f) \Downarrow$,

iii) for all $\lambda \in \Lambda$, $(\Lambda(f))/\lambda \simeq f(\lambda)$,

iv) $\Lambda(f)$ must Λ' *iff* $\Lambda \cap \Lambda' \neq \emptyset$.

Proof: All of these are straightforward from the definition. □

This added structure is now applied to prove any sort-finite $\boldsymbol{\mathcal{M}}^T$ complete:

Theorem 5.26 *Any sort-finite $\boldsymbol{\mathcal{M}}^T$ is complete.*

Proof: First for 1-completeness. First it is easy to see that $\boldsymbol{\mathcal{M}}^T$ is 1-finitely generated by the set

$$\mathrm{Ap}_1^T = \{\{\lambda\} \mid \lambda \in \mathcal{L}\} \cup \{\emptyset\}.$$

Moreover $\{\lambda\}^1$ holds for any state iff $\langle\lambda\rangle\top$ does, so we can in the present context completely disregard atomic propositions. We build now for each $\phi \in$ TRNF a rooted testing model σ_ϕ by letting $\sigma_\top = \Omega$ and $\sigma_{\langle\lambda\rangle\phi'} = \lambda(\sigma_{\phi'})$. It is straightforward to check that $\sigma_\phi \models \phi$, that σ_ϕ is least among those satisfying ϕ under $\sqsubseteq_1$ and that if σ_ϕ can λ then $\sigma_\phi/\lambda \simeq_1 \sigma_{\phi/\lambda}$. Let $\Phi \subseteq$ TRNF and $\phi = \bigwedge \Phi \in \mathrm{PNF}_1$. We then let

$$\sigma_\phi = \sum\nolimits_{\phi' \in \Phi} \sigma_{\phi'},$$

and it follows from 4.8 and 5.8 that $\sigma_\phi \models \phi$ and that σ_ϕ is least among those satisfying ϕ. Moreover, if σ_ϕ can λ then

$$\begin{aligned} \sigma_\phi/\lambda &\simeq_1 \textstyle\sum_{\langle\lambda\rangle\phi' \in \Phi} \sigma_{\phi'} \quad \text{by 4.7.iii)} \\ &\simeq_1 \textstyle\sum_{\phi' \in \phi/\lambda} \sigma_{\phi'} \\ &= \sigma_{\phi/\lambda} \end{aligned}$$

and we are done. Note in particular that all $\phi \in \mathrm{PNF}_1$ are consistent.
Next for $i = 2$. We proceed by induction on the structure of $\phi \in \mathrm{PNF}_2$. If $\phi = \bot$ we are done as then ϕ is inconsistent. If $\phi = \top$ then let $\sigma_\phi = \Omega$. Now $\Omega \models \top$, Ω is least under $\sqsubseteq_2$ and $\Omega \Uparrow$ so i)–iv) of 5.10 are satisfied. Let then

$$\phi = \bigwedge_{j \in J} \Lambda_j^2 \wedge \bigwedge_{\lambda \in \Lambda} [\lambda]\phi_\lambda,$$

with either J or Λ nonempty.
Suppose first that $J = \emptyset$. By the induction hypothesis we find for each $\lambda \in \Lambda$ s.t. ϕ_λ is consistent a σ_λ satisfying i)–iv) of 5.10. For $\lambda \in \mathcal{L} \setminus \Lambda$ let $\sigma_\lambda = \Omega$. Let then

$$\sigma_\phi = \sum \{\lambda(\sigma_\lambda) \oplus 0 \mid \lambda \in \mathcal{L} \text{ and if } \lambda \in \Lambda \text{ then } \phi_\lambda \text{ consistent}\}$$

To check that $\sigma_\phi \models \phi$ note that $\sigma_\phi \Downarrow$ and that if σ_ϕ can λ, $\lambda \in \Lambda$ then ϕ_λ is consistent and $\sigma_\phi/\lambda \simeq_2 \sigma_\lambda \models \phi_\lambda$. Also if $\sigma \models \phi$ as well then $\sigma \Downarrow$ and for any $\Lambda' \in \mathrm{Ap}^T$ and if σ can λ then if $\lambda \in \Lambda$, ϕ_λ is consistent and whether or not $\lambda \in \Lambda$, σ_ϕ can λ and $\sigma_\lambda \sqsubseteq_2 \sigma/\lambda$—so $\sigma_\phi \sqsubseteq_2 \sigma$. We have already checked conditions ii)–iv) of 5.10.
So assume instead that $J \neq \emptyset$. Suppose now that there is some $j \in J$ s.t. $\Lambda_j \subseteq \Lambda$ and for all $\lambda' \in \Lambda_j$, $\phi_{\lambda'}$ is inconsistent. Then ϕ is inconsistent, for if $\Sigma \models \phi$ then Σ must Λ_j and Σ c~~a~~n λ' for all $\lambda' \in \Lambda_i$. So assume instead that for all $j \in J$ s.t. $\Lambda_j \subseteq \Lambda$ there is some $\lambda' \in \Lambda_i$ s.t. $\phi_{\lambda'}$ is consistent. Let a $\Lambda'' \in \mathrm{Ap}^T$ be *admissible*, if for all $\lambda'' \in \Lambda'' \cap \Lambda$, $\phi_{\lambda''}$ is consistent and for all $j \in J$, $\Lambda_j \cap \Lambda'' \neq \emptyset$. Clearly admissible sets exist—take $\mathcal{L} \setminus \{\lambda \in \mathcal{L} \mid \phi_\lambda \text{ inconsistent}\}$ for example. Note that the set of admissible sets form an acceptance set in the terminology of Hennessy [50]. Define now for any $\lambda \in \mathcal{L}$ s.t. ϕ_λ is consistent if $\lambda \in \Lambda$ $f(\lambda)$ by

i) if $\lambda \in \Lambda$ then $f(\lambda) = \sigma_{\phi_\lambda}$ where by the induction hypothesis, σ_{ϕ_λ} satisfy conditions i)–iv) of 5.10, and

ii) otherwise $f(\lambda) = \Omega$.

Then we let

$$\sigma_\phi = \sum \{\Lambda''(f) \mid \Lambda'' \in \mathrm{Ap}^T \text{ admissible}\}.$$

Clearly σ_ϕ is well-defined as $\boldsymbol{\mathcal{M}}^T$ is sort-finite. We check first that $\sigma_\phi \models \phi$. Certainly $\sigma_\phi \Downarrow$. Let $j \in J$ and $\sigma_\phi \Rightarrow \sigma'$ and σ' stable. Then $\sigma' \simeq_2 \Lambda''(f)$ for some admissible Λ'' and then σ' can λ for some $\lambda \in \Lambda_j$. So σ_ϕ must Λ_j. Let then $\lambda \in \Lambda$ and assume σ_ϕ can λ. Then ϕ_λ is consistent and

$$\sigma_\phi/\lambda \simeq_2 f(\lambda) \models \phi_\lambda$$

so indeed $\sigma_\phi \models [\lambda]\phi_\lambda$—and thus $\sigma_\phi \models \phi$ as desired. Assume then that $\sigma \models \phi$ as well and assume that σ_ϕ must Λ'''. Then for some $j \in J$,

$$\Lambda_j \setminus \{\lambda \in \Lambda \mid \phi_\lambda \text{ inconsistent}\} \subseteq \Lambda'''.$$

For assume not. If $J = \emptyset$ then σ_ϕ m$\!\!\!/$ust Λ''' so assume instead that $J \neq \emptyset$. Let then

$$\Lambda'' = \{\lambda \in \mathcal{L} \mid \lambda \notin \Lambda''', \phi_\lambda \text{ consistent if} \lambda \in \Lambda\}.$$

Then Λ'' is admissible as for any $j \in J$, $\Lambda_j \cap \Lambda'' \neq \emptyset$. But $\Lambda'' \cap \Lambda''' = \emptyset$ so σ_ϕ m$\!\!\!/$ust Λ'''— a contradiction. Thus for some $j \in J$,

$$\Lambda_j \setminus \{\lambda \in \Lambda \mid \phi_\lambda \text{ inconsistent}\} \subseteq \Lambda''',$$

and σ must Λ_i—hence

$$\sigma \text{ must } \Lambda_j \setminus \{\lambda \in \Lambda \mid \phi_\lambda \text{ inconsistent}\}$$

and hence σ must Λ'''.
Next $\sigma_\phi \Downarrow$ and $\sigma \Downarrow$ and if σ can λ then if $\lambda \in \Lambda$, ϕ_λ is consistent and $\sigma_\lambda \models \phi_\lambda$. But if $\lambda \in \Lambda$ then $\sigma_\phi/\lambda = f(\lambda) \sqsubseteq_2 \sigma/\lambda$ and otherwise $\sigma_\phi/\lambda \simeq_2 \Omega \sqsubseteq_2 \sigma/\lambda$; thus we have shown $\sigma_\phi \sqsubseteq_2 \sigma$.
Finally the check that σ_ϕ satisfies 5.10.iii)–iv) is immediate at this stage and the proof is complete. $\Box$

It is interesting that constructions so closely related to the acceptance- and representation trees of De Nicola and Hennessy [30, 50] turn up from—at least superficially—quite different, logical considerations here. Also it may give some comfort as to the extent to which the notion of complete model classes is a natural and useful one.

5.7 The testing logics

Having proved that for any finite synchronisation algebra $\mathcal{L}$, the class of all testing models obtained from some frame over $\mathcal{L}$ is a finitely-based, complete model class, we turn to the axiomatising the associated consequence relation $\models_{\mathcal{M}^T}$. In view of the relative completeness result for $\mathbf{FL}^i$, 5.20, it suffices to

i) axiomatise the ground fragment, i.e. the relation $\phi \models_{\mathcal{M}^T} \psi$ for ϕ, ψ ground formulas,

ii) syntactically characterise the predicate conϕ for $\phi \in \mathrm{PNF}_i$,

iii) syntactically characterise the relation $\phi \models_{\mathcal{M}^T} \bigwedge \boldsymbol{\Lambda}$ for $\boldsymbol{\Lambda}$ a finite, non-empty set of Λ^i with $\Lambda \in \mathrm{Ap}^T$ and $\phi \in \mathrm{PNF}_i$, and

iv) syntactically characterise the relation $\phi \models_{\mathcal{M}^T} \langle\lambda\rangle\top$ for $\phi \in \mathrm{PNF}_1$.

Solutions to these four points will immediately yield unrelativised complete axiomatisations for the whole of $\models_{\mathcal{M}^T}$ for both the TFm_i^+ fragments. The only point worth noting here are axiom $2 \rightarrow_i$ and rule $18 \rightarrow_2$ where con is applied to arbitrary formulas. It is, however, very easy to see, by inspecting the proof of lemma 5.19, that it is in fact sufficient to apply $2 \rightarrow_i/18 \rightarrow_2$ only when in fact ϕ is in prime normal form, whence they can be safely restricted to that case.

We first address the point i) above. We refine the axiomatisation of $\mathbf{GL}^i$, and the corresponding normal form theorems by taking into account the specific nature of atomic propositions in the case of sort-finite classes of testing models. So let us fix for the remainder of the section some sort-finite testing model class $\mathcal{M}^T$. For $i \in \{1,2\}$ let the logic $\mathbf{TGL}^i$ be axiomatised by the $\mathbf{GL}^i$ axioms and rules plus the following axioms:

Axioms for $\vdash_1$:

$1T_1$:	$\emptyset_1 \vdash_1 \bot$
$2T_1$:	$\Lambda_1^1 \vee \Lambda_2^1 \vdash_1 (\Lambda_1 \cup \Lambda_2)^1$
$3T_1$:	$(\Lambda_1 \cup \Lambda_2)^1 \vdash_1 \Lambda_1^1 \vee \Lambda_2^1$
$4T_1$:	$\{\lambda\}^1 \vdash_1 \langle\lambda\rangle\top$
$5T_1$:	$\langle\lambda\rangle\top \vdash_1 \{\lambda\}^1$

Axioms for $\vdash_2$:

$1T_2$:	$\emptyset^2 \vdash_2 \bot$	
$2T_2$:	$\Lambda_1^2 \vdash_2 (\Lambda_1 \cup \Lambda_2)^2$	
$3T_2$:	$\Lambda^2 \wedge [\lambda]\bot \vdash_2 (\Lambda \setminus \{\lambda\})^2$	provided $\Lambda \neq \{\lambda\}$
$4T_2$:	$\Lambda^2 \wedge \bigwedge_{\lambda\in\Lambda}[\lambda]\bot \vdash_2 \bot$	

The $\mathbf{TGL}^1$ axioms merely serves to explain away sets Λ in terms of the disjunction of formulas $\langle\lambda\rangle\top$ with $\lambda \in \Lambda$. For the $\mathbf{TGL}^2$ axioms, $2T_2$ expresses the fact that bigger sets are weaker assertions, $3T_2$ that labels that can never be performed are irrelevant, and $4T_2$ that if any Λ^2 holds then at least one of the elements of Λ is "performable". Soundness of these axiomatisations is easily obtained.

Theorem 5.27 (Soundness of $\mathbf{TGL}^i$). *For $i \in \{1,2\}$ and $\phi, \psi \in \mathrm{GTFm}_i^+$, if $\phi \vdash_{\mathbf{TGL}^i} \psi$ then $\phi \models_{\mathcal{M}^T} \psi$.*

PROOF: As usual. □

We next refine the notion of normal form. Let TTRNF denote the subset of TRNF not containing any atomic propositions Λ^1 and let TPNF_1 and TNF_1

denote the corresponding subsets of PNF_1, NF_1. For TPNF_2 fix $\lambda \in \mathcal{L}$ and let TPNF_2 be the least subset of PNF_2 s.t.

i) $\perp \in \mathrm{TPNF}_2$,

ii) $(\bigwedge_{j\in J}(\Lambda_j)^2) \wedge (\bigwedge_{\lambda\in\Lambda}[\lambda]\phi_\lambda) \in \mathrm{TPNF}_2$, if

1)–4) of 4.25.v.b),

5) for all $j \in J$, if $\Lambda_j \subseteq \Lambda'$ and $\Lambda_j \neq \Lambda'$ then $\Lambda_k \neq \Lambda'$ for all $k \in J$,

6) for all $j \in J$ and $\lambda \in \Lambda_j \cap \Lambda$, $\phi_\lambda \neq \perp$.

Then TNF_2 is the corresponding subset of NF_2.

To prove completeness we proceed by specialising the results of section 4.4. Note that propositions 4.26 and 4.27 goes through unchanged in the present setting. We then prove the $\mathcal{M}^T$ correlate of lemma 4.28.

Lemma 5.28 *For $i \in \{1,2\}$,$X \in \{(\mathrm{TR}),\mathrm{P},()\}$ and $\phi \in \mathrm{T}X\mathrm{NF}_{(i)}$,*

i) ϕ is $\mathcal{M}^T$-consistent iff $\phi \neq \perp$,

ii) ϕ is $\mathcal{M}^T$-valid iff $\phi = \top$.

PROOF: i) It suffices to note first that any $\phi \in \mathrm{TTRNF}$ is consistent and second that a $\phi \in \mathrm{TPNF}_2$ is inconsistent iff either $\phi = \perp$ (but then we are done) or $\perp$ can be proved from ϕ using only the axiom $4T_2$. That this is so is easily seen from the construction in the proof of 5.26. But this latter property can not hold for ϕ as it is in TPNF_2.
ii) It suffices to show for all $\phi \in \mathrm{T}X\mathrm{NF}_{(i)}$ that if $\Omega \models \phi$ then $\phi = \top$. This is as straightforward as i). □

And we go on to the $\mathcal{M}^T$ correlate of proposition 4.29:

Proposition 5.29 *For $i \in \{1,2\}$, $X \in \{(\mathrm{TR}),\mathrm{P},()\}$ and $\phi,\psi \in \mathrm{T}X\mathrm{NF}_{(i)}$, if $\phi \models \psi$ then $\phi \leq \psi$.*

PROOF: Assume that $\phi,\psi \in \mathrm{T}X\mathrm{NF}_{(i)}$ and that $\phi \not\leq \psi$, and we show that there then exist a rooted model σ in $\mathcal{M}^T$ s.t. $\sigma \models \phi$ and $\sigma \not\models \psi$. We appeal throughout to the construction of 5.26. We need only consider the case for $X = \mathrm{P}$ and $i = 2$—the case for $X = \mathrm{TR}$ and $i = 1$ is straightforward, and the remaining cases mimic exactly the corresponding cases in the proof of proposition 4.29.

