University of Edinburgh

Department of Computer Science

Constructive λ-models

by

Andreas Knobel

CST-68-90
(also published as ECS-LFCS-90-120)

Thesis

October, 1990

James Clerk Maxwell Building,
The King's Buildings,
Mayfield Road,
Edinburgh,
EH9 3JZ.

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Andreas Knobel

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(graduation date July 1991)

Abstract

We study λ-models in a constructive setting.

We present two novel ways of deriving λ-models. These two definitions make sense classically, but yield nothing of interest. The first extends the structure of a λ-model to its space of *singletons.* These two models and all the models in between have the same equational theory. The second takes a full function space hierarchy and defines a λ-submodel whose universe consists of those points in the hierarchy that satisfy a *logical relation.* Call a model obtained in this way *extension model.* We prove that, given a 'classical' λ-model, it is consistent with **IZF** that it be isomorphic to an extension model. Also, this extension model has the same equational theory as the full function space hierarchy from which it was obtained. We prove these claims by building a fairly simple model of **IZF** in which these statements hold. This set theoretic model only depends on the *cardinality* of the original λ-model. We deduce that there is a model of **IZF** in which there exists a full function space hierarchy for *every* classical model such that the two have the same theory.

We go on to explore the logic of the world where these λ-models exist.

Acknowledgements

I wish to thank my supervisors Gordon Plotkin and Eugenio Moggi for their help and encouragement. The main results in this thesis were only obtained after long and intense discussions with Eugenio.

David C. McCarty had the idea to look at full abstraction in a constructive setting. Gerard Berry spent a whole afternoon explaining to me his thoughts on sequentiality. I am indebted to them and the following people for helping me along: Simon Ambler, Inge Bethke, Rod Burstall, George Cleland, Mads Dam, Mike Fourman, Martin Illsley, Mark Jerrum, Claire Jones, Eleanor Kerse, Michael Mendler, Faron Moller, Colin Stirling and Allen Stoughton.

I should like to thank my friends here in Edinburgh for having been so nice and generous: Charlotte, Agnieszka and Pawel, Sun, Jordi, Bob, Leslie, Julian, Anna, Carl, Bruce, Elena, Janet, Moira, Marina and Emmie in London.

During the first three years of my stay here I benefitted from a grant offered by Edinburgh University and through all these years from the generosity of my parents.

I dedicate this thesis to them and Petra, for her patience.

Declaration

This thesis was composed by myself. The work reported herein, unless otherwise stated, is my own.

Contents

Chapter 0

Introduction

0.1 Extensional λ-models

The extensional λ-calculus was created as a theory of *functions*, and it should therefore be expected that the natural models of a λ-theory are function spaces. Yet a description of such function spaces has in some cases turned out to be singularly elusive. Plotkin found a first example of this kind. He describes in [Plo77] a simple programming language **PCF** that is based on the typed λ-calculus and defines an equivalence relation on the set of terms of the language: two terms are identified if they show the same behaviour in all program contexts, i.e. if the outcome of all programs remains the same if one of the terms is replaced by the other. Plotkin goes on to show that the continuous function hierarchy with the natural interpretation is *not* a model of **PCF**. This failure is due to the existence of 'parallel functions' in the hierarchy that are not represented in the language. If a constant *por* and equations that describe its parallel behaviour are added to **PCF**, the continuous function hierarchy becomes a *fully abstract model*, i.e. two terms are equivalent iff they are mapped to the same value in the model.

Milner [Mil77] then proved that there was a unique (modulo isomorphism) fully abstract model of **PCF** with certain natural cpos at ground type. However his definition of the model is syntactic (the term model) and gives no information on the nature of the functions involved.

Ever since, attempts have been made to eliminate the unwanted functions from the continuous function hierarchy. The most successful approach to date was taken by Berry [Ber79]. He places a further simple condition 'stability' on the continuous functions and proves that all functions in the fully abstract model of **PCF** are stable. But the function *por* is not. Unfortunately this is still not enough: there are even functions taking arguments of ground type which are stable but not sequential.

It seems that the definition of sequential function must not only take into account their extensional, i.e. input-output behaviour, but also *intensional* aspects, the way they are computed. This insight led to a number of definitions that attempt to explain sequentiality 'at machine level'. Examples are the *concrete data structures* by Kahn and Plotkin [KP78], the more general notion of *event structure* by Winskel [Win80], and the concept of *sequential algorithm* by Berry and Curien [BC82]. In all these cases a function is defined to be sequential if it is the i-o function of a sequential process. This definition at last works at ground level, but fails higher up. The history of **PCF** and the search for a fully abstract functional model is recorded in [BCL85]. Stoughton [Sto88] investigates the conditions for the existence of fully abstract models not only when there is an equivalence relation on the set of terms (equational and contextual full abstraction), but also when the terms are partially ordered (inequational full abstraction). He treats **PCF** in great detail in his book.

0.2 Constructive help

It is well known among constructivists and viewed with suspicion by their classical colleagues that there are models of constructive set theory where all numeric functions are recursive (see McCarty [McC88]) or all endofunctions on the reals are continuous (see volume I of Troelstra and van Dalen's book [TvD88]), i.e. where there are fewer functions in some function space than can be *proved* to exist classically. With this in mind, it is natural to ask whether there is a model of constructive set theory in which there exists a fully abstract model of **PCF** based on sets and function spaces without additional structure, i.e. a model where all functions are sequential.

In his paper [Sco80], Dana Scott gives a hint of how such a model could be constructed. His method is very general. He builds a model of intuitionistic set theory from the fully abstract syntactic model of a λ-theory, and embeds the syntactic model in the set theoretic universe. Types are now interpreted as simple sets and functions. Full abstraction is preserved, so in the case of **PCF** all functions are sequential.

This thesis proposes an alternative solution. As a motivation, we shall have a closer look at the intuitionistic reals. Since constructive set theory is strictly weaker than its classical counterpart, many classical equivalences are not provable constructively. We conclude that also classically equivalent definitions no longer describe the same constructive objects. Reals are normally either defined as equivalence classes of Cauchy sequences of rationals, i.e. as the ω-completion of the space of rationals, or in terms of Dedekind cuts. Each alternative admits a host of different constructive definitions (see again [TvD88]). Now let us assume that we can embed a model of classical set theory in a model of constructive set theory. The constructive model will contain a copy $\mathbf{R}^*$ of the classical reals. Let $\mathbf{R}$ be the set of reals in the constructive universe according to one of the constructive definitions. Then $\mathbf{R}^*$ will be a subset of $\mathbf{R}$. In general not every function in $\mathbf{R}^* \Rightarrow \mathbf{R}^*$ will have an extension in $\mathbf{R} \Rightarrow \mathbf{R}$. By changing the constructive definition of reals, we can in fact to some extent *determine* which functions in $\mathbf{R}^* \Rightarrow \mathbf{R}^*$ should be represented in $\mathbf{R} \Rightarrow \mathbf{R}$. Of course, if we now aim for the continuous functions and succeed in finding a definition of real that will ensure that only they have an extension, we can still not be sure that now all functions in $\mathbf{R} \Rightarrow \mathbf{R}$ are continuous.

Fortunately for us, it turns out that this idea can be made to work in the case of sequential functions. We shall prove a general theorem which states that there is a fairly simple model of constructive set theory inside which for any given countable extensional λ-model a full function space model with the same equational theory can be constructed. This function space model is built following the ideas in the previous paragraph. In the case of **PCF**, the term model is copied into the set theoretic universe. Then supersets of the sets at ground type are found, such that exactly the sequential functions have extensions. At higher types this process is repeated.

This method has some advantages over the one using the Yoneda embedding. First and foremost, Scott's approach yields a different set theoretic universe for every λ-theory, whereas our construction caters for all of them. Furthermore our universe is a simple Kripke style model, whose characteristics are well known. This would ultimately permit us to ask questions about the relative consistency of statements. Finally, work on these models has led to a number of new and—hopefully—useful concepts. They indicate the direction an axiomatic approach might take.

We now give a brief overview over the contents of this thesis.

0.3 Overview

Chapter 1 contains all the material from category theory that will later be needed. The most important definition here is that of a *topos*. Toposes will be used to build models of constructive set theory, in which most of the action in the subsequent chapters takes place.

Chapter 2 is a concise introduction to constructive set theory. We present the axiomatic system **IZF**, which is the constructive equivalent to Zermelo-Fraenkel. We introduce a number of constructive definitions. As will be apparent from the introductory remarks, functions and extensions of functions will occupy a special place here. We construct a class of models of **IZF** to highlight the differences between the two systems, and show how to embed the classical von Neumann hierarchy in each of these models. We introduce the concept of 'classical set' in an attempt to describe the sets in the image of this embedding.

Chapter 3 focuses on typed and untyped combinatory algebras. We are interested in a particular variety of ca's, the extensional λ-models, which are essentially just function spaces. We then consider two non classical ways of deriving new ca's from given ones. First we look at the ca of *stable* $\neg\neg$*-singletons* of a combinatory algebra. It turns out that under certain conditions the same equations hold in both, i.e. that they have the same equational theory. Some first order properties are also preserved. Next, we introduce extension models. For the untyped case an extension model is obtained by selecting two sets $X_0 \subset X_1$. The universe of the model is defined to be the set functions $f : X_1 \to X_1$ that are the extension of some function in $X_0 \Rightarrow X_0$, or equivalently those functions f for which $fX_0 \subset X_0$.

Chapter 4 establishes the main result of this thesis for the typed calculus. We start by looking at two λ-theories, a simple theory **MON** of monotonic functions and at the above mentioned **PCF**. Then we prove the main theorem which states that for every extensional λ-model there is a model of **IZF** that contains a full function space model which has the same equational theory as the λ-model we started from. The set theoretic model only depends on the *cardinality* of the λ-model. One of the consequences of this theorem is that there exists a model of constructive set theory which contains a fully abstract full function space model for *every* λ-theory that has a fully abstract classical model. We end this chapter by discussing the main result and mentioning possible improvements.

Chapter 5 proves an equivalent theorem for the untyped calculus. Again we start by looking at a classical example. This will be Scott's inverse limit construction. Thereafter we state and prove the main result, which is a straightforward adaption of the typed case. Again we point to some consequences.

Chapter 6 finally explores the strange constructive world where models as in Chapters 4 and 5 exist. We talk about the various degrees of 'fuzziness' which a set can exhibit. We end this thesis by giving some hints as to how the whole subject could be treated axiomatically.

Chapter 1

Some concepts from category theory

In this chapter we shall review some notions from category theory. Mac Lane [ML71] provides a thorough introduction. For information about toposes we refer to the exhaustive book by Johnstone [Joh77]. A gentler introduction is Goldblatt's book [Gol79]. We also recommend a recent work by Barr and Wells [BW85].

A note on foundations. Categories tend to be large, i.e. classes, although most of the categories we shall be concerned with are sets. Recall that a category $\mathbf{C}$ is *small* if the class of all morphisms in $\mathbf{C}$ is a set. □

Notation. We shall write $C \in \mathbf{C}$ for $C \in \mathrm{Obj}(\mathbf{C})$ and $f \in \mathbf{C}(C_0, C_1)$ or $f : C_0 \to C_1$ for $f \in \mathrm{Hom}_{\mathbf{C}}(C_0, C_1)$. Given functors $F_0, F_1 : \mathbf{C}_0 \to \mathbf{C}_1$, let $[F_0, F_1]$ denote the set of natural transformations between them. For $\mathbf{C}$ a category, let $\mathbf{C}^{\mathrm{op}}$ denote its 'dual', i.e. the category with the same objects as $\mathbf{C}$ and all arrows reversed. □

1.1 Cartesian closed categories

Fix categories $\mathbf{C}, \mathbf{J}$. A $\mathbf{J}$-*diagram in* $\mathbf{C}$ is a functor $F \in \mathbf{C}^{\mathbf{J}}$. Define the *diagonal functor* $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ by

$$\begin{array}{rcll} \Delta C(J) & = & C & \text{for all } J \in \mathbf{J} \\ \Delta C(f) & = & \mathrm{id}_C & \text{for all } f : J_0 \to J_1 \\ \Delta f(J) & = & f & \text{for all } J \in \mathbf{J} \end{array}$$

Let $C \in \mathbf{C}$ and $F \in \mathbf{C}^{\mathbf{J}}$. A *cone* from the vertex C to the base F is a natural transformation $\gamma : \Delta C \to F$. Conversely, a cone *from* the base F *to* the vertex C or *cocone* is a natural transformation $\gamma : F \to \Delta C$.

Definition 1.1 *The* limit *of a* $\mathbf{J}$*-diagram* F *in* $\mathbf{C}$ *is an object* $\mathrm{lim}F \in \mathbf{C}$ *and a cone* $\gamma : \Delta\mathrm{lim}F \to F$ *such that for every object* $C_0 \in \mathbf{C}$ *and cone* $\gamma_0 : \Delta C_0 \to F$ *there is a unique* $f : C_0 \to \mathrm{lim}F$ *such that* $\gamma \circ \Delta f = \gamma_0$.

In this definition γ is the *limiting cone.*

Definition 1.2 *The* colimit *of a* $\mathbf{J}$*-diagram* F *in* $\mathbf{C}$ *is an object* $\mathrm{colim}F \in \mathbf{C}$ *and a cone* $\gamma : F \to \Delta\mathrm{colim}F$ *(the* colimiting cone*) such that for every object* $C_0 \in \mathbf{C}$ *and cone* $\gamma_0 : F \to \Delta C_0$ *there is a unique* $f : \Delta\mathrm{colim}F \to C_0$ *such that* $\Delta f \circ \gamma = \gamma_0$.

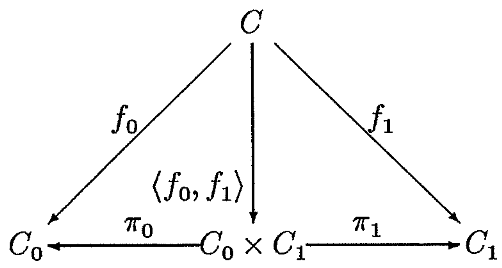
Limits and colimits—if they exist—are unique up to isomorphism in $\mathbf{C}$.

Definition 1.3 *The* terminal object 1 *is the limit of the empty diagram.*

This means that for every $C \in \mathbf{C}$ there is a unique arrow $! : C \to 1$.

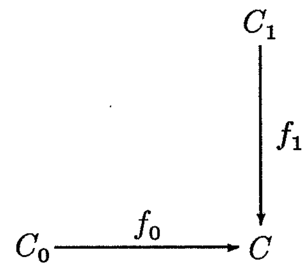
Definition 1.4 *The* product $C_0 \times C_1$ *of two objects* $C_0, C_1 \in \mathbf{C}$ *is the limit of the diagram* $\{C_0, C_1\}$.

This is the same as saying that there are morphisms $\pi_0 : C_0 \times C_1 \to C_0$ and $\pi_1 : C_0 \times C_1 \to C_1$ (the components of the limiting cone) such that for any object $C \in \mathbf{C}$ and morphisms $f_0 : C \to C_0$ and $f_1 : C \to C_1$ there exists a unique morphism $\langle f_0, f_1 \rangle$ that makes the following diagram commute.



π_0 and π_1 are the *projections* on C_0 and C_1.

In this connection we should mention the construction of a *pullback* or generalized product. This is simply the limit of the diagram



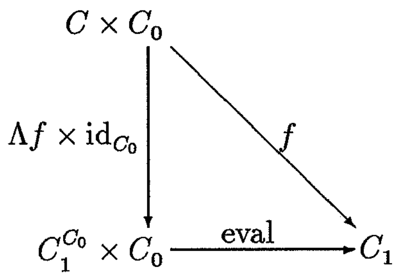
Note that we get the definition of product if we substitute 1 for C.

Definition 1.5 *The* coproduct $C_0 + C_1$ *of two objects* $C_0, C_1 \in \mathbf{C}$ *is the colimit of the diagram* $\{C_0, C_1\}$.

The components of the colimiting cone are the *inclusions*

$$\iota_0 : C_0 \to C_0 + C_1 \text{ and } \iota_1 : C_1 \to C_0 + C_1$$

Definition 1.6 *Let* $\mathbf{C}$ *have finite products. The* exponential *of two objects* C_0, C_1 *in* $\mathbf{C}$ *is an object* $C_1^{C_0} \in \mathbf{C}$ *together with a morphism* eval : $\mathrm{C}_1^{\mathrm{C}_0} \times \mathrm{C}_0 \to \mathrm{C}_1$ *such that for all objects* $C \in \mathbf{C}$ *and morphisms* $f : C \times C_0 \to C_1$ *there exists a unique morphism* $\Lambda f : C \to C_1^{C_0}$ *such that the following diagram commutes.*



Again $C_1^{C_0}$ is unique up to isomorphism—if it exists at all.

We now come to the most important definition of this section.

Definition 1.7 *A* cartesian closed category *(ccc) has a terminal object, finite products and exponentials.*

Example. The category of all sets **Set** is a ccc. It has a terminal object $\{\emptyset\}$, the usual set theoretic products and function spaces as exponentials. eval is function application. □

Example. A binary relation $\leq_X$ on a set X is a *preorder* if it is reflexive and transitive. $\leq_X$ is a *partial order* if also

$$x_0 \leq_X x_1 \wedge x_1 \leq_X x_0 \rightarrow x_0 = x_1$$

Now let $\mathbf{X} = \langle X, \leq_X \rangle$ be a partially ordered set. Then the relation $\leq_X^{\mathrm{op}}$ on X defined by

$$x_0 \leq_X^{\mathrm{op}} x_1 \overset{\Delta}{\leftrightarrow} x_1 \leq_X x_0$$

is also a partial order ($\leq_X$ 'reversed'). Let $\mathbf{X}^{\mathrm{op}}$ stand for $\langle X, \leq_X^{\mathrm{op}} \rangle$.

An *(ascending or descending) chain* in X is a totally ordered subset of X. Let α be an ordinal. A *($\leq_X$-)α-chain* in X is an α-sequence $\{x_{\alpha_0}\}_{\alpha_0 \in \alpha}$ such that

$$\forall \alpha_0, \alpha_1 \in \alpha \; (\alpha_0 \subset \alpha_1 \rightarrow x_{\alpha_0} \leq_X x_{\alpha_1})$$

An *antichain* in X is a set of pairwise incomparable elements of X. $\mathbf{X} = \langle X, \leq_X \rangle$ is a *complete partial order* (*cpo* for short) if every ascending ω-chain has a supremum in X. A function between two cpos is *continuous* if it preserves the suprema of all ω-chains. A cpo $\mathbf{X} = \langle X, \leq_X \rangle$ with a least point $\perp_X$ ('bottom') is called *pointed.*

Definition 1.8 *Let* **CPO** *denote the category of all cpos and continuous functions, and* **CPPO** *the category of all complete pointed partial orders and continuous functions.*

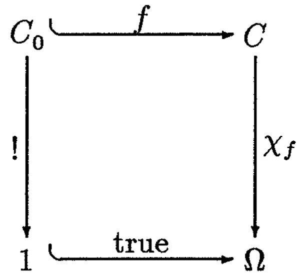
CPO and **CPPO** are cartesian closed: the product of two cpos is their *set-theoretic* product with the product ordering. The exponential of two cpos is the set of all continuous functions between them, ordered pointwise. eval is function application. **CPPO** is a full subcategory of **CPO**. □

1.2 Toposes

Set has another important property. Subsets of a set X stand in a 1-1 correspondence with the characteristic functions on X, i.e. $\mathcal{P}(X) \cong 2^X$. These notions can be generalized.

Definition 1.9 *Let* $C \in \mathbf{C}$. *A* subobject of C *is a monomorphism* $f : C_0 \hookrightarrow C$.

Definition 1.10 *Let* **C** *be a ccc. A* subobject classifier *is an object* $\Omega \in \mathbf{C}$ *and a subobject* true $: 1 \hookrightarrow \Omega$ *such that for all subobjects* $f : C_0 \hookrightarrow C$ *there exists exactly one morphism* $\chi_f : C \to \Omega$ *that makes the following diagram into a pullback.*



Ω is likewise unique up to isomorphism.

Example continued. 2 is a subobject classifier in **Set**. As true we can take the constant function $\Delta 1$. Given sets $X_0 \subset X$ the characteristic function of X_0 is defined on X as

$$\chi_{X_0}(x) = 1 \text{ if } x \in X_0$$

$$\chi_{X_0}(x) = 0 \text{ if } x \notin X_0$$

□

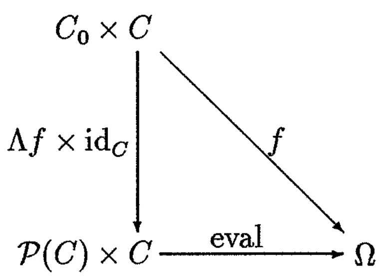
We now come to the main definition in this chapter.

Definition 1.11 *A ccc is an* (elementary) topos *if it has a subobject classifier.*

In **Set** we had $\mathcal{P}(X) \cong 2^X$. Similarly, in a general topos we can define the *powerobject* of $C \in \mathbf{C}$ to be an object $\mathcal{P}(C)$ together with a map

$$\text{eval} : \mathcal{P}(C) \times C \to \Omega$$

such that for every $C_0 \in \mathbf{C}$ and every map $f : C_0 \times C \to \Omega$ there is a unique map $\Lambda f : C_0 \to \mathcal{P}(C)$ such that the following diagram commutes.



Note that this diagram is just a special case of the diagram for exponentials. We see that $\mathcal{P}(1) \cong \Omega^1 \cong \Omega$.

Example. We have already seen that **Set** is a topos. More generally, any model **V** of set theory can be turned into a topos. We give an indication of how this is done without going into details.

- Objects are equivalence classes of sets in **V**. For $v_0, v_1 \in \mathbf{V}$

$$v_0 \approx v_1 \leftrightarrow \mathbf{V} \models v_0 = v_1$$

- Morphisms in $\mathrm{Hom}_{\mathbf{V}}([v_0], [v_1])$ are equivalence classes of functions in **V**. That is, for $f \in \mathbf{V}$

$$f \in \mathrm{Hom}_{\mathbf{V}}([v_0], [v_1]) \text{ iff } \mathbf{V} \models f \text{ is a function from } v_0 \text{ to } v_1$$

- The terminal object is the (equivalence class of the) set $\{\emptyset\}$, products and exponentials are (the equivalence classes of) set theoretic products and function spaces as in **Set**.

- The subobject classifier is defined as $\Omega \triangleq [\mathcal{P}(1)]$.

If ϕ is a set theoretic formula we have

$$\mathbf{V} \models \phi \leftrightarrow 0 \in \{\, 0 \mid \phi \,\}$$

There is a 1-1 correspondence between 'truth values' and subsets of 1, i.e. Ω. Ω is therefore also referred to as the *truth value object.* □

1.3 Presheaves

We are now going to concentrate on a particular class of toposes. A functor F from a category **C** into **Set** can be viewed as a variable set, $F(C)$ giving its value at a stage $C \in \mathbf{C}$ and $F(f)$ the transformation of $F(C_0)$ into $F(C_1)$ for $f : C_0 \to C_1$. For reasons we do not wish to explain here, we shall consider *contravariant* functors, or functors $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Definition 1.12 *Given a category* **C**, *the category* $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ *of* presheaves over **C** *has functors* $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ *as objects and natural transformations* $\gamma : F_0 \to F_1$ *as morphisms.*

We shall now give some examples of presheaves and show how a terminal object, products, exponentials and a subobject classifier can be defined in $\mathbf{Set}^{\mathbf{C}^{op}}$ making it into a topos. We omit the proofs that these objects have the right universal properties.