Now we cannot have $\phi = \bot$ so let

$$\phi = \bigwedge_{j \in J} (\Lambda_j)^2 \wedge \bigwedge_{\lambda \in \Lambda_1} [\lambda]\phi_\lambda.$$

If $\psi = \bot$ then we are done by lemma 5.28.i), so assume instead that

$$\psi = \bigwedge_{k \in K} (\Lambda'_k)^2 \wedge \bigwedge_{\lambda \in \Lambda_2} [\lambda]\psi_\lambda.$$

Then either

i) there is some $k \in K$ s.t. for all $j \in J$, $\Lambda_j \not\subseteq \Lambda'_k$, or

ii) there is a $\lambda' \in \Lambda_2$ s.t. $\psi_{\lambda'} = \top$, $J = \emptyset$ and $\Lambda_1 = \emptyset$,

iii) for all $\lambda' \in \Lambda_2$, $\psi_{\lambda'} \neq \top$ and $\Lambda_2 \not\subseteq \Lambda_1$,

iv) there is some $\lambda' \in \Lambda_1 \cap \Lambda_2$ s.t. $\phi_{\lambda'} \not\leq \psi_{\lambda'}$.

i) If $\phi \models \psi$ then $\phi \models (\Lambda'_k)^2$. By lemma 5.28.i), $\phi \not\models \bot$. By the proof of 5.26 there is some $j \in J$ s.t.

$$\Lambda_j \setminus \{\lambda \in \Lambda_1 \mid \phi_\lambda \text{ inconsistent}\} \subseteq \Lambda'_k,$$

hence as $\phi \in \mathrm{TPNF}_2$, $\Lambda_j \subseteq \Lambda'_k$—a contradiction.
ii) If $J = \emptyset$ and $\Lambda_1 = \emptyset$ then $\Omega \models \phi$; but $\Omega \not\models \psi$.
iii) If for all $\lambda' \in \Lambda_2$, $\psi_{\lambda'} \neq \top$ then by lemma 4.28.ii) for all $\lambda' \in \Lambda_2$, $\Omega \not\models \psi_{\lambda'}$. Now we apply the construction of 5.26 to find a least σ_ϕ s.t. $\sigma_\phi \models \phi$. If $\phi = \top$ then $\sigma_\phi = \Omega$ and $\Omega \not\models \psi$ as by the assumption Λ_2 is nonempty. Otherwise $\sigma_\phi \Downarrow$, and as there is some $\lambda \in \Lambda_2 \setminus \Lambda_1$, σ_ϕ can λ and $\sigma_\phi/\lambda \simeq_2 \Omega$. But then $\sigma_\phi \not\models \psi$ and we are done.
iv) Use the induction hypothesis and the construction of 5.26. □

It thus remains to be checked that each formula can be rewritten into an equivalent normal form using the axioms and rules.

Lemma 5.30 *For $i \in \{1,2\}$ and $\phi \in \mathrm{GTFm}_i^+$ (over $\boldsymbol{\mathcal{M}}^T$) there is a $\phi' \in \mathrm{TNF}_i$ s.t. $\phi \equiv_{\mathbf{TGL}^i} \phi'$.*

PROOF: The only difference w.r.t. the proof of 4.30 is a few straightforward applications of the new axioms. □

And we thus obtain

Corollary 5.31 (Completeness of $\mathbf{TGL}^i$). *For $i \in \{1,2\}$ and $\phi, \psi \in \mathrm{GTFm}_i^+$ (over $\boldsymbol{\mathcal{M}}^T$), $\phi \vdash_{\mathbf{TGL}^i} \psi$ iff $\phi \models_{\boldsymbol{\mathcal{M}}^T} \psi$.*

PROOF: See the proof of corollary 4.31. □

Notice that along the way we have essentially solved the remaining three issues brought out in the introduction of the present section. Concerning ii), the syntactical characterisation of i-consistent prime normal forms, this is given by lemma 5.28. For iii), the syntactical characterisation of the relation $\phi \models_{\boldsymbol{\mathcal{M}}^T} \bigwedge \boldsymbol{\Lambda}$ with $\phi \in \mathrm{PNF}_i$ for $i = 1$ is in fact redundant, and for $i = 2$ it can be seen that for any $\phi \in \mathrm{TPNF}_2$, $\phi \models_{\boldsymbol{\mathcal{M}}^T} \bigwedge \boldsymbol{\Lambda}$ iff $\phi = \bigwedge_{j \in J} (\Lambda_j)^2 \wedge \bigwedge_{\lambda \in \Lambda} [\lambda] \phi_\lambda$, and for each $\Lambda' \in \boldsymbol{\Lambda}$ there is a $j \in J$ s.t. $\Lambda_j \subseteq \Lambda'$. Finally, for iv), the syntactical characterisation of the relation $\phi \models_{\boldsymbol{\mathcal{M}}^T} \langle \lambda \rangle \top$ for $\phi \in \mathrm{TPNF}_1$ is trivial: let $\phi \in \mathrm{TPNF}_1$, $\phi = \bigvee \Phi$, $\Phi \subseteq \mathrm{TTRNF}$. Then $\phi \models_{\boldsymbol{\mathcal{M}}^T} \langle \lambda \rangle \top$ iff for some ϕ', $\langle \lambda \rangle \phi' \in \Phi$.

We have thus completed our enterprise, and a complete, unrelativised axiomatisation of the full logics over the class of all testing models over some some finite synchronisation algebra is obtained from the axiomatisations of $\mathbf{FL}^i$ and $\mathbf{TGL}^i$ using the above three additional syntactical characterisations.

5.8 Concluding remarks

We have presented relevant extensions of the two ground logics introduced in chapter 4. The implication is interpreted over a parallel composition of models determined by a synchronisation algebra and decomposition maps of atomic propositions. We have presented axiomatisations and shown them to be sound and relatively complete.

This result relies on model classes being suitably closed under parallel composition and a notion of disjoint union of models. Furthermore we require the existence of least models satisfying consistent prime normal forms. These properties provide the basis needed for allowing completeness results for the full fragments based on rewriting techniques to go through, in particular rewriting any formula into an equivalent formula in the ground fragment. Hence the implication is in a technical sense redundant, in that any formula in the full fragments can be replaced by an equivalent one in the ground fragments, and the axiomatisations can be viewed as kinds of very elaborate "expansion theorems" for the relevant implication, borrowing Milner's terminology for CCS [76]. On the other hand the implication is far from redundant in that in the ground fragment compositional reasoning w.r.t. the parallel composition seems unattainable. That is, whereas the implication is redundant with respect to formulas this seems not to be so w.r.t. formula *schemes*. This situation is reminiscent for instance of the incompleteness results in modal logic (c.f. [108]), and, nearer to hand, of the situation in the equational theory for CCS (see [76]) where all closed instances of the associative and commutative laws for $|$ are provable, even though they are not provable as equation schemas themselves. It also explains the slightly puzzling situation concerning the $\top/\bot$-related axioms.

It is quite possible that the present axiomatisation can be simplified. On the essential points, however—the reference to the syntactically prime and coprime formulas, and the reference to prime normal forms it is not clear how much of a simplification can be gained, apart from purely cosmetic changes. The syntactical characterisation of primes and coprimes seems indispensable, and for the normal forms, simplification is tied up with more general versions of the elimination theorems 5.14, 5.16 and 5.15.

It should be noted that these problems makes proof in practice extremely cumbersome (but also mechanical) in that one becomes forced to adopt strictly delimited reduction- or proof- strategies. Thus for practical purposes (and less for machine implementation) simplifications become imperative; but—and this is a point we wish to stress—these issues also serve to illustrate the importance of adopting a more logical rather than as here a purely semantical approach to program logics; for instance in relation to the logically far more attractive approach of chapter 3.

Finally we have considered testing models as a concrete example. These are models derived from frames by adding to them atomic propositions in the form of "may-" and "must-sets" respectively. The check that for any sort-finite synchronisation algebra the class of all testing models over that algebra indeed has the desired properties involve in particular the provision of a decomposition of the must-sets, and an introduction of in particular a notion of guarded choice.

Lacking, and left for future work, are axiomatisations of the logics covering both the may- and must-fragments in one go. For the ground fragment this is unlikely to be very difficult—due to our approach, however, we need for the full logic very strong normal form theorems. Left also, is the consideration of the logics with classical negation, characterising the equivalences induced by the weak preorders.

It should be noted that implicit in our treatment of testing models is a restriction to a conception of parallel composition as synchronous. We can in our general framework deal with asynchronous parallellism by admitting a special "idling" label (which—it must be noted—will be regarded as observable) which is possible from any state in any model. This last property, however, is violated in the classes of testing models as considered here. A similar example, maintaining this property, remains to be worked out, although no essential difficulties are envisaged.

More serious is the more general problem of admitting model classes where unobservable transitions may result from the joint occurrence of observable ones, corresponding to the interpretation of the τ-label in CCS as unobservable. In this latter case our framework breaks down and much more fundamental work is required to remedy this situation.

Chapter 6

Conclusion

The idea underlying the work reported in the present thesis is to use relevance logic to put structure in concurrent computation on a logical footing. Modal and temporal logics, for instance, provide natural accounts of the dynamic behaviour of systems, but with respect to aspects of program structure such as parallel composition they prove more problematic. We have shown that the presence of such program combinators induce a deduction structure on the logics that is naturally captured by a relevant implication. The bulk of the thesis, chapters 3 to 5, has been dedicated to fleshing out this idea for two concrete examples; the first emphasising the logical and algebraic account of programs and their properties, whereas the second is much more based on operational considerations. Partly in preparation for these examples, chapter 2 contains a general introduction to the model theory of (in particular positive) relevance logics where we introduced a notion of model based on semilattices with an inf-preserving binary operation, generalising the ternary relation model of Sylvan and Meyer [98, 97].

In the concluding sections of each chapter we have commented on the work reported there, and given suggestions for future work specific to those chapters. Here we discuss more broadly other possible viewpoints on the problems we have been dealing with and which problems are particularly important for future work.

Immediate concerns

Much more work is needed to determine the viability of the ideas put forward in the present thesis. More examples should be worked out in the spirit of the work reported in chapters 3 to 5. In particular the emphasis on testing-like equivalences here should be relaxed and relevant versions of in particular Hennessy-Milner logic should be investigated, aiming towards a logical account of CCS-type asynchronous parallel composition. Also more expressive temporal logics such as fixed point extensions of the logics considered here as well as Hennessy-Milner logic should be considered.

Structure as "first class" connectives

At the outset our motivation was the problem of obtaining compositionality in program logics w.r.t. parallel composition. Our suggestion certainly solves this problem or rather, provides a specific logical handle for it. But it does more than that: It adds structural properties to the program logics as full-blown connectives. We argue briefly in the introduction that this may not be as unnatural as one might think at first glance; but in the end it must be practical application studies that decides this issue. This does, however, make a qualitative difference on comparison to the more specialised approaches of for instance Stirling and Winskel [106, 115] described in the introduction. Whereas in their framework only "ground consequences" of structural assertions need be captured, in our setting we need to capture also "higher-degree" consequences—that this makes a significant difference may be seen by comparing our "initial algebra"-axiomatisations of chapters 3 and 5 to the axiomatisations of [106, 115]. This could be an unnecessary complication: Once the ways of introducing a given connective is fixed one should in principle also know the ways of eliminating that connective again—see for instance [99] for an application of this idea to intuitionistic logic. Maybe relatively simple characterisations of the "initial algebra"-axiomatisations of ch. 3 and 5 can be found along similar lines.

Implication and fusion

We have chosen to focus interest throughout on the relevant implication—this, however, is by no means the only choice possible. Winskel [115], for instance, suggests that his "parallel fusion" connective could be admitted as a full-blown connective just as we consider the implication. There is, as we saw in the introduction, a close connection between these connectives—Winskel's parallel fusion connective is just an intensional conjunction w.r.t. the notion of deduction captured by our relevant implication. There is, however, a reason for our choice of the implication as the primary connective, which maybe has not been brought out sufficiently clearly in the main text. Recall our satisfaction condition for implication:

$$p \models \phi \rightarrow \psi \text{ iff for all } q, \text{ if } q \models \phi \text{ then } p \times q \models \psi, \tag{6.1}$$

A corresponding condition for the fusion, or intensional conjunction, could read (c.f. chapter 2):

$$p \models \phi \circ \psi \text{ iff } \exists p_1, p_2 \text{ s.t. } p_1 \models \phi, p_2 \models \psi, \text{ and } p_1 \times p_2 R p, \tag{6.2}$$

where R is some suitable preorder or equivalence. In the abstract algebraic setting this is of course perfectly okay, but if we think of p, q as ranging over terms in some programming language, the desired interpretation of R is as some suitable behavioral preorder, or equivalence. But *the logic itself* has a story to tell about what R should be—if we are taking the logic seriously we should expect, indeed require, R to be exactly the preorder $\leq$ induced by the logic by

$$p \leq q \text{ iff for all } \phi, \text{ if } p \models \phi \text{ then } q \models \phi. \tag{6.3}$$

Thus no uniform interpretation of the fusion, for instance over different fragments of program logics as in chapters 3 and 5, is possible. In this sense, the implication could be argued to be more fundamental (but at any rate "nicer") than the fusion in that the former is interpreted, as in 6.1, avoiding the if not technical, then conceptual circularity of 6.2 and 6.3.

Other connectives

There is some interest in developing process-based interpretations embracing richer fragments—of particular interest is the relevant De-Morgan negation and the linear logic modal operators. For the negation it is clearly possible, as in Girard's "phase semantics" [43], to pick out some constant $\perp$ and then restrict attention to those properties ϕ for which $[\![(\phi \rightarrow \perp) \rightarrow \perp]\!] = [\![\phi]\!]$. Abramsky [6] suggests relating this to testing equivalence. Let $\perp$ be the set of processes that may be seen as successful such that p passes the test q iff $p \times q \in [\![\perp]\!]$ (we assume as in the introduction $\times$ to be commutative and associative). Let $\phi \equiv \psi$ iff $[\![\phi \rightarrow \perp]\!] = [\![\psi \rightarrow \perp]\!]$ (iff $[\![(\phi \rightarrow \perp) \rightarrow \perp]\!] = [\![(\psi \rightarrow \perp) \rightarrow \perp]\!]$). Then $\phi \equiv \psi$ iff any test passed by all $p \in [\![\phi]\!]$ is passed by all $q \in [\![\psi]\!]$ and vice versa. So $\equiv$ may be thought of as a kind of testing equivalence.

For the linear $\Box$ (or ! in [43]) the models of chapter 2 may be extended by a binary relation R which is reflexive and s.t.

i) if $1Rx$ then $1 \leq x$,

ii) if $x \sqcap yRz$ then there are x_1, y_1 s.t. xRx_1, yRy_1 and $x_1 \sqcap y_1 \leq z$,

iii) if $xyRz$ then there are x_1, y_1 s.t. xRx_1, yRy_1 and $x_1y_1 \leq z$.

Then satisfaction is extended by $x \models \Box\phi$ iff there is some y s.t. $1 \leq y \leq x$, y idempotent (i.e. $yy = y$) and for all z, if yRz then $z \models \phi$. This interpretation is sound and complete for modal linear logic w.r.t. extended $\mathbf{LL}^+$-frames. We haven't, however, found any natural interpretation of R in terms of processes.

Contexts

Of course parallel composition may not be the only process combinator causing logical difficulties. Another example is the restriction operator of CCS. This causes Stirling to consider versions of the ternary consequence relation $\models'$ of the introduction indexed by restriction sets [106]. How this extends to the present set-up remains to be seen. An alternative is to consider more general contexts instead of just "multiplication to the right". Abramsky [6] suggests "birelevant" logics of processes with one implication interpreted over parallel and another over sequential composition. Larsen [64] considers contexts as modal property transformers. He considers, for instance, the problem of given a CCS context (i.e. a CCS term with a "slot" in it) $\mathrm{C}[\cdot]$ and a property ϕ to find another property ψ s.t. a necessary and sufficient condition for $\mathrm{C}[p]$ to satisfy ϕ is for p to satisfy ψ. Now contexts form a monoid with the monoid operation $\cdot$ defined by $\mathrm{C}_1[\cdot] \cdot \mathrm{C}_2[\cdot] = \mathrm{C}_1[\mathrm{C}_2[\cdot]]$ and the empty context $[\cdot]$ as the unit. Thus we may use contexts to interpret a relevant implication by

$$\mathrm{C}_1[\cdot] \models \phi \rightarrow \psi \text{ iff for all } \mathrm{C}_2[\cdot], \text{ if } \mathrm{C}_2[\cdot] \models \phi \text{ then } \mathrm{C}_1[\cdot] \cdot \mathrm{C}_2[\cdot] \models \psi$$

with ϕ valid iff $[\cdot] \models \phi$. Constants, in particular, may be considered as nullary contexts. To account for dynamic behaviour we could appeal for instance to Larsen's idea of processes as action transducers.

"True" concurrency

We have focused throughout on the conception of processes as transition systems, or modal frames, and algebraically on CCS/SCCS-style process calculi and their (in-) equational theories. An alternative is to consider models such as Petri nets or event structures. It is clearly possible to interpret linear logic with implication, fusion, intensional truthhood and extensional conjunction in terms of Petri nets by using for instance markings in place of processes, multiunion as parallel composition and the empty marking as the unit process. It is even possible that such a free monoid interpretation will be complete. Other approaches could be based on Meseguer and Montanari's monoids of nets [74], or Winskel's categorical accounts of Petri nets and event structures [116].

Appendix A

Proof of theorem 3.7

We provide here a proof of theorem 3.7, namely that for $i \in \{1,2,3\}$, $\mathcal{F}_i \cong \mathcal{D}_i$. We start by showing that $\mathcal{D}_i \in \boldsymbol{\mathcal{C}}_i$. Note first the following basic properties of paths and sets of paths:

Proposition A.1

i) $\leq$ *is a partial order on n-paths and for* $i \in \{1,2,3\}$, $\leq_i$ *is a partial order on i-closed sets of n-paths.*

ii) Any 1- or 3-closed f.g. set is finite,

iii) If S *is i-f.g. then there is a least set* $\mathrm{gen}_i(S)$ *i-generating* S.

PROOF: i) Straigthforward.
ii) If S is 1- or 3-f.g. let $S = \mathrm{cl}_i(S')$ for S' finite, $i \in \{1,3\}$. The length of paths $\langle \overline{\lambda}, j \rangle$, defined as the length of the string $\overline{\lambda}$, contained in S' is bounded above by some n. The set of all paths of length not exceeding n is finite and $S \subseteq S_n$.
iii) Assume that S is i-f.g. and generated by S_1 finite, say. Let

$$\mathcal{S} = \{S' \mid S' \text{ generates } S\}$$

and we show that $\bigcap \mathcal{S}$ generates S as well. Thus let $s \in S$. Then, as S_1 generates S, there is some $s_1 \in S_1$ s.t. $s \leq s_1$, for the case $i = 1$. If $s_1 \in \cap\mathcal{S}$ we are done so assume not. Then we find some $S_2 \in \mathcal{S}$ s.t. $s_1 \notin S_2$. But $s_1 \in S$ and S_2 generates S so there must be some $s_2 \in S_2$ s.t. $s_1 \leq s_2$, in fact $s_1 \leq s_2$ and $s_2 \not\leq s_1$. Then there must be some $s_3 \in S_1$ s.t. $s_2 \leq s_3$, as $s_2 \in S_1$ and by i), $s_3 \not\leq s_1$. Now if $\bigcap \mathcal{S}$ does not generate S we can continue this process indefinitely thus obtaining an infinite, strictly increasing chain of distinct members of S_1 which was supposed to be finite. □

Next we consider the operations of path prefixing and multiplication.

Proposition A.2

i) Path prefixing is monotone w.r.t. $\leq$.

ii) $\langle \mathrm{NP}, \cdot, \langle \varepsilon, 1\rangle\rangle$ *forms a commutative monoid.*

iii) For all $\lambda, \mu \in \mathcal{L}$, $s_1, s_2 \in \mathrm{NP}$, $\lambda(s_1) \cdot \mu(s_2) = (\lambda\mu)(s_1 \cdot s_2)$.

iv) Path multiplication is monotone in both arguments and preserve $\langle \varepsilon, 0\rangle$.

v) For all n-paths s, s_1, s_2, *if* $s \leq s_1 \cdot s_2$ $(s_1 \cdot s_2 \leq s)$ *then there are n-paths* s_1', s_2' *s.t.* $s = s_1' \cdot s_2'$ *and* $s_1' \leq s_1$ $(s_1 \leq s_1')$ *and* $s_2' \leq s_2$ $(s_2 \leq s_2')$.

vi) Prefixing and multiplication of paths preserve normality.

PROOF: i) Let $\lambda \in \mathcal{L}$, $s_1 = \langle \overline{\lambda}_1, j_1\rangle$, $s_2 = \langle \overline{\lambda}_2, j_2\rangle \in \mathrm{NP}$ and $s_1 \leq s_2$. If $j_1 = j_2 = 1$—thus $\overline{\lambda}_1 = \overline{\lambda}_2$—then trivially $\lambda(s_1) \leq \lambda(s_2)$, so assume instead that $j_1 = 0$ and $\overline{\lambda}_1$ is a prefix of some *unrolling* of $\langle \overline{\lambda}_2, j_2\rangle$, i.e. a prefix of $\langle \overline{\lambda}_2[e]^n, j_2\rangle$ for some n. Then $\lambda(s_1) = \langle \lambda\overline{\lambda}_1, j_1\rangle$ and either $\lambda(s_2) = \langle \lambda\overline{\lambda}_2, j_2\rangle$ and we are done, or $\overline{\lambda}_2 = \varepsilon$, $j_2 = 1$ and $\lambda = e$, whence $\lambda(s_2) = \langle \varepsilon, 1\rangle$. But then also $e\overline{\lambda}_1$ is a prefix of some unrolling of $\langle \varepsilon, 1\rangle$ and we are done.
ii) That $\langle \varepsilon, 1\rangle$ is left- and right unit for path multiplication is trivial. Commutativity follows by a straightforward induction on the length of paths. For associativity, let $s_1, s_2, s_3 \in \mathrm{NP}$ and we show $s_1 \cdot (s_2 \cdot s_3) = (s_1 \cdot s_2) \cdot s_3$ by induction on the maximal length of s_1, s_2 and s_3. When one of s_1, s_2 or s_3 are either of $\langle \varepsilon, 0\rangle$ or $\langle \varepsilon, 1\rangle$ the result is trivial, so let instead $s_k = \langle \lambda_k\overline{\lambda}_k, j_k\rangle$, $k \in \{1, 2, 3\}$. Let now

$$\begin{aligned} s &= \langle \lambda_1\overline{\lambda}_1, j_1\rangle \cdot (\langle \lambda_2\overline{\lambda}_2, j_2\rangle \cdot \langle \lambda_3\overline{\lambda}_3, j_3\rangle) \\ &= \langle \lambda_1\overline{\lambda}_1, j_1\rangle \cdot (\lambda_2\lambda_3)(\langle \overline{\lambda}_2, j_2\rangle \cdot \langle \overline{\lambda}_3, j_3\rangle). \end{aligned}$$

Let further $s' = \langle \overline{\lambda}_2, j_2\rangle \cdot \langle \overline{\lambda}_3, j_3\rangle$. If $s' = \langle \varepsilon, 1\rangle$ and $\lambda_2\lambda_3 = e$ then

$$\begin{aligned} s &= \langle \lambda_1\overline{\lambda}_1, j_1\rangle \cdot \langle \varepsilon, 1\rangle \\ &= s_1 \\ &= (\lambda_1\lambda_2\lambda_3)((\langle \overline{\lambda}_1, j_1\rangle \cdot \langle \overline{\lambda}_2, j_2\rangle) \cdot \langle \overline{\lambda}_3, j_3\rangle) \end{aligned}$$

by the induction hypothesis. Then

$$\begin{aligned} s &= ((\lambda_1\lambda_2)(\langle\overline{\lambda}_1, j_1\rangle \cdot \langle\overline{\lambda}_2, j_2\rangle)) \cdot \langle\lambda_3\overline{\lambda}_3, j_3\rangle \\ &= (s_1 \cdot s_2) \cdot s_3 \end{aligned}$$

and we are done. Else if either $s' \neq \langle\varepsilon, 1\rangle$ or $\lambda_2\lambda_3 \neq e$ then

$$s = (\lambda_1\lambda_2\lambda_3)(\langle\overline{\lambda}_1, j_1\rangle \cdot (\langle\overline{\lambda}_2, j_2\rangle \cdot \langle\overline{\lambda}_3, j_3\rangle))$$

and we are done by the induction hypothesis as before.
iii) Similar to ii).
iv) Clearly $\langle\varepsilon, 0\rangle$ is preserved under path multiplication, and by ii) we need only check monotonicity in one argument. So let $s, s_1, s_2 \in \mathrm{NP}$ and $s_1 \leq s_2$, and we show $s \cdot s_1 \leq s \cdot s_2$. We prove this again by induction on the largest length of s, s_1 and s_2. If $s = \langle\varepsilon, 0\rangle$ or $s = \langle\varepsilon, 1\rangle$ we are done immediately, so assume that $s = \langle\lambda\overline{\lambda}, j\rangle$. Also if $s_1 = \langle\varepsilon, 0\rangle$ we are done and if $s_1 = \langle\varepsilon, 1\rangle$ then also $s_2 = \langle\varepsilon, 1\rangle$ and we are done as well, so assume that $s_1 = \langle\lambda_1\overline{\lambda}_1, j_1\rangle$. If $j_1 = 1$ then $s_1 = s_2$ and we are done. Otherwise $j_1 = 0$ and $\lambda_1\overline{\lambda}_1$ is a prefix of some unrolling of s_2. If $s_2 = \langle\varepsilon, 0\rangle$ we have a contradiction and if $s_2 = \langle\varepsilon, 1\rangle$ then $\lambda_1\overline{\lambda}_1 = [e]^n$ for some $n \geq 1$. Then $s \cdot s_2 = s$ and $s \cdot s_1 = \langle\lambda\overline{\lambda}', 0\rangle$, where $\lambda\overline{\lambda}'$ is either identical to $\lambda\overline{\lambda}$ or the prefix of $\lambda\overline{\lambda}$ of length n—in either case $s \cdot s_1 \leq s \cdot s_2$. Finally assume that $s_2 = \langle\lambda_2\overline{\lambda}_2, j_2\rangle$. Then $s \cdot s_1 = (\lambda\lambda_1)(\langle\overline{\lambda}, j\rangle \cdot \langle\overline{\lambda}_1, j_1\rangle)$ and $s \cdot s_2 = (\lambda\lambda_2)(\langle\overline{\lambda}, j\rangle \cdot \langle\overline{\lambda}_2, j_2\rangle)$ and $\langle\overline{\lambda}_1, j_1\rangle \leq \langle\overline{\lambda}_2, j_2\rangle$ whence we obtain $s \cdot s_1 \leq s \cdot s_2$ by the induction hypothesis and i).
v) Suppose $s, s_1, s_2 \in \mathrm{NP}$, $s = \langle\overline{\lambda}, j\rangle$, $s_k = \langle\overline{\lambda}_k, j_k\rangle$, $k \in \{1, 2\}$, and assume first that $s \leq s_1 \cdot s_2$. If $j = 1$ then $s = s_1 \cdot s_2$ and we are done. Assume instead that $j = 0$. Then $\overline{\lambda}$ is a prefix of some unrolling of $s_1 \cdot s_2$. It is now a straightforward matter to replace s_1, s_2 with an s_1', s_2' of the same length as s, ending in 0, s.t. $s = s_1' \cdot s_2'$. The second case where $s_1 \cdot s_2 \leq s$ is similar but slightly more involved, due to the potential need of replacing trailing 0's in s_1, s_2 with a 1 and consequently removing suffixes of the form $[e]^n$, $n \geq 0$, in order to maintain normality.
vi) Straightforward. □

It is not hard to see now that the $\mathcal{D}_i$ are indeed well-defined.

Proposition A.3 *For $i \in \{1, 2, 3\}$, D_i is closed under the constants and operations.*

PROOF: The only thing to check is that the constants and operations preserve the property of being i-f.g.—but this is straightforward using theorem A.2; except perhaps for $\cdot_3$. But there note that if S_1, S_2 are 3-f.g. then by A.1.ii), S_1, S_2 are finite, and then so is $S_1 \cdot_3 S_2$. □

We next go on to simplify the constants and operations.

Proposition A.4 *For $i \in \{1, 2, 3\}$, S_i, $S_{1,i}, S_{2,i} \in D_i$,*

i) $0_1 = \{\langle \varepsilon, 0 \rangle\} = 0_3$,

$0_2 = \mathrm{NP}$,

ii) $1_1 = \{\langle \varepsilon, 1 \rangle\} \cup \{\langle [e]^n \mid n \geq 0\}$,

$1_2 = \{\langle \varepsilon, 1 \rangle\} = 1_3$,

iii) $(\lambda)_1(S_1) = \{\lambda(s) \mid s \in S_1\} \cup \{\langle \varepsilon, 0 \rangle\}$,

$(\lambda)_i = \{\lambda(s) \mid s \in S_i\}$, $i \in \{2, 3\}$,

iv) $S_{1,1} \oplus_1 S_{2,1} = S_{1,1} \cup S_{2,1}$,

$S_{1,2} \oplus_2 S_{2,2} = S_{1,2} \cup S_{2,2}$,

v) $S_{1,1} \cdot_1 S_{2,1} = \{s_1 \cdot s_2 \mid s_1 \in S_{1,1}, s_2 \in S_{2,1}\}$,

$S_{1,2} \cdot_2 S_{2,2} = \{s_1 \cdot s_2 \mid s_1 \in S_{1,2}, s_2 \in S_{2,2}\}$,

vi) $S_{1,1} \leq_1 S_{2,1}$ *iff* $S_{1,1} \subseteq S_{2,1}$,

$S_{1,2} \leq_2 S_{2,2}$ *iff* $S_{2,2} \subseteq S_{1,2}$.

PROOF: i) For 0_3 the result follows by A.1.i). For 0_1, whenever $\langle \overline{\lambda}, j \rangle \leq \langle \varepsilon, 0 \rangle$ then $\langle \overline{\lambda}, j \rangle = \langle \varepsilon, 0 \rangle$. For 0_2 note that $\langle \varepsilon, 0 \rangle \leq s$ for all $s \in \mathrm{NP}$.
ii) For 1_1, whenever $\langle \overline{\lambda}, j \rangle \leq \langle \varepsilon, 1 \rangle$ then either $j = 1$ and $\overline{\lambda} = \varepsilon$ or $j = 0$ and $\overline{\lambda}$ is a prefix of some unrolling of $\langle \varepsilon, 1 \rangle$—i.e. $\overline{\lambda} = [e]^n$ for some $n \geq 0$. For 1_2, 1_3 the result is straightforward.
iii) For $(\lambda)_1$ let $s_1 \in S_1$ and assume $s_1 = \langle \overline{\lambda}_1, j_1 \rangle$ and let $\langle \overline{\lambda}, j \rangle \leq \lambda(\langle \overline{\lambda}_1, j_1 \rangle)$. If $j = 1$ then $\langle \overline{\lambda}, j \rangle = \lambda(\langle \overline{\lambda}_1, j_1 \rangle)$, and we are done, and if $j = 0$ then $\overline{\lambda}$ is a prefix of some unrolling of $\lambda(\langle \overline{\lambda}_1, j_1 \rangle)$. Then either $\overline{\lambda} = \varepsilon$ and we are done, or $\overline{\lambda} = \lambda \overline{\lambda}'$ with $\langle \overline{\lambda}', j \rangle \leq \langle \overline{\lambda}_1, j_1 \rangle$ i.e. $\langle \overline{\lambda}', j \rangle \in S_1$—and we are done as well. For

$j \in \{2,3\}$ it suffices to note that if $s_2 \in S_2$ and $\lambda(s_2) \leq s$ then $s = \lambda(s')$ for some $s' \geq s_2$—thus $s' \in S_2$.
iv) It suffices to note that the union of lower- (upper-) sets is lower (upper).
v) Use theorem A.2.v).
vi) Immediate. □

Let us then check that the algebras $\mathcal{D}_i$ indeed have the desired properties.