The constant functors. For every set X the constant functor ΔX maps objects $C \in \mathbf{C}$ to X and morphisms $f : C_0 \to C_1$ to the identity on X. □

The Terminal object. This is defined to be the constant functor $\Delta 1$. □

Subobjects. Subobjects are natural transformations $\gamma : F_0 \hookrightarrow F_1$ that are monomorphisms in the category $\mathbf{Set}^{\mathbf{C}^{op}}$. The simplest subobjects are those where the natural transformation involved is the *inclusion*. We then write $F_0 \subset F_1$. □

Products. Let $F_0, F_1 \in \mathbf{Set}^{\mathbf{C}^{op}}$. Their product $F_0 \times F_1$ is defined pointwise.

$$\begin{array}{rcl} F_0 \times F_1 : C_0 & \mapsto & F_0(C_0) \times F_1(C_0) \\ f \downarrow & \mapsto & \uparrow F_0(f) \times F_1(f) \\ C_1 & \mapsto & F_0(C_1) \times F_1(C_1) \end{array}$$

□

Coproducts. Let $F_0, F_1 \in \mathbf{Set}^{\mathbf{C}^{op}}$. Their coproduct is simply the co-construction of their product, namely

$$\begin{array}{rcl} F_0 + F_1 : C_0 & \mapsto & F_0(C_0) + F_1(C_0) \\ f \downarrow & \mapsto & \uparrow F_0(f) + F_1(f) \\ C_1 & \mapsto & F_0(C_1) + F_1(C_1) \end{array}$$

□

The same construction applies to limits and colimits in general.

Now let α be an ordinal and $\{F_{\alpha_0}\}_{\alpha_0 \in \alpha}$ an $\subset$-α-chain in $\mathbf{Set}^{\mathbf{C}^{op}}$. Define $\bigcup_{\alpha_0 \in \alpha} F_{\alpha_0}$ by setting for every $C \in \mathbf{C}$

$$(\bigcup_{\alpha_0 \in \alpha} F_{\alpha_0})(C) \triangleq \bigcup_{\alpha_0 \in \alpha} (F_{\alpha_0}(C))$$

To describe the behaviour of $\bigcup_{\alpha_0 \in \alpha} F_{\alpha_0}$ on morphisms pick $f : C_0 \to C_1$ and $x \in (\bigcup_{\alpha_0 \in \alpha} F_{\alpha_0})(C_1)$. Then for some $\alpha_1 \in \alpha$

$$x \in F_{\alpha_1}(C_1)$$

Set

$$(\bigcup_{\alpha_0 \in \alpha} F_{\alpha_0})(f)(x) = F_{\alpha_1}(f)(x)$$

It is easy to see that this definition works.

We have the following

Lemma 1.13 *Let $\{F_{\alpha_0}\}_{\alpha_0 \in \alpha}$ be as described above. Then*

$$\text{colim}_{\alpha_0 \in \alpha} F_{\alpha_0} = \bigcup_{\alpha_0 \in \alpha} F_{\alpha_0}$$

Proof. $\bigcup_{\alpha_0 \in \alpha} F_{\alpha_0}$ is the vertex of a co-cone with the inclusions as components. We have to verify that it has the universal property. So pick $F \in \mathbf{Set}^{\mathbf{C}^{op}}$ and transformations $\{\eta_{\alpha_0} : F_{\alpha_0} \to F\}_{\alpha_0 \in \alpha}$ such that the corresponding diagram commutes. Pick $C \in \mathbf{C}$. We define the function

$$\eta_\alpha(C) : \bigcup_{\alpha_0 \in \alpha} F_{\alpha_0}(C) \to F(C)$$

Take $x \in \bigcup_{\alpha_0 \in \alpha} F_{\alpha_0}(C)$. There is some $\alpha_1 \in \alpha$ such that $x \in F_{\alpha_1}(C)$. Put

$$\eta_\alpha(C)(x) = \eta_{\alpha_1}(C)(x)$$

η_α thus defined is a natural transformation and unique. □

Representable functors. Let $\mathbf{C}$ be a small category. Then for every $C \in \mathbf{C}$ the functor

$$\text{Hom}_{\mathbf{C}}(-, C) : \mathbf{C}^{op} \to \mathbf{Set}$$

for which

$$\begin{array}{rcl} \text{Hom}_{\mathbf{C}}(-, C) : C_0 & \mapsto & \text{Hom}_{\mathbf{C}}(C_0, C) \\ f \downarrow & \mapsto & \uparrow - \circ f \\ C_1 & \mapsto & \text{Hom}_{\mathbf{C}}(C_1, C) \end{array}$$

is a presheaf over $\mathbf{C}$. We call a presheaf over $\mathbf{C}$ *representable* if it is isomorphic to $\text{Hom}_{\mathbf{C}}(-, C)$ for some $C \in \mathbf{C}$. □

Exponentials. Let $F_0, F_1 \in \mathbf{Set}^{\mathbf{C}^{op}}$ and $C \in \mathbf{C}$. Define

$$F_1^{F_0}(C) \triangleq [\text{Hom}_{\mathbf{C}}(-, C) \times F_0, F_1]$$

Thus $F_1^{F_0}(C)$ are families $\{f_{f_0} : F_0(C_0) \to F_1(C_0)\}_{f_0 : C_0 \to C}$ of *compatible* functions indexed by morphisms with codomain C.

On morphisms $F_1^{F_0}$ acts as follows.

$$F_1^{F_0}(f : C_0 \to C_1)(\{g_{f_1}\}_{f_1 : C \to C_1}) = \{g_{f \circ f_0}\}_{f_0 : C \to C_0}$$

The functions in $F_1^{F_0}(C)$ can be thought of as *local sections* or *approximations* of a function as seen from C.

eval : $\mathrm{F}_1^{\mathrm{F}_0} \times \mathrm{F}_0 \to \mathrm{F}_1$ is defined as

$$\mathrm{eval}(C)\langle\{f_{f_0}\}_{f_0 : C_0 \to C}, x\rangle = f_{\mathrm{id}_C}(x)$$

□

The subobject classifier. In order to define this presheaf, we need the following concept.

Definition 1.14 *Let $C \in \mathbf{C}$. A C-*crible *is a family $\mathcal{C}$ of maps with codomain C which is closed under right composition, i.e. if $f_1 : C_1 \to C$ and $f : C_0 \to C_1$ then*

$$f_1 \in \mathcal{C} \to f_1 \circ f \in \mathcal{C}$$

Now we let $\Omega(C)$ be the set of C-cribles and define

$$\Omega(f : C_0 \to C_1)(\mathcal{C}) = \{\, f_0 : C \to C_0 \mid f \circ f_0 \in \mathcal{C} \,\}$$

The cribles will turn out—under the right interpretation—to serve as truth values. Indeed for any object $C \in \mathbf{C}$ the C-cribles ordered by inclusion form a complete Heyting algebra (c.f. Section 1.5). $\Omega(f) : \Omega(C_1) \to \Omega(C_0)$ is a structure preserving map for all $f : C_0 \to C_1$.

We have yet to take care of true : $1 \hookrightarrow \Omega$. At every C, this map simply picks $\{\, f_0 \mid f_0 : C_0 \to C \,\}$ from $\Omega(C)$, i.e. the largest C-crible.

□

Powerpresheaves. From $\mathcal{P}(F) \cong \Omega^F$ and the construction of the exponentials we get for $C \in \mathbf{C}$

$$\begin{aligned} \mathcal{P}(F)(C) &\cong \Omega^F(C) \\ &= [\mathrm{Hom}_{\mathbf{C}}(-, C) \times F, \Omega] \end{aligned}$$

In other words, $\mathcal{P}(F)(C)$ contains all families of sets

$$\{X_{f_0} \subset F(C_0)\}_{f_0 : C_0 \to C}$$

such that if $f : C_0 \to C_1$ and $f_1 : C_1 \to C$ then

$$F(f)(X_{f_1}) \subset X_{f_1 \circ f}$$

$\mathcal{P}(F)$ acts as follows on morphisms $f : C_0 \to C_1$.

$$\mathcal{P}(F)(f)(\{X_{f_1}\}_{f_1 : C \to C_1}) = \{X_{f \circ f_0}\}_{f_0 : C \to C_0}$$

□

1.4 The Yoneda lemma

We know that $\mathrm{Hom}_{\mathbf{C}}(-, C)$ is a presheaf over $\mathbf{C}$ for every $C \in \mathbf{C}$. This fact can be used to embed $\mathbf{C}$ in $\mathbf{Set}^{\mathbf{C}^{op}}$. The embedding preserves limits and exponentials that might exist in $\mathbf{C}$.

Definition 1.15 *The* Yoneda functor $Y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ *is defined by*

$$\begin{array}{rcl} Y : C_0 & \mapsto & \mathrm{Hom}_{\mathbf{C}}(-, C_0) \\ f \downarrow & \mapsto & \downarrow f \circ - \\ C_1 & \mapsto & \mathrm{Hom}_{\mathbf{C}}(-, C_1) \end{array}$$

Recall that a functor is *full* if it is surjective on the hom-sets, it is *faithful* if it is injective on the hom-sets. A subcategory of some category is full if the inclusion is full. We can now state the following lemma. The proof first appeared in [Yon54].

Lemma 1.16 *The Yoneda embedding* $Y : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{op}}$ *is full and faithful.*

Proof. Pick $\eta : \mathrm{Hom}_{\mathbf{C}}(-, C_0) \to \mathrm{Hom}_{\mathbf{C}}(-, C_1)$ and $f : C \to C_0$. The following diagram commutes by naturality of η.

$$\begin{array}{rcl} \eta(C_0) : \mathrm{Hom}_{\mathbf{C}}(C_0, C_0) & \to & \mathrm{Hom}_{\mathbf{C}}(C_0, C_1) \\ - \circ f \downarrow & & \downarrow - \circ f \\ \eta(C) : \mathrm{Hom}_{\mathbf{C}}(C, C_0) & \to & \mathrm{Hom}_{\mathbf{C}}(C, C_1) \end{array}$$

By chasing id_{C_0} around the square we obtain

$$\eta(C)(f) = \eta(C)(\mathrm{id}_{C_0} \circ f) = (\eta(C_0)(\mathrm{id}_{C_0})) \circ f$$

So η is determined by its value at $(\eta(C_0))(\mathrm{id}_{C_0})$. □

We have already remarked that Y preserves limits, hence terminal objects and products. It does not necessarily preserve colimits though.

1.5 Presheaves over partial orders

A preorder $\mathbf{P} = \langle P, \leq_P \rangle$ can be seen as a particularly simple category. P is the set of objects and $\leq_P$ is the set of morphisms, i.e. $\mathrm{Hom}_{\mathbf{P}}(p_0, p_1)$ contains one morphism if $p_0 \leq_P p_1$, and none otherwise.

Composition of maps is defined by

$$(p_0 \leq_P p_1) \circ (p_1 \leq_P p_2) \triangleq (p_0 \leq_P p_2)$$

and total by transitivity of $\leq_P$. Reflexivity assures that the identity morphisms exist.

If $\mathbf{P}$ is a partial order, isomorphic objects are equal.

Here are some partial orders that will be used as categories.

Von Neumann ordinals. For α an ordinal, set

$$\boldsymbol{\alpha} \triangleq \langle \alpha, \subset_\alpha \rangle$$

Note that the points of an ordinal are ordered by inclusion, and not by $\in$. □

Lifted sets. Given a set X, its *lifting* $\mathbf{X}_\perp = \langle X_\perp, \leq_\perp \rangle$ is defined by

$$X_\perp \triangleq X \cup \{\perp\}$$

and

$$x_0 \leq_\perp x_1 \text{ iff } x_0 = x_1 \text{ or } x_0 = \perp$$

□

Open sets. The open sets $\mathcal{O}$ of a topological space $\langle X, \mathcal{O} \rangle$ ordered by inclusion. □

Finite subsets. Given a set X let $[X]^{<\omega}$ denote its finite subsets. Let

$$\mathbf{X}_{\mathrm{fin}} \triangleq \langle [X]^{<\omega}, \subset \rangle$$

$x_0 \subset x_1$ is read 'x_1 *extends* x_0' or 'x_1 *refines* x_0'. □

Sequences. For α an ordinal and X a set, let X^α denote the set of all α-*sequences.* Set

$$X^{<\alpha} \triangleq \bigcup_{\alpha_0 \in \alpha} X^{\alpha_0}$$

Let s_0, s_1 be sequences from $X^{<\alpha}$. If $s_0 \subset s_1$, s_1 is said to *extend* s_0. Let $\mathbf{X}^{<\alpha}$ denote the set of all sequences in $X^{<\alpha}$, ordered by extension. □

$\mathbf{X}^{<\alpha}$ is a tree.

Trees. A partial order $\mathbf{P} = \langle P, \leq_P \rangle$ is a *tree* if for all $p_0 \in P$ the set

$$\{ p \in P \mid p \leq_P p_0 \}$$

is well ordered. A tree is *rooted* if it has a minimum. The supremum on the cardinalities of its chains is its *height*, the supremum of the cardinalities of its antichains is its *width*. Let $p_0, p_1 \in P$. Then p_1 is a *successor* of p_0 if

$$\{p_1\} = \{ p \mid p \leq p_1 \} \setminus \{ p \mid p \leq p_0 \}$$

$\mathbf{P}$ is α-*branching* if the set of successors of every point is of cardinality α. A two-branching tree is called *binary.*

Often we need to know more about a tree than its global width. $\omega_\perp$ has global width ω, yet for every $n_0 \in \omega$ the subtrees with underlying set $\{ n \mid n \geq_\perp n_0 \} = \{n_0\}$ is now only 1 wide. We shall therefore say that a tree $\mathbf{P} = \langle P, \leq_P \rangle$ is of *hereditary width* α if for all $p_0 \in P$ the subtree $\{ p \in P \mid p \geq_P p_0 \}$ is of width α.

In this parlance $\mathbf{2}^{<\omega}$ is a binary tree of height ω and hereditarily ω wide. □

Heyting algebras.

Definition 1.17 *Let* $\mathbf{\Omega} = \langle \Omega, \leq_\Omega \rangle$ *be a partial order.* $\mathbf{\Omega}$ *is a* complete Heyting algebra *(cHa) if every subset of* Ω *has a supremum and an infimum and* $\wedge$ *distributes over* $\bigvee$, *i.e. if whenever* $x_0 \in \Omega$, $X \subset \Omega$

$$x_0 \wedge (\bigvee X) = \bigvee \{ x_0 \wedge x \mid x \in X \}$$

Let 1 (or $\top$) and 0 (or $\perp$) denote the largest and least point in Ω. Define a new binary operation $\rightarrow$ on Ω by setting

$$x_0 \rightarrow x_1 \triangleq \bigvee \{ x \in \Omega \mid x \wedge x_0 \leq_\Omega x_1 \}$$

and a unary operation $\neg$ by

$$\neg x \triangleq x \to \bot$$

Similarly we can define the unary operation $\neg\neg$ on Ω by

$$\neg\neg x \triangleq \neg(\neg x)$$

Double negation is an instance of a *topology.*

Definition 1.18 *A* topology *is a unary operation* $j : \Omega \to \Omega$ *such that*

1. $j\top = \top$
2. $\forall x \in \Omega \; (jjx = jx)$
3. $\forall x_0, x_1 \in \Omega \; (j(x_0 \wedge x_1) = jx_0 \wedge jx_1)$

For any set X, its powerset $\mathcal{P}(X)$ with inclusion as order is clearly a cHa. A topology on $\mathcal{P}(X)$ is a monotone, idempotent and $\cap$-preserving function. □

We quickly review some of the constructs in a presheaf topos where now the underlying category is a partial order $\mathbf{P}$. First two helpful definitions.

Definition 1.19 *The* lower closure $\downarrow P_0$ *of a set* $P_0 \subset P$ *is defined by*

$$\downarrow P_0 \triangleq \{\, p \in P \mid \exists p_0 \in P_0 \; (p \leq_P p_0) \,\}$$

For $p \in P$ let $\downarrow p$ denote $\downarrow\{p\}$.

Definition 1.20 $P_0 \subset P$ *is* lower closed *if* $P_0 = \downarrow P_0$.

If $\mathbf{P}$ is seen as a category, the lower closed subsets of $\downarrow p$ stand in a 1-1 relation with the p-cribles.

Now let $F_0, F_1 \in \mathbf{Set}^{\mathbf{P}^{op}}$ and $p \in P$. Then $F_1^{F_0}(p)$ is simply the set of $\downarrow p$-indexed families of compatible functions. $\Omega \in \mathbf{Set}^{\mathbf{P}^{op}}$ maps $p \in P$ to the set of all lower closed subsets of $\downarrow p$. On morphisms $p_0 \leq_P p_1$ the functor Ω acts as follows. Pick $P_1 \in \Omega(p_1)$ then

$$\Omega(p_0 \leq_P p_1)(P_1) = P_1 \cap \downarrow p_0$$

It is time to look at an example.

Example (Continuous R-valued functions). Let $\langle X, \mathcal{O} \rangle$ be a topological space with open sets $\mathcal{O}$ ordered by inclusion. Define the presheaf R of real-valued functions over $\mathcal{O}$ by setting for $X_0 \in \mathcal{O}$

$$R(X_0) \triangleq \{\text{continuous real valued functions on } X_0\}$$

and for $X_0 \subset X_1$ and $f : X_1 \to \mathbf{R}$

$$R(X_0 \subset X_1)(f) = f|X_0$$

Here $f|X_0$ is the restriction of f to X_0 (c.f. Definition 2.17). □

Chapter 2

Constructive set theory

In this chapter we will give a brief presentation of that part of constructive set theory which we will use later on. For a general introduction to constructivism we refer to the comprehensive book by Troelstra and van Dalen [TvD88], which in its second volume also has a chapter on sets. The most complete bibliography of the constructive literature up to date is [Mul87].

2.1 The constructive predicate calculus IQC

Remark. **IQC** stands for 'Intuitionistic Quantified Calculus' (rather than 'Predicate') to distinguish it from the 'Intuitionistic Propositional Calculus'. We use 'Intuitionistic' and not 'Constructive' to avoid the classical 'C'. The names and much of the notation in this chapter are taken from [TvD88]. □

2.1.1 Syntax and axioms

The language $\mathcal{L}_{IQC}$ of **IQC** consists of an infinite supply of individual variables Vars, the logical symbols $\bot, \vee, \wedge, \rightarrow, \exists, \forall$, brackets $(,)$ and function and relation symbols $\{f_i^n\}_{i\in\omega}$ and $\{R_i^n\}_{i\in\omega}$ for every arity $n \in \omega$.

Let us say a set is defined *inductively* if it is the *smallest* set meeting some condition.

Thus define inductively the set of $\mathcal{L}_{IQC}$*-terms* $\mathcal{T}$ by

- Vars $\subset \mathcal{T}$
- if $\{t_0, \ldots, t_{n-1}\} \subset \mathcal{T}$ then $f_i^n(t_0, \ldots, t_{n-1}) \in \mathcal{T}$

Similarly, the set of *atomic formulas* $\mathcal{A}$ is the smallest set such that

- $\bot \in \mathcal{A}$
- if $\{t_0, \ldots, t_{n-1}\} \subset \mathcal{T}$ then $R_i^n(t_0, \ldots, t_{n-1}) \in \mathcal{A}$

Finally, the set of *formulas* $\mathcal{F}$ includes all atomic formulas and

- if $\{\phi, \psi\} \subset \mathcal{F}$ then $\{\phi \wedge \psi, \phi \vee \psi, \phi \to \psi\} \subset \mathcal{F}$
- if $x \in$ Vars and $\phi \in \mathcal{F}$ then $\{\forall x \phi, \exists x \phi\} \subset \mathcal{F}$

We abbreviate $\phi \to \bot$ to $\neg\, \phi$. Let $\top \overset{\Delta}{\leftrightarrow} \neg\, \bot$.

The usual notions of binding, open and closed formulas apply. If we write

$$\phi(x_0, \ldots, x_{n-1})$$

it is implied that the free variables of ϕ are a subset of $\{x_0, \ldots, x_{n-1}\}$. A *sentence* is a closed formula. *Constants* are function symbols of arity 0. $\phi(y/x)$ means that every free x in ϕ has been replaced by the new variable y.

Now let $\mathcal{S}$ be a set of sentences. The *consequence relation*

$$\vdash_{IQC} \ \subset\ [\mathcal{S}]^{<\omega} \times \mathcal{S}$$

will be given by a natural deduction system. For details see [TvD88]. Put

$$\langle \{\phi_0, \ldots, \phi_{n-1}\}, \phi_n \rangle \ \in \vdash_{IQC}$$

if ϕ_n can be deduced from $\phi_0, \ldots, \phi_{n-1}$ using the following rules:

$$\frac{\begin{array}{c}\Gamma\\ |\\ \bot\end{array}}{\phi}\ <\bot\mathrm{E}>$$

$$\frac{\begin{array}{c}\Gamma\\ |\\ \phi \wedge \psi\end{array}}{\phi}\ <\wedge\mathrm{E}> \qquad \frac{\begin{array}{c}\Gamma\\ |\\ \phi \wedge \psi\end{array}}{\psi}\ <\wedge\mathrm{E}> \qquad \frac{\begin{array}{cc}\Gamma_0 & \Gamma_1\\ | & |\\ \phi & \psi\end{array}}{\phi \wedge \psi}\ <\wedge\mathrm{I}>$$

$$\frac{\begin{array}{ccc}\Gamma_0 & \Gamma_1[\phi] & \Gamma_2[\psi]\\ | & | & |\\ \phi \vee \psi & \zeta & \zeta\end{array}}{\zeta}\ <\vee\mathrm{E}> \qquad \frac{\begin{array}{c}\Gamma\\ |\\ \phi\end{array}}{\phi \vee \psi}\ <\vee\mathrm{I}> \qquad \frac{\begin{array}{c}\Gamma\\ |\\ \psi\end{array}}{\phi \vee \psi}\ <\vee\mathrm{I}>$$

$$\frac{\begin{array}{c}\Gamma_0\\ |\\ \phi\to\psi\end{array}\quad\begin{array}{c}\Gamma_1\\ |\\ \phi\end{array}}{\psi}<\to\mathrm{E}> \qquad\qquad \frac{\begin{array}{c}\Gamma[\phi]\\ |\\ \psi\end{array}}{\phi\to\psi}<\to\mathrm{I}>$$

$$\frac{\begin{array}{c}\Gamma\\ |\\ \forall x\ \phi\end{array}}{\phi(t/x)}<\forall\mathrm{E}> \qquad\qquad \frac{\begin{array}{c}\Gamma\\ |\\ \phi(y/x)\end{array}}{\forall x\phi}<\forall\mathrm{I}>$$

$$\frac{\begin{array}{c}\Gamma_0\\ |\\ \exists x\phi\end{array}\quad\begin{array}{c}\Gamma_1[\phi(y/x)]\\ |\\ \psi\end{array}}{\psi}<\exists\mathrm{E}> \qquad\qquad \frac{\begin{array}{c}\Gamma\\ |\\ \phi(t/x)\end{array}}{\exists x\phi}<\exists\mathrm{I}>$$

$\Gamma[\phi]$ means that $\phi\in\Gamma$ can be discarded. The following restrictions are placed on these rules:

1. In $<\forall\mathrm{E}>$ and $<\exists\mathrm{I}>$ t must be free for x in ϕ, i.e. no free variables in t must get bound in ϕ when it is substitued for x.
2. In $<\forall\mathrm{I}>$ y must be new for Γ and ϕ.
3. In $<\exists\mathrm{E}>$ y must be new for $\Gamma_1\setminus\{\phi(y/x)\}$, ϕ and ψ.

$<\bot\mathrm{E}>$ is the *absurdity rule*: if the absurd follows from a set of assumptions, anything follows. In the course of a not quite formal proof we shall use $\lightning$ when we arrive at $\bot$, i.e. a contradiction.

Of sentences $\phi_0,\ldots,\phi_n$ we say that ϕ_n is a *consequence* of $\phi_0,\ldots,\phi_{n-1}$, or that ϕ_n can be *proved from* assumptions $\phi_0,\ldots,\phi_{n-1}$ if

$$\langle\{\phi_0,\ldots,\phi_{n-1}\},\phi_n\rangle\in\ \vdash_{IQC}$$

or 'infix'

$$\phi_0,\ldots,\phi_{n-1}\vdash_{IQC}\phi_n$$

Usually we omit the outmost universal quantifiers of a sentence.