Theorem A.5 *For $i \in \{1,2,3\}$, $\mathcal{D}_i \in \mathcal{C}_i$.*

Proof: We must check for each $i \in \{1,2,3\}$ that $\mathcal{D}_i$ satisfies the (in-)equations required from def. 3.5. Note first that the special conditions regarding may- and must-algebras—namely that $\oplus_1$ ($\oplus_2$) is the sup (inf) w.r.t. $\leq_1$ ($\leq_2$)—follow immediately from theorem A.4.iv) and vi). Next we check the conditions 3.5.i)–iv). i) and ii) are straightforward.
iii) $\langle D_i, \cdot_i, 1_i \rangle$ is a commutative monoid with $\cdot_i$ preserving $\oplus_i$, $\leq_i$ and 0_i. For $i \in \{1,2\}$ the result follows from A.2.ii), iv) and A.4.ii) and v). So let $i = 3$. The commutative and unit laws follow directly from A.2.ii) and A.4.ii). For associativity suffice it to note that for all $S_1, S_2, S_3 \in D_3$,

$$\begin{aligned}
S_1 \cdot_3 (S_2 \cdot_3 S_3) &= \mathrm{cl}_3\{s_1 \cdot s \mid s_1 \in S_1, s \in \mathrm{cl}_3\{s_2 \cdot s_3 \mid s_2 \in S_2, s_3 \in S_3\}\} \\
&= \mathrm{cl}_3\{s_1 \cdot s \mid s_1 \in S_1, \\
&\qquad \exists s_{2,1}, s_{2,2} \in S_2, s_{3,1}, s_{3,2} \in S_3 . s_{2,1} \cdot s_{3,1} \leq s \leq s_{2,2} \cdot s_{3,2}\} \\
&= \mathrm{cl}_3\{s_1 \cdot (s_2 \cdot s_3) \mid s_1 \in S_1, s_2 \in S_2, s_3 \in S_3\}
\end{aligned}$$

by monotonicity of path multiplication, and the result then follows by its associativity, A.2.ii). Next, that $\cdot_3$ preserves 0_3 follows from A.2.iv) and A.4.i). The monotonicity of $\cdot_3$ w.r.t. $\leq_3$ follow from A.2.iv) and finally for preservation of $\oplus_3$ note that for all $S_1, S_2, S_3 \in D_3$,

$$\begin{aligned}
S_1 \cdot_3 (S_2 \oplus_3 S_3) &= \mathrm{cl}_3\{s_1 \cdot s \mid s_1 \in S_1, s \in \mathrm{cl}_3(S_2 \cup S_3)\} \\
&= \mathrm{cl}_3\{s_1 \cdot s \mid s_1 \in S_1, \exists s_2, s_3 \in S_2 \cup S_3 . s_2 \leq s \leq s_3\} \\
&= \mathrm{cl}_3\{s_1 \cdot s_2, s_1 \cdot s_3 \mid s_1 \in S_1, s_2 \in S_2, s_3 \in S_3\} \\
&\qquad \text{(by monotonicity of path multiplication)} \\
&= (S_1 \cdot_3 S_2) \oplus_3 (S_1 \cdot_3 S_3).
\end{aligned}$$

iv) For each $\lambda \in \mathcal{L}$, $(\lambda)_i$ preserves $\leq_i$ and $\oplus_i$ and the equations a), b) and c) of def. 3.5.iv) holds. The monotonicity of each $(\lambda)_i$ w.r.t. $\leq_i$ and for $i \in \{1,2\}$

the preservation of $\oplus_i$ follows from the monotonicity of path prefixing A.2.i) and A.4.iii) and iv). For $i = 3$ let $S_1, S_2 \in D_3$ and we calculate

$$\begin{aligned}
(\lambda)_3(S_1 \oplus_3 S_2) &= \{\lambda(s) \mid s \in \mathrm{cl}_3(S_1 \cup S_2)\} && \text{(by A.4.ii))} \\
&= \{\lambda(s) \mid \exists s_1, s_2 \in S_1 \cup S_2 . s_1 \leq s \leq s_2\} \\
&= \mathrm{cl}_3((\lambda)_3(S_1) \cup (\lambda)_3(S_2)) && \text{(by A.2.i))} \\
&= (\lambda)_3(S_1) \oplus_3 (\lambda)_3(S_2).
\end{aligned}$$

Equation c) follows directly from the definition of path prefixing and A.4.ii). For equation a), $S \in D_i$ and $i = 1$ note that $S \oplus_1 0_1 = S$ and for $i = 2$ that $S \oplus_2 0_2 = 0_2$ (we have already shown this). For $i = 3$ note that

$$S \oplus_3 0_3 = \mathrm{cl}_3(S \cup \{\langle \varepsilon, 0 \rangle\}) = \mathrm{cl}_1(S).$$

Finally for equation b), $S_1, S_2 \in D_i$ we obtain for $i \in \{2, 3\}$

$$\begin{aligned}
(\lambda)_i(S_1) \cdot_i (\mu)_i(S_2) &= \mathrm{cl}_i\{\lambda(s_1) \cdot \mu(s_2) \mid s_1 \in S_1, s_2 \in S_2\} \\
&= \mathrm{cl}_i\{(\lambda\mu)(s_1 \cdot s_2) \mid s_1 \in S_1, s_2 \in S_2\} && \text{(by A.2.iii))} \\
&= (\lambda\mu)_i(S_1 \cdot_i S_2) && \text{(by A.4.iii)).}
\end{aligned}$$

The proof for $i = 1$ is almost identical, and we are done. □

We can go on then to check that our representations $\mathcal{D}_i$ are indeed initial in $\boldsymbol{\mathcal{C}}_i$. Before embarking on the main proof we state a little lemma concerning least generating sets.

Lemma A.6 *For $i \in \{1, 2, 3\}$, $S, S_1, S_2 \in D_i$,*

i) $(\lambda)_i(S) = \mathrm{cl}_i\{\lambda(s) \mid s \in \mathrm{gen}_i(S)\}$,

ii) $S_1 \oplus_i S_2 = \mathrm{cl}_i(\mathrm{gen}_i(S_1) \cup \mathrm{gen}_i(S_2))$,

iii) $S_1 \cdot_i S_2 = \mathrm{cl}_i\{s_1 \cdot s_2 \mid s_1 \in \mathrm{gen}_i(S_1), s_2 \in \mathrm{gen}_i(S_2)\}$

Proof: Straightforward. □

We can now go on to prove the representation theorem proper:

Proof: (of theorem 3.7): We have just checked that each $\mathcal{D}_i$ is indeed an algebra of the appropriate type. To prove the result we then need for every $\mathcal{A} \in \boldsymbol{\mathcal{C}}_i$ to establish a unique homomorphism $f : \mathcal{D}_i \to \mathcal{A}$. Note first that every map $f : \mathcal{D}_i \to \mathcal{A}$ for $\mathcal{A} \in \mathcal{C}_i$ determines a map $f^\dagger$ from finite, nonempty sets of n-paths to $\mathcal{A}$, defined by

$$\begin{aligned} f^\dagger(\{s_1, \ldots, s_n\}) &= f(\mathrm{cl}_i\{s_1, \ldots, s_n\}) \\ &= f(\mathrm{cl}_i\{s_1\} \oplus_i \cdots \oplus_i \mathrm{cl}_i\{s_n\}) \end{aligned}$$

for $n \geq 1$. Further, f is a homomorphism iff $f^\dagger$ satisfies

i) $f^\dagger\{s_1, \ldots, s_n\} = f^\dagger\{s_1\} \oplus_{\mathcal{A}} \cdots \oplus_{\mathcal{A}} f^\dagger\{s_n\}$, $n \geq 1$,

ii) $f^\dagger\{\langle \varepsilon, 0\rangle\} = 0_{\mathcal{A}}$,

iii) $f^\dagger\{\langle \varepsilon, 1\rangle\} = 1_{\mathcal{A}}$,

iv) $f^\dagger\{\langle \lambda\overline{\lambda}, j\rangle\} = (\lambda)_{\mathcal{A}}(f^\dagger\{\langle \overline{\lambda}, j\rangle\})$,

and any such $f^\dagger$ determines f.

The only-if direction here is straightforward, and clearly conditions i)–iv) defines $f^\dagger$, so if f is a homomorphism it is also unique. So it remains to check existence. For $\mathcal{A}$ fixed, let the set of *traces* be the least s.t. $0_{\mathcal{A}}$, $1_{\mathcal{A}}$ are traces, and if $x \in \mathcal{A}$ is a trace and $\lambda \in \mathcal{L}$ then so is $(\lambda)_{\mathcal{A}}(x)$. Note that for every $x \in \mathcal{A}$ in the range of $f^\dagger$, $x = \sum A$ for $A \subseteq \mathcal{A}$ a finite, nonempty set of traces. Next note that $f^\dagger$ has the properties that

a) $f^\dagger\{\lambda(s)\} = (\lambda)_{\mathcal{A}}(f^\dagger\{s\})$, for $s \in \mathrm{NP}$,

b) $f^\dagger(S) = \sum_{\mathcal{A}}\{f^\dagger\{s\} \mid s \in S\}$, for $S \subseteq \mathrm{NP}$, S finite,

c) $f^\dagger\{s_1 \cdot s_2\} = f^\dagger\{s_1\} \cdot_{\mathcal{A}} f^\dagger\{s_2\}$, for $s_1, s_2 \in \mathrm{NP}$.

The verification of a)–c) is straightforward. For c) it involves a simple induction on the length of paths, using a).

Now there is little difficulty in verifying the homomorphism properties of f. First $f(0_i) = f^\dagger\{\langle \varepsilon, 0\rangle\} = 0_{\mathcal{A}}$, and $f(1_i) = f^\dagger\{\langle \varepsilon, 1\rangle\} = 1_{\mathcal{A}}$. For $S \in D_i$:

$$\begin{aligned} f((\lambda)_i(S)) &= f^\dagger(\mathrm{gen}_i((\lambda)_i(S))) \\ &= f^\dagger\{\lambda(s) \mid s \in \mathrm{gen}_i(S)\} \quad \text{(by A.6.i))} \\ &= \textstyle\sum_{\mathcal{A}}\{f^\dagger\{\lambda(s)\} \mid s \in \mathrm{gen}_i(S)\} \quad \text{(by b) above)} \\ &= \textstyle\sum_{\mathcal{A}}\{(\lambda)_{\mathcal{A}}(f^\dagger\{s\}) \mid s \in \mathrm{gen}_i(S)\} \quad \text{(by a) above)} \\ &= (\lambda)_{\mathcal{A}}(\textstyle\sum_{\mathcal{A}}\{f^\dagger\{s\} \mid s \in \mathrm{gen}_i(S)\}) \quad \text{(by equational reasoning)} \\ &= (\lambda)_{\mathcal{A}}(f^\dagger(\mathrm{gen}_i(S))) \quad \text{(by b))} \end{aligned}$$

$$= (\lambda)_{\mathcal{A}}(f(S))$$

For $S_1, S_2 \in D_i$:

$$\begin{aligned}
f(S_1 \oplus_i S_2) &= f^\dagger(\mathrm{gen}_i(S_1 \oplus_i S_2)) \\
&= f^\dagger(\mathrm{gen}_i(S_1) \cup \mathrm{gen}_i(S_2)) \quad \text{(by A.6.ii))} \\
&= f^\dagger(\mathrm{gen}_i(S_1)) \oplus_{\mathcal{A}} f^\dagger(\mathrm{gen}_i(S_2)) \quad \text{(by b))} \\
&= f(S_1) \oplus_{\mathcal{A}} f(S_2)
\end{aligned}$$

For $S_1, S_2 \in D_i$:

$$\begin{aligned}
f(S_1 \cdot_i S_2) &= f^\dagger(\mathrm{gen}_i(S_1 \cdot_i S_2)) \\
&= f^\dagger(\mathrm{gen}_i(S_1) \cdot \mathrm{gen}_i(S_2)) \quad \text{(by A.6.iii))} \\
&= \textstyle\sum_{\mathcal{A}}\{f^\dagger\{s_1 \cdot s_2\} \mid s_1 \in \mathrm{gen}_i(S_1), s_2 \in \mathrm{gen}_i(S_2)\} \quad \text{(by b))} \\
&= \textstyle\sum_{\mathcal{A}}\{f^\dagger\{s_1\} \cdot_{\mathcal{A}} f^\dagger\{s_2\} \mid s_1 \in \mathrm{gen}_i(S_1), s_2 \in \mathrm{gen}_i(S_2)\} \quad \text{(by c))} \\
&= (f^\dagger(\mathrm{gen}_i(S_1))) \cdot_{\mathcal{A}} (f^\dagger(\mathrm{gen}_i(S_2))) \quad \text{(by equational reasoning)} \\
&= f(S_1) \cdot_{\mathcal{A}} f(S_2).
\end{aligned}$$

The check that f is monotone is straightforward, and we have thus established the homomorphism property of f, and the proof is complete. $\square$

Appendix B

Proofs of lemmas 3.9 and 3.11

In this appendix we provide proofs of lemma 3.9, relating the operational notions on processes and their representations, and for lemma 3.11, stating that for all $i \in \{1,2,3\}$, $\leq_i = \preceq_i$.

We start by proving a lemma concerning the computational behaviour of paths. The relations $\mathrm{can}^{\mathrm{D}}$ and $\mathrm{live}^{\mathrm{D}}$ on paths are obtained by specialisation as usual, and we further let the partial operation $\mathrm{succ}^{\mathrm{D}}(s)$ be defined iff s $\mathrm{live}^{\mathrm{D}}$ and then $\mathrm{succ}^{\mathrm{D}}(s)$ is the unique s' s.t. for some λ, $s \xrightarrow{\lambda} s'$.

Lemma B.1 *Let $s, s_1, s_2 \in \mathrm{NP}$, $i \in \{1,2,3\}$ and $S \in D_i$.*

i) If s $\mathrm{can}^{\mathrm{D}}$ λ then $\lambda(\mathrm{succ}^{\mathrm{D}}(s)) = s$,

ii) $\lambda(s)$ $\mathrm{can}^{\mathrm{D}}$ λ and $\mathrm{succ}^{\mathrm{D}}(\lambda(s)) = s$,

iii) $s_1 \cdot s_2$ $\mathrm{live}^{\mathrm{D}}$ iff s_1 $\mathrm{live}^{\mathrm{D}}$ and s_2 $\mathrm{live}^{\mathrm{D}}$, and if s_1, s_2 $\mathrm{live}^{\mathrm{D}}$ then $\mathrm{succ}^{\mathrm{D}}(s_1 \cdot s_2) = \mathrm{succ}^{\mathrm{D}}(s_1) \cdot \mathrm{succ}^{\mathrm{D}}(s_2)$,

iv) If S $\mathrm{can}^{\mathrm{D}}$ λ then $S/_{\mathrm{D}}\lambda \in D_i$.

PROOF: i) and ii) and the first half of iii) are immediate. The second half of iii) is a straightforward induction on the length of $s_1 \cdot s_2$. For iv) assume $i = 3$—the other cases are easily derived from this. Assume that $S \in D_3$ and S $\mathrm{can}^{\mathrm{D}}$ λ. Clearly $S/_{\mathrm{D}}\lambda$ is nonempty. To check it is convex, let $s_1, s_2 \in S$, s_1 $\mathrm{can}^{\mathrm{D}}$ λ, s_2 $\mathrm{can}^{\mathrm{D}}$ λ and $\mathrm{succ}^{\mathrm{D}}(s_1) \leq s \leq \mathrm{succ}^{\mathrm{D}}(s_2)$. By i), $\lambda(\mathrm{succ}^{\mathrm{D}}(s_1)) = s_1 \leq \lambda(s) \leq \lambda(\mathrm{succ}^{\mathrm{D}}(s_2)) = s_2$. Then $\lambda(s) \in S$ as S is convex. But by ii), $\lambda(s)$ $\mathrm{can}^{\mathrm{D}}$ λ and $\mathrm{succ}^{\mathrm{D}}(\lambda(s)) = s$—thus $s \in S/_{\mathrm{D}}\lambda$. Note finally that $(\mathrm{gen}_3(S))/_{\mathrm{D}}\lambda$ generates $S/_{\mathrm{D}}\lambda$—this is seen similarly—thus $S/_{\mathrm{D}}\lambda$ is f.g. and $S/_{\mathrm{D}}\lambda \in D_3$. □

We then proceed to prove a slightly extended version of lemma 3.9 stating that *i)* $[\![p]\!]_i = \sum_i\{[\![p']\!]_i \mid p \twoheadrightarrow p'\}$,

ii) P can λ *iff* $[\![P]\!]_i$ $\mathrm{can}^{\mathrm{D}}$ λ,

iii) For $i \in \{2,3\}$, P live *iff* $[\![P]\!]_i$ $\mathrm{live}^{\mathrm{D}}$,

iv) For $i \in \{2,3\}$, P must Λ *iff* $[\![P]\!]_i$ $\mathrm{must}^{\mathrm{D}}$ Λ,

v) For all stable p, *if* p live *then* $[\![\mathrm{succ}(p)]\!]_i = \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in [\![p]\!]_i, s\ \mathrm{live}^{\mathrm{D}}\}$,

vi) If P can λ *then* $[\![p/\lambda]\!]_i = [\![P]\!]/_{\mathrm{D}}\lambda$.

Proof: (Of lemma 3.9): i) By induction on the structure of p. For $p = 0$, $p = 1$, $p = \lambda(p')$ the result is trivial. So calculate

$$\begin{aligned}
[\![p \oplus q]\!]_i &= [\![p]\!]_i \oplus_i [\![q]\!]_i \ \ \text{(by the definition of } [\![\cdot]\!]_i) \\
&= (\textstyle\sum_i\{[\![p']\!]_i \mid p \twoheadrightarrow p'\}) \oplus_i (\textstyle\sum_i\{[\![q']\!]_i \mid q \twoheadrightarrow q'\}) \\
&\quad \text{(by the induction hypothesis)} \\
&= \textstyle\sum_i\{[\![p']\!]_i \mid p \oplus q \twoheadrightarrow p'\} \ \ \text{(by equational reasoning).} \\
[\![p \times q]\!]_i &= [\![p]\!]_i \cdot_i [\![q]\!]_i \\
&= (\textstyle\sum_i\{[\![p']\!]_i \mid p \twoheadrightarrow p'\}) \cdot_i (\textstyle\sum_i\{[\![q']\!]_i \mid q \twoheadrightarrow q'\}) \\
&= \textstyle\sum_i\{[\![p']\!]_i \cdot_i [\![q']\!]_i \mid p \twoheadrightarrow p', q \twoheadrightarrow q'\} \\
&\quad \text{(by equational reasoning)} \\
&= \textstyle\sum_i\{[\![p' \cdot q']\!]_i \mid p \twoheadrightarrow p', q \twoheadrightarrow q'\} \ \ \text{(By } [\![\cdot]\!]_i) \\
&= \textstyle\sum_i\{[\![p']\!]_i \mid p \times q \twoheadrightarrow p'\}
\end{aligned}$$

ii) In view of i) it suffices to consider stable terms only, and we proceed by structural induction again:

a) 0 $\not\mathrm{can}$ λ for all λ, $[\![0]\!]_i = \mathrm{cl}_i(\langle\varepsilon, 0\rangle)$ $\not\mathrm{can}^{\mathrm{D}}$ λ, $i \in \{1,3\}$ and not $[\![0]\!]_i$ $\mathrm{live}^{\mathrm{D}}$.

b) 1 can λ iff $\lambda = e$. $[\![1]\!]_i = \mathrm{cl}_i(\langle\varepsilon, 1\rangle)$ $\mathrm{can}^{\mathrm{D}}$ λ iff $\lambda = e$.

c) $\lambda(p)$ can μ iff $\lambda = \mu$. $[\![\lambda(p)]\!]_i = (\lambda)_i([\![p]\!]_i)$ $\mathrm{can}^{\mathrm{D}}$ μ iff $\lambda = \mu$.

d) $p_1 \times p_2$ can λ iff $\exists \lambda_1, \lambda_2 \in \mathcal{L}$ s.t. $\lambda = \lambda_1\lambda_2$, p_1 can λ_1, p_2 can λ_2 iff $\exists \lambda_1, \lambda_2 \in \mathcal{L}$ s.t. $\lambda = \lambda_1\lambda_2$, $[\![p_1]\!]_i$ $\mathrm{can}^{\mathrm{D}}$ λ_1 and $[\![p_2]\!]_i$ $\mathrm{can}^{\mathrm{D}}$ λ_2 iff $[\![p_1 \times p_2]\!]_i$ $\mathrm{can}^{\mathrm{D}}$ λ.

iii) This is shown similarly as ii), by induction on the structure on stable terms.
iv) This is a corollary of i), ii) and iii).
v) Assume that p live and we proceed by induction on the structure of p. We calculate:

$$
\begin{aligned}
[\![\mathrm{succ}(1)]\!]_i &= [\![1]\!]_i \\
&= \mathrm{cl}_i\{\langle\varepsilon, 1\rangle\} \\
&= \mathrm{cl}_i\{\mathrm{succ}^{\mathrm{D}}(\langle\varepsilon, 1\rangle)\} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in [\![1]\!]_i\}
\end{aligned}
$$

$$
\begin{aligned}
[\![\mathrm{succ}(\lambda(p))]\!]_i &= [\![p]\!]_i \\
&= \{\mathrm{succ}^{\mathrm{D}}(\lambda(s)) \mid s \in [\![p]\!]_i\} \quad \text{(by B.1.ii))} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in (\lambda)_i([\![p]\!]_i)\} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in [\![\lambda(p)]\!]_i\}
\end{aligned}
$$

$$
\begin{aligned}
[\![\mathrm{succ}(p_1 \times p_2)]\!]_i &= [\![\mathrm{succ}(p_1) \times \mathrm{succ}(p_2)]\!]_i \\
&= [\![\mathrm{succ}(p_1)]\!]_i \cdot_i [\![\mathrm{succ}(p_2)]\!]_i \\
&= \{\mathrm{succ}^{\mathrm{D}}(s_1) \mid s_1 \in [\![p_1]\!]_i\} \cdot_i \{\mathrm{succ}^{\mathrm{D}}(s_2) \mid s_2 \in [\![p_2]\!]_i\} \\
&\quad \text{(by the induction hypothesis)} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s_1) \cdot \mathrm{succ}^{\mathrm{D}}(s_2) \mid s_1 \in [\![p_1]\!]_i, s_2 \in [\![p_2]\!]_i\} \\
&= \{\mathrm{succ}^{\mathrm{D}}(p_1 \cdot p_2) \mid s_1 \in [\![p_1]\!]_i, s_2 \in [\![p_2]\!]_i\} \quad \text{(by B.1)} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in [\![p_1]\!]_i \cdot [\![p_2]\!]_i\} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in [\![p_1 \times p_2]\!]_i\}
\end{aligned}
$$

vi) Assume P can λ and we calculate

$$
\begin{aligned}
[\![P/\lambda]\!]_i &= [\![\{\mathrm{succ}(p') \mid \exists p \in P \exists p'.p \twoheadrightarrow p', p' \text{ can } \lambda\}]\!]_i \\
&= \textstyle\sum_i\{[\![\mathrm{succ}(p')]\!]_i \mid \exists p \in P \exists p'.p \twoheadrightarrow p', p' \text{ can } \lambda\} \\
&= \textstyle\sum_i\{\{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in [\![p']\!]_i\} \mid \exists p \in P, \exists p'.p \twoheadrightarrow p', p' \text{ can } \lambda\} \\
&= \textstyle\sum_i\{\{\mathrm{succ}^{\mathrm{D}}(s)\} \mid \exists p \in P \exists p'.p \twoheadrightarrow p', p' \text{ can } \lambda, s \in [\![p']\!]_i\} \\
&= \textstyle\sum_i\{\{\mathrm{succ}^{\mathrm{D}}(s)\} \mid \exists p \in P \exists p'.p \twoheadrightarrow p', s \in [\![p']\!]_i, s \text{ can}^{\mathrm{D}} \lambda\} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \text{ can}^{\mathrm{D}} \lambda, s \in \textstyle\sum\{[\![p']\!]_i \mid \exists p \in P.p \twoheadrightarrow p'\}\} \\
&= \{\mathrm{succ}^{\mathrm{D}}(s) \mid s \in [\![P]\!]_i, s \text{ can}^{\mathrm{D}} \lambda\} \\
&= [\![P]\!]_i/_{\mathrm{D}}\lambda.
\end{aligned}
$$

□

Next for the proof of lemma 3.11.

PROOF: (of lemma 3.11). We prove 3.11 for $i = 2$ only. The case for $i = 1$ is similar and easier, and these two cases suffice to establish the result for $i = 3$. First for $\subseteq$: Assume $S_1, S_2 \in D_3$. We show by induction on n that $S_1 \preceq_2^n S_2$. The base case for $n = 0$ is trivial, so assume that $n = n' + 1$. Assume S_1 must$^{\mathrm{D}}$ Λ. Let $s_2 \in S_2$ and we must find an $\lambda \in \Lambda$ s.t. s_2 can$^{\mathrm{D}}$ λ. If no such λ exist either s_2 live$^{\mathrm{D}}$ fails or s_2 can$^{\mathrm{D}}$ μ for some $\mu \notin \Lambda$. In either case there must be some

$s_1 \in S_1$ s.t. either s_1 live$^{\mathrm{D}}$ fails or s_1 can$^{\mathrm{D}}$ $\mu \notin \Lambda$—but then S_1 must$^{\mathrm{D}}$ Λ fails as well. Assume then that S_1 live$^{\mathrm{D}}$ and S_2 can$^{\mathrm{D}}$ λ. Then there is some $s_2 \in S_2$ s.t. s_2 can$^{\mathrm{D}}$ λ and then—as S_1 live$^{\mathrm{D}}$—there is some $s_1 \in S_1$ s.t. s_1 can$^{\mathrm{D}}$ λ as well, thus S_1 can$^{\mathrm{D}}$ λ. Let now $s_2 \in S_2/_{\mathrm{D}}\lambda$. Then $\lambda(s_2) \in S_2$, and then there is some $s_1' \in S_1$ s.t. $s_1' \leq \lambda(s_2)$. As S_1 live$^{\mathrm{D}}$ also s_1' live$^{\mathrm{D}}$, thus s_1' can$^{\mathrm{D}}$ λ and then $s_1' = \lambda(s_1)$ with $s_1 \leq s_2$, $s_1 \in s_1/_{\mathrm{D}}\lambda$.

Next for the converse containment. Assume again that $S_1, S_2 \in D_3$ and we show that if $S_1 \not\lesssim_2 S_2$ then for some $n \geq 0$, $S_1 \not\precsim_2^n S_2$. If $S_1 \not\lesssim_2 S_2$ there is some $s_2 \in S_2$ s.t. for all $s_1 \in S_1$, $s_1 \not\leq s_2$. We proceed as before by induction on the length of s_2. For the base case, if $s_2 = \langle \varepsilon, 0\rangle$ then $\langle \varepsilon, 0\rangle \notin S_1$. Let

$$\Lambda = \{\lambda \mid \exists s_1 \in S_1 . s_1 \text{ can}^{\mathrm{D}}\ \lambda\}.$$

As S_1 is f.g. Λ is finite. Now S_1 live$^{\mathrm{D}}$ so S_1 must$^{\mathrm{D}}$ Λ, but S_2 must$^{\mathrm{D}}$ Λ fails—hence $S_1 \not\precsim_2^1 S_2$. Suppose then $s_2 = \langle \varepsilon, 1\rangle$. Then for all $s_1 \in S_1$, $s_1 \neq \langle \varepsilon, 1\rangle$ and $s_1 \neq \langle [e]^m, 0\rangle$ for all $m \geq 0$. Let now k be the largest s.t. $[e]^k$ is a prefix of some $s_1 \in S_1$. We proceed by induction on k. If $k = 0$ then S_1 $\cancel{\text{can}}^{\mathrm{D}}$ e. Also S_1 live$^{\mathrm{D}}$, as otherwise $\langle \varepsilon, 0\rangle \in S_1$. Let Λ be as above. Then S_1 must$^{\mathrm{D}}$ Λ—and S_2 must$^{\mathrm{D}}$ Λ fails, so $S_1 \not\precsim_2^1 S_2$. If $k = k'+1$ we obtain again S_1 live$^{\mathrm{D}}$, S_2 can$^{\mathrm{D}}$ e and S_1 can$^{\mathrm{D}}$ e and we show $S_1/_{\mathrm{D}}e \not\precsim_2^{k'} S_2/_{\mathrm{D}}e$. For again $\langle \varepsilon, 1\rangle \in S_2/_{\mathrm{D}}e$ and k' is the largest s.t. $[e]^{k'}$ is a prefix of some $s_1' \in S_1/_{\mathrm{D}}e$ so the conclusion follows; and thus we have shown $S_1 \not\precsim_2^k S_2$.

Assume then that $s_2 = \langle \lambda\overline{\lambda}, j_1\rangle$. Then for all $s_1 \in S_1$, $s_1 \neq \langle \varepsilon, 0\rangle$ and if $s_1 = \langle \lambda\overline{\mu}, j_2\rangle$ then $\langle \overline{\mu}.j_2 \not\leq \langle \overline{\lambda}, j_1\rangle$. Then once more S_1 live$^{\mathrm{D}}$ and S_2 can$^{\mathrm{D}}$ λ. If S_1 $\cancel{\text{can}}^{\mathrm{D}}$ λ then we are done as then $S_1 \not\precsim_2^1 S_2$. On the other hand if S_1 can$^{\mathrm{D}}$ λ then $S_1/_{\mathrm{D}}\lambda \not\lesssim_2 S_2/_{\mathrm{D}}\lambda$, because $\langle \overline{\lambda}, j_1\rangle \in S_2/_{\mathrm{D}}\lambda$ and for all $s_1' \in S_1/_{\mathrm{D}}\lambda$, $s_1' \not\leq \langle \overline{\lambda}, j_1\rangle$. Then by the induction hypothesis, $S_1/_{\mathrm{D}}\lambda \not\precsim_2^n S_2/_{\mathrm{D}}\lambda$ for some $n \geq 0$ and then $S_1 \not\precsim_2^{n+1} S_2$, and the proof is complete. $\square$

Appendix C

Proof of theorem 3.35

In this appendix we provide a proof of the normal form theorem, theorem 3.35, stating that for each variable-free $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$

i) there is a $\phi' \in \mathrm{NF}$ s.t. $\phi \equiv_{\mathbf{PL}^{1,D}} \phi'$,

ii) there is a $\phi' \in \mathrm{NF}_2$ s.t. $\phi \equiv_{\mathbf{PL}^{2,D}} \phi'$.