For completeness we also add the following (standard) definitions: ϕ is a theorem of **IQC** if

$$\vdash_{IQC}\ \phi$$

A *theory* T is a set of sentences. The *closure* of T is the smallest set of sentences $\overline{T}$ such that

- $T \subset \overline{T}$
- $\{\phi_0, \ldots, \phi_{n-1}\} \subset \overline{T}$ and $\phi_0, \ldots, \phi_{n-1} \vdash_{IQC} \phi_n$ entail $\phi_n \in \overline{T}$

A set of sentences is a set of *axioms* for a theory if they have the same closure.

A theory T may only make use of part of the function and relation symbols of **IQC**. They will be referred to as the *language* $\mathcal{L}_T$ *of* T.

Note that if the rule 'Reductio Ad Absurdum'

$$\frac{\neg\neg\,\phi}{\phi} \;\text{<RAA>}$$

is added to **IQC** we get the classical predicate calculus **QC**.

Alternatively, we could add the 'Principle of the Excluded Middle'

$$\frac{}{\phi \vee \neg\,\phi} \;\text{<PEM>}$$

2.1.2 Two interpretations of IQC

Behind the first interpretation—the *Kripke* interpretation—lies the idea of the creative mathematician who never errs, why not call her P.? During her creative hours, P. can construct new objects and add them to her domain of discourse, or she can verify statements about objects that are already in that domain. A day in the life of P. might therefore be charted as a tree $\mathbf{T} = \langle T, \leq_T \rangle$ where $\leq_T$ is interpreted as temporal, and every point $t \in T$ has associated with it a domain D_t and a list of statements Φ_t that have been verified (or proved) at this moment. We say a sentence ϕ *is forced at* t if $\phi \in \Phi_t$. The fact that P. never makes mistakes is reflected in the requirement that

- $\forall t_0, t_1 \in T\ (t_0 \leq_T t_1 \rightarrow D_{t_0} \subset D_{t_1})$
- $\forall t_0, t_1 \in T\ (t_0 \leq_T t_1 \rightarrow \Phi_{t_0} \subset \Phi_{t_1})$

Logical operators are interpreted in the following way.

- $\perp$ is never forced.
- $\phi \wedge \psi$ is forced at t if both ϕ and ψ are forced at t.
- $\phi \vee \psi$ is forced at t if either ϕ or ψ is forced at t.
- $\phi \rightarrow \psi$ is forced at t_0 if for all $t_1 \geq_T t_0$ ψ is forced at t_1 whenever ϕ is forced at t_1.
- $\forall x\ \phi$ is forced at t_0 if whenever $t_1 \geq_T t_0$ and $d \in D_{t_1}$ then $\phi(d/x)$ is forced at t_1.
- $\exists x\ \phi$ is forced at t if there is a $d \in D_t$ such that $\phi(d/x)$ is forced at t.

In this interpretation <PEM> can fail. For **T** take **2**. Let ϕ be a formula forced only at 1. Then $\phi \vee \neg\, \phi$ is not forced at 0. If it were, either ϕ or $\neg\, \phi$ would be forced at 0. ϕ is not forced by assumption. So assume $\neg\, \phi$ is forced at 0. Then $\neg\, \phi$ is forced at 1, and by <→ E> $\perp$. For a strict definition we refer to section 2.2.7.

To illustrate the second interpretation consider for every natural number the statement $\phi(n)$

> $2n + 4$ is the sum of two primes.

The *Goldbach conjecture* is the statement

> $\phi(n)$ holds for all natural numbers n.

For any $n \in \omega$, we can decide with the help of a computer *in finite time* whether $\phi(n)$ or $\neg\, \phi(n)$. Yet it is undecided—on 30 January 1990—whether $\forall n \in \omega\ \phi(n)$ or $\neg\, \forall n \in \omega\ \phi(n)$.

One might wonder whether simpler relations might still behave classically. This is not the case, and an example to the contrary can easily be constructed: define for every $n \in \omega$ the natural number $G(n)$ by

$$\begin{array}{rcll} G(n) & \triangleq & 0 & \text{if } \phi(n) \\ G(n) & \triangleq & n & \text{if } \neg\, \phi(n) \end{array}$$

The relation $G(n) = 0$ is decidable on ω, yet we do not know whether

$$\forall n \in \omega\ (G(n) = 0)$$

or not.

The idea that a proof should be seen as an *algorithm* leads to the *realizability interpretation.* For details see McCarty's thesis [McC84].

Note that in the Kripke interpretation we *do* have

$$\text{Goldbach} \vee \neg\, \text{Goldbach}$$

2.1.3 Some constructive theorems

We shall give some constructively provable sentences. In general none of the implications below is reversible.

Lemma 2.1 *1.* $\phi \to \neg\neg\, \phi$

2. $(\phi \to \psi) \to (\neg\, \psi \to \neg\, \phi)$

3. $(\neg\, \phi \vee \neg\, \psi) \to \neg\, (\phi \wedge \psi)$

4. $\neg\, (\phi \vee \psi) \to (\neg\, \phi \wedge \neg\, \psi)$

5. $(\exists x \neg\, \phi) \to (\neg\, \forall x\, \phi)$

6. $(\forall x \neg\, \phi) \to (\neg\, \exists x\, \phi)$

Proof.

1. Assume ϕ and $\neg\, \phi$. By $<\to \mathrm{E}>$ we get $\perp$, and by $<\to \mathrm{I}>$ $\neg\neg\, \phi$ with $\neg\, \phi$ discharged.

2. Assume $\phi \to \psi$, ϕ and $\neg\, \psi$. From the first two we get ψ by $<\to \mathrm{E}>$, then by the same rule $\perp$ from the last assumption. We get $\neg\, \phi$ by $<\to \mathrm{I}>$ discharging ϕ and by the same rule the result, discharging $\neg\, \psi$.

3. Assume $\neg\, \phi \vee \neg\, \psi$, $\phi \wedge \psi$ and $\neg\, \phi$. From the second assumption we get ϕ by $<\wedge \mathrm{E}>$. Similarly, if we assume $\neg\, \psi$. Hence by $<\vee \mathrm{E}>$ $\perp$. Therefore, $\neg\, (\phi \wedge \psi)$ by $<\to \mathrm{I}>$.

4. Assume $\neg\, (\phi \vee \psi)$ and ϕ. Then $\phi \vee \psi$ by $<\vee \mathrm{I}>$. Therefore $\neg\, \phi$. In the same way we get $\neg\, \psi$. The result follows by $<\wedge \mathrm{I}>$.

5. Assume $\exists x \neg\, \phi$, $\neg\, \phi(y/x)$ and $\forall x\, \phi$. From the latter we get $\phi(y/x)$ by $<\forall \mathrm{E}>$, and $\perp$ by $<\to \mathrm{E}>$ and can discharge $\neg\, \phi(y/x)$ by $<\exists \mathrm{E}>$. We get the result by $<\to \mathrm{I}>$.

6. Assume $\forall x \neg \phi$ and $\exists x\ \phi$. Assume $\phi(y/x)$. Then $\neg \phi(y/x)$ by $<\forall\text{E}>$. We discharge $\phi(y/x)$ by $<\exists\text{E}>$. The result follows by $<\rightarrow \text{I}>$.

□

We add another list of sentences, omitting the (straightforward) proofs. We shall want to refer to it later.

Lemma 2.2 *1.* $\neg\neg (\phi \vee \neg \phi)$

2. $(\phi \wedge \psi) \vee \xi \leftrightarrow (\phi \vee \xi) \wedge (\psi \vee \xi)$

3. $(\phi \vee \psi) \wedge \xi \leftrightarrow (\phi \wedge \xi) \vee (\psi \wedge \xi)$

4. $(\phi \rightarrow \psi) \wedge (\phi \rightarrow \xi) \leftrightarrow (\phi \rightarrow \psi \wedge \xi)$

5. $\neg\neg (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \neg\neg \psi)$

6. $\neg\neg \forall x\ \phi \rightarrow \forall x \neg\neg \phi$

7. $\exists x \neg\neg \phi \rightarrow \neg\neg \exists x\ \phi$

2.2 Constructive Zermelo-Fraenkel set theory

In this section we shall present the constructive version of Zermelo-Fraenkel set theory **IZF** and define a class of models for it. For a concise introduction to classical set theory and a list of the axioms of **ZF** we refer to Kunen's book [Kun80]. The material about **IZF** is widely scattered; the most complete account can again be found in [TvD88]. Another source is [McC84].

2.2.1 The system IZF

The language of **IZF** has two binary relation symbols $\in$ and $=$ and an unspecified supply C of constant symbols. We write $x_0 \neq x_1$ for $\neg\, x_0 = x_1$ and $x \notin X$ for $\neg\, x \in X$. Also, $\forall x \in X$ and $\exists x \in X$ stand for $\forall x\, (x \in X \rightarrow \ldots)$ and $\exists x\, (x \in X \wedge \ldots)$, respectively. A formula is *restricted* if it is provably equivalent to a formula in which all quantifiers are of the form $\forall x \in X$ or $\exists x \in X$. For readability's sake, we use a 'typed' notation, sets appearing to the left of $\in$ (elements) will be denoted by small, those to the right (sets) by capital letters, and those one level up (families of sets) in caligraphy style. Of course this notation occasionally breaks down.

IZF is a first order theory with equality, i.e. the following axioms hold.

1. $\exists x\ (x = x)$ <Existence>
2. $x = x$ <Reflexivity>
3. $x_0 = x_1 \rightarrow x_1 = x_0$ <Symmetry>
4. $(x_0 = x_1 \wedge x_1 = x_2) \rightarrow x_0 = x_2$ <Transitivity>
5. $(x_0 = x_1 \wedge x_0 \in X) \rightarrow x_1 \in X$
6. $(X_0 = X_1 \wedge x \in X_0) \rightarrow x \in X_1$

These five axioms will be referred to as **EQ**.

Lemma 2.3 *For ϕ a formula of* **IZF** *and x_1 free for x_0 in ϕ*

$$\mathbf{EQ} \vdash_{IQC} x_0 = x_1 \rightarrow (\phi \leftrightarrow \phi(x_1/x_0))$$

Proof. By structural induction. □

Here now is the list of axioms that are specific to **IZF**. We shall make use of the usual abbreviations. If x is a set let $\{x\}$ denote the *singleton* containing x as its only point. It exists by <Pairing> and <Separation>. If X is a set and ϕ a formula of **IZF**, there exists a unique subset X_0 of X by <Separation>, such that

$$x_0 \in X_0 \leftrightarrow x_0 \in X \wedge \phi(x_0/x)$$

This will be denoted by

$$\{\ x \in X \mid \phi\ \}$$

Next, for x_0 a set, let $\{\ x_0 \mid \phi\ \}$ denote

$$\{\ x \in \{x_0\} \mid \phi\ \}$$

$\emptyset$ is the set $\{\ x \mid \bot\ \}$, which exists by <Existence> and <Separation>. x^+ is $x \cup \{x\}$, the 'successor' of x. $X_0 \subset X_1$ means

$$\forall x \in X_0\ (x \in X_1)$$

1. $\forall x\,(x \in X_0 \leftrightarrow x \in X_1) \rightarrow X_0 = X_1$ <Extensionality>
2. $\exists X\,(x_0 \in X \wedge x_1 \in X)$ <Pairing>
3. $\exists x_0\,\forall x_1\,(x_1 \in X \rightarrow x_1 \subset x_0)$ <Union>
4. $\exists X_0\,\forall x\,(x \in X_0 \leftrightarrow (x \in X_1 \wedge \phi))$ <Separation>
5. $\exists X\,\forall x_0\,(x_0 \subset x_1 \rightarrow x_0 \in X)$ <Powerset>
6. $\exists X\,(\emptyset \in X \wedge \forall x \in X\,(x^+ \in X))$ <Infinity>
7. $(\forall x_0 \in X_0\,\exists x_1\,\phi) \rightarrow \exists X_1\,\forall x_0 \in X_0\,\exists x_1 \in X_1\,\phi$ <Collection>
8. $\forall X\,(\forall x \in X\,\phi(x/X) \rightarrow \phi(X)) \rightarrow \forall x\,\phi(x/X)$ <Induction>

<Separation>, <Collection> and <Induction> are actually lists of axioms, namely one for every formula ϕ.

We use the normal representation of the natural numbers in **IZF**, i.e. $0 = \emptyset$ and $n + 1 = n^+$. The *set* of natural numbers, which exists by <Infinity> and <Separation>, will be denoted ω. Finally, put $\Omega \triangleq \mathcal{P}(1)$.

Before turning to a discussion of **IZF**, we fix some definitions. Since **IZF** is strictly weaker than **ZF**, many classically equivalent notions break down into different constructive ones, and some care has to be taken to find the most appropriate.

A first example of this kind is the notion of being inhabited.

Definition 2.4 *A set X is* nonempty *if $X \neq \emptyset$. It is* inhabited *if*

$$\exists x\,(x \in X)$$

Note that $X \neq \emptyset \leftrightarrow \neg\neg\,\exists x\,(x \in X)$.

2.2.2 Decidability

Properties that behave classically deserve a name. We have

Definition 2.5 *Let $\phi(\vec{x})$ be a formula of* **IZF** *and X a set. Then ϕ is* decidable *on X if*

$$\forall \vec{x} \in X\,(\phi \vee \neg\,\phi)$$

By separation formulas determine subsets of sets. Hence the following

Definition 2.6 *Let $X_0 \subset X$. Then X_0 is a* decidable subset *of X if*

$$\forall x \in X \ (x \in X_0 \lor x \notin X_0)$$

Since we shall be most concerned with *equality* on sets, we add one more definition for this special case.

Definition 2.7 *A set is* discrete *if equality is decidable on it.*

Decidable formulas have nice 'closure' properties. We have

Lemma 2.8 *Given decidable properties $\phi(\vec{x}), \psi(\vec{x})$ on a set X, each of the following is also decidable on X.*

1. $\phi \land \psi$

2. $\phi \lor \psi$

3. $\phi \rightarrow \psi$

Also

4. $\bot$ *is decidable, and therefore also* $\neg\, \phi$.

Proof.

1. Assume ϕ and ψ are decidable on X. Pick $x'_i \in X$. Instantiate to

$$\phi(x'_i/x_i) \lor \neg\, \phi(x'_i/x_i)$$

and

$$\psi(x'_i/x_i) \lor \neg\, \psi(x'_i/x_i)$$

By Lemma 2.2.3 we have either of

- $\phi(x'_i/x_i) \land \psi(x'_i/x_i)$
- $\phi(x'_i/x_i) \land \neg\, \psi(x'_i/x_i)$
- $\neg\, \phi(x'_i/x_i) \land \psi(x'_i/x_i)$
- $\neg\, \phi(x'_i/x_i) \land \neg\, \psi(x'_i/x_i)$

The latter three imply $\neg\, (\phi(x'_i/x_i) \land \psi(x'_i/x_i))$, so always

$$(\phi(x'_i/x_i) \land \psi(x'_i/x_i)) \lor \neg\, (\phi(x'_i/x_i) \land \psi(x'_i/x_i))$$

By $<\forall \text{I}>$ we obtain the result.

2. Similar to 1. The first three cases in 1. imply $\phi(x_i'/x_i) \vee \psi(x_i'/x_i)$, the last $\neg\, (\phi(x_i'/x_i) \vee \psi(x_i'/x_i))$, so again

$$(\phi(x_i'/x_i) \vee \psi(x_i'/x_i)) \vee \neg\, (\phi(x_i'/x_i) \vee \psi(x_i'/x_i))$$

3. Again glancing at 1., we have

$$\neg\, (\phi(x_i'/x_i) \rightarrow \psi(x_i'/x_i))$$

if

$$\phi(x_i'/x_i) \wedge \neg\, \psi(x_i'/x_i)$$

and

$$\phi(x_i'/x_i) \rightarrow \psi(x_i'/x_i)$$

in the other cases. Hence always

$$(\phi(x_i'/x_i) \rightarrow \psi(x_i'/x_i)) \vee \neg\, (\phi(x_i'/x_i) \rightarrow \psi(x_i'/x_i))$$

The result follows by <∀I>.

4. follows from 3.

□

2.2.3 Stability and j-properties

A somewhat weaker property than decidability is stability.

Definition 2.9 *Let $\phi(\vec{x})$ be a formula of* **IZF** *and X a set. Then ϕ is* stable *on X if*

$$\forall \vec{x} \in X\ (\neg\neg\, \phi \rightarrow \phi)$$

Definition 2.10 *Let $X_0 \subset X$. Then X_0 is a* stable subset *of X if*

$$\forall x \in X\ (\neg\neg\, x \in X_0 \rightarrow x \in X_0)$$

We want to be more general. Let $j\ :\ \mathcal{P}(1) \rightarrow \mathcal{P}(1)$ be a topology. Define a unary operator j (same name and also called 'topology'!) on the formulas of **IZF** by

$$j(\phi) \stackrel{\Delta}{\leftrightarrow} 0 \in j(\{\ 0 \mid \phi\ \})$$

The proof of the following lemma is easy.

Lemma 2.11 *1.* $j\top \leftrightarrow \top$

2. For all formulas of **IZF** ϕ *we have*

$$jj\phi \leftrightarrow j\phi$$

3. For all formulas of **IZF** ϕ *and* ψ *we have*

$$j(\phi \wedge \psi) \leftrightarrow j\phi \wedge j\psi$$

We can introduce some useful concepts.

Definition 2.12 *1.* $f : X \hookrightarrow Y$ *is* j-dense *in* Y *if*

$$\forall y \in Y\ (j(y \in f(X)))$$

2. Let $f : X \hookrightarrow Y$*. The* j-closure *of* X *in* Y *is the set*

$$X^j \triangleq \{\, y \in Y \mid j(y \in f(X))\,\}$$

3. X *is* j-closed *if* $X = X^j$.

So a stable subset is the same as a $\neg\neg$-closed subset.

Definition 2.13 *A set* X *is* j-separated *if*

$$\forall x_0, x_1 \in X\ (j(x_0 = x_1) \rightarrow x_0 = x_1)$$

For a discussion of these definitions we refer to de Vries' thesis [dV89]. We should also mention that the concepts here defined in set theory have first been described 'externally' in the language of category theory. The same is true of the notion of *sheaf* later on. For the categorical definitions see [Joh77].

2.2.4 Functions

Definition 2.14 *Let* X, Y *be sets. A* partial function f *from* X *to* Y*, denoted by* $f : X \rightharpoonup Y$*, is a subset of* $X \times Y$ *such that*

$$(\langle x_0, y_0\rangle \in f \wedge \langle x_1, y_1\rangle \in f \wedge x_0 = x_1) \rightarrow y_0 = y_1 \quad \text{<Functionality>}$$

$\langle x_0, y_0\rangle \in f$ *is written* $f(x_0) = y_0$ *or* $fx_0 = y_0$. f *is* defined *at* $x \in X$ *if*

$$\exists y \in Y\ (fx = y)$$

This will be denoted by $fx\downarrow$ *and* $\neg\, fx\downarrow$ *by* $fx\uparrow$. *The* domain *of* f *is*

$$\text{dom}f \triangleq \{\, x \in X \mid fx\downarrow \,\}$$

The range $\text{ran}f$ *of* f *is* Y. *The* image *of* f *is*

$$\text{im}f \triangleq \{\, y \in Y \mid \exists x \in X\ (fx = y)\,\}$$

f *is a* (total) function *if* $\text{dom}f = X$. *This will be written as* $f : X \to Y$.

Hence $f \subset X \times Y$ is a function if

$$\forall x \in X\ \exists ! y \in Y\ (fx = y)$$

Here we made use of the abbreviation

$$\exists ! y \in Y\ \phi$$

for

$$\exists y \in Y\ (\phi \wedge \forall y_0 \in Y\ (\phi(y_0/y) \to y_0 = y))$$

We shall denote the space of functions $f : X \to Y$ by Y^X or $X \Rightarrow Y$.

Definition 2.15 *Let* $f : X \to Y$

- f *is* injective *(*$f : X \hookrightarrow Y$*) if*

$$\forall x_0, x_1 \in X\ (fx_0 = fx_1 \to x_0 = x_1)$$

- f *is* surjective *(*$f : X \twoheadrightarrow Y$*) if*

$$\forall y \in Y\ \exists x \in X\ (fx = y)$$

- f *is* bijective *(*$f : X \hookrightarrow\!\!\!\!\!\rightarrow Y$*) if it is both injective and surjective.*

The definition of an injection is an example of how alternative formulations of a classical definition yield different constructive concepts. We can weaken the definition to

$$\forall x_0, x_1 \in X\ (fx_0 = fx_1 \to \neg\neg\, x_0 = x_1)$$

or strengthen it to

$$\forall x_0, x_1 \in X\ (\neg\neg\, fx_0 = fx_1 \to x_0 = x_1)$$

and talk of *weakly* and *strongly injective* functions.

Bijections preserve certain properties of sets. As an example we prove

Lemma 2.16 *The image of a discrete set under a bijection is discrete.*

Proof. Take $f : X \hookrightarrow\!\!\!\!\rightarrow Y$ where X is discrete. Pick $y_0', y_1' \in f(X)$. Then

$$y_0' = y_1' \leftrightarrow f^{-1}(y_0') = f^{-1}(y_1')$$

and since equality is decidable on X, so it is on $f(X)$. □

2.2.5 Restrictions and extensions

Now we arrive at the definition of an innocuous looking (through classical eyes!) concept which will later exhibit a very unclassical behaviour. First,

Definition 2.17 *Let $X_0 \subset X_1$ and $f : X_1 \to Y$. The* restriction *of f to X_0 is*

$$f|X_0 \triangleq f \cap (X_0 \times Y)$$

Conversely, we have

Definition 2.18 *Assume $X_0 \subset X_1, Y_0 \subset Y_1$ and $f_0 : X_0 \to Y_0$. Then the function $f_1 : X_1 \to Y_1$* extends *f_0 if*

$$f_0 \subset f_1$$

f_0 has an extension in *$X_1 \Rightarrow Y_1$ if there is a $f_1 : X_1 \to Y_1$ extending f_0.*

Note that f_1 extends f_0 if

$$\forall x \in X_1\ (x \in X_0 \to f_0x = f_1x)$$

and that $f : X_1 \to Y_1$ is an extension of a function in $X_0 \Rightarrow Y_0$ (namely $f \cap (X_0 \times Y_0)$) if

$$f(X_0) \subset Y_0$$

In classical theory, if $X_0 \subset X_1$ and Y is not empty, then all functions $f : X_0 \to Y$ have extensions in $X_1 \Rightarrow Y$ and they may have many different ones. This contrasts with the situation in constructive set theory where functions may have no or only one extension. Therefore the following definition will prove useful.

Definition 2.19 *Let $X_0 \subset X_1$. Then $f : X_1 \to Y$ is* determined by its values on *X_0 if it is the unique extension of $f|X_0$.*

Note that in Definition 2.18 we talk of f having an extension *in* $X_1 \Rightarrow Y_1$ rather than f having an extension *to* X_1. The distinction is important: while there may not be enough points in Y_0 to map the elements in X_1 to, there might well be sufficiently many in Y_1.

Definition 2.20 *Let $X_0 \subset X_1, Y_0 \subset Y_1$ be sets. Put*

$$\text{Ext}(X_0 \subset X_1, Y_0 \subset Y_1) \triangleq \{ f : X_1 \to Y_1 \mid fX_0 \subset Y_0 \}$$

In order to get used to the notions introduced above and for future reference we prove the following

Lemma 2.21 *Let X_0 be a subset of X. Then $f : X \to Y$ is determined by its values on X_0 if*

1. $\forall x \in X\ (\neg\neg\ x \in X_0)$.
2. *equality is stable on Y.*

Proof. Let $f'_0, f'_1 : X \to Y$ be such that $f'_0|X_0 = f'_1|X_0$, i.e.

$$\forall x \in X\ (x \in X_0 \to f'_0 x = f'_1 x)$$

Take $x' \in X$. We have $\neg\neg\ x' \in X_0$, hence $\neg\neg\ f'_0 x' = f'_1 x'$, so $f'_0 x' = f'_1 x'$. □

We shall revise more definitions later. Now let us turn to a brief comparison of **IZF** with **ZF**.

2.2.6 IZF vs. ZF

When trying to set up an axiom system for constructive set theory, one might be tempted to use **ZF** and reason in **IQC**. However, it turns out that some of the axioms in **ZF** are too strong, at least in the way they are usually stated.

Foundation

Looking at **IZF** one quickly notices the absence of the Axiom of Foundation

$$\exists x\ (x \in X) \to \exists x \in X\ (x \cap X = \emptyset) \quad \text{<Foundation>}$$

While <Foundation> may not be of relevance to the development of most of classical set theory, its absence becomes a necessity constructively.

Lemma 2.22

$$\mathbf{IZF} + \text{<Foundation>} \vdash_{IQC} \text{<RAA>}$$

Proof. For ϕ a formula of **IZF** consider

$$X = \{ 0 \mid \phi \} \cup \{1\}$$

Assume <Foundation> and $\neg\neg\,\phi$. X is inhabited, hence we have

$$\exists x\, (x \in X \wedge x \cap X = \emptyset)$$

The only candidates for this are 0 and 1. 1 is ruled out since $\neg\neg\,\phi$ implies

$$\neg\neg\, 0 \in (1 \cap X)$$

Therefore

$$0 \in X$$

and ϕ. □

<Foundation> has in fact been replaced by the classically equivalent, but constructively weaker <Induction>. Grayson explains in [Gra75] what can still be proved.

Replacement

The Axiom of Replacement in **ZF**

$$(\forall x_0 \in X_0\ \exists! x_1\ \phi) \rightarrow \exists X_1\ \forall x_0 \in X_0\ \exists x_1 \in X_1\ \phi \quad \text{<Replacement>}$$

has in **IZF** been strengthened to <Collection>. Friedman and Scedrov show in [FS83] that this strengthening is strict.

Axiom of Choice

Next we consider the Axiom of Choice. As one might by now expect, many of its versions are no longer provably equivalent in **IZF**. The following formulation is too strong.

$$(\forall X \in \mathcal{X}\ \exists x (x \in X)) \rightarrow \exists f : \mathcal{X} \rightarrow \bigcup \mathcal{X}\ (\forall X \in \mathcal{X}\ (fX \in X)) \quad \text{<AC>}$$

We have

Lemma 2.23 ([Dia85])

$$\mathbf{IZF} + \langle \mathrm{AC} \rangle \vdash_{IQC} \langle \mathrm{RAA} \rangle$$

Proof. Assume <AC> and $\neg\neg\,\phi$. Let

$$X_0 = \{0\} \cup \{\, 1 \mid \phi \,\}$$

and

$$X_1 = \{\, 0 \mid \phi \,\} \cup \{1\}$$

Put $\mathcal{X} = \{X_0, X_1\}$. By <AC> there is a function

$$f \,:\, \mathcal{X} \to \bigcup \mathcal{X} = \{0,1\}$$

such that

$$f(X_i) \in X_i \quad \text{for } i = 0,1$$

$\neg\neg\,\phi$ implies

$$\neg\neg\, X_0 = X_1$$

hence

$$\neg\neg\, f(X_0) = f(X_1)$$

Equality is decidable on $\{0,1\}$, so either

$$f(X_0) = f(X_1) = 0$$

or

$$f(X_0) = f(X_1) = 1$$

If the former is true,

$$0 \in \{\, 0 \mid \phi \,\} \cup \{1\}$$

hence ϕ. In the other case

$$1 \in \{0\} \cup \{\, 1 \mid \phi \,\}$$

and again ϕ. □

<AC> has another formulation, which is equivalent even constructively.

Definition 2.24 *Let X, Y be sets. <AC-XY> is the following statement.*

For all **IZF**-*formulas ϕ the following sentence holds.*

$$\forall x \in X\ \exists y \in Y\ \phi(x,y) \to \exists f : X \to Y\ (\forall x \in X\ \phi(x, f(x)))$$

With this we have

Definition 2.25 <AC> *is the following statement.*

For all sets X, Y <AC-XY>

Note that while in a model the Axiom of Choice might fail, particular instances will hold.

It is high time to introduce some models of **IZF** and produce examples where classical logic fails.

2.2.7 A class of models of IZF

We shall now associate with every every category **C** a model $\mathbf{V}^{\mathbf{C}}$ of **IZF**. First we define the domain V by building a set theoretic hierarchy inside $\mathbf{Set}^{\mathbf{C}^{op}}$. We first recall the classical construction by von Neumann.

Define by induction on **On** the following sequence in **Set**.

Definition 2.26

$$
\begin{aligned}
\mathbf{V}_0 &\triangleq \emptyset \\
\mathbf{V}_{\alpha+1} &\triangleq \mathcal{P}(\mathbf{V}_\alpha) \\
\mathbf{V}_\alpha &\triangleq \bigcup_{\alpha_0 \in \alpha} \mathbf{V}_{\alpha_0} \quad \textit{for } \alpha \textit{ a limit}
\end{aligned}
$$

Write $v \in \mathbf{V}$ if

$$\exists \alpha \in \mathbf{On}\ (v \in \mathbf{V}_\alpha)$$

For an interpretation of **ZF** in **V** we refer to Kunen [Kun80]. From the same source we cite the following result.

Lemma 2.27

$$\mathbf{V} \models \mathbf{ZF}$$

Recall that **V** is in fact a *transitive* model of **ZF**, i.e. we have

$$v \in \mathbf{V} \to v \subset \mathbf{V}$$

We now mimic this procedure and define by induction on **On** a chain of functors V_α in $\mathbf{Set}^{\mathbf{C}^{op}}$. This will be an instance of the general method of defining a universe and an interpretation of **IZF** in a topos, which can be found in Fourman's paper [Fou80].

Definition 2.28

$$\begin{array}{rcl} V_0 & \triangleq & \emptyset(-) \\ V_{\alpha+1} & \triangleq & \mathcal{P}(V_\alpha) \\ V_\alpha & \triangleq & \mathrm{colim}_{\alpha_0 \in \alpha} V_{\alpha_0} \quad \textit{for } \alpha \textit{ a limit} \end{array}$$

Lemma 2.29

$$\forall \alpha_0, \alpha_1 \in \mathbf{On}\ (\alpha_0 \subset \alpha_1 \rightarrow V_{\alpha_0} \subset V_{\alpha_1})$$

Proof. Let $\phi(\alpha)$ be the formula $\forall \alpha_0 \in \mathbf{On}\ (\alpha_0 \subset \alpha \rightarrow V_{\alpha_0} \subset V_\alpha)$. We prove $\forall \alpha \in \mathbf{On}\ \phi$ by induction. The cases $\phi(0)$ and $\phi(1)$ are obvious. Now assume that $\forall \alpha_0 \in \alpha\ \phi(\alpha_0)$. We prove $\phi(\alpha)$ by case inspection.

1. If α is a limit, $\phi(\alpha)$ holds by Lemma 1.13.

2. Let $\alpha \geq 2$ be the successor of α^-. Assume $\phi(\alpha^-)$. First, let us show that for all $C \in \mathbf{C}$

$$V_{\alpha^-}(C) \subset V_\alpha(C)$$

Take $C \in \mathbf{C}$ and $v \in V_{\alpha^-}(C)$. It is of the form

$$\{X_{f_0} \subset V_{\alpha_0}(C_0)\}_{f_0 : C_0 \rightarrow C}$$

for some $\alpha_0 \in \alpha^-$. By hypothesis we have for all $f_0 : C_0 \rightarrow C$

$$X_{f_0} \subset V_{\alpha^-}(C_0)$$

so

$$v \in \mathcal{P}(V_{\alpha^-})(C) = V_\alpha(C)$$

Next, we prove that for all morphisms $f : C_0 \rightarrow C_1$

$$V_{\alpha^-}(f) = V_\alpha(f) | V_{\alpha^-}(C_1)$$

To see this, pick $v \in V_{\alpha^-}(C_1)$. v is of the form

$$\{X_{f_2} \subset V_{\alpha_0}(C_2)\}_{f_2 : C_2 \rightarrow C_1}$$

for some $\alpha_0 \in \alpha^-$. We have by induction hypothesis

$$\begin{array}{l} V_{\alpha^-}(f)(v) = \\ \quad \mathcal{P}(V_{\alpha_0})(f)(v) = \\ \quad \{X_{f \circ f_2}\}_{f_2 : C_2 \rightarrow C_0} \end{array}$$

but also

$$\begin{aligned}&V_\alpha(f)(v) = \\ &\quad \mathcal{P}(V_{\alpha^-})(f)(v) = \\ &\quad \{X_{f \circ f_2}\}_{f_2 : C_2 \to C_0}\end{aligned}$$

□

It emerges that for all limit ordinals α

$$V_\alpha = \mathrm{colim}_{\alpha_0 \in \alpha} V_{\alpha_0} = \bigcup_{\alpha_0 \in \alpha} V_{\alpha_0}$$

Lemma 2.29 permits the introduction of the following convention.

Convention. We write $v \in V(C)$ for

$$\exists \alpha \in \mathbf{On}\ (v \in V_\alpha(C))$$

The generalization to more complex contexts is straightforward.

We turn to the interpretation of **IZF** in $\mathbf{V}^{\mathbf{C}}$. Set

$$\mathbf{V}^{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{op}}(1, V)$$

Constants c of $\mathcal{L}_{IZF}$ are interpreted as global elements

$$c \in \mathbf{V}^{\mathbf{C}}$$

Conversely, we talk about the members of $\mathbf{V}^{\mathbf{C}}$ in the language of set theory.

Convention. We shall write $c(C)$ instead of the cumbersome $c(C)(0)$.

The interpretation of *formulas* ϕ is defined by a *forcing relation* $\Vdash_A$ between 'stages' $C \in \mathbf{C}$ and ϕ and with respect to an assignment A, i.e. for all x free in ϕ

$$A(x) \in V(C)$$

If A is an assignment of the free variables in ϕ at C, then $A|f$ denotes its *restriction* along $f : C_0 \to C$, i.e. for all x free in ϕ

$$(A|f)(x) = V(f)(A(x))$$

Also, if A is an assignment of the free variables in ϕ at C and $v \in V(C)$, then $A\{v/x\}$ is A with the value of x replaced by v.

Define inductively on the structure of ϕ

$$C \Vdash_{\mathrm{A}} x_0 \in x_1 \quad \text{iff} \quad \mathrm{A}(x_0) \in (\mathrm{A}(x_1))_{\mathrm{id_C}}$$
$$C \Vdash_{\mathrm{A}} c_0 \in c_1 \quad \text{iff} \quad c_0(C) \in (c_1(C))_{\mathrm{id_C}}$$

similarly for the cases $x \in c$ and $c \in x$

$$C \Vdash_{\mathrm{A}} x_0 = x_1 \quad \text{iff} \quad \mathrm{A}(x_0) = \mathrm{A}(x_1)$$
$$C \Vdash_{\mathrm{A}} c_0 = c_1 \quad \text{iff} \quad c_0(C) = c_1(C)$$

similarly for $x = c$

$$C \Vdash_{\mathrm{A}} \bot \quad \text{never}$$
$$C \Vdash_{\mathrm{A}} \phi \wedge \psi \quad \text{iff} \quad C \Vdash_{\mathrm{A}} \phi \text{ and } C \Vdash_{\mathrm{A}} \psi$$
$$C \Vdash_{\mathrm{A}} \phi \vee \psi \quad \text{iff} \quad C \Vdash_{\mathrm{A}} \phi \text{ or } C \Vdash_{A} \psi$$
$$C \Vdash_{\mathrm{A}} \phi \to \psi \quad \text{iff} \quad \forall f_0 : C_0 \to C\ (C_0 \Vdash_{\mathrm{A}|f_0} \phi \to C_0 \Vdash_{\mathrm{A}|f_0} \psi)$$
$$C \Vdash_{\mathrm{A}} \exists x\, \phi \quad \text{iff} \quad \exists v \in V(C)\ (C \Vdash_{\mathrm{A}\{v/x\}} \phi)$$
$$C \Vdash_{\mathrm{A}} \forall x\, \phi \quad \text{iff} \quad \forall f_0 : C_0 \to C\ \forall v \in V(C_0)\ (C_0 \Vdash_{(\mathrm{A}|f_0)\{v/x\}} \phi)$$

We write

$$\mathbf{V}^{\mathbf{C}} \models \phi$$

if $C \Vdash_A \phi$ for every $C \in \mathbf{C}$ and assignment A. We have

Theorem 2.30 *For all categories* $\mathbf{C}$

$$\mathbf{V}^{\mathbf{C}} \models \mathbf{IZF}$$

Proof. See [Fou80]. □

Note on double negation. Let $C \in \mathbf{C}$. There is a natural preorder $\preceq_C$ on the arrows C in $\mathbf{C}$ with codomain C defined by

$$f_0 \preceq_C f_1 \overset{\Delta}{\Leftrightarrow} \exists f_2\, (f_1 = f_0 \circ f_2) \tag{2.1}$$

Set also

$$f_0 \prec_C f_1 \overset{\Delta}{\Leftrightarrow} f_0 \preceq_C f_1 \wedge f_0 \neq f_1$$

We have

$$C \Vdash_A \neg\neg\, \phi$$

if the set $\mathcal{C}_0$ of morphisms $f_0 : C_0 \to C$ with

$$C_0 \Vdash_{A|f_0} \phi$$

is *cofinal* in $\mathcal{C}$, i.e.

$$\forall f \in \mathcal{C}\, \exists f_0 \in \mathcal{C}_0\, (f \preceq_C f_0)$$

Therefore we shall say that *cofinally* ϕ if $\neg\neg\,\phi$, rather than the clumsy *not not* ϕ. □

2.2.8 Heyting algebras and forcing

We shall indicate how 'forcing' is related to Heyting algebras.

Let **C** be a category. For $C \in \mathbf{C}$ let $\Omega(C)$ be the set of C-cribles. Clearly $\Omega(C)$ is a cHa. Take a (constant free) formula $\phi(x_0, \ldots, x_{n-1})$ of **IZF**. We can define its truth value at C

$$[\![\phi]\!](C) \;:\; V^n(C) \to \Omega(C)$$

by setting

$$[\![\phi]\!](C)(v_0, \ldots, v_{n-1}) \triangleq \{\, f_0 : C_0 \to C \mid C_0 \Vdash_{\{v_i/x_i\}|f_0} \phi \,\}$$

$\Vdash$ was defined in such a way that we have

$$\begin{aligned}
[\![\bot]\!](C)(\vec{v}) &= \emptyset \\
[\![\phi \wedge \psi]\!](C)(\vec{v}) &= [\![\phi]\!](C)(\vec{v}) \wedge [\![\psi]\!](C)(\vec{v}) \\
[\![\phi \vee \psi]\!](C)(\vec{v}) &= [\![\phi]\!](C)(\vec{v}) \vee [\![\psi]\!](C)(\vec{v}) \\
[\![\phi \to \psi]\!](C)(\vec{v}) &= [\![\phi]\!](C)(\vec{v}) \to [\![\psi]\!](C)(\vec{v}) \\
[\![\forall x\, \phi]\!](C)(\vec{v}) &= \bigwedge_{f_0 : C_0 \to C} \Big(\bigwedge_{v \in V(C_0)} [\![\phi]\!](C_0)(v/x, \vec{v}|f_0) \Big) \\
[\![\exists x\, \phi]\!](C)(\vec{v}) &= \bigvee_{v \in V(C)} [\![\phi]\!](C)(v/x, \vec{v})
\end{aligned}$$

2.2.9 Embedding V in $\mathbf{V}^{\mathbf{C}}$

We are now going to indicate how the classical universe **V** can be regarded as isomorphic to a submodel of $\mathbf{V}^{\mathbf{C}}$. Theorem 2.32 has been copied from [Bel77] and its proof amended to suit our needs.

We define for every $v \in \mathbf{V}$ and $C \in \mathbf{C}$

$$v^*(C) \triangleq \{\{\, v_0^*(C_0) \mid v_0 \in v \,\}_{f_0}\}_{f_0 : C_0 \to C}$$

We prove

Lemma 2.31 *For all $v \in \mathbf{V}$*

$$v^* \in \mathbf{V}^{\mathbf{C}}$$

Proof. By induction on the membership relation on $\mathbf{V}$.

- For the base case pick $C \in \mathbf{C}$. Then

$$\emptyset^*(C) = \{\emptyset_{f_0}\}_{f_0 : C_0 \to C} \in \mathcal{P}(\emptyset)(C) \subset V(C)$$

To establish naturality we calculate for $f : C_0 \to C_1$

$$\begin{aligned} & V(f)(\emptyset^*(C_1)) \\ & \quad V(f)\{\emptyset_{f_1}\}_{f_1 : C \to C_1} = \\ & \quad \{\emptyset_{f \circ f_0}\}_{f_0 : C \to C_0} = \\ & \quad \emptyset^*(C_0) \end{aligned}$$

- For the induction step assume that

$$\forall v_0 \in v_1 \, (v_0^* \in \mathbf{V}^{\mathbf{C}})$$

We have at $C \in \mathbf{C}$

$$v_1^*(C) = \{\{\, v_0^*(C_0) \mid v_0 \in v_1 \,\}_{f_0}\}_{f_0 : C_0 \to C}$$

First, for all $C_0 \in \mathbf{C}$ by hypothesis

$$\{\, v_0^*(C_0) \mid v_0 \in v_1 \,\} \subset V(C_0)$$

Also, for $f : C_0 \to C_1$ by hypothesis

$$\begin{aligned} & V(f)\{\, v_0^*(C_1) \mid v_0 \in v_1 \,\} = \\ & \quad \{\, v_0^*(C_0) \mid v_0 \in v_1 \,\} \end{aligned}$$

therefore

$$v_1^*(C) \in V(C)$$

Again we have for $f : C_0 \to C_1$

$$\begin{aligned} & V(f)(v_1^*(C_1)) = \\ & \quad V(f)\{\{ v_0^*(C) \mid v_0 \in v_1 \}_{f_1}\}_{f_1:C\to C_1} = \\ & \quad \{\{ v_0^*(C) \mid v_0 \in v_1 \}_{f \circ f_0}\}_{f_0:C\to C_0} = \\ & \quad v_1^*(C_0) \end{aligned}$$

□

Members of $\mathbf{V}^{\mathbf{C}}$ of the form v^* will be called *standard.*

We have

Theorem 2.32 *1. For* $v_0, v_1 \in \mathbf{V}$

$$v_0 \in v_1 \text{ iff } \mathbf{V}^{\mathbf{C}} \models v_0^* \in v_1^*$$

$$v_0 = v_1 \text{ iff } \mathbf{V}^{\mathbf{C}} \models v_0^* = v_1^*$$

2. The map $-^*$ *is 1-1 from* $\mathbf{V}$ *into* $\mathbf{V}^{\mathbf{C}}$*.*

3. For any restricted formula $\phi(x_0, \ldots, x_{n-1})$ *and any* $v_0, \ldots, v_{n-1} \in \mathbf{V}$

$$\phi(v_i/x_i) \text{ iff } \mathbf{V}^{\mathbf{C}} \models \phi(v_i^*/x_i)$$

Proof.

1. (a) If $v_0 \in v_1$ then $\mathbf{V}^{\mathbf{C}} \models v_0^* \in v_1^*$.
   Assume $v_0 \in v_1$. Then for all $C \in \mathbf{C}$

   $$v_0^*(C) \in \{ v^*(C) \mid v \in v_1 \} = (v_1^*(C))_{\mathrm{id}_C}$$

   therefore

   $$C \Vdash_{\emptyset} v_0^* \in v_1^*$$

   and

   $$\mathbf{V}^{\mathbf{C}} \models v_0^* \in v_1^*$$

   (b) If $v_0 = v_1$ then $\mathbf{V}^{\mathbf{C}} \models v_0^* = v_1^*$.
   Assume $v_0 = v_1$. Then for all $C \in \mathbf{C}$

   $$\begin{aligned} v_0^*(C) &= \{ v^*(C_0) \mid v \in v_0 \}_{f:C_0\to C} \\ &= \{ v^*(C_0) \mid v \in v_1 \}_{f:C_0\to C} = v_1^*(C) \end{aligned}$$

So

$$C \Vdash_{\emptyset} v_0^* = v_1^*$$

and again

$$\mathbf{V}^{\mathbf{C}} \models v_0^* = v_1^*$$

(c) If $\mathbf{V}^{\mathbf{C}} \models v_0^* \in v_1^*$ then $v_0 \in v_1$.
If $\mathbf{V}^{\mathbf{C}} \models v_0^* = v_1^*$ then $v_0 = v_1$.
In fact we prove the following, stronger statement. Let $C \in \mathbf{C}$ and $v_0, v_1 \in \mathbf{V}$. Then

$$C \Vdash_{\emptyset} v_0^* \in v_1^* \to v_0 \in v_1$$

and

$$C \Vdash_{\emptyset} v_0^* = v_1^* \to v_0 = v_1$$

The proof is by double induction on the (well-founded) $\in$-relation on v_1. Set

$$\phi(v_1) \equiv \forall v_0 \in \mathbf{V}\ (C \Vdash_{\emptyset} v_0^* \in v_1^* \to v_0 \in v_1)$$

and

$$\psi(v_1) \equiv \forall v_0 \in \mathbf{V}\ (C \Vdash_{\emptyset} v_0^* = v_1^* \to v_0 = v_1)$$

We have to prove

$$\forall v \in \mathbf{V}\ \phi(v)$$

and

$$\forall v \in \mathbf{V}\ \psi(v)$$

It is clear that $\phi(\emptyset)$ and $\psi(\emptyset)$. For the induction step pick $C \in \mathbf{C}$ and assume

$$\forall v_0 \in v_1\ \phi(v_0)$$

$$\forall v_0 \in v_1\ \psi(v_0)$$

and for $v_0 \in \mathbf{V}$

i.

$$C \Vdash_{\emptyset} v_0^* \in v_1^*$$

This last assumption is equivalent to

$$v_0^*(C) \in (v_1^*(C))_{\mathrm{id}_C} = \{\ v^*(C) \mid v \in v_1\ \}$$

Therefore for some $v \in v_1$

$$C \Vdash_\emptyset v_0^* = v^*$$

so by induction hypothesis $v_0 = v$ and $v_0 \in v_1$.

ii.

$$C \Vdash_\emptyset v_0^* = v_1^*$$

This is equivalent to

$$v_0^*(C) = v_1^*(C)$$

or unscrambled

$$\{\{ v^*(C_0) \mid v \in v_0 \}_f\}_{f:C_0 \to C} = \{\{ v^*(C_0) \mid v \in v_1 \}_f\}_{f:C_0 \to C}$$

which implies specifically that

$$\{ v^*(C) \mid v \in v_0 \}_{\mathrm{id}_C} = \{ v^*(C) \mid v \in v_1 \}_{\mathrm{id}_C}$$

It follows that

$$\forall v \in v_0\, (v^*(C) \in \{ v^*(C) \mid v \in v_1 \}_{\mathrm{id}_C})$$

or

$$\forall v \in v_0\, (C \Vdash_\emptyset v^* \in v_1^*)$$

hence by induction hypothesis

$$\forall v \in v_0\, (v \in v_1)$$

In the same manner we also obtain

$$\forall v \in v_1\, (v \in v_0)$$

therefore by extensionality $v_0 = v_1$.

2. follows from 1.

3. We shall prove the following statement.

For any restricted formula $\phi(x_i)$, any $v_i \in \mathbf{V}$ and $C \in \mathbf{C}$

$$\phi(v_i/x_i) \text{ iff } C \Vdash_\emptyset \phi(v_i^*/x_i) \tag{2.2}$$

From this claim 3. clearly follows.

We prove (2.2) by structural induction on ϕ.

(a) For ϕ atomic the statement holds by 1.

(b) If ϕ is $\perp$, a conjunction, disjunction or implication, the step is easy.