The proof proceed by induction first on the structure and then on the modal depth of formulas. For variable-free $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ the *modal depth*, $|\phi|$, of ϕ is defined inductively by

i) $|\mathrm{t}| = |\top| = |\underline{0}| = |\bot| = 0$,

ii) $|(\lambda)\phi| = |\overline{(\lambda)}\phi| = |\phi| + 1$,

iii) $|\phi \wedge \psi| = |\phi \vee \psi| = |\phi \to \psi| = \max(|\phi|, |\psi|)$.

Proof: (of theorem 3.35.i). We prove the slightly more general statement that for each variable-free $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ there is a $\phi' \in \mathrm{NF}$ s.t. $\phi \equiv_{\mathbf{PL}^{1,D}} \phi'$ and $|\phi'| \leq |\phi|$. First by induction on ϕ. i) $\phi = \phi_1 \to \phi_2$. By the outer induction hypothesis there are ϕ_1', ϕ_2' s.t. $\phi_1', \phi_2' \in \mathrm{NF}$, $\phi_1 \equiv_{\mathbf{PL}^{1,D}} \phi_1'$, $\phi_2 \equiv_{\mathbf{PL}^{1,D}} \phi_2'$ and $|\phi_i'| \leq |\phi_i|$ for $i \in \{1,2\}$ (For the remainder of this case we omit subscripting $\equiv$ by $\mathbf{PL}^{1,D}$). By 3.22 $\phi \equiv \phi_1' \to \phi_2'$ and $|\phi_1' \to \phi_2'| \leq |\phi|$ (we rely heavily on the replacement theorem 3.22 in the sequel and shall usually omit references to it). We proceed by induction on the size of ϕ and by cases on ϕ_1' and—when necessary—ϕ_2'.

a) $\phi_1' = \bot$. By 3.29.v), $\bot \to \phi_2' \equiv \top$, and $|\top| \leq |\bot \to \phi_2'|$.

b) $\phi_1 = \top$. We proceed by cases on ϕ_2':

1) $\phi_2' = \bot$. Here $(\top \to \bot) \equiv \bot$ by 21 and by $\vdash_{\mathbf{PL}^{1,D}} \top$ and

$$\vdash_{\mathbf{PL}^{1,D}} \top \to ((\top \to \bot) \to \bot).$$

Moreover, $|\bot| \leq |\top \to \bot|$.

2) $\phi_2' = \top$. We obtain $\top \to \top \equiv \top$ by theorem 3.29.ix).

3) $\phi_2' = \underline{0}$. We have $\top \to \underline{0} \equiv \underline{0}$ by $\vdash_{\mathbf{PL}^{1,D}} \top$ and $\vdash_{\mathbf{PL}^{1,D}} \top \to ((\top \to \underline{0}) \to \underline{0})$ for one direction and by reflexivity and 18 for the other.

4) $\phi_2' = \mathrm{t}$. We obtain $\top \to \mathrm{t} \equiv \underline{0}$ by theorem 3.30.viii) and axioms 16 and 18.

5) $\phi_2' = \bigvee_{\lambda \in \Lambda_1} (\lambda)\phi_\lambda$. Here $\top \to \phi_2' \equiv \underline{0}$ by axiom 32 and theorems 3.33, 3.29.xiv).

c) $\phi_1' = \underline{0}$. If $\phi_2' = \bot$ then we obtain $\underline{0} \to \bot \equiv \bot$ by axioms 21 and 22.

If $\phi \neq \bot$ then $\phi_2' \in \mathrm{satNF}$, thus $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \phi_2'$ by theorems 3.33 and 3.29.xiv). But then $\underline{0} \to \phi_2' \equiv \top$ by axioms 18 and 20.

d) $\phi_1' = \mathrm{t}$. Here $\mathrm{t} \to \phi_2' \equiv \phi_2'$ by axiom 10, and $|\phi_2'| \leq |\mathrm{t} \to \phi_2'|$.

e) $\phi_1' = \bigvee_{\lambda \in \Lambda_1} (\lambda)\psi_\lambda$. We proceed again by induction on the structure of ϕ_2':

1) $\phi_2' = \bot$. Here $(\phi_1' \to \bot) \equiv \bot$ by theorem 3.33 and axiom 21, and $|\bot| \leq |\phi_1' \to \bot|$.

2) $\phi_2' = \top$. Here $(\phi_1' \to \top) \equiv \top$ by 3.29.ix) and axiom 20.

3) $\phi_2' = \underline{0}$. We obtain first $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to (\phi_1' \to \underline{0})$ by reflexivity and axiom 18. For the converse direction note that

$$\vdash_{\mathbf{PL}^{1,D}} (\phi_1' \to \underline{0}) \to \bigwedge_{\lambda \in \Lambda_1} ((\lambda)\psi_\lambda \to \underline{0})$$

by 3.21.vii). Note secondly that $\vdash_{\mathbf{PL}^{1,D}} \neg\neg(\lambda)\psi_\lambda$ for all $\lambda \in \Lambda_1$ by 3.33, and then that

$$\vdash_{\mathbf{PL}^{1,D}} (\bigwedge_{\lambda \in \Lambda_1} ((\lambda)\psi_\lambda \to \underline{0})) \to \underline{0}$$

by axiom 33 and 3.29.xi) (and $\wedge$-elim-1/-2). Thus $\phi_1' \to \underline{0} \equiv \underline{0}$ (and moreover $|\underline{0}| \leq |\phi_1' \to \underline{0}|$).

4) $\phi_2' = \mathrm{t}$. We obtain $\phi \equiv \bigwedge_{\lambda\in\Lambda_1}((\lambda)\psi_\lambda \to \mathrm{t})$ by 3.21.vii). By 3.21.v),

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1}((\lambda)\psi_\lambda \to (e)\mathrm{t}).$$

Now $\mathrm{t} \in \mathrm{satNF}$ and for each $\lambda \in \Lambda_1$, $\psi_\lambda \in \mathrm{satNF}$, so $\vdash_{\mathbf{PL}^{1,D}} \neg\neg(\lambda)\psi_\lambda$ and $\vdash_{\mathbf{PL}^{1,D}} \neg\neg\mathrm{t}$ by 3.33. Then by axiom 31 and 3.21.ix),

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1}(\lambda^{-1})(\psi_\lambda \to \mathrm{t}).$$

For each $\lambda \in \Lambda_1$, $|\psi_\lambda \to \mathrm{t}| < |\phi|$ so for each $\lambda \in \Lambda$ we find a $\gamma_\lambda \in \mathrm{NF}$ s.t. $(\psi_\lambda \to \mathrm{t}) \equiv \gamma_\lambda$, and $|\gamma_\lambda| \leq |\psi_\lambda \to \mathrm{t}|$ and thus $\phi \equiv \bigwedge_{\lambda\in\Lambda_1}(\lambda^{-1})\gamma_\lambda$. Suppose now that for some $\lambda \in \Lambda_1$, $\gamma_\lambda = \bot$. Then by 3.29.viii) and $\wedge$-elim-1/-2 we get $\phi \equiv \bot$.

Otherwise for all $\lambda \in \Lambda_1$, $\lambda^{-1}\gamma_\lambda \in \mathrm{satNF}$ so

$$\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \bigwedge_{\lambda\in\Lambda_1}(\lambda^{-1})\gamma_\lambda$$

by 3.33, 3.29.xiv) and $\wedge$-intro. Now if $\mathrm{card}(\Lambda_1) > 1$ then $\phi \equiv \underline{0}$ by axiom 26. Otherwise let $\Lambda_1 = \{\lambda\}$ and we obtain

$$\phi \equiv (\lambda^{-1})\gamma_\lambda \in \mathrm{NF}$$

and

$$|(\lambda^{-1})\gamma_\lambda| \leq |(\lambda)\psi_\lambda \to \mathrm{t}| \leq |\phi_1' \to \mathrm{t}|$$

and we are done.

5) $\phi_2' = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu$. Now by 3.21.vii), $\phi \equiv \bigwedge_{\lambda\in\Lambda_1}((\lambda)\psi_\lambda \to (\mu)\gamma_\mu)$, and then by axiom 30 and 3.21.viii),

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1}(\bigvee_{\mu\in\Lambda_2}((\lambda)\psi_\lambda \to (\mu)\gamma_\mu)).$$

As in the previous case, we note that for each $\lambda \in \Lambda_1$ and $\mu \in \Lambda_2$, $\psi_\lambda, \gamma_\mu \in \mathrm{satNF}$, thus $\vdash_{\mathbf{PL}^{1,D}} \neg\neg\psi_\lambda$ and $\vdash_{\mathbf{PL}^{1,D}} \neg\neg\gamma_\mu$ by 3.33. Hence by axiom 31, 3.29.xi) and 3.21.ix),

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1}(\bigvee_{\mu\in\Lambda_2}((\lambda^{-1}\mu)(\psi_\lambda \to \gamma_\mu))).$$

Now $|\psi_\lambda \to \gamma_\mu| < |\phi|$ for all $\lambda \in \Lambda_1$, $\mu \in \Lambda_2$, and hence by the induction hypothesis we can for each λ, μ find a $\delta_{\lambda,\mu} \in \mathrm{NF}$ s.t. $\delta_{\lambda,\mu} \equiv \psi_\lambda \to \gamma_\mu$ and moreover $|\delta_{\lambda,\mu}| \leq |\psi_\lambda \to \gamma_\mu|$. Thus

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1}(\bigvee_{\mu\in\Lambda_2}((\lambda^{-1}\mu)\delta_{\lambda,\mu})).$$

Using 3.30.v) we get $\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2}(\bigwedge_{\lambda\in\Lambda_1}(\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)})$. If there is some $\lambda \in \Lambda_1$ s.t. for all $\mu \in \Lambda_2$, $\delta_{\lambda,\mu} = \bot$ then $\phi \equiv \bot$ (and $|\bot| < |\phi|$), so assume not. Say then that an $f : \Lambda_1 \to \Lambda_2$ is OK, if for all $\lambda \in \Lambda_1$, $\delta_{\lambda,f(\lambda)} \neq \bot$. Then, noting that OK f's exist,

$$\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2, f \text{ OK}} (\bigwedge_{\lambda\in\Lambda_1} (\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)}).$$

Fix some OK f. We rewrite each term $\bigwedge_{\lambda\in\Lambda_1}((\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)})$ into a term of the form either $\bot$ or $\underline{0}$ or $(\lambda')\theta_{\lambda'}$ with $\theta_{\lambda'} \in$ satNF. Suppose first that there is some $\lambda_1, \lambda_2 \in \Lambda_1$ s.t. $\lambda_1^{-1}f(\lambda_1) \neq \lambda_2^{-1}f(\lambda_2)$. Then by axiom 26, $\vdash_{\mathbf{PL}^{1,D}} \bigwedge_{\lambda\in\Lambda_1}((\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)}) \to \underline{0}$. Conversely, as for each $\lambda \in \Lambda$, $(\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)} \in$ satNF we obtain by 3.33 and 3.29.xiv),

$$\bigwedge_{\lambda\in\Lambda_1} ((\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)}) \equiv \underline{0}$$

If on the other hand for all $\lambda_1, \lambda_2 \in \Lambda_1$, $\lambda_1^{-1}f(\lambda_1) = \lambda_2^{-1}f(\lambda_2) = \lambda_f$, say, then by 3.30.i),

$$\bigwedge_{\lambda\in\Lambda_1} ((\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)}) \equiv (\lambda_f)(\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)}).$$

For each $\lambda \in \Lambda_1$, $|\delta_{\lambda,f(\lambda)}| \leq |\psi_\lambda \to \gamma_{f(\lambda)}|$, so

$$|\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)}| = \max\{|\delta_{\lambda,f(\lambda)}| \mid \lambda \in \Lambda_1\} < |\phi|$$

and then by the inner induction hypothesis we obtain a $\theta_{\lambda_f} \in$ NF s.t. $\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)} \equiv \phi_{\lambda_f}$ and $|\theta_{\lambda_f}| \leq |\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)}|$, and then

$$\bigwedge_{\lambda\in\Lambda_1} ((\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)}) \equiv (\lambda_f)\theta_{\lambda_f}.$$

If $\theta_{\lambda_f} = \bot$ then $\bigwedge_{\lambda\in\Lambda_1}((\lambda^{-1}f(\lambda))\delta_{\lambda,f(\lambda)}) \equiv \bot$ and if not, $(\lambda_f)\theta_{\lambda_f} \in$ satNF. So we have rewritten ϕ into a formula of the form $\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2, f \text{ OK}}(\theta_f)$ where for each f OK either $\theta_f = \bot$ or $\theta_f = \underline{0}$ or $\theta_f = (\lambda_f)\theta_{\lambda_f}$, and $|\theta_{\lambda_f}| < |\phi|$.

If for all $f : \Lambda_1 \to \Lambda_2$, f OK, $\theta_f = \bot$ then $\phi \equiv \bot$ and $|\bot| < |\phi|$ so assume not. Let f be D(oubly) OK, if f OK and $\theta_f \neq \bot$. Then $\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2, f \text{ DOK}}(\theta_f)$. Now if for all $f : \Lambda_1 \to \Lambda_2$, f DOK, $\theta_f = \underline{0}$ then $\phi \equiv \underline{0}$ and $|\underline{0}| < |\phi|$, and otherwise say an f is T(riply) OK, if f DOK and $\theta_f \neq \underline{0}$. Then $\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2, f \text{ TOK}}(\theta_f)$. For $\phi \vee \underline{0} \equiv \phi$

whenever $\phi \in$ satNF, by 3.33 and 3.29.xiv). But for all f TOK, $\theta_f = (\lambda_f)\theta_{\lambda_f}$ with $\theta_{\lambda_f} \in$ satNF, so $\phi' = \bigvee_{f:\Lambda_1\to\Lambda_2, f \text{ TOK}}(\theta_f) \in$ satNF. Now for each $f : \Lambda_1 \to \Lambda_2$,

$$\begin{aligned} |\,\theta_{\lambda_f}\,| &\leq |\textstyle\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)}| \\ &\leq \max\{\,|\,\psi_\lambda \to \gamma_{f(\lambda)}\,|\mid \lambda \in \Lambda_1.\} \end{aligned}$$

Hence for each $f : \Lambda_1 \to \Lambda_2$,

$$|\,(\lambda_f)\theta_{\lambda_f}\,| \leq \max\{\,|\,(\lambda)\psi_\lambda \to (f(\lambda))\gamma_{f(\lambda)}\,|\mid \lambda \in \Lambda_1\}$$

and hence $|\,\phi'\,| \leq |\,\phi\,|$ as desired, and we have completed case i).

ii) $\phi = \phi_1 \wedge \phi_2$. By the outer induction hypothesis we find $\phi_1', \phi_2' \in$ NF s.t. $\phi_i \equiv \phi_i'$ and $|\,\phi_i'\,| \leq |\,\phi_i\,|$ for $i \in \{1, 2\}$. Thus $\phi \equiv \phi_1' \wedge \phi_2'$ and $|\,\phi_1' \wedge \phi_2'\,| \leq |\,\phi\,|$. We proceed as before by induction on the size of ϕ, and then by cases on ϕ_1', ϕ_2':

a) $\phi_1' = \bot$. Then $\phi \equiv \bot$.

b) $\phi_1' = \top$. Then $\phi \equiv \phi_2'$.

c) $\phi_1' = \underline{0}$. If $\phi_2' = \bot$ then $\phi \equiv \bot$ as in case a). Otherwise $\phi_2' \in$ satNF and then by 3.33 and 3.29.xiv), $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \phi_2'$. Thus $\phi \equiv \underline{0} \in$ NF (and $|\,\underline{0}\,| \leq |\,\phi\,|$).