(c) Let

$$\phi \equiv \exists x \in x_0\, \psi \equiv \exists x\, (x \in x_0 \wedge \psi)$$

i. For $v_i \in \mathbf{V}$ assume $\phi(v_i/x_i)$. Then there is a $v \in \mathbf{V}$ such that

$$(x \in x_0 \wedge \psi)(v/x, v_i/x_i)$$

By (a) and (b) and the induction hypothesis we obtain

$$C \Vdash_{\emptyset} (x \in x_0 \wedge \psi)(v^*/x, v_i^*/x_i)$$

hence

$$C \Vdash_{\{v^*(C)/x\}} (x \in x_0 \wedge \psi)(v_i^*/x_i)$$

and

$$C \Vdash_{\emptyset} \exists x\, (x \in x_0 \wedge \psi)(v_i^*/x_i)$$

ii. Now assume for $v_i \in \mathbf{V}$

$$C \Vdash_{\emptyset} \exists x\, (x \in x_0 \wedge \psi)(v_i^*/x_i)$$

Then there is a x' such that

$$x' \in (v_0^*(C))_{\mathrm{id}_C} = \{\, v^*(C) \mid v \in v_0 \,\}$$

and

$$C \Vdash_{\{x'/x\}} (x \in x_0 \wedge \psi)(v_i^*/x_i)$$

Hence there is a $v \in v_0$ which by transitivity is in $\mathbf{V}$ and such that such that

$$C \Vdash_{\{v^*(C)/x\}} (x \in x_0 \wedge \psi)(v_i^*/x_i)$$

therefore

$$C \Vdash_{\emptyset} (x \in x_0 \wedge \psi)(v^*/x, v_i^*/x_i)$$

and again by (a), (b) and the induction hypothesis

$$(x \in x_0 \wedge \psi)(v/x, v_i/x_i)$$

The result follows.

(d) Let

$$\phi \equiv \forall x \in x_0\, \psi \equiv \forall x\ (x \in x_0 \rightarrow \psi)$$

i. For $v_i \in \mathbf{V}$ assume

$$\forall x\ (x \in x_0 \rightarrow \psi)(v_i/x_i) \tag{2.3}$$

Pick $f\ :\ C_0 \rightarrow C$ and x' such that

$$x' \in (v_0^*(C_0))_{\mathrm{id}_{C_0}} = \{\ v^*(C_0) \mid v \in v_0\ \}$$

There is a $v \in v_0$ such that

$$x' = v^*(C_0) \tag{2.4}$$

We have by (2.3)

$$\psi(v/x, v_i/x_i)$$

hence by induction hypothesis

$$C_0\ \Vdash_{\emptyset}\ \psi(v^*/x, v_i^*/x_i)$$

and

$$C_0\ \Vdash_{\{v^*(C_0)/x\}}\ \psi(v_i^*/x_i)$$

therefore by (2.4)

$$C_0\ \Vdash_{\{x'/x\}}\ \psi(v_i^*/x_i)$$

and the result follows.

ii. Now assume

$$C\ \Vdash_{\emptyset}\ \forall x\ (x \in x_0 \rightarrow \psi)(v_i^*/x_i)$$

Pick $v \in \mathbf{V}$ such that $v \in v_0$. We have

$$C\ \Vdash_{\emptyset}\ (x \in x_0)(v^*/x, v_i^*/x_i)$$

therefore

$$C\ \Vdash_{\emptyset}\ \psi(v^*/x, v_i^*/x_i)$$

and by induction hypothesis

$$\psi(v/x, v_i/x_i)$$

The result follows.

□

We make explicit a property of v^* which we have repeatedly used in the above proof.

Lemma 2.33 *Let $C \in \mathbf{C}$, $v \in \mathbf{V}$ and y such that*

$$C \Vdash_{\{y/x\}} x \in v^*$$

Then there exists a $v_0 \in v$ such that

$$C \Vdash_{\{y/x\}} x = v_0^*$$

Proof. Immediate from the definition of v^*. □

2.2.10 Classical sets

Can the standard sets in a universe $\mathbf{V}^{\mathbf{C}}$ be described axiomatically? I do not know. A step towards a description is the following definition.

Definition 2.34 *A set X is* pseudo-classical *if*

$$\forall X_0 \subset X\ (\forall x \in X\ (x \in X_0 \vee x \notin X_0) \rightarrow \\ \exists x \in X\ (x \in X_0) \vee \neg\, \exists x \in X\ (x \in X_0))$$

A pseudo-classical set also satisfies the following condition.

Lemma 2.35 *Let X be pseudo-classical. Then*

$$\forall X_0 \subset X\ (\forall x \in X\ (x \in X_0 \vee x \notin X_0) \rightarrow \\ \forall x \in X\ (x \in X_0) \vee \neg\, \forall x \in X\ (x \in X_0))$$

Proof. Assume X to be pseudo-classical. Pick $X_0' \subset X$. Assume

$$\forall x \in X\ (x \in X_0 \vee x \notin X_0) \tag{2.5}$$

Then also

$$\forall x \in X\ (x \in X_0^c \vee x \notin X_0^c)$$

Therefore by pseudo-classicality of X

$$\exists x \in X\ (x \in X_0^c) \vee \neg\, \exists x \in X\ (x \in X_0^c)$$

The first option implies

$$\neg \forall x \in X \ (x \in X_0)$$

the second

$$\forall x \in X \ \neg\neg \ (x \in X_0)$$

so by (2.5)

$$\forall x \in X \ (x \in X_0)$$

□

Lemma 2.36 *Let X, Y be pseudo-classical. Then $X \times Y$ is pseudo-classical.*

Proof. Pick $Z' \subset X \times Y$ such that Z' is decidable. Define

$$X_0 \triangleq \{ \ x \in X \mid \exists y \in Y \ (\langle x, y \rangle \in Z') \ \}$$

We have by pseudo-classicality of Y and decidability of Z' that for every $x \in X$ the statement

$$\exists y \in Y \ (\langle x, y \rangle \in Z')$$

is decidable, therefore X_0 is a decidable subset of X. We get by pseudo-classicality of X

$$\exists x \in X \ (x \in X_0) \lor \neg \exists x \in X \ (x \in X_0)$$

which implies

$$\exists x \in X \ \exists y \in Y \ (\langle x, y \rangle \in Z') \lor \neg \exists x \in X \ \exists y \in Y \ (\langle x, y \rangle \in Z')$$

□

The following lemmas will help to explain the term 'pseudo-classical'.

Lemma 2.37 *Let X be pseudo-classical and $\phi(x)$ be decidable on X. Then*

1. $\exists x \in X \ \phi \lor \neg \exists x \in X \ \phi$

2. $\forall x \in X \ \phi \lor \neg \forall x \in X \ \phi$

Proof. Immediate from Definition 2.34 and Lemma 2.35. □

Lemma 2.38 *Let X be pseudo-classical, $R_i^{n_i} \subset X^{n_i}$ decidable relations on X and ϕ a formula having R_i as relation symbols and quantifiers of the form $\exists x \in X$ and $\forall x \in X$. Then*

$$\forall \vec{x} \in X\ (\phi \vee \neg\, \phi)$$

Proof. By induction on the structure of ϕ and Lemmas 2.8, 2.36 and 2.37. □

In order to get used to pseudo-classical sets, we prove some lemmas that indicate their usefulness.

Lemma 2.39 *If X is pseudo-classical and X_0 a decidable subset of X then*

$$\neg\neg\, \exists x \in X\ (x \in X_0) \to \exists x \in X\ (x \in X_0)$$

Proof. Assume $\neg\neg\, \exists x \in X\ (x \in X_0)$. X_0 is a decidable subset of X, therefore

$$\exists x \in X\ (x \in X_0) \vee \neg\, \exists x \in X\ (x \in X_0)$$

The latter case is ruled out since X_0 is nonempty. Therefore it is inhabited. □

Since X is a decidable subset of itself, we get as an immediate consequence

Corollary 2.40 *If X is pseudo-classical then*

$$\neg\neg\, \exists x\ (x \in X) \to \exists x\ (x \in X)$$

Lemma 2.41 *For any sets X_0, X_1, if X_0 is pseudo-classical and X_1 is discrete, then $X_1^{X_0}$ is discrete.*

Proof. Take $f_0', f_1' : X_0 \to X_1$. We have

$$\forall x \in X_0\ (f_0'(x) = f_1'(x) \vee f_0'(x) \neq f_1'(x))$$

By pseudo-classicality of X_0 we have

$$\forall x \in X_0\ (f_0'(x) = f_1'(x)) \vee \neg\, \forall x \in X_0\ (f_0'(x) = f_1'(x))$$

hence the result. □

Lemma 2.42 *Let X_0 be pseudo-classical, X_1 discrete. Then for any two functions $f_0, f_1 : X_0 \to X_1$*

$$f_0 \neq f_1 \to \exists x \in X_0\ (f_0(x) \neq f_1(x))$$

Proof. Pick $f'_0, f'_1 \in X_0 \Rightarrow X_1$. Assume $f'_0 \neq f'_1$, i.e.

$$\neg \forall x \in X_0 \, (f'_0(x) = f'_1(x)) \tag{2.6}$$

We have by discreteness of X_1

$$\forall x \in X_0 \, (f'_0(x) \neq f'_1(x) \vee \neg\neg \, f'_0(x) = f'_1(x))$$

hence by pseudo-classicality of X_0

$$\exists x \in X_0 \, (f'_0(x) \neq f'_1(x)) \vee \neg \, \exists x \in X_0 \, (f'_0(x) \neq f'_1(x))$$

The second option and discreteness of X_1 entail

$$\forall x \in X_0 \, (f'_0(x) = f'_1(x))$$

which is ruled out by (2.6). The result follows. □

Remark. The definition of a 'pseudo-classical set' appears to be new. The weaker *Markov Principle* however is well known in constructive mathematics.

Definition 2.43 *For X a set let MP(X) be the statement*

$$\begin{aligned} &\forall X_0 \subset X \; (\forall x \in X \; (x \in X_0 \vee x \notin X_0) \rightarrow \\ &\qquad \neg\neg \, \exists x \in X \; (x \in X_0) \rightarrow \exists x \in X \; (x \in X_0)) \end{aligned}$$

□

We have already seen (Lemma 2.39) that MP(X) holds for all pseudo-classical sets. It may be interesting to note that in the realizability model ω satisfies MP although it is not pseudo-classical (cf. [McC84]).

Note that in spite of the lemmas above, pseudo-classical sets may still be quite confused. A set X may only have the empty set and itself as decidable subsets. In this case Definition 2.34 does not say very much about X. We therefore add a minimal requirement.

Definition 2.44 *A set X is* classical *if it is pseudo-classical and discrete.*

We have

Lemma 2.45 *Standard sets are classical.*

Proof. Standard sets are discrete by Lemma 2.32.3. To prove pseudo-classicality let $\mathbf{C}$ be a category. Pick $v \in \mathbf{V}$. We have to prove that in $\mathbf{V}^{\mathbf{C}}$

$$\forall X \subset v^* \, (\forall x \in v^* \, (x \in X \lor x \notin X) \rightarrow \\ \exists x \in v^* \, (x \in X) \lor \neg \exists x \in v^* \, (x \in X))$$

So pick $C \in \mathbf{C}$, $f_0 : C_0 \rightarrow C$ and Y such that

$$C_0 \Vdash_{\{Y/X\}} X \subset v^*$$

Let $f_1 : C_1 \rightarrow C_0$ and assume

$$C_1 \Vdash_{\{Y/X\}|f_1} \forall x \in v^* \, (x \in X \lor x \notin X)$$

We have two cases:

1. for all $f_2 : C_2 \rightarrow C_1$ and $y \in v^*(C_2)$

   $$C_2 \not\Vdash_{\{Y/X\}|(f_1 \circ f_2)\{y/x\}} x \in X$$

   Then

   $$C_1 \Vdash_{\{Y/X\}|f_1} \neg \exists x \in v^* \, (x \in X)$$

2. there is a $f_2 : C_2 \rightarrow C_1$ and a $y \in v^*(C_2)$ such that

   $$C_2 \Vdash_{\{Y/X\}|(f_1 \circ f_2)\{y/x\}} x \in X$$

   By Lemma 2.33 there is a $v_0 \in v$ such that

   $$C_2 \Vdash_{\{Y/X\}|(f_1 \circ f_2)} v_0^* \in X$$

   and by decidability of X

   $$C_1 \Vdash_{\{Y/X\}|f_1} v_0^* \in X$$

   hence

   $$C_1 \Vdash_{\{Y/X\}|f_1} \exists x \in v^* \, (x \in X)$$

□

Conjecture. The category of classical sets is a full subcategory of the category of all sets, i.e. when two sets X and Y are classical then so are $X \times Y$ and Y^X.

□

2.3 Some counterexamples

We are now in a position to prove the independence from **IZF** of some classical theorems. This will be done by giving models $\mathbf{V}^{\mathbf{C}}$ of **IZF** in which the sentences fail. We introduce the following notation.

Notation. Let **P** be a partial order with a top and P_0 a downward closed subset of P. Let $f : P_0 \to \omega$ be such that

$$p_0 \leq_P p_1 \to f(p_0) = f(p_1)$$

Then $\{ f \mid P_0 \}$ denotes the smallest set X in $\mathbf{V}^{\mathbf{P}}$ such that

$$p \Vdash f(p) \in X \text{ if } p \in P_0$$

□

Definition 2.46 *Let* ϕ *be a classical theorem and* $\mathbf{V}^{\mathbf{C}}$ *a model of* **IZF**.

- $\mathbf{V}^{\mathbf{C}}$ *is a* weak counterexample *to* ϕ *if*

$$\mathbf{V}^{\mathbf{C}} \not\models \phi$$

- $\mathbf{V}^{\mathbf{C}}$ *is a* strong counterexample *to* ϕ *if*

$$\mathbf{V}^{\mathbf{C}} \models \neg \phi$$

Lemma 2.47

$$\mathbf{IZF} \nvdash \forall\phi\, ((\neg\neg\, \phi \to \phi) \to (\neg\, \phi \vee \neg\neg\, \phi))$$

Proof. Consider the model $\mathbf{V}^{2_\perp^{\mathrm{op}}}$. Let ϕ be a sentence forced at 0. Then $(\neg\neg\, \phi \to \phi)$ is forced at $\perp$, but $(\neg\, \phi \vee \neg\neg\, \phi)$ is not. □

Lemma 2.48

$$\mathbf{IZF} \nvdash \forall\phi\, ((\neg\, \phi \vee \neg\neg\, \phi) \to (\neg\neg\, \phi \to \phi))$$

Proof. Consider the model $\mathbf{V}^{2^{\mathrm{op}}}$. Let ϕ be a sentence forced at 1. Then $(\neg\, \phi \vee \neg\neg\, \phi)$ is forced at 0, but $(\neg\neg\, \phi \to \phi)$ is not. □

Lemma 2.49

$$\mathbf{IZF} \not\vdash \neg \forall n \in \omega\, \phi(n) \rightarrow \exists n \in \omega \neg \phi(n)$$

Proof. Take the model $\mathbf{V}^{\omega^{op}}$ and $\phi(n)$ such that $\phi(n)$ is forced at $\downarrow n$. Then $\neg \forall n \in \omega\, \phi(n)$ is forced everywhere, but $\exists n \in \omega \neg \phi(n)$ nowhere. □

Lemma 2.50 *There is a model* $\mathbf{V}^{\mathbf{C}}$ *and a set* $X \in \mathbf{V}^{\mathbf{C}}$ *such that*

$$\mathbf{V}^{\mathbf{C}} \models \neg \forall x_0, x_1 \in X\ (x_0 = x_1 \vee x_0 \neq x_1)$$

Proof. Consider the model $\mathbf{V}^{\omega^{op}}$. For $n \in \omega$ define $X_n \in \mathbf{V}^{\omega^{op}}$ by

$$X_n \triangleq \{\, 1 \mid \downarrow n \,\}$$

We have for all X_n

$$\mathbf{V}^{\omega^{op}} \models X_n \in \mathcal{P}(1)$$

Also for every $n \in \omega$

$$n \not\Vdash_{\emptyset} X_n = X_{n+1} \vee X_n \neq X_{n+1}$$

hence for all $n \in \omega$

$$n \not\Vdash_{\emptyset} \forall x_0, x_1 \in \mathcal{P}(1)\ (x_0 = x_1 \vee x_0 \neq x_1)$$

and

$$\mathbf{V}^{\omega^{op}} \models \neg \forall x_0, x_1 \in \mathcal{P}(1)\ (x_0 = x_1 \vee x_0 \neq x_1)$$

□

What can be said about the categories $\mathbf{C}$ that have the property that there exists an $X \in \mathbf{V}^{\mathbf{C}}$ such that

$$\mathbf{V}^{\mathbf{C}} \models \neg \forall x_0, x_1 \in X\ (x_0 = x_1 \vee \neg\, x_0 = x_1)?$$

This is answered by the following

Lemma 2.51 *If for a category* $\mathbf{C}$ *there is an* $X \in \mathbf{V}^{\mathbf{C}}$ *such that*

$$\mathbf{V}^{\mathbf{C}} \models \neg \forall x_0, x_1 \in X\ (x_0 = x_1 \vee x_0 \neq x_1)$$

then for all $C \in \mathbf{C}$ *and for all* $f_0 : C_0 \rightarrow C$ *there exists a* $f_2 : C_1 \rightarrow C$ *such that* $f_2 \succ_C f_0$.

Proof. Recall the definition of $\succeq_C$ (2.1). Assume

$$C \Vdash_{\emptyset} \neg \forall x_0, x_1 \in X \ (x_0 = x_1 \vee \neg \, x_0 = x_1)$$

This means that for all $f_0 : C_0 \to C$

$$C_0 \not\Vdash_{\emptyset} \forall x_0, x_1 \in X \ (x_0 = x_1 \vee \neg \, x_0 = x_1)$$

i.e. there exist a $f_1 : C_1 \to C$, $f_1 \succeq_C f_0$ and $y_0, y_1 \in X(C_1)$ such that

$$C_1 \not\Vdash_{\{y_0/x_0; y_1/x_1\}} x_0 = x_1 \tag{2.7}$$

and

$$C_1 \not\Vdash_{\{y_0/x_0; y_1/x_1\}} x_0 \neq x_1 \tag{2.8}$$

From (2.7) and (2.8) we deduce that there is a morphism $f_2 : C_2 \to C$ such that $f_2 \succ_C f_1 \succeq_C f_0$ and

$$C_2 \Vdash_{\{y_0/x_0; y_1/x_1\}|f} x_0 = x_1$$

where $f_2 = f_1 \circ f$. □

Similarly we could prove the next

Lemma 2.52 *If for a category* $\mathbf{C}$ *there is an* $X \in \mathbf{V}^{\mathbf{C}}$ *such that*

$$\mathbf{V}^{\mathbf{C}} \models \neg \forall x_0, x_1 \in X \ (x_0 \neq x_1 \vee \neg\neg \, x_0 = x_1)$$

then for all $C \in \mathbf{C}$ *and* $f_0 : C_0 \to C$ *there exist* $f_1 : C_1 \to C$ *and* $f_2 : C_2 \to C$ *such that* $f_1 \succ_C f_0$, $f_2 \succ_C f_0$ *and* f_1 *and* f_2 *are incompatible, i.e. there is no* $f_3 : C_3 \to C$ *such that* $f_3 \succeq_C f_1$ *and* $f_3 \succeq_C f_2$.

So, if $\mathbf{C}^{\mathrm{op}}$ is a tree, it will have hereditary width at least ω.

There is a class of sentences that will impose an even greater lower bound. We state without proof

Lemma 2.53 *Let* $\mathbf{P}$ *be a tree,* $\alpha \in \mathbf{V}$ *an infinite cardinal. Then if*

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models \exists X \ (\alpha^* \subset X \wedge \neg \forall x_0 \in X \ \exists x_1 \in X \ (x_0 \neq x_1))$$

then $\mathbf{P}$ *has hereditary width at least* α.

2.4 Further revision of some classical concepts

We use functions to compare the *size* of sets.

Definition 2.54 *Let X, Y be sets.*

$$\begin{aligned} X \preceq Y & \quad \textit{iff} \quad \exists f\,(f : X \hookrightarrow Y) \\ X \approx Y & \quad \textit{iff} \quad \exists f\,(f : X \hookrightarrow\!\!\!\!\rightarrow Y) \\ X \prec Y & \quad \textit{iff} \quad X \preceq Y \wedge X \not\approx Y \end{aligned}$$

Classically we have the Schröder-Bernstein Lemma.

Lemma 2.55

$$\mathbf{ZF} \vdash_{QC} X \preceq Y \wedge Y \preceq X \rightarrow X \approx Y \quad \text{<SB>}$$

Proof. See for instance [Kun80]. □

Constructively <SB> does not hold. As a counterexample take $\mathbf{V}^{2^{op}}$. Pick a sentence ϕ forced at 1 and consider the sets ω and $\omega \cup \{\, \omega \mid \phi \,\}$. Evidently the inclusion is a monomorphism from ω to $\omega \cup \{\, \omega \mid \phi \,\}$. In the other direction define f by

$$f(n) \triangleq n + 1 \quad \text{for } n \in \omega$$

and

$$f(\omega) \triangleq 0$$

It is clear that there cannot be a bijection between the two sets.

We can prove something stronger.

Lemma 2.56

$$\mathbf{IZF} + \text{<SB>} + \omega \textit{ is classical} \vdash_{IQC} \text{<PEM>}$$

Proof. Given ϕ, take again the sets ω and $\omega \cup \{\, \omega \mid \phi \,\}$. There are injections in both directions. Now assume there is a bijection

$$f : \omega \hookrightarrow\!\!\!\!\rightarrow \omega \cup \{\, \omega \mid \phi \,\}$$

We have

$$\phi \leftrightarrow \omega \in \mathrm{im}(f) \leftrightarrow \exists n \in \omega\, (fn = \omega)$$

Equality is decidable on $\omega \cup \{\omega\}$, hence $\phi(n) \equiv fn = \omega$ is a decidable relation on the classical set ω, therefore

$$\exists n \in \omega\ (fn = \omega)$$

is decidable and so is ϕ. □

It may now almost come as a pleasant surprise that another lemma related to size still holds.

Lemma 2.57 (Cantor) *For all sets* X

$$X \prec \mathcal{P}(X)$$

Proof. It is clear that X can be mapped injectively into $\mathcal{P}(X)$. Assume there exists a bijection $f : X \hookrightarrow\!\!\!\!\rightarrow \mathcal{P}(X)$. Consider the subset of X

$$X_0 \triangleq \{\, x \in X \mid x \notin fx \,\}$$

X_0 is the image of a point $x_0 \in X$. Assume

$$x_0 \in fx_0 = \{\, x \in X \mid x \notin fx \,\}$$

We have a contradiction. Therefore

$$x_0 \notin fx_0$$

but then

$$x_0 \in \{\, x \in X \mid x \notin fx \,\} = fx_0$$

Another contradiction. We see that there is no bijection. □

Now we go over to the question of ordinals. We remark that the usual classical definition does not work. First two auxiliary notions.

Definition 2.58 *A set* X *is* transitive *if*

$$\forall x \in X\ (x \subset X)$$

Definition 2.59 *Let* R *be an anti-reflexive, transitive binary relation on a set* X*.* R *is a* well ordering *if every nonempty subset of* X *has an* R*-least member.*

Definition 2.60 (classical) *An* ordinal *is a* $\in$*-well ordered, transitive set.*

The reason for avoiding this definition lies in the somewhat surprising fact that constructively only the empty set can be well ordered.

Lemma 2.61 *Let X be a nonempty set. Then*

$$\mathbf{IZF} + \exists R \text{ such that } R \text{ well orders } X \vdash_{IQC} <\mathrm{RAA}>$$

Proof. Let R be a well ordering on X. There is an R-minimum $x \in X$. Assume $\neg\neg\,\phi$. Then

$$X_0 \triangleq X \setminus \{x\} \cup \{\, x \mid \phi \,\}$$

is a nonempty subset of X and has an R-minimum x_0. $\neg\neg\,\phi$ implies $\neg\neg\, X = X_0$, so $\neg\neg\, x = x_0$. As x is the minimum and we do not have xRx_0, it follows that $x = x_0$. Therefore $x \in X_0$ and ϕ. □

We propose an alternative—and as always classically equivalent— definition of our ordinals. At least we manage to salvage one of their most important characteristics.