d) $\phi_1' = \mathrm{t}$. By symmetry we have already dealt with the cases $\phi_2' = \bot$, $\phi_2' = \top$ and $\phi_2' = \underline{0}$. If $\phi_2' = \mathrm{t}$ then $\phi \equiv \mathrm{t}$ and $|\,\mathrm{t}\,| \leq |\,\phi\,|$. So assume instead that $\phi_2' = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu$. We obtain

$$\begin{aligned} \phi &\equiv \textstyle\bigvee_{\mu\in\Lambda_2}(\mathrm{t} \wedge (\mu)\gamma_\mu) && \text{(by distribution)} \\ &\equiv \textstyle\bigvee_{\mu\in\Lambda_2}((e)\mathrm{t} \wedge (\mu)\gamma_\mu) && \text{(by 3.21.v))} \end{aligned}$$

Let $\mu \in \Lambda_2$. Note that $(\mu)\gamma_\mu$, $(e)\mathrm{t} \in$ satNF and hence by 3.33 and 3.29.xiv), $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to ((e)\mathrm{t} \wedge (\mu)\gamma_\mu)$ and thus

$$\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \bigvee_{\mu\in\Lambda_2} ((e)\mathrm{t} \wedge (\mu)\gamma_\mu).$$

If $\mu \in \Lambda_2$ and $\mu \neq e$ then $\vdash_{\mathbf{PL}^{1,D}} ((e)\mathrm{t} \wedge (\mu)\gamma_\mu) \to \underline{0}$ by 26. Hence if $e \notin \Lambda_2$ we obtain $\phi \equiv \underline{0}$ (and $|\,\underline{0}\,| \leq |\,\phi\,|$), and otherwise

$$\phi \equiv \underline{0} \vee ((e)\mathrm{t} \wedge (e)\gamma_e).$$

Now $(e)\mathrm{t} \wedge (e)\gamma_e \equiv (e)(\mathrm{t} \wedge \gamma_e)$ by 3.30.i), and as $|\,\mathrm{t} \wedge \gamma_e\,| < |\,\phi\,|$ we find by the inner induction hypothesis a $\delta \in \mathrm{NF}$ s.t. $\mathrm{t} \wedge \gamma_e \equiv \delta$ and $|\,\delta\,| < |\,\phi\,|$. Then $\phi \equiv \underline{0} \vee (e)\delta$. If $\delta = \bot$ then $(e)\delta \equiv \bot$ by 3.29.viii) and otherwise $\delta \in \mathrm{satNF}$, thus $(e)\delta \in \mathrm{satNF}$ and by 3.33, 3.29.xiv), $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to (e)\delta$. But then $\phi \equiv (e)\delta$ and $|\,(e)\delta\,| = |\,\delta\,| + 1 \leq |\,\phi\,|$.

e) $\phi'_1 = \bigvee_{\lambda\in\Lambda_1}(\lambda)\psi_\lambda$. The only remaining case is when $\phi'_2 = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu$. Now $\phi \equiv \bigvee_{\lambda\in\Lambda_1}(\bigvee_{\mu\in\Lambda_2}((\lambda)\psi_\lambda \wedge (\mu)\gamma_\mu))$ by distribution, and the proof proceeds as in the previous subcase.

iii) $\phi = \phi_1 \vee \phi_2$. Again by the outer induction hypothesis we obtain $\phi'_1, \phi'_2 \in \mathrm{NF}$ s.t. $\phi_i \equiv \phi'_i$ and $|\,\phi'_i\,| \leq |\,\phi_i\,|$ for $i \in \{1, 2\}$. Thus $\phi \equiv \phi'_1 \vee \phi'_2$ and $|\,\phi'_1 \vee \phi'_2\,| \leq |\,\phi\,|$. We proceed by cases on ϕ'_1, ϕ'_2:

a) $\phi'_1 = \bot$. Then $\phi \equiv \phi'_2$.

b) $\phi'_1 = \top$. Then $\phi \equiv \top$.

c) $\phi'_1 = \underline{0}$. If $\phi'_2 = \bot$ we are done as in case a). Otherwise $\phi'_2 \in \mathrm{satNF}$ and then $\vdash_{\mathbf{PL}^{1,D}} \underline{0} \to \phi'_2$ as now seen several times. Then $\phi \equiv \phi'_2$.

d) $\phi'_1 = \mathrm{t}$. By symmetry we have already dealt with the cases $\phi'_2 = \bot$, $\phi'_2 = \top$ and $\phi'_2 = \underline{0}$. If $\phi'_2 = \mathrm{t}$ then $\phi \equiv \mathrm{t}$ and if $\phi'_2 = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu$ then by 3.21.v),

$$\phi \equiv (e)\mathrm{t} \vee \bigvee_{\mu\in\Lambda_2} (\mu)\gamma_\mu \in \mathrm{satNF},$$

and

$$|\,(e)\mathrm{t} \vee \bigvee_{\mu\in\Lambda_2} (\mu)\gamma_\mu\,| \leq |\,\phi'_1 \vee \phi'_2\,| \leq |\,\phi\,| \;.$$

e) $\phi'_1 = \bigvee_{\lambda\in\Lambda_1}(\lambda)\psi_\lambda$. The only case remaining is for $\phi'_2 = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu$. But then $\phi'_1 \vee \phi'_2 \in \mathrm{satNF}$ and we are done.

iv) $\phi = (\lambda)\phi_1$. By the outer induction hypothesis we find as usual $\phi'_1 \in \mathrm{NF}$ s.t. $\phi_1 \equiv \phi'_1$ and $|\,\phi'_1\,| \leq |\,\phi_1\,|$. If $\phi'_1 = \bot$ then by 3.29.viii) we obtain $\phi \equiv \bot$. Otherwise $\phi'_1 \in \mathrm{satNF}$ and then $\phi \equiv (\lambda)\phi'_1 \in \mathrm{satNF}$ as well, and $|\,(\lambda)\phi'_1\,| \leq |\,\phi\,|$.
v) $\phi = \overline{(\lambda)}\phi_1$. Again we find $\phi'_1 \in \mathrm{NF}$ s.t. $\phi_1 \equiv \phi'_1$ and $|\,\phi'_1\,| \leq |\,\phi_1\,|$. We proceed by cases on ϕ'_1, noting that we have $\phi \equiv \overline{(\lambda)}\phi'_1$:

a) $\phi'_1 = \bot$. We obtain $\phi \equiv \bot$ by 3.29.xv).

b) $\phi'_1 = \top$. We get $\phi \equiv \top$ by 3.29.vii).

c) $\phi'_1 = \underline{0}$. By 3.30.vi) we get $\phi \equiv \bot$.

d) $\phi'_1 = \mathrm{t}$. By 3.21.v) we get $\phi \equiv \overline{(\lambda)}(e)\mathrm{t}$. Now if $\lambda \neq e$ then $\phi \equiv \bot$ by axiom 29. Otherwise we get $\phi \equiv \mathrm{t}$ by theorem 3.30.iii) and $|\,\mathrm{t}\,| < |\,\phi\,|$.

e) $\phi'_1 = \bigvee_{\lambda' \in \Lambda_1} (\lambda')\psi_{\lambda'}$. Now by 3.30.ii) we get $\phi \equiv \bigvee_{\lambda' \in \Lambda_1} \overline{(\lambda)}(\lambda')\psi_{\lambda'}$. Let now $\lambda' \in \Lambda_1$ and $\lambda \neq \lambda'$. Then $\overline{(\lambda)}(\lambda')\psi_{\lambda'} \equiv \bot$ by axiom 29, hence if $\lambda \notin \Lambda_1$ we obtain $\phi \equiv \bot$. Otherwise, by 3.30.iii), $\phi \equiv \psi_\lambda \in \mathrm{NF}$ and $|\,\psi_\lambda\,| < |\,\phi\,|$.

vi)–ix). The cases for $\phi = \bot$, $\phi = \top$, $\phi = \underline{0}$, $\phi = \mathrm{t}$ are trivial as then $\phi \in \mathrm{NF}$ already. □

We then proceed to the very similar proof for the must-case:

Proof: (of theorem 3.35.ii): As for i) we prove that for each variable-free $\phi \in \mathrm{PFm}_{\{\underline{0},\top,\bot\}}$ there is some $\phi' \in \mathrm{NF}_2$ s.t. $\phi \equiv_{\mathbf{PL}^{2,D}} \phi'$ and $|\,\phi'\,| \leq |\,\phi\,|$. Again we start by induction on the structure of ϕ.

i) $\phi = \phi_1 \to \phi_2$. By the outer induction hypothesis there are $\phi'_1, \phi'_2 \in \mathrm{NF}_2$ s.t. $\phi_i \equiv_{\mathbf{PL}^{2,D}} \phi'_i$ and $|\,\phi'_i\,| \leq |\,\phi_i\,|$ for $i \in \{1,2\}$ (again we suppress immediately the subscripting of $\equiv$). We then obtain $\phi \equiv \phi'_1 \to \phi'_2$ and $|\,\phi'_1 \to \phi'_2\,| \leq |\,\phi\,|$. We proceed by induction on the size of ϕ, then by cases on ϕ'_1, and then—as necessary—on ϕ'_2:

a) $\phi'_1 = \bot$. By 20 and 3.29.v), $\bot \to \phi'_2 \equiv \top$.

b) $\phi'_1 = \top$. Proceed by cases on ϕ'_2:

1) $\phi'_2 = \bot$. $\top \to \bot \equiv \bot$.

2) $\phi'_2 = \top$. $\top \to \top \equiv \top$.

3) $\phi'_2 = \mathrm{t}$. We obtain $\top \to \mathrm{t} \equiv \bot$ by 3.31.xi).

4) $\phi'_2 = \bigvee \mu \in \Lambda_2 (\mu)\gamma_\mu$. We obtain $\top \to \phi'_2 \equiv \bot$ by axiom 38.

c) $\phi'_1 = \mathrm{t}$. By 10 we get $\mathrm{t} \to \phi'_2 \equiv \phi'_2$.

d) $\phi'_1 = \bigvee_{\lambda \in \Lambda_1} (\lambda)\psi_\lambda$. We proceed again by cases on ϕ'_2:

1) $\phi'_2 = \bot$. By 3.33, $\vdash_{\mathbf{PL}^{2,D}} \neg\neg\phi'_1$ thus $\phi'_1 \to \bot \equiv \bot$.

2) $\phi'_2 = \top$. We obtain $\phi'_1 \to \top \equiv \top$ by 3.29.ix).

3) $\phi_2' = \mathrm{t}$. By 3.21.vii), $\phi \equiv \bigwedge_{\lambda\in\Lambda_1}((\lambda)\psi_\lambda \to \mathrm{t})$ and by 3.21.v),

$$\phi = \bigwedge_{\lambda\in\Lambda_1} ((\lambda)\psi_\lambda \to (e)\mathrm{t}).$$

Now for each $\lambda \in \Lambda_1$, $(\lambda)\psi_\lambda \in \mathrm{satNF}$, by 3.33, $\vdash_{\mathbf{PL}^{2,D}} \neg\neg(\lambda)\psi_\lambda$, whence by axiom 36, 3.29.xi) and 3.21.ix) we obtain

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1} (\lambda^{-1})(\psi_\lambda \to \mathrm{t}),$$

and for each $\lambda \in \Lambda_1$, $|\,\psi_\lambda \to \mathrm{t}\,| < |\,\phi\,|$, so for each $\lambda \in \Lambda_1$ we obtain by the inner induction hypothesis a $\gamma_\lambda \in \mathrm{NF}_2$ s.t. $\psi_\lambda \to \mathrm{t} \equiv \gamma_\lambda$ and $|\,\gamma_\lambda\,| < |\,\psi_\lambda \to \mathrm{t}\,|$, and thus $\phi \equiv \bigwedge_{\lambda\in\Lambda_1}(\lambda^{-1})\gamma_\lambda$.

If for some $\lambda \in \Lambda_1$, $\gamma_\lambda = \bot$ then $\phi \equiv \bot$ by 3.31.vii). If $\mathrm{card}(\Lambda_1) > 1$ then we also obtain $\phi \equiv \bot$ by 3.31.x). Otherwise let $\Lambda_1 = \{\lambda\}$ and we obtain

$$\phi \equiv (\lambda^{-1})\gamma_\lambda \in \mathrm{satNF}$$

and $|\,(\lambda^{-1})\gamma_\lambda\,| \leq |\,\phi\,|$.

4) $\phi_2' = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu$. As before we obtain

$$\begin{aligned}\phi &\equiv \bigwedge\nolimits_{\lambda\in\Lambda_1}((\lambda)\psi_\lambda \to \bigvee\nolimits_{\mu\in\Lambda_2}(\mu)\gamma_\mu) \\ &\equiv \bigwedge\nolimits_{\lambda\in\Lambda_1}(\bigvee\nolimits_{\mu\in\Lambda_2}((\lambda)\psi_\lambda \to (\mu)\gamma_\mu))\end{aligned}$$

by axiom 30 and 3.21.viii). Now we note that for each $\lambda \in \Lambda_1$, $\mu \in \Lambda_2$, $\psi_\lambda, \gamma_\mu \in \mathrm{satNF}$, thus $\vdash_{\mathbf{PL}^{2,D}} \neg\neg\psi_\lambda$ and $\vdash_{\mathbf{PL}^{2,D}} \neg\neg\gamma_\mu$ by 3.33. Hence by axiom 36, 3.29.xi) and 3.21.ix) we obtain

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1} (\bigvee_{\mu\in\Lambda_2} ((\lambda^{-1}\mu)(\psi_\lambda \to \gamma_\mu))).$$

Now $|\,\psi_\lambda \to \gamma_\mu\,| < |\,\phi\,|$ for all $\lambda \in \Lambda_1$, $\mu \in \Lambda_2$ and thus by the induction hypothesis we find for each such λ, μ a $\delta_{\lambda,\mu} \in \mathrm{NF}_2$ s.t. $\delta_{\lambda,\mu} \equiv \psi_\lambda \to \gamma_\mu$ and $|\,\delta_{\lambda,\mu}\,| \leq |\,\psi_\lambda \to \gamma_\mu\,|$. Hence

$$\phi \equiv \bigwedge_{\lambda\in\Lambda_1} (\bigvee_{\mu\in\Lambda_2} ((\lambda^{-1}\mu)\delta_{\lambda,\mu})).$$

By distribution,

$$\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2} (\bigwedge_{\lambda\in\Lambda_1} (\lambda^{-1}\mu)\delta_{\lambda,\mu}).$$

If for some $\lambda \in \Lambda_1$ for all $\mu \in \Lambda_2$, $\delta_{\lambda,\mu} = \perp$ then $\phi \equiv \perp$, so assume not. Say that an $f : \Lambda_1 \to \Lambda_2$ is OK, if for all $\lambda \in \Lambda_1$, $\delta_{\lambda,f(\lambda)} \neq \perp$. Then—noting OK f's exist,

$$\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2, f\ \mathrm{OK}} (\bigwedge_{\lambda\in\Lambda_1} (\lambda^{-1} f(\lambda))\delta_{\lambda,f(\lambda)}).$$

Fix some OK f. Suppose that there is some $\lambda_1, \lambda_2 \in \Lambda_1$ s.t.

$$\lambda_1^{-1} f(\lambda_1) \neq \lambda_2^{-1} f(\lambda_2).$$

Then

$$\bigwedge_{\lambda\in\Lambda_1} (\lambda^{-1} f(\lambda))\delta_{\lambda,f(\lambda)} \equiv \perp$$

by 3.31.x). If on the other hand for each $\lambda_1, \lambda_2 \in \Lambda_1$,

$$\lambda_1^{-1} f(\lambda_1) = \lambda_2^{-1} f(\lambda_2) = \lambda_f,$$

say, then by 3.31.viii) we obtain

$$\bigwedge_{\lambda\in\Lambda_1} ((\lambda^{-1} f(\lambda))\delta_{\lambda,f(\lambda)}) \equiv (\lambda_f)(\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)}).$$

Now

$$|\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)}| \leq \max\{|\psi_\lambda \to \gamma_{f(\lambda)}| \mid \lambda \in \Lambda_1\} < |\phi|$$

so by the induction hypothesis we obtain a $\theta_{\lambda_f} \in \mathrm{NF}_2$ s.t.