Definition 2.62 (constructive) *Let X be a set. X is an* ordinal *if it is transitive and induction can be done on it, i.e. for all set theoretic formulas $\phi(x)$ with x_0 free for x*

$$\forall x \in X\; (\forall x_0 \in x\; \phi(x_0/x) \rightarrow \phi) \rightarrow \forall x \in X\; \phi$$

The class of ordinals will be denoted **On**.

The natural numbers and ω are ordinals.

We might now be tempted to define some notion of *cardinality* of sets in terms of ordinals. The following definitions and notes show that there are already considerable problems at a low level.

Definition 2.63 *Let X be a set. It is*

- strictly finite *if* $\exists n \in \omega\;\; \exists f\; (f\, :\, n \hookrightarrow\!\!\!\!\!\rightarrow X)$
- finite *if* $\exists n \in \omega\;\; \exists f\; (f\, :\, n \twoheadrightarrow X)$
- subfinite *if* $\exists n\; \exists N\; (N \subset n\;\; \wedge \exists f\; (f\, :\, N \hookrightarrow\!\!\!\!\!\rightarrow X))$
- countable *if* $\exists f\; (f\, :\, \omega \twoheadrightarrow X)$
- subcountable *if* $\exists N\; (N \subset \omega \wedge \exists f\; (f\, :\, N \twoheadrightarrow X))$

- infinite *if* $\exists f\ (f\ :\ \omega \hookrightarrow X)$

Subfinite sets are—as their name suggests—subsets of finite sets. They have the closure properties classically attributed to finite sets. For more information we refer to Grayson [Gra78].

The distinction between the various degrees of finiteness is relevant. We have for example

Lemma 2.64

$$\mathbf{IZF} + \forall x\ (x\,is\ subfinite \rightarrow x\,is\ finite)\ \vdash_{IQC}\ <\mathrm{RAA}>$$

The proof of this lemma appears in [McC84].

Chapter 3

λ-calculus and combinatory logic

In this chapter we are going to look at the languages of the λ-calculus and combinatory logic, and we define some theories and models.

The standard text on the (untyped) λ-calculus is the book by Barendregt [Bar84]. Koymans worked on models ([Koy82] and [Koy84]). See also Bethke's thesis [Bet88]. The connection between the typed theory and ccc's is established in the work of Lambek and Scott [LS86].

3.1 The syntax of the λ-calculus

We start with the typed calculus.

Definition 3.1 *Let* Γ *be a nonempty set of (ground) type symbols. The* set of types Σ over Γ *is defined inductively by*

- $\Gamma \subset \Sigma$
- *if* $\sigma_0, \sigma_1 \in \Sigma$ *then* $(\sigma_0 \to \sigma_1) \in \Sigma$

For the rest of the chapter let Σ be a fixed set of types.

Definition 3.2 *A* Σ-typed set *is a* Σ*-indexed familiy of mutually disjoint sets.*

If X is a Σ-typed set, write $x \in X$ for $x \in \bigcup_{\sigma\in\Sigma} X^\sigma$. x is *of type* σ $(x : \sigma)$ if $x \in X^\sigma$.

Remark. We shall use the term 'Σ-typed' in various contexts. Thus we shall for example speak of a Σ-typed function to mean a Σ-typed set $f = \{f^\sigma\}_{\sigma\in\Sigma}$ such that each f^σ is a function. □

Now let C be a Σ-typed set of constants, Vars a Σ-typed infinite set of variables. The *language* $\mathcal{L}_{\lambda^\Sigma(C)}$ of the Σ-typed λ-calculus is made up of C, Vars, brackets (,) and the abstractor λ.

The terms $\Lambda^\Sigma(C)$ of the calculus are Σ-typed and defined inductively by

- $C^\sigma \subset \Lambda^\sigma(C)$ for all $\sigma \in \Sigma$
- $\text{Vars}^\sigma \subset \Lambda^\sigma(C)$ for all $\sigma \in \Sigma$
- if $t_0 \in \Lambda^{\sigma_0 \to \sigma_1}(C)$ and $t_1 \in \Lambda^{\sigma_0}(C)$ then $(t_0 t_1) \in \Lambda^{\sigma_1}(C)$ for every $\sigma_0, \sigma_1 \in \Sigma$
- if $x \in \text{Vars}^{\sigma_0}$ and $t \in \Lambda^{\sigma_1}(C)$ then $(\lambda x.t) \in \Lambda^{\sigma_0 \to \sigma_1}(C)$ for every $\sigma_0, \sigma_1 \in \Sigma$

We omit the type of a term if it is of no importance or can be inferred from context.

There is a notion of *binding* of variables in λ-terms. The set of *free variables* $\text{FV}(t)$ of a term t is defined inductively on the structure of t as

- $\text{FV}(c) = \emptyset$ for $c \in C$
- $\text{FV}(x) = \{x\}$ for $x \in \text{Vars}$
- $\text{FV}(t_0 t_1) = \text{FV}(t_0) \cup \text{FV}(t_1)$
- $\text{FV}(\lambda x.t) = \text{FV}(t) \setminus \{x\}$

Let $(\Lambda^\Sigma)^0(C)$ denote the set of *closed terms*, i.e. terms t with no free variables.

Terms that only differ by the names of the bound variables will be treated as identical (α-equivalence).

Terms can be substituted for free variables in other terms. Let $t_0(t_1/x)$ denote t_0 with every free occurrence of x replaced by t_1. Rename variables in t_1 to avoid capture.

Let us agree to write $t_0 t_1 t_2$ for $(t_0 t_1) t_2$, and $\lambda x.t_0 t_1$ for $\lambda x.(t_0 t_1)$.

3.1.1 The untyped calculus

We shall regard this as having only one type, hence there is only one set of constants, variables etc. Terms $\Lambda(C)$ are defined as above, without mention of types, so that one of the term forming rules will now read

- if $t_0 \in \Lambda(C)$ and $t_1 \in \Lambda(C)$ then $(t_0 t_1) \in \Lambda(C)$

3.1.2 $\lambda^{\Sigma}(C)$-theories

Remark. The definitions and results in the remainder of this chapter, although stated for the typed calculus, apply also to the untyped version, and we shall use them in an untyped context. □

Definition 3.3 *A* $\lambda^{\Sigma}(C)$-theory *is a set of equations between* $\Lambda^{\Sigma}(C)$*-terms of the same type.*

Purely to enhance readability, we shall sometimes place equations inside the delimiters $||$. Here now is the definition of the basic $\lambda^{\Sigma}(C)$-theory.

Definition 3.4 $\boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *is the* $\lambda^{\Sigma}(C)$*-theory inductively defined by*

1. $|t = t| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ <Reflexivity>
2. *if* $|t_0 = t_1| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *and* $|t_1 = t_2| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *then* $|t_0 = t_2| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ <Transitivity>
3. *if* $|t_0 = t_1| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *then* $|t_1 = t_0| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ <Symmetry>
4. *if* $|t_0 = t_1| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *and* $|t_2 = t_3| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *then* $|t_0 t_2 = t_1 t_3| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ <Application>
5. *if* $|t_0 = t_1| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *then* $|\lambda x.t_0 = \lambda x.t_1| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ <$\boldsymbol{\xi}$>
6. $|(\lambda x.t_0)t_1 = t_0(t_1/x)| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ <$\boldsymbol{\beta}$>
7. $|\lambda x.tx = t| \in \boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ *if* $x \notin \mathrm{FV}(t)$ <$\boldsymbol{\eta}$>

If T is a theory

$$\boldsymbol{\lambda\beta\eta}^{\Sigma}(C) + T \vdash t_0 = t_1$$

means that $|t_0 = t_1|$ can be derived from the axioms in $\boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ and T using the rules of equality and $\boldsymbol{\xi}$. Call a theory T *inconsistent* if for all terms t_0, t_1

$$\boldsymbol{\lambda\beta\eta}^{\Sigma}(C) + T \vdash t_0 = t_1$$

Two terms are *incompatible relative to* T if T is consistent and $T \cup \{t_0 = t_1\}$ is not. Two terms are *incompatible* if they are incompatible relative to $\emptyset$.

Remark. We defined a theory as a 'set' of equations without specifiying where this set 'lives'. It would be quite reasonable to assume that a $\lambda^{\Sigma}(C)$-theory is a set in a *constructive* universe. Hence we cannot in general assume that, e.g.

$$\lambda\beta\eta^{\Sigma}(C) + T \ \vdash \ t_0 = t_1 \ \text{ or } \ \lambda\beta\eta^{\Sigma}(C) + T \ \not\vdash \ t_0 = t_1$$

i.e. neither T nor the set of its consequences need be a decidable subset of the set of all equations. However, since we wish to start out from a classical theory and define constructive *models* for it, we shall assume that T is a decidable. □

We shall now give a well known result that can be proved constructively. It concerns the existence of fixed points in the untyped theory.

Lemma 3.5 *Every term $t_0 \in \Lambda(C)$ has a fixed point, i.e. there is a term t_1 such that*

$$\lambda\beta\eta(C) \ \vdash \ t_0 t_1 = t_1$$

Proof. Let

$$t_1 \equiv (\lambda x.t_0(xx))(\lambda x.t_0(xx))$$

Then

$$\lambda\beta\eta(C) \ \vdash \ t_1 \equiv (\lambda x.t_0(xx))(\lambda x.t_0(xx)) = t_0((\lambda x.t_0(xx))(\lambda x.t_0(xx))) \equiv t_0 t_1$$

□

In the next section we shall show how the abstractor λ can be eliminated by translating λ-terms into terms of combinatory logic.

3.2 Combinatory logic

Combinatory logic provides an alternative to the λ-calculus. It is equational (although we shall occasionally trespass into first order logic), hence all model theoretic results from equational logic can be applied to it. Again we start by describing the language.

Let C be a Σ-typed set of constants and Vars a Σ-typed set of variables. The language $\mathcal{L}^{\Sigma}_{CL}(C)$ of combinatory logic is made up of C, Vars, brackets (,) and

- a constant k^{σ_0,σ_1} of type $(\sigma_0 \to (\sigma_1 \to \sigma_0))$ for every $\sigma_0, \sigma_1 \in \Sigma$
- a constant $s^{\sigma_0,\sigma_1,\sigma_2}$ of type $(\sigma_0 \to (\sigma_1 \to \sigma_2)) \to ((\sigma_0 \to \sigma_1) \to (\sigma_0 \to \sigma_2))$ for every $\sigma_0, \sigma_1, \sigma_2 \in \Sigma$

Σ-typed terms $CL^{\Sigma}(C)$ of combinatory logic are defined inductively by

- $C^{\sigma} \subset CL^{\sigma}(C)$ for every $\sigma \in \Sigma$
- $\mathrm{Vars}^{\sigma} \subset CL^{\sigma}(C)$ for every $\sigma \in \Sigma$
- $k^{\sigma_0,\sigma_1} \in CL^{\sigma_0 \to (\sigma_1 \to \sigma_0)}(C)$ for every $\sigma_0, \sigma_1 \in \Sigma$
- $s^{\sigma_0,\sigma_1,\sigma_2} \in CL^{(\sigma_0 \to (\sigma_1 \to \sigma_2)) \to ((\sigma_0 \to \sigma_1) \to (\sigma_0 \to \sigma_2))}(C)$ for every $\sigma_0, \sigma_1, \sigma_2 \in \Sigma$
- if $t_0 \in CL^{\sigma_0 \to \sigma_1}(C)$ and $t_1 \in CL^{\sigma_0}(C)$ then $(t_0 t_1) \in CL^{\sigma_1}(C)$ for every $\sigma_0, \sigma_1 \in \Sigma$

Analogous to the definition of $\lambda^{\Sigma}(C)$-theory is that of a theory of combinatory logic.

Definition 3.6 *A* $CL^{\Sigma}(C)$-theory *is a set of equations between* $CL^{\Sigma}(C)$*-terms of the same type.*

Definition 3.7 *Let* $\mathbf{CL}^{\Sigma}(C)$ *be the* $CL^{\Sigma}(C)$*-theory having as axioms and rules*

1. $|t = t| \in \mathbf{CL}^{\Sigma}(C)$ <Reflexivity>
2. *if* $|t_0 = t_1| \in \mathbf{CL}^{\Sigma}(C)$ *and* $|t_1 = t_2| \in \mathbf{CL}^{\Sigma}(C)$ *then* $|t_0 = t_2| \in \mathbf{CL}^{\Sigma}(C)$ <Transitivity>
3. *if* $|t_0 = t_1| \in \mathbf{CL}^{\Sigma}(C)$ *then* $|t_1 = t_0| \in \mathbf{CL}^{\Sigma}(C)$ <Symmetry>
4. *if* $|t_0 = t_1| \in \mathbf{CL}^{\Sigma}(C)$ *and* $|t_2 = t_3| \in \mathbf{CL}^{\Sigma}(C)$ *then* $|t_0 t_2 = t_1 t_3| \in \mathbf{CL}^{\Sigma}(C)$
5. $|k^{\sigma_0,\sigma_1} t_0 t_1 = t_0| \in \mathbf{CL}^{\Sigma}(C)$ *for* $t_0 : \sigma_0 \to \sigma_1$, $t_1 : \sigma_0$
6. $|s^{\sigma_0,\sigma_1,\sigma_2} t_0 t_1 t_2 = t_0 t_2 (t_1 t_2)| \in \mathbf{CL}^{\Sigma}(C)$ *for* $t_0 : \sigma_0 \to (\sigma_1 \to \sigma_2)$, $t_1 : \sigma_0 \to \sigma_1$, $t_2 : \sigma_0$

The CL-equivalent to rule <η> in the λ-calculus is as follows.

7. If $|t_0 x = t_1 x| \in \mathbf{CL}^{\Sigma}(C)$ then $|t_0 = t_1| \in \mathbf{CL}^{\Sigma}(C)$ provided that $x \notin \mathrm{FV}(t_0 t_1)$ <**ext**>

If T is a $CL^{\Sigma}(C)$-theory write

$$\mathbf{CL}^{\Sigma}(C) + <\mathbf{ext}> + T \ \vdash \ t_0 = t_1$$

if $t_0 = t_1$ can be deduced from the axioms in T and the axioms and rules in $\mathbf{CL}^{\Sigma}(C)$ and <**ext**>.

Define $i = skk$. This is actually a list of definitions. From now on we shall assume that equations like this one are well typed. We have

$$\mathbf{CL}^{\Sigma}(C) \vdash ix = x$$

so i acts as a left identity on terms.

We can simulate abstraction in $CL^{\Sigma}(C)$ by defining inductively for every $x \in$ Vars a function

$$<x>: CL^{\Sigma}(C) \rightarrow CL^{\Sigma}(C)$$

by

- $<x> x = i$
- $<x> t = kt$ if t does not contain x
- $<x> t_0 t_1 = s(<x> t_0)(<x> t_1)$

Now we can give a translation of $\Lambda^{\Sigma}(C)$-terms into $CL^{\Sigma}(C)$-terms and vice versa.

Definition 3.8 *The function* $-_{CL} : \Lambda^{\Sigma}(C) \rightarrow CL^{\Sigma}(C)$ *is given by*

- $c_{CL} = c$
- $x_{CL} = x$
- $(t_0 t_1)_{CL} = (t_0)_{CL}(t_1)_{CL}$
- $(\lambda x.t)_{CL} = <x> (t_{CL})$

Definition 3.9 *The function* $-_{\lambda} : CL^{\Sigma}(C) \rightarrow \Lambda^{\Sigma}(C)$ *is given by*

- $c_{\lambda} = c$
- $x_{\lambda} = x$
- $k_{\lambda} = \lambda xy.x$
- $s_{\lambda} = \lambda xyz.xz(yz)$
- $(t_0 t_1)_{\lambda} = (t_0)_{\lambda}(t_1)_{\lambda}$

We have (see [Mog88])

Lemma 3.10 *1.* $\mathbf{CL}^{\Sigma}(C) \vdash t = (t_\lambda)_{CL}$

2. $\boldsymbol{\lambda\beta\eta}^{\Sigma}(C) \vdash t = (t_{CL})_\lambda$

3. $\mathbf{CL}^{\Sigma}(C) + <\mathbf{ext}> + T \vdash t_0 = t_1 \leftrightarrow \boldsymbol{\lambda\beta\eta}^{\Sigma}(C) + T_\lambda \vdash (t_0)_\lambda = (t_1)_\lambda$

An analogous result exists for the untyped calculus.

3.3 Combinatory algebras

The λ-calculus is supposed to be a theory of functions. Unfortunately the presence of abstraction makes the definition of models rather awkward. Combinatory logic on the other hand lacks this intuitive appeal. However, since it is an equational theory (possibly upgraded to first order), we already have some information about its models. The following definitions are taken from [Mey82]. Again, we shall only treat the typed case. The 'untyped' definitions are then straightforward.

For the remainder of this chapter let C be a fixed Σ-typed set of constants.

Definition 3.11 *A* Σ-typed combinatory algebra (ca) over C *is a structure*

$$\mathcal{U} = \langle U, \cdot_U, k_U, s_U, C_U \rangle$$

where

- U *is an inhabited* Σ*-typed set, the* underlying universe
- $\cdot_U = \{ \cdot_U^{\sigma_0,\sigma_1} : U^{\sigma_0 \to \sigma_1} \times U^{\sigma_0} \to U^{\sigma_1} \mid \sigma_0, \sigma_1 \in \Sigma \}$ *is* application
- $k_U = \{ k_U^{\sigma_0,\sigma_1} \in U^{\sigma_0 \to (\sigma_1 \to \sigma_0)} \mid \sigma_0, \sigma_1 \in \Sigma \}$
- $s_U = \{ s_U^{\sigma_0,\sigma_1,\sigma_2} \in U^{(\sigma_0 \to (\sigma_1 \to \sigma_2)) \to ((\sigma_0 \to \sigma_1) \to (\sigma_0 \to \sigma_2))} \mid \sigma_0, \sigma_1, \sigma_2 \in \Sigma \}$
- $c_U \in U^\sigma$ *for* $c \in C^\sigma$

and

$$\mathbf{CL}^{\Sigma}(C) \vdash t_0 = t_1 \to \mathcal{U} \models t_0 = t_1$$

that is

$$\forall u_0 \in U^{\sigma_0} \ldots u_{n-1} \in U^{\sigma_{n-1}} \; (t_0(u_i/x_i))_U = (t_1(u_i/x_i))_U$$

where x_i *is of type* σ_i *and* $\mathrm{FV}(t_0 t_1) \subset \{x_0, \ldots, x_{n-1}\}$.

If $\mathcal{U}$ is a ca, we sometimes use $|\mathcal{U}|$ to denote the universe of $\mathcal{U}$.

Remark. This definition of a combinatory algebra $\mathcal{U}$ does not assume that the underlying universe is a family of classical sets. In the following chapters U will be a family of sets in some model **V** of **IZF**.

$$\mathcal{U} \models t_0 = t_1$$

will then be interpreted as

$$\mathbf{V} \models \forall u_0 \in U^{\sigma_0} \ldots u_{n-1} \in U^{\sigma_{n-1}} \ (t_0(u_i/x_i))_U = (t_1(u_i/x_i))_U$$

□

Definition 3.12 *Let $\mathcal{U}$ be a Σ-typed ca over C. The* theory *Th($\mathcal{U}$) of $\mathcal{U}$ is the set of equations that hold in $\mathcal{U}$.*

We add a related definition.

Definition 3.13 *Let T be a $CL^{\Sigma}(C)$-theory, $\mathcal{U}$ a Σ-typed ca over C. Then $\mathcal{U}$ is a* fully abstract model of *T if $T =$ Th($\mathcal{U}$).*

In model theory the usual expression for this state of affairs is that $\{\mathcal{U}\}$ is *complete for T*.

A combinatory algebra may have attributes not expressible in pure equations. We shall give here two important ones.

Definition 3.14 *A $CL^{\Sigma}(C)$-*model *is a structure*

$$\mathcal{U} = \langle U, \cdot_U, k_U, s_U, \epsilon_U, C_U \rangle$$

where $\langle U, \cdot_U, k_U, s_U, C_U \rangle$ is a combinatory algebra and

1. $\mathcal{U} \models (\epsilon \cdot x_0) \cdot x_1 = x_0 \cdot x_1$

2. $\mathcal{U} \models \epsilon \cdot \epsilon = \epsilon$

3.

$$\mathcal{U} \models \forall x \ (x_0 \cdot x = x_1 \cdot x) \rightarrow \epsilon \cdot x_0 = \epsilon \cdot x_1 \quad \text{<Weak Extensionality>} \tag{3.1}$$

A stronger property is expressed in the following

Definition 3.15 *A* $CL^{\Sigma}(C)$*-model* $\mathcal{U}$ *is an* extensional $\lambda^{\Sigma}(C)$-model *if it satisfies*

$$\mathcal{U} \models \forall x\,(x_0 \cdot x = x_1 \cdot x) \rightarrow x_0 = x_1 \quad \text{<Extensionality>}$$

$\mathcal{U}$ is a λ-model because of the following fact.

Lemma 3.16 *If* $\mathcal{U}$ *is an extensional* $\lambda^{\Sigma}(C)$*-model, then*

$$\mathbf{CL}^{\Sigma}(C) + \text{<ext>} \ \vdash\ t_0 = t_1 \rightarrow \mathcal{U} \models t_0 = t_1$$

and the fact that $\mathbf{CL}^{\Sigma}(C) +$ <ext> and $\boldsymbol{\lambda\beta\eta}^{\Sigma}(C)$ are equivalent.

Weak extensionality essentially means that we can pick a canonical representative among those points with the same applicative behaviour. Extensionality says that a point is *determined* by its applicative behaviour. Note that a $CL^{\Sigma}(C)$-model is extensional if $i = \epsilon$.

We now recall some definitions from universal algebra and model theory. Let $\mathcal{U}, \mathcal{V}$ be Σ-typed ca's over C.

Definition 3.17 $\mathcal{U}$ *and* $\mathcal{V}$ *are* equationally equivalent *(*$\mathcal{U} \equiv \mathcal{V}$*) if they have the same theory.*

Definition 3.18 *A* ca-homomorphism from $\mathcal{U}$ to $\mathcal{V}$ *is a Σ-typed function*

$$\phi = \{\ \phi^{\sigma} : U^{\sigma} \rightarrow V^{\sigma} \mid \sigma \in \Sigma\ \}$$

such that for $\sigma_0, \sigma_1, \sigma_2 \in \Sigma$

- $\phi^{\sigma_1}(u_0 \cdot_U^{\sigma_0,\sigma_1} u_1) = \phi^{\sigma_0 \rightarrow \sigma_1}(u_0) \cdot_V^{\sigma_0,\sigma_1} \phi^{\sigma_0}(u_1)$
- $\phi^{\sigma_0 \rightarrow (\sigma_1 \rightarrow \sigma_0)}(k_U^{\sigma_0,\sigma_1}) = k_V^{\sigma_0,\sigma_1}$
- $\phi^{(\sigma_0 \rightarrow (\sigma_1 \rightarrow \sigma_2)) \rightarrow ((\sigma_0 \rightarrow \sigma_1) \rightarrow (\sigma_0 \rightarrow \sigma_2))}(s_U^{\sigma_0,\sigma_1,\sigma_2}) = s_V^{\sigma_0,\sigma_1,\sigma_2}$
- $\phi^{\sigma}(c_U) = c_V$ *for* $c \in C^{\sigma}$

ϕ *is an* ca-isomorphism *if it is bijective.* $\mathcal{U}$ *and* $\mathcal{V}$ *are* isomorphic *(*$\mathcal{U} \cong_{\phi} \mathcal{V}$ *or simply* $\mathcal{U} \cong \mathcal{V}$*) if there is an isomorphism* ϕ *between them.*

ϕ *is an* embedding *if it is an injection.* $\mathcal{U}$ *is a* sub-combinatory algebra *of* $\mathcal{V}$ *(*$\mathcal{U} \subset \mathcal{V}$*) if* U *is a subset of* V *and the inclusion is a homomorphism. Write* $\mathcal{U} \,\tilde{\subset}\, \mathcal{V}$ *if* $\mathcal{U}$ *is isomorphic to a sub-ca of* $\mathcal{V}$*.*

Lemma 3.19 *$\mathcal{U}$ is a sub-ca of $\mathcal{V}$ if*

- $U \subset V$
- $\cdot_U^{\sigma_0,\sigma_1} = \cdot_V^{\sigma_0,\sigma_1} \cap (U^{\sigma_0 \to \sigma_1} \times U^{\sigma_0})$
- $k_U = k_V$
- $s_U = s_V$
- $c_U = c_V \quad$ *for all* $c \in C$

Proof. Clear. □

The next lemma is immediate.

Lemma 3.20 *Let $\mathcal{F}$ be a family of sub-ca's of $\mathcal{V}$. Then $\bigcap \mathcal{F}$ is a sub-ca of $\mathcal{V}$.*

This lemma justifies the following definition.

Definition 3.21 *Let X be a Σ-typed set, such that $X \subset V$. Then the* sub-ca of $\mathcal{V}$ generated by X *is defined by*

$$[X] \triangleq \bigcap \{ \mathcal{U} \mid X \subset U \wedge \mathcal{U} \subset \mathcal{V} \}$$

Example. The sub-ca of $\mathcal{V}$ generated by the empty set is the *interior* of $\mathcal{V}$, denoted by $\mathcal{V}^0$. It consists of all those elements in V that can be expressed by application of the constants in V. □

3.4 Sheaves and singletons

Everything done so far has been perfectly classical. In this section we look at a way of deriving new algebras from a given one which is only interesting in a constructive context.

The family of singletons of a classical ca can easily itself be turned into a combinatory algebra which will be isomorphic to the original one by $\{-\}$. Constructively however there is a number of competing definitions of a singleton. Only in some cases can the space of such singletons be equipped with the right structure. The definition of a singleton we are about to give—that of a 'stable $\neg\neg$-singleton'—will ensure that this can be done. Moreover this new ca will inherit many other properties the original one might possess.

De Vries' thesis [dV89] contains some useful definitions and fact. We refer to him for credits. Let j be a topology.

Definition 3.22 *A set X is a j-*sheaf *if*

$$\forall X_0 \subset X \; (j\exists! x \in X \; (x \in X_0) \rightarrow \exists! x \in X \; j(x \in X_0))$$

Denote with $\mathbf{Sh}_j$ the category of j-sheaves and set theoretic functions. It is a full subcategory of **Set**. We have a first result.

Lemma 3.23 *A j-sheaf X is j-separated.*

Proof. Pick $x_0', x_1' \in X$. Assume

$$j(x_0' = x_1') \tag{3.2}$$

Consider the set $\{x_0', x_1'\}$. By (3.2) we have

$$j\exists! x \in X \; (x \in \{x_0', x_1'\})$$

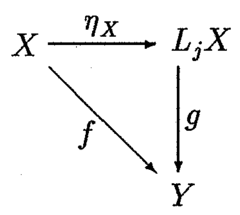
therefore

$$\exists! x \in X \; j(x \in \{x_0', x_1'\})$$

which implies $x_0' = x_1'$. □

We can associate a sheaf with every set.

Definition 3.24 *Let X be a set. The j-*sheaf associated with X *is a sheaf L_jX and function $\eta_X : X \rightarrow L_jX$ such that for every $f : X \rightarrow Y$ with Y a j-sheaf there exists exactly one function g such that the following diagram commutes.*



L_j can be explicitly constructed with the help of singletons. The classical definition is well known.

Definition 3.25 (classical) *A set $X_0 \subset X$ is a* singleton *if*

$$\exists! x \in X \; (x \in X_0)$$

There is more choice constructively. De Vries lists thirty-odd notions that are classically equivalent to the one given above, but whose equivalence cannot be proved constructively. We shall not detain ourselves but give the definition of singleton used in defining the associated sheaves.

Definition 3.26 (constructive) *A set* $X_0 \subset X$ *is a* j-singleton *if*

$$j\exists x_0 \in X\ \forall x \in X\ (x \in X_0 \leftrightarrow j(x_0 = x))$$

We define L_j.

For X a set put

$$L_jX \triangleq \{\ X_0 \subset X \mid X_0 = X_0^j \wedge X_0 \text{is a } j\text{-singleton}\ \}$$

and

$$\begin{array}{rccc} \eta_X : & X & \rightarrow & L_jX \\ & x & \mapsto & \{x\}^j \end{array}$$

The proof that L_jX is a j-sheaf with the universal property for every X can be found in [dV89].

Now let $f : X \rightarrow Y$. Define L_jf to be the unique function that makes the following diagram commute.

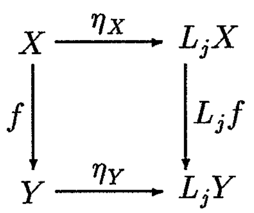


Diagram 1

It is clear that L_j is a functor.

Definition 3.27 L_j : **Set** $\rightarrow$ $\mathbf{Sh}_j$ *is the* associated j-sheaf functor.

We have an important lemma.

Lemma 3.28 L_j *preserves finite limits.*

Proof. See [Joh77]. □

In particular L_j preserves finite products.

3.4.1 The space of $\neg\neg$-singletons

We shall now become more specific and restrict ourselves to double negation as topology. The definition of a singleton becomes simpler in the case where X is $\neg\neg$-separated.

Lemma 3.29 *Let X be $\neg\neg$-separated. Then $X_0 \subset X$ is a $\neg\neg$-singleton iff*

$$\forall x_0, x_1 \in X_0\ (x_0 = x_1) \tag{3.3}$$

$$X_0 \neq \emptyset \tag{3.4}$$

Proof.

- $\rightarrow$

  Assume X_0 to be a $\neg\neg$-singleton. Pick $x_0', x_1' \in X_0$, assume

  $$x_0' \neq x_1' \tag{3.5}$$

  and

  $$\exists x_0 \in X\ \forall x \in X\ (x \in X_0 \leftrightarrow x = x_0) \tag{3.6}$$

  Further assume

  $$y \in X \land \forall x \in X\ (x \in X_0 \leftrightarrow x = y) \tag{3.7}$$

  From (3.7) we get $x_0' = x_1'$, contradicting (3.5). Therefore *not* (3.6), but this contradicts the fact that X_0 is a $\neg\neg$-singleton. Therefore (3.5) must be false. We get $x_0' = x_1'$ since X is $\neg\neg$-separated.

  Further assume

  $$\neg\, \exists x \in X\ (x \in X_0) \tag{3.8}$$

  and

  $$\exists x_0 \in X\ \forall x \in X\ (x \in X_0 \leftrightarrow x = x_0) \tag{3.9}$$

  Assume

  $$y \in X \land \forall x \in X\ (x \in X_0 \leftrightarrow x = y) \tag{3.10}$$

  Then $y \in X_0$, hence

  $$\exists x \in X\ (x \in X_0)$$

  This contradicts (3.8), therefore *not* (3.9). But this contradicts the fact that X_0 is a $\neg\neg$-singleton, therefore

  $$\neg\neg\, \exists x \in X\ (x \in X_0)$$

- $\leftarrow$

  Assume
  $$\neg\, \exists x_0 \in X\, \forall x \in X\, (x \in X_0 \leftrightarrow x = x_0) \tag{3.11}$$
  and
  $$\exists x \in X\, (x \in X_0) \tag{3.12}$$
  From (3.3) and (3.12) we get
  $$\exists x_0 \in X\, \forall x \in X\, (x \in X_0 \leftrightarrow x = x_0)$$
  which contradicts (3.11). Hence *not* (3.12). But this contradicts (3.4). Hence *not* (3.11).

□

So a $\neg\neg$-singleton of a $\neg\neg$-separated set X is a nonempty *partial element*.
From now on we shall write $-^s$ for $L_{\neg\neg}$.
We recapitulate.

Lemma 3.30 *Let X be $\neg\neg$-separated and $X_0 \subset X$. Then $X_0 \in X^s$ iff X_0 is a stable $\neg\neg$-singleton.*

Lemma 3.31 *If X is $\neg\neg$-separated, then for all $x \in X$*
$$\{x\}^{\neg\neg} = \{x\}$$

Proof. Follows from the definition of $\{x\}^{\neg\neg}$. □

Convention. Let X be $\neg\neg$-separated. We shall identify X with its image in X^s under $\eta_X = \{-\}$.

Lemma 3.32 *Let X be $\neg\neg$-separated. Then*
$$\forall X_0 \in X^s\, (\neg\neg\, X_0 \in X) \tag{3.13}$$

Proof. If $X_0 \in X^s$ then $\neg\neg\, \exists x \in X\, (x \in X_0)$ which with our convention is the same as (3.13). □

Lemma 3.33 *Let X, Y be $\neg\neg$-separated, $f : X \to Y$ and $X_0 \in X^s$. Then*
$$f^s(X_0) = \{\, f(x) \mid x \in X_0 \,\}^{\neg\neg}$$

Proof. To prove this we must show:

1. $\{ f(x) \mid x \in X_0 \}^{\neg\neg} \in Y^s$.

2. Diagram 1 commutes.

So

1. X_0 is nonempty, hence the same holds for $\{ f(x) \mid x \in X_0 \}^{\neg\neg}$. Further assume that

$$y'_0, y'_1 \in \{ f(x) \mid x \in X_0 \}^{\neg\neg}$$

We have

$$\neg\neg\, \exists! x \in X_0\, (y'_0 = f(x)) \wedge \neg\neg\, \exists! x \in X_0\, (y'_1 = f(x))$$

hence $\neg\neg\, y'_0 = y'_1$ and $y'_0 = y'_1$ since Y is $\neg\neg$-separated.

2. For all $x_0 \in X$ clearly

$$\{f(x_0)\} = \{ f(x) \mid x \in \{x_0\} \}^{\neg\neg}$$

□

In classical set theory, $\neg\neg$-singletons of a set are of no interest whatsoever; they are quite simply the normal singletons, i.e. they are inhabited. How far must we err from the classical path to get something of interest? Quite a bit.

Lemma 3.34 *Let X be classical. Assume*

$$\forall\phi\, (\neg\, \phi \vee \neg\neg\, \phi)$$

Then

$$\forall X_0 \in X^s\, (X_0 \textit{is inhabited})$$

Proof. Pick $X'_0 \in X^s$. Assume

$$\forall\phi\, (\neg\, \phi \vee \neg\neg\, \phi)$$

Instantiate this to

$$\forall x \in X\, (\neg\, x \in X'_0 \vee \neg\neg\, x \in X'_0)$$

By stability of X'_0 we get

$$\forall x \in X\, (x \in X'_0 \vee \neg\, x \in X'_0)$$

X is classical, X'_0 is a decidable subset of X, hence

$$\exists x \in X\ (x \in X'_0) \vee \neg\, \exists x \in X\ (x \in X'_0)$$

X'_0 is nonempty, therefore inhabited. □

We see that if

$$\forall \phi\ (\neg\, \phi \vee \neg\neg\, \phi)$$

then

$$\{-\}\ :\ X \rightarrow X^s$$

is a bijection.

This lemma tells us where to look for something more unusual. In any model where the $\neg\neg$- singletons are not just singletons in the classical sense, we have

$$\neg\, \forall \phi\ (\neg\, \phi \vee \neg\neg\, \phi)$$

Before we actually consider such a model, we prove a partial converse of Lemma 3.34.

Lemma 3.35 *Let X be discrete and $2 \hookrightarrow X$. Then*

$$\exists f\ (f\ :\ X \hookrightarrow\!\!\!\!\rightarrow X^s)$$

implies

$$\forall \phi\ (\neg\, \phi \vee \neg\neg\, \phi)$$

Proof. The image of a discrete set under a bijection is discrete (Lemma 2.16). Pick $x'_0, x'_1 \in X$ with $x'_0 \neq x'_1$ and consider

$$\{\ x'_0 \mid \neg\, \phi\ \} \cup \{\ x'_1 \mid \neg\neg\, \phi\ \}$$

This is certainly a partial element of X. Moreover it is nonempty since

$$\neg\neg\, (\neg\, \phi \vee \neg\neg\, \phi)$$

and stable because for any $x \in X$

$$\begin{aligned}
&\neg\neg\, x \in \{\ x'_0 \mid \neg\, \phi\ \} \leftrightarrow \\
&\qquad \neg\neg\, x = x'_0 \wedge \neg\neg\neg\, \phi \leftrightarrow \quad \text{(by discreteness of } X\text{)} \\
&\qquad x = x'_0 \wedge \neg\, \phi \leftrightarrow \\
&\qquad x \in \{\ x'_0 \mid \neg\, \phi\ \}
\end{aligned}$$

Similarly

$$\neg\neg\, x \in \{\, x_1' \mid \neg\neg\, \phi \,\} \leftrightarrow x \in \{\, x_1' \mid \neg\neg\, \phi \,\}$$

As X^s is discrete, either $\{x_0'\}$ or $\{x_1'\}$ is equal to

$$\{\, x_0' \mid \neg\, \phi \,\} \cup \{\, x_1' \mid \neg\neg\, \phi \,\}$$

therefore either x_0' or x_1' is in

$$\{\, x_0' \mid \neg\, \phi \,\} \cup \{\, x_1' \mid \neg\neg\, \phi \,\}$$

and therefore $\neg\, \phi$ or $\neg\neg\, \phi$. □

Lemma 3.36 *Let $V \in \mathbf{V}$, $2 \hookrightarrow V$. Then*

$$\mathbf{V}^{(2^{<\omega})^{op}} \models \neg\, \exists f\, (f \,:\, V^* \hookrightarrow\!\!\!\!\to (V^*)^s)$$

Proof. Pick $v_0, v_1 \in V$, $v_0 \neq v_1$ and $s_0, s_1 \in 2^{<\omega}$, $s_0 \subset s_1$. Assume

$$s_1 \,\Vdash_\emptyset\, \exists f\, (f \,:\, V^* \hookrightarrow\!\!\!\!\to (V^*)^s)$$

Then

$$s_1 \,\Vdash_\emptyset\, (V^*)^s \text{ is discrete}$$

But consider some $X \in \mathbf{V}^{(2^{<\omega})^{op}}$, such that

$$s_1 0 \,\Vdash_\emptyset\, v_0^* \in X \in (V^*)^s$$

and

$$s_1 1 \,\Vdash_\emptyset\, v_1^* \in X \in (V^*)^s$$

We have

$$s_1 \,\not\Vdash_\emptyset\, \{v_0^*\} = X \vee \{v_0^*\} \neq X \;\lightning$$

hence

$$s_0 \,\Vdash_\emptyset\, \neg\, \exists f\, (f \,:\, V^* \hookrightarrow\!\!\!\!\to (V^*)^s)$$

□

Let me add the following thought.

Conjecture. The following statement is provable in **IZF**: Let X be such that $X \not\approx X^s$. Then there is an infinite sequence

$$X = X_0 \subset X_1 \subset \ldots \subset X^s$$

of subsets of X^s such that

$$X_0 \neq X_1 \neq \ldots \neq X^s$$

□

3.4.2 Extensions of relations

If X is $\neg\neg$-separated X can be regarded as a subspace of X^s. We are going to investigate in this section which properties of X are inherited by X^s.

First recall that if $X^0, \ldots, X^{n-1}$ are $\neg\neg$-separated then

$$\prod_{i \in n} (X^i)^s \cong (\prod_{i \in n} X^i)^s$$

We shall make this bijection explicit.

Lemma 3.37 *The function*

$$\begin{array}{rccl} \phi & : & \prod_{i \in n} (X^i)^s & \rightarrow & (\prod_{i \in n} X^i)^s \\ & & \langle X_0, \ldots, X_{n-1} \rangle & \mapsto & \{ \langle x_0, \ldots, x_{n-1} \rangle \mid x_i \in X_i \} \end{array}$$

is a bijection.

Proof. Straightforward. □

Assume $X^0, \ldots, X^{n-1}$ are $\neg\neg$-separated. Let $R \subset \prod_{i \in n} X^i$. Then

$$R^s \subset (\prod_{i \in n} X^i)^s \cong \prod_{i \in n} (X^i)^s$$

is an extension of R. We wish to give an explicit description of R^s, in the case where R is stable. As a preparation we prove the following result.

Lemma 3.38 *Let $X^0, \ldots, X^{n-1}$ be $\neg\neg$-separated, R a stable relation on $\prod_{i \in n} X^i$ and $X_i \in (X^i)^s$ for $i \in n$. Then*

$$\neg\neg\ \exists x_0 \in X_0 \ldots \exists x_{n-1} \in X_{n-1}\ R x_0 \ldots x_{n-1} \tag{3.14}$$

$$\leftrightarrow$$

$$\forall x_0 \in X_0 \ldots \forall x_{n-1} \in X_{n-1}\ R x_0 \ldots x_{n-1} \tag{3.15}$$

Proof.

- $\rightarrow$

  Assume (3.14). For $i \in n$ pick $x'_i \in X_i$. Assume

  $$\neg\ R x'_0 \ldots x'_{n-1} \tag{3.16}$$

  Further assume

  $$\bigwedge_{i \in n} \forall x_i, y_i \in X_i\ (x_i = y_i) \tag{3.17}$$

Finally, assume

$$\exists x_0 \in X_0 \ldots \exists x_{n-1} \in X_{n-1}\, Rx_0 \ldots x_{n-1} \tag{3.18}$$

and

$$\bigwedge_{i \in n} z_i \in X_i \wedge Rz_0 \ldots z_{n-1} \tag{3.19}$$

We instantiate (3.17) to

$$\bigwedge_{i \in n} x'_i = z_i$$

therefore by (3.16)

$$\neg\, Rz_0 \ldots z_{n-1}$$

This contradicts (3.19). Therefore *not* (3.18), but this contradicts (3.14), therefore *not*(3.17); this in turn contradicts the fact that $X_0, \ldots, X_{n-1}$ are $\neg\neg$-singletons; we therefore reject Assumption (3.16) and the result follows by stability of R.

- $\leftarrow$

Assume (3.15) and

$$\neg\, \exists x_0 \in X_0 \ldots \exists x_{n-1} \in X_{n-1}\, Rx_0 \ldots x_{n-1} \tag{3.20}$$

Further assume

$$\bigwedge_{i \in n} \exists x_i\, (x_i \in X_i) \tag{3.21}$$

and

$$\bigwedge_{i \in n} y_i \in X_i \tag{3.22}$$

By 3.15 we have $Ry_0 \ldots y_{n-1}$. Therefore

$$\exists x_0 \in X_0 \ldots \exists x_{n-1} \in X_{n-1}\, Rx_0 \ldots x_{n-1}$$

This contradicts (3.20). Therefore *not* (3.21), but this contradicts the fact that $X_0, \ldots, X_{n-1}$ are singletons. Hence *not* (3.20).

□

Lemma 3.39 *Let* $X^0, \ldots, X^{n-1}$ *and* R *be as in Lemma 3.38. For* $i \in n$ *let* $X_i \in (X^i)^s$. *Then*

$$R^s X_0 \ldots X_{n-1} \leftrightarrow \forall x_0 \in X_0 \ldots \forall x_{n-1} \in X_{n-1}\, (Rx_0 \ldots x_{n-1})$$

Proof. We have

$$\begin{aligned}
&R^s X_0 \dots X_{n-1} \leftrightarrow \\
&\quad \langle X_0, \dots, X_{n-1} \rangle \in R^s \leftrightarrow \quad \text{(by Lemma 3.37)} \\
&\quad \{ \langle x_0, \dots, x_{n-1} \rangle \mid x_i \in X_i \} \in R^s \leftrightarrow \quad \text{(by definition)} \\
&\quad \neg\neg\, \exists x_0 \in X_0 \dots \exists x_{n-1} \in X_{n-1}\, R x_0 \dots x_{n-1}
\end{aligned}$$

The result follows from Lemma 3.38. □

We move on to more complex formulas. Not every sentence that holds for the points in X also holds for the points in X^s. To see what is preserved we need the following definition.

Definition 3.40 *Let ϕ be a formula of* **IZF**. *It is* negative *if it is built up from atomic formulas using only $\wedge, \rightarrow$ and $\forall$.*

In fact we shall want to be slightly more general. For the next three lemmas let $X^0, \dots, X^{n-1}$ be $\neg\neg$-separable and $\mathcal{X}_0, \dots, \mathcal{X}_{n-1}$ families of singletons such that

$$X^i \subset \mathcal{X}_i \subset (X^i)^s$$

for $i \in n$. For $i \in m$ let

$$(R_i)_X \subset \prod_{i \in n} X^i$$

be stable. Furthermore let ϕ be a negative formula built from relation symbols R_i.

Interpret R_i on $\prod_{i\in n} \mathcal{X}_i$ as $(R_i)^s_X \cap \prod_{i\in n} (X^i)^s$.

First we shall argue that ϕ is stable.

Lemma 3.41

$$\forall X_0 \in \mathcal{X}_0 \dots X_{n-1} \in \mathcal{X}_{n-1}\, (\neg\neg\, \phi \rightarrow \phi)$$

Proof. This will be by structural induction on ϕ.

1. $\phi \equiv R$

   Pick $X'_i \in \mathcal{X}_i$. We have

$$\begin{aligned}
&\neg\neg\, R^s_X X'_0 \dots X'_{n-1} \leftrightarrow \quad \text{(by Lemma 3.39)} \\
&\quad \neg\neg\, \forall x_0 \in X'_0 \dots \forall x_{n-1} \in X'_{n-1}\, R_X x_0 \dots x_{n-1} \rightarrow \\
&\quad \forall x_0 \in X'_0 \dots \forall x_{n-1} \in X'_{n-1}\, \neg\neg\, R_X x_0 \dots x_{n-1} \leftrightarrow \quad \text{(by stability of } R) \\
&\quad \forall x_0 \in X'_0 \dots \forall x_{n-1} \in X'_{n-1}\, R_X x_0 \dots x_{n-1} \leftrightarrow \quad \text{(by Lemma 3.39)} \\
&\quad R^s_X X'_0 \dots X'_{n-1}
\end{aligned}$$

2. $\phi \equiv \bot$
   clear.

3. $\phi \equiv \psi \wedge \xi$

$$\begin{aligned} &\neg\neg\,(\psi \wedge \xi) \leftrightarrow \\ &\quad (\neg\neg\,\psi \wedge \neg\neg\,\xi) \leftrightarrow \quad \text{(by induction hypothesis)} \\ &\quad \psi \wedge \xi \end{aligned}$$

4. $\phi \equiv \psi \to \xi$

$$\begin{aligned} &\neg\neg\,(\psi \to \xi) \to \\ &\quad (\neg\neg\,\psi \to \neg\neg\,\xi) \leftrightarrow \quad \text{(by induction hypothesis)} \\ &\quad \psi \to \xi \end{aligned}$$

5. $\phi \equiv \forall x\, \psi$

$$\begin{aligned} &\neg\neg\,\forall x\,\psi \to \\ &\quad \forall x\,\neg\neg\,\psi \leftrightarrow \quad \text{(by induction hypothesis)} \\ &\quad \forall x\,\psi \end{aligned}$$

□

Next we prove that ϕ as a relation on $\prod_{i\in n} \mathcal{X}_i$ is an extension of its interpretation on $\prod_{i\in n} X^i$.

Lemma 3.42

$$\forall x_0 \in X^0 \ldots x_{n-1} \in X^{n-1}\ (\phi(x_0,\ldots,x_{n-1}) \leftrightarrow \phi(\{x_0\},\ldots,\{x_{n-1}\}))$$

Proof. This proof is also by induction on the structure of ϕ.