$$\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)} \equiv \theta_{\lambda_f}$$

and $|\theta_{\lambda_f}| \leq |\bigwedge_{\lambda\in\Lambda_1} \delta_{\lambda,f(\lambda)}|$ and then

$$\bigwedge_{\lambda\in\Lambda_1} ((\lambda^{-1} f(\lambda))\delta_{\lambda,f(\lambda)}) \equiv (\lambda_f)\theta_{\lambda_f}.$$

If $\theta_{\lambda_f} = \perp$ then $(\lambda_f)\theta_{\lambda_f} \equiv \perp$ and if not, then $(\lambda_f)\theta_{\lambda_f} \in \mathrm{satNF}$. Thus we have rewritten ϕ into a formula of the form

$$\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2, f\ \mathrm{OK}} (\theta_f)$$

where for each f OK, $\theta_f \in \mathrm{NF}_2$ and $|\theta_f| \leq |\phi|$, and moreover either $\theta_f = \perp$ or $\theta_f = (\lambda_f)\theta_{\lambda_f} \in \mathrm{satNF}$. Now if for all f OK, $\theta_f = \perp$ then $\phi \equiv \perp$, and otherwise let $f : \Lambda_1 \to \Lambda_2$ be D(oubly) OK, if f OK and $\theta_f \neq \perp$. Then

$$\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2, f\ \mathrm{DOK}} \theta_f = \phi'.$$

Now $\phi' \in \mathrm{NF}_2$ and $|\phi'| \leq |\phi|$ and the case is completed.

ii) $\phi = \phi_1 \wedge \phi_2$. Again we obtain $\phi_1', \phi_2' \in \mathrm{NF}_2$ s.t. $\phi_i \equiv \phi_i'$, $|\,\phi_i'\,| \leq |\,\phi_i\,|$ for $i \in \{1,2\}$ and again we proceed by induction on the size of ϕ and case-analysis:

a) $\phi_1' = \bot$. $\bot \wedge \phi_2' \equiv \bot$.

b) $\phi_1' = \top$. $\top \wedge \phi_2' \equiv \phi_2'$.

c) $\phi_1' = \mathrm{t}$. Proceed as usual, ignoring the cases $\phi_2' = \bot$, $\phi_2' = \top$ by symmetry. If $\phi_2' = \mathrm{t}$ then $\phi \equiv \mathrm{t}$, so assume instead $\phi_2' = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu$. Here

$$\phi \equiv (e)\mathrm{t} \wedge \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu \equiv \bigvee_{\mu\in\Lambda_2}((e)\mathrm{t} \wedge (\mu)\gamma_\mu).$$

If $e \notin \Lambda_2$ then $\phi \equiv \bot$ by 3.31.x) and otherwise $\phi \equiv (e)\mathrm{t} \wedge (e)\gamma_e$. Then, by 3.31.viii), $\phi \equiv (e)(\mathrm{t} \wedge \gamma_e)$ and by the inner induction hypothesis we obtain a $\delta \in \mathrm{NF}_2$ s.t. $\delta \equiv \mathrm{t} \wedge \gamma_e$ and $|\,\delta\,| < |\,\phi\,|$. Then $\phi \equiv (e)\delta$, so if $\delta = \bot$ then $\phi \equiv \bot$ and otherwise $(e)\delta \in \mathrm{NF}_2$ and $|\,(e)\delta\,| \leq |\,\phi\,|$.

d) $\phi_1' = \bigvee_{\lambda\in\Lambda_1}(\lambda)\psi_\lambda$. The only subcase left to consider is for

$$\phi_2' = \bigvee_{\mu\in\Lambda_2}(\mu)\gamma_\mu.$$

The proof proceeds as the previous case.

iii) $\phi = \phi_1 \vee \phi_2$. Straightforward.
iv) $\phi = (\lambda)\phi_1$. We obtain a $\phi_1' \in \mathrm{NF}_2$ s.t. $\phi_1' \equiv \phi_1$ and $|\,\phi_1'\,| \leq |\,\phi_1\,|$. If $\phi_1' = \bot$ then $\phi \equiv \bot$, and otherwise $\phi_1' \in \mathrm{satNF}$ thus $\phi \equiv (\lambda)\phi_1' \in \mathrm{NF}_2$ and we are done.
v) $\phi = \overline{(\lambda)}\phi_1$. We obtain $\phi_1' \in \mathrm{NF}_2$ s.t. $\phi_1' \equiv \phi_1$ and $|\,\phi_1'\,| \leq |\,\phi_1\,|$. We proceed by case-analysis as usual:

a) $\phi_1' = \bot$. Then $\phi \equiv \bot$ by 3.31.v).

b) $\phi_1' = \top$. Then $\phi \equiv \top$ by 3.31.vii).

c) $\phi_1' = \mathrm{t}$. We get $\phi \equiv \overline{(\lambda)}(e)\mathrm{t}$ thus if $\lambda \neq e$, $\phi \equiv \bot$ by axiom 29 and otherwise $\phi \equiv \mathrm{t}$ by 3.31.i).

d) $\phi_1' = \bigvee_{\lambda'\in\Lambda_1}(\lambda')\psi_{\lambda'}$. By 3.31.ix) we obtain $\phi \equiv \bigvee_{\lambda'\in\Lambda_1}\overline{(\lambda)}(\lambda')\psi_{\lambda'}$. If $\lambda \notin \Lambda_1$ then by axiom 29 and $\vee$-elim, $\phi \equiv \bot$ and otherwise by 3.31.i), $\phi \equiv \psi_\lambda \in \mathrm{NF}_2$ and $|\,\psi_\lambda\,| \leq |\,\phi\,|$.

vi) $\phi = \underline{0}$. Here $\phi \equiv \top$.
vii)–ix) are trivial as then $\phi \in \mathrm{NF}_2$. $\square$

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Notation and symbols

Number refers to the page on which the symbol is defined or of its first major occurrence.

Sets, lattices and algebras

$\sum_{i \in I} A_i$	Disjoint union of sets A_i, $i \in I$	116
in_i	Injection from A_i to $\sum_{i \in I} A_i$	142
$\mathcal{A}_1 \cong \mathcal{A}_2$	Isomorphism of algebras $\mathcal{A}_1$ and $\mathcal{A}_2$	65
$\uparrow x, \downarrow x$	Upper, downwards closure of x	37
ξ	Upper closed set	42
∇	Filter in semilattice	28
Δ	Ideal in supsemilattice	68
C	Chain in poset	44
$\mathrm{K}(L)$	Compact elements of algebraic lattice L	43
U	Scott-open subset of poset	44

Logical syntax

Ap	Countable set of atomic proposition symbols	18
Ap_1	1-prime atomic proposition symbols	157
α, β, ζ	Atomic proposition symbols	18
$\rightarrow$	Relevant implication	18
$\wedge, \vee$	Extensional conjunction, disjunction	18
$\top, \bot$	Extensional truthhood,falsehood	18
t	Intensional truthhood	18
$\circ$	Fusion	18
$\leftrightarrow$	Double implication	26
$\neg$	Negation	18
$\phi, \psi, \gamma, \delta, \theta$	Formulas	10

$\Phi, \Psi, \Theta, \Gamma$	Finite sequences of formulas	18
ε	The empty sequence	12
Fm	Language of propositional formulas	18
X	Sublanguage	18
X^+	Positive ($\{\mathrm{t}, \rightarrow, \wedge, \vee\}$-) fragment of X	18
X_A	A-fragment of X	18
X_A^+	$A \cup \{\mathrm{t}, \rightarrow, \wedge, \vee\}$-fragment of X	18
$\mathcal{L}$	Universe of labels	48
λ, μ, η, ν	Labels	48
$\lambda \cdot \mu,\ \lambda\mu$	Multiplication of labels λ and μ	48
e	Identity label for $\cdot$	48
Λ	Set of labels	59
$\boldsymbol{\Lambda}$	Subset of $\mathcal{P}(\mathcal{L})$	171
$\simeq$	Kleene equality of label expressions	148
$\underline{0}$	Zero-constant (chapter 3)	67
$(\lambda), \overline{(\lambda)}$	Forwards, reverse indexed modal operator (chapter 3)	68
PFm	Language of modal process formulas (chapter 3)	68
$\langle\lambda\rangle, [\lambda]$	Indexed possibility, necessity operator	119
GTFm	Language of ground modal formulas (chapter 4)	119
TFm	Language of relevant modal formulas (chapter 5)	152
$\mid\phi\mid$	Modal depth of ϕ	119
SPR_i	Syntactically i-prime formulas, $i \in \{1,2\}$	153
SCPR_i	Syntactically i-coprime formulas, $i \in \{1,2\}$	154

Frames and models

$\mathcal{F}$	Frame	27
$\boldsymbol{\mathcal{F}}$	Class of frames	29
$\mathcal{M}$	Model	27
$\boldsymbol{\mathcal{M}}$	Class of models	29
$\boldsymbol{\mathcal{M}}_{\mathcal{F}}$	Class of all $\boldsymbol{\mathcal{F}}$-based models	29
S	Carrier set (state set) of frames	27
σ, τ	States in S	108
Σ	Subset of S	109
L	Set of labels	108
$\boldsymbol{\mathcal{C}}_{(1/2/3)}$	Class of synchronous (may/must/general) algebras	62

$\mathcal{F}_{(1/2/3)}$	The initial sync. (may/must/general) algebra	62
$\cdot$	Binary, inf-preserving operation on frames	27
$\oplus$	Semilattice operation in synchronous algebras	62
1	Unit for $\cdot$	62
0	Zero for $\cdot$	62
V	Valuation (map from Ap to $\mathcal{P}(S)$)	27
$\models$	Satisfaction, or entailment	10
$[\![\phi]\!]$	Subset of S satisfying ϕ	30
$\twoheadrightarrow$	Unindexed transition relation	57
$\Rightarrow$	Reflexive, transitive closure of $\twoheadrightarrow$	110
$\xrightarrow{\lambda}$	λ-indexed transition relation	58
$\stackrel{\lambda}{\Rightarrow}$	$\Rightarrow \circ \xrightarrow{\lambda} \circ \Rightarrow$	110
$\uparrow$	Partiality predicate on states	108
$\Uparrow, \Downarrow$	Divergence, convergence predicate on states	110
Σ can λ	$\exists \sigma \in \Sigma, \tau.\ \sigma \stackrel{\lambda}{\Rightarrow} \tau$	60
$\mathrm{init}(\sigma)$	$\{\lambda \mid \exists \tau.\ \sigma \xrightarrow{\lambda} \tau\}$	110
$\mathrm{init}^{\Rightarrow}(\sigma)$	$\{\lambda \mid \exists \tau.\ \sigma \stackrel{\lambda}{\Rightarrow} \tau\}$	110
Σ/λ	$\{\tau \mid \exists \sigma \in \Sigma.\ \sigma \stackrel{\lambda}{\Rightarrow} \tau\}$	60
ρ	Derivation in modal frame	123
ρ_i	i'th state of ρ	123
$\mathrm{string}(\rho)$	String of labels labelling ρ	126
$\simeq_b, \sqsubseteq_b$	Bisimulation equivalence, preorder	49
$\simeq_s, \sqsubseteq_s$	Simulation equivalence, preorder	49
$\simeq_{(i)}, \sqsubseteq_{(i)}$	(i-) testing equivalence, preorder, $i \in \{1,2,3\}$	60
$\stackrel{\sqsubset}{\sim}_{(i)}, \cong_{(i)}$	(i-) testing embedding, isomorphism of models, $i \in \{1,2\}$	113
$\mathcal{M}^{\dagger}$	Linearisation of $\mathcal{M}$	111
$\stackrel{\lambda}{\rightrightarrows}$	Linearised transition relation	109
$\mathrm{lin}(\mathcal{S})$	Linear closure of $\mathcal{S}$	111
$\mathcal{M}_1 \parallel \mathcal{M}_2$	Parallel composition of $\mathcal{M}_1$ and $\mathcal{M}_2$	147
$\boldsymbol{\delta}_i$	Decomposition map Ap $\rightarrow \mathcal{P}(\mathrm{GTFm}_i^+ \times \mathrm{GTFm}_i^+)$	147
$\mathrm{con}\phi$	ϕ consistent (relative to model class $\boldsymbol{\mathcal{M}}$	162
$(\mathcal{M})^*$	Embedding from 3R-models to models	42
$(\mathcal{M})^{3R}$	Embedding from models to 3R-models	45

Logics

C	Logical calculus	18
$\mathbf{L}$	Metavariable over logics determined by calculi	19
$\vdash_{\mathbf{L}}$	$\mathbf{L}$ provability predicate, or relation	19
$\equiv_{\mathbf{L}}$	Equivalence in $\mathbf{L}$	76
$\mathrm{th}(\mathbf{L})$	Set of all $\mathbf{L}$-theories	33
∇	Theory	32
$\mathrm{th}_{\mathbf{L}}(A)$	Least $\mathbf{L}$-theory containing A	33
$\uparrow \phi$	Least theory containing ϕ	33
$\mathrm{C}(\mathbf{L})$	Canonical $\mathbf{L}$-model	32

Processes and tests

$\oplus$	Internal choice operator	57
$+$	CCS/SCCS $+$-operator	48
$\Box$	CSP choice operator	100
$\times$	Synchronous parallel composition	10
0	The inactive, convergent process	57
Ω	The inactive, divergent process	172
1	The neutral process	57
$\lambda()$	Prefixing by λ	57
$\mathrm{Pc}^{\oplus}$	Language of synchronous processes with internal choice	57
Pc^{+}	Language of synchronous processes with $+$	48
p, q, r, s	Processes	48
P, Q	Finite, nonempty sets of processes	57
St	Set of stable processes	57
$\mathrm{st}(p)$	Stable states of p	58
p live	$\forall \sigma \in \mathrm{st}(p) \exists \lambda, q.\ \sigma \xrightarrow{\lambda} q$	60
$\mathrm{succ}(p)$	Successor of live, stable p	58
$\mathrm{C}[\cdot]$	Process context	98
$0, \surd$	Failing, or successful test	59
$\mathcal{T}$	Language of tests	59
t	Test	59
P may t	Potential test acceptance relation	59
P must t	Necessary test acceptance relation	59

Normal forms and representations

$\overline{\lambda}$	String of labels	63
s	Pair $\langle\overline{\lambda}, j\rangle$, $j \in \{0,1\}$	63
S	Set of paths	63
NP	Set of all normal paths	64
$s_1 \cdot s_2$	Multiplication of paths s_1 and s_2	63
$\mathrm{cl}_i(S)$	i-closure of S, $i \in \{1,2,3\}$	64
$\mathrm{gen}_i(S)$	Least i-generating set of i-closed S, $i \in \{1,2,3\}$	64
$\mathcal{D}_i$	Representation of initial synchronous i-algebra	65
D_i	Carrier set of $\mathcal{D}_i$	65
$S\ \mathrm{can}^{\mathrm{D}}\ \lambda$	Can-relation on sets of paths	66
$S\ \mathrm{live}^{\mathrm{D}}$	Liveness predicate on sets of paths	66
$\mathrm{succ}^{\mathrm{D}}(s)$	Partial successor operation on paths	195
$S/_{\mathrm{D}}\lambda$	Partial successor map on sets of paths	66
$S\ \mathrm{must}^{\mathrm{D}}\ \Lambda$	Label acceptance relation on sets of paths	66
$S_1 \preceq_i S_2$	Computational i-ordering on D_i	66
$r_i(S)$	Embedding from D_i to PFm	71
satNF	Satisfiable normal forms	94
valNF	Valid normal forms	95
TRNF	Trace normal forms	135
PNF_i	Prime i-normal forms, $i \in \{1,2\}$	136
NF_i	i-normal forms, $i \in \{1,2\}$	136
$\mathrm{nf}(\phi)$	Normal form of ϕ	143
TTRNF	Testing trace normal forms	176
TPNF_i	Testing prime i-normal forms, $i \in \{1,2\}$	176
TNF_i	Testing i-normal forms, $i \in \{1,2\}$	176