1. $\phi \equiv R$
   Pick $x_i' \in X^i$. We have

$$\begin{aligned} &R_X^s\{x_0'\}\ldots\{x_{n-1}'\} \leftrightarrow \quad \text{(by Lemma 3.39)} \\ &\quad \forall x_0 \in \{x_0'\}\ \ldots \forall x_{n-1} \in \{x_{n-1}'\}\ R_X x_0 \ldots x_{n-1} \leftrightarrow \\ &\quad R x_0' \ldots x_{n-1}' \end{aligned}$$

2. $\phi \equiv \bot$
   from 1.

3. $\phi \equiv \psi \wedge \xi$
   clear.

4. $\phi \equiv \psi \to \xi$
   Pick $x'_i \in X^i$. Assume

$$(\psi \to \xi)(\{x'_0\}, \dots, \{x'_{n-1}\}) \tag{3.23}$$

   and $\psi(x'_0, \dots, x'_{n-1})$. By induction hypothesis then $\psi(\{x'_0\}, \dots, \{x_{n-1}\})$, therefore by (3.23) $\xi(\{x'_0\}, \dots, \{x'_{n-1}\})$ and again by induction hypothesis $\xi(x'_0, \dots, x'_{n-1})$.

   The other direction is analogous.

5. $\phi \equiv \forall x_0\, \psi$
   Pick $x'_1 \in X^1, \dots, x'_{n-1} \in X^{n-1}$.

   - $\to$
     Assume

$$\forall x_0 \in X^0\ \psi(x'_1, \dots, x'_{n-1})$$

     Pick $X'_0 \in \mathcal{X}_0$. Assume

$$\neg\, \psi(X'_0, \{x'_1\}, \dots, \{x'_{n-1}\})$$

     and

$$\exists x_0\, (x_0 \in X'_0) \tag{3.24}$$

     as well as

$$y_0 \in X'_0$$

     i.e.

$$\{y_0\} = X'_0$$

     But we have

$$\psi(y_0, x'_1, \dots, x'_{n-1})$$

     hence by induction hypothesis

$$\psi(\{y_0\}, \{x'_1\}, \dots, \{x'_{n-1}\})\ \lightning$$

therefore *not* (3.24). This on the other hand contradicts the fact that X_0' is a singleton, therefore

$$\neg\neg\, \psi(X_0', \{x_1'\}, \ldots, \{x_{n-1}'\})$$

and the result by Lemma 3.41.

- $\leftarrow$

  Assume

  $$\forall X_0 \in \mathcal{X}_0\ \psi(X_0, \{x_1'\}, \ldots, \{x_{n-1}'\})$$

  Then in particular for $x_0 \in X^0$

  $$\psi(\{x_0\}, \{x_1'\}, \ldots, \{x_{n-1}'\})$$

  therefore by induction hypothesis

  $$\psi(x_0, x_1', \ldots, x_{n-1}')$$

□

With these two facts established we can finally prove the result we have been aiming for.

Lemma 3.43

$$\forall X_0 \in \mathcal{X}_0 \ldots \forall X_{n-1} \in \mathcal{X}_{n-1}\ \phi(X_0, \ldots, X_{n-1}) \tag{3.25}$$

$$\leftrightarrow$$

$$\forall x_0 \in X^0 \ldots \forall x_{n-1} \in X^{n-1}\ \phi(x_0, \ldots, x_{n-1}) \tag{3.26}$$

Proof.

- $\rightarrow$

  Assume (3.25). Pick $x_i \in X^i$. We have

  $$\phi(\{x_0'\}, \ldots, \{x_{n-1}\})$$

  hence by the previous lemma

  $$\phi(x_0', \ldots, x_{n-1}')$$

- $\leftarrow$

  Assume (3.26). Pick $X'_i \in \mathcal{X}^i$. Assume

$$\neg\, \phi(X'_0, \ldots, X'_{n-1})$$

and

$$\bigwedge_{i \in n} \exists x_i\; (x_i \in X'_i) \tag{3.27}$$

as well as

$$\bigwedge_{i \in n} y_i \in X'_i$$

that is

$$\{y_i\} = X'_i \text{ for all } i \in n$$

We have $\phi(y_0, \ldots, y_{n-1})$, therefore by the previous lemma

$$\phi(X'_0, \ldots, X'_{n-1}) \lightning$$

hence *not* (3.27), but the X'_i are nonempty, therefore

$$\neg\neg\, \phi(X'_0, \ldots, X'_{n-1})$$

and the result follows by Lemma 3.41.

□

We see that a relation R on X that is *defined* by negative formulas has an extension R^s on X^s that fits the same definition.

Example. The three axioms used in defining partial orders are all negative formulas. Partial orders with a least element can likewise be defined using negative formulas. □

Counterexample. The definition of a cpo uses existential quantification. Therefore if X is a cpo with respect to a partial order $\leq_X$, then X^s will not in general be a cpo with respect to $\leq^s_X$. We construct a counterexample.

Take the model $\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}}$. Consider in $\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}}$ the unit interval $[0,1] \subset \mathbf{R}$ with the normal ordering. We have

$$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \models [0,1] \text{ is a cpo}$$

Define the sequence $\{X_n\}_{n \in \omega} \subset [0,1]^s$ by

$$X_n = \{\ \textstyle\sum_{i \leq n} s(i) 2^{-i} \mid \{\ s \mid |s| > n\ \}\ \}$$

We have

$$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \models \forall x_n \in X_n \, \forall x_{n+1} \in X_{n+1} \, (x_n \leq x_{n+1})$$

therefore

$$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \models X_n \leq^s_{[0,1]} X_{n+1}$$

but

$$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \models \{X_n\}_{n\in\omega} \text{ does not have a supremum in } [0,1]^s.$$

To see this, pick $s_0, s_1 \in 2^{<\omega}$ such that $s_0 \subset s_1$. Assume

$$s_1 \Vdash_\emptyset \exists X_\omega \in [0,1]^s \, (X_\omega \text{ is the supremum of } \{X_n\}_{n\in\omega}) \tag{3.28}$$

X_ω is nonempty, therefore there exists a $s_2 \in 2^{<\omega}, s_1 \subset s_2$ such that

$$s_2 \Vdash_\emptyset \exists r \in [0,1] \, (r \in X_\omega)$$

On the one hand

$$s_2 \Vdash_\emptyset r \geq \sum_{i\in|s_2|+1} (s_2(i)2^{-i}) + 2^{-(|s_2|+1)}$$

since r is an upper bound for $\{ \sum_{n\in\omega} s(n)2^{-n} \mid s_2 \subset s \}$, on the other

$$s_2 \Vdash_\emptyset r = \sum_{i\in|s_2|+1} (s_2(i)2^{-i})$$

since it has to be the smallest. We have arrived at a contradiction. Therefore *not* (3.28), hence

$$s_0 \Vdash_\emptyset \neg \exists X_\omega \in [0,1]^s \, (X_\omega \text{ is the supremum of } \{X_n\}_{n\in\omega})$$

□

Example. An equivalence relation is reflexive, symmetric and transitive. These properties are defined by negative formulas. Therefore the maximal extension of an equivalence relation will also be one. Equality is of course a prime example of an equivalence relation. □

We shall now investigate a special kind of relations: functions. We are interested in the question whether for every function

$$f : X \to Y$$

its extension f^s is a function in $X^s \Rightarrow Y^s$.

Lemma 3.44 *Let X, Y be $\neg\neg$-separated and*

$$f : X \to Y$$

Then $f^s \subset X^s \times Y^s$ is a function.

Proof. We have to check two things: whether f^s is functional and whether it is total.

1. Functionality is expressed by the sentence

   $$fxy_0 = fxy_1 \to y_0 = y_1$$

   This is a negative formula. It follows that f^s is functional.

2. The definition of totality uses an existential quantifier, so Lemma 3.43 cannot be applied.

   Define for $X_0 \in X^s$

   $$Y_0 \triangleq f(X_0)^{\neg\neg} = \{\, y \in Y \mid \neg\neg\, \exists x \in X_0\, (y = f(x_0))\,\}$$

   We have already seen (Lemma 3.33) that $Y_0 \in Y^s$.

   Now pick $x_0' \in X_0, y_0' \in Y_0$. We have

   $$\neg\neg\, \exists x \in X_0\, (y_0' = f(x))$$

   therefore by Lemma 3.38

   $$\forall x \in X_0\, (y_0' = f(x))$$

   hence

   $$y_0' = f(x_0')$$

   But this means $f^s X_0 Y_0$.

□

So $f^s : X^s \to Y^s$ as the extension of $f : X \to Y$ is the same as the image of f under $L_{\neg\neg}$. The ambiguous use of f^s is therefore justified.

Corollary 3.45 *Let X, Y be $\neg\neg$-separated. Then every function $f : X \to Y$ has a unique extension in $X^s \Rightarrow Y^s$.*

Proof. Uniqueness follows from Lemmas 2.21, 3.23 and 3.32. □

Constants are of course just a particularly simple kind of function. What is the extension of a constant c_X from a set X to X^s? We calculate

$$c_X^s = X_0 \leftrightarrow \forall x \in X_0 \, (c_X = x)$$

which simply means

$$c_X^s = \{c_X\}$$

As a last point in this section we investigate the behaviour of homomorphisms.

Definition 3.46 *Let X, Y be sets, R a relational symbol and R_X, R_Y its interpretation in X and Y. Then a function*

$$f \, : \, X \to Y$$

is an R-homomorphism if

$$\forall x \in X(R_X x \to R_Y f(x))$$

We can prove the following statement.

Lemma 3.47 *Let X, Y be $\neg\neg$-separated, R a relational symbol and R_X, R_Y stable. Interpret R on X^s and Y^s as R_X^s and R_Y^s, respectively. Let $f \, : \, X \to Y$ be an R-homomorphism. Then f^s is an R-homomorphism.*

Proof. Pick $X_0' \in X^s$. Assume $R^s X_0'$. This means

$$\forall x \in X_0' \, (Rx)$$

f is an R-homomorphism, therefore

$$\forall x \in X_0' \, (Rf(x))$$

which is equivalent to $R^s f^s(X_0')$. □

Example. Let $\mathbf{P}_0, \mathbf{P}_1$ be partial orders. Then the extension of a monotone function from P_0 to P_1 will again be monotone with respect to $\leq_{P_0}^s$ and $\leq_{P_1}^s$. □

3.5 New ca's from old

Convention. We shall from now let combinatory algebras inherit the attributes that their universes possess and speak, e.g., of *¬¬-separated* ca's rather than of algebras whose universe is ¬¬-separated.

Let $\mathcal{U} = \langle U, \cdot_U, k_U, s_U, C_U \rangle$ be a fixed ¬¬-separated combinatory algebra. With all the work done in the previous section it is now easy to turn U^s into a ca.

Definition 3.48 *Let*

$$\mathcal{U}^s = \langle U^s, \cdot_{U^s}, k_{U^s}, s_{U^s}, C_{U^s} \rangle$$

where $\cdot_{U^s}, k_{U^s}, s_{U^s}$ *are* $\overset{s}{\cdot}_U, \overset{s}{k}_U, \overset{s}{s}_U$ *and* c_{U^s} *is* $\overset{s}{c}_U$ *for all* $c \in C$.

Theorem 3.49 $\mathcal{U}^s$ *is a combinatory algebra.*

Proof. By Lemma 3.44 application is well defined and total. Also, since application and the constants have in U^s been defined as the extensions of their interpretations in U, $\mathcal{U}^s$ satisfies all equations that hold in $\mathcal{U}$. □

This last remark can be strengthened.

Lemma 3.50 *Let* $\mathcal{V}$ *by a* Σ*-typed ca over* C *and* $\mathcal{U} \mathrel{\tilde{\subset}} \mathcal{V} \mathrel{\tilde{\subset}} \mathcal{U}^s$. *Then*

$$Th(\mathcal{U}) = Th(\mathcal{V})$$

Proof. Easy. □

We add the proofs that any ¬¬-separated $CL^{\Sigma}(C)$-algebra $\mathcal{U}$ is a $CL^{\Sigma}(C)$-model, respectively an extensional $\lambda^{\Sigma}(C)$-model whenever $\mathcal{U}^s$ is.

Lemma 3.51 *Let*

$$\mathcal{U} = \langle U, \cdot_U, k_U, s_U, \epsilon_U, C_U \rangle$$

be a ¬¬-separated ca over C *and* $\mathcal{V}$ *a ca such that*

$$\mathcal{U} \mathrel{\tilde{\subset}} \mathcal{V} \mathrel{\tilde{\subset}} \mathcal{U}^s$$

Then

1. $\mathcal{V}$ *is a* $CL^{\Sigma}(C)$*-model iff* $\mathcal{U}$ *is.*

2. $\mathcal{V}$ *is an extensional* $\lambda^{\Sigma}(C)$*-model iff* $\mathcal{U}$ *is.*

Proof. Weak extensionality and extensionality are both defined by negative formulas and therefore preserved. □

3.6 Functional models

We already mentioned that the λ-calculus is a theory of functions. It is therefore natural to look for models in which types of the form $\sigma_0 \to \sigma_1$ are interpreted as subspaces of the space of total functions $U^{\sigma_0} \Rightarrow U^{\sigma_1}$ and application $\cdot_U^{\sigma_0,\sigma_1}$ is simple function application. This leads to the following concept.

Definition 3.52 *Let $\mathcal{U}$ be a Σ-typed ca. A function $f : U^{\sigma_0} \to U^{\sigma_1}$ is* representable *if*

$$\exists u_0 \in U^{\sigma_0 \to \sigma_1}\ \forall u_1 \in U^{\sigma_0}\ (f(u_1) = u_0 \cdot_U^{\sigma_0,\sigma_1} u_1)$$

u_1 *then* represents f.

We use $[U^{\sigma_0} \to U^{\sigma_1}]$ to denote the representable functions in $U^{\sigma_0} \Rightarrow U^{\sigma_1}$. Define a Σ-indexed function F by setting for γ a ground type

$$F^{\gamma} : U^{\gamma} \hookrightarrow\!\!\!\!\to U^{\gamma}$$

$$F^{\gamma}(u) = u$$

and at higher types for $\sigma_0, \sigma_1 \in \Sigma$

$$F^{\sigma_0 \to \sigma_1} : U^{\sigma_0 \to \sigma_1} \to [U^{\sigma_0} \to U^{\sigma_1}]$$

$$(F^{\sigma_0,\sigma_1}(u_0))(u_1) = u_0 \cdot_U^{\sigma_0,\sigma_1} u_1$$

If all functions are uniquely representable, i.e. F is a bijection, then $\mathcal{U}$ is essentially just a family of functionspaces.

Definition 3.53 *A Σ-typed set V is a* full function space hierarchy *if for all $\sigma_0, \sigma_1 \in \Sigma$*

$$V^{\sigma_0 \to \sigma_1} = (V^{\sigma_0} \Rightarrow V^{\sigma_1})$$

Let $\mathrm{Fsp}(\{X^{\gamma}\}_{\gamma\in\Gamma})$ *denote the full function space hierarchy induced by* $\{X^{\gamma}\}_{\gamma\in\Gamma}$.

It may be wise to add explicitly the definition for the untyped case.

Definition 3.54 *A set V is a* full function space hierarchy *if*

$$V \approx V^V$$

This paves the way for the following definition.

Definition 3.55 *An extensional $\lambda^{\Sigma}(C)$-model is a* full function space model *if it is isomorphic to a $\lambda^{\Sigma}(C)$-model whose universe is a full function space hierarchy, and whose application is function application.*

Now assume that $\mathcal{U}$ is a *CL*-model. The Axiom of Weak Extensionality states that for every representable function there is a *canonical* representative. This enables us to define a Σ-typed function G such that

$$F \circ G = \mathrm{id}_{[U^{\sigma_0} \to U^{\sigma_1}]}$$

without appealing to the Axiom of Choice. For $f \in [U^{\sigma_0} \to U^{\sigma_1}]$ we simply let $G(f)$ be the unique element in $F(\epsilon)(F^{-1}(\{f\}))$ (c.f. (3.1)).

If $\mathcal{U}$ is extensional, every representable function is represented by a *unique* element. In terms of F and G this means that $G = F^{-1}$. So in the extensional case $\mathcal{U}$ is isomorphic to a *CL*-model $\mathcal{V}$ that has been obtained by taking some sets as ground types and at types $\sigma_0 \to \sigma_1$ for $V^{\sigma_0 \to \sigma_1}$ a subset of the total function space $V^{\sigma_1} \Rightarrow V^{\sigma_0}$. Call such models *functional.*

Evidently, when we have an extensional λ-model, we are interested in the nature of the representable functions and wish to describe them without reference to the λ-model. In the following chapter we shall look at an example that shows that in many cases this is far from easy. Of course everything that has been said also applies to the untyped models. They will be examined in Chapter 5.

For the moment we shall stay with the typed calculus and drop the assumption that $\mathcal{U}$ be extensional. Can $\mathcal{U}$ still be seen to have some functional character? In order to find an answer, we shall look at yet another way of deriving new combinatory algebras from old ones.

We need a new concept.

Definition 3.56 *Let V be a full function space hierarchy. A relation $R \subset V^n$ is* logical *if for all $\sigma_0, \sigma_1 \in \Sigma$ and for all $f_0, \ldots, f_{n-1} \in V^{\sigma_0 \to \sigma_1}$*

$$\begin{aligned} &R^{\sigma_0 \to \sigma_1} f_0 \ldots f_{n-1} \leftrightarrow \\ &\qquad \forall x_0, \ldots, x_{n-1} \in V^{\sigma_0}\ (R^{\sigma_0} x_0 \ldots x_{n-1} \to R^{\sigma_1} f_0(x_0) \ldots f_{n-1}(x_{n-1})) \end{aligned}$$

Hence a logical relation is determined by $\{R^{\gamma}\}_{\gamma \in \Gamma}$.

Now assume $\mathcal{V}$ is a full function space model, and R an inhabited unary logical relation on V. Recall that we have

$$\begin{aligned} (k_V(v_0))(v_1) &= v_0 \\ ((s_V(v_0))(v_1))(v_2) &= (v_0(v_2))(v_1(v_2)) \end{aligned}$$

We see that whenever Rv_0, then $Rk_V(v_0)$, hence Rk_V, and similarly Rs_V. Assume that also Rc for all $c \in C$. We can define a sub-ca of $\mathcal{V}$ by taking as its universe all those points $v \in V$ for which Rv. Note that the fact that R is logical ensures that application is total.

Let us be more explicit. A unary relation R on V is a *subset* of V. It is logical if we have at types $\sigma_0, \sigma_1 \in \Sigma$

$$\begin{aligned} R^{\sigma_0 \to \sigma_1} &= \{ f : V^{\sigma_0} \to V^{\sigma_1} \mid R^{\sigma_0 \to \sigma_1} f \} \\ &= \{ f : V^{\sigma_0} \to V^{\sigma_1} \mid f(R^{\sigma_0}) \subset R^{\sigma_0} \} \end{aligned}$$

i.e. $R^{\sigma_0 \to \sigma_1}$ is the set of functions in $V^{\sigma_0} \Rightarrow V^{\sigma_1}$ that are the *extensions* of some function in $R^{\sigma_0} \Rightarrow R^{\sigma_1}$. These remarks lead to the following definition.

Definition 3.57 *Let $\mathcal{V}$ be a full function space model and $\{R^\gamma\}_{\gamma \in \Gamma}$ a family of inhabited sets such that*

$$R^\gamma \subset V^\gamma \text{ for all } \gamma \in \Gamma$$

and Rc_V for all $c \in C$ where R is the logical relation on V determined by $\{R^\gamma\}$. Let $\mathcal{E}(\{R^\gamma\}_{\gamma \in \Gamma}, \mathcal{V})$ denote the Σ-typed sub-ca of $\mathcal{V}$ over C whose universe is defined by

$$\begin{aligned} E^\gamma &\triangleq R^\gamma \text{ for } \gamma \in \Gamma \\ E^{\sigma_0 \to \sigma_1} &\triangleq \mathrm{Ext}(R^{\sigma_0} \subset V^{\sigma_0}, R^{\sigma_1} \subset V^{\sigma_1}) \end{aligned}$$

$\mathcal{E}$ is called an *extension model.*

We add the analogous definition for the untyped case.

Definition 3.58 *Let $\mathcal{V}$ be a full function space model and R an inhabited logical relation on V such that Rc_V for all $c \in C$. Let $\mathcal{E}(\{R\}, \mathcal{V})$ denote the sub-ca of $\mathcal{V}$ over C whose universe is R.*

When is $\mathcal{E}$ extensional? We see from the definition that then for all $\sigma_0, \sigma_1 \in \Sigma$

$$\forall f_0, f_1 \in E^{\sigma_0 \to \sigma_1} \ (\forall x \in E^{\sigma_0}(f_0(x) = f_1(x)) \to f_0 = f_1 \ )$$

or

$$\begin{aligned} &\forall f_0, f_1 : V^{\sigma_0} \to V^{\sigma_1} \\ &\quad (f_0(E^{\sigma_0}) \subset E^{\sigma_1} \wedge f_1(E^{\sigma_0}) \subset E^{\sigma_1} \wedge \forall x \in E^{\sigma_0} \ (f_0(x) = f_1(x)) \to f_0 = f_1) \end{aligned}$$

which simply means that whenever a function in $E^{\sigma_1} \Rightarrow E^{\sigma_0}$ has an extension in $V^{\sigma_1} \Rightarrow V^{\sigma_0}$, this extension is unique!

Now assume we have an extensional λ-model $\mathcal{U}$. Is it an extension model? Classically certainly not, except in the trivial case where $\mathcal{U}$ is a full function space model. Constructively, however, the situation is different. We shall prove in the following two chapters, once for the typed, once for the untyped calculus, that for *every* (classical) extensional λ-model $\mathcal{M}$ there is a tree $\mathbf{P}$ such that

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models \mathcal{M}^* \text{is isomorphic to an extension model.}$$

$\mathbf{P}$ only depends on the *cardinality* of $\mathcal{M}$. Furthermore $\mathcal{M}$ has the same theory as the full function space model used in the construction of the extension model.

If we are only interested in *theories* we may recall that for every extensional λ-model there is a *countable* extensional λ-model with the same theory. We obtain: there is a partial order $\mathbf{P}$ such that for *every* classical extensional λ-model $\mathcal{M}$ there is a full function space model $\mathcal{V}$ in $\mathbf{V}^{\mathbf{P}^{\mathrm{op}}}$ such that $\mathcal{V}$ has the same theory as $\mathcal{M}$. Surely heaven for λ-calculators!

Chapter 4

Typed extensional λ-models

We shall start this chapter by giving two examples of λ-models, the first quite simple, the second more complicated. After this we prove the typed version of the main theorem in this thesis. We then go back to the two examples and apply the results of the theorem to them.

4.1 Monotone functions MON

Let Σ be the set of types with only one ground type γ. Define a Σ-typed partial order $\mathbf{P}$ by setting

- $\mathbf{P}^{\gamma} = \mathbf{2}$
- $\mathbf{P}^{\sigma_0 \to \sigma_1} = \langle P^{\sigma_0 \to \sigma_1}, \leq_P^{\sigma_0 \to \sigma_1} \rangle$ where
  - $P^{\sigma_0 \to \sigma_1} = \{ f : P^{\sigma_0} \to P^{\sigma_1} \mid f \text{is monotone} \}$ and
  - $f_0 \leq_P^{\sigma_0 \to \sigma_1} f_1 \leftrightarrow$

$$\forall p_0, p_1 \in P^{\sigma_0} \ (p_0 \leq_P^{\sigma_0} p_1 \to f_0(p_0) \leq_P^{\sigma_1} f_1(p_1))$$

Note that the ordering of the function spaces is simply the pointwise one and in fact a logical relation.

We can now describe the theory of monotone functions **MON**. Let C contain two constants $\top, \bot$ of ground type. Define the structure

$$\mathcal{P} = \langle P, \cdot_P, k_P, s_P, C_P \rangle$$

by setting

- $f \cdot_P^{\sigma_0, \sigma_1} p = f(p)$ for every $\sigma_0, \sigma_1 \in \Sigma$

- $k_P^{\sigma_0,\sigma_1}(p_0)(p_1) = p_0$ for every $\sigma_0, \sigma_1 \in \Sigma$
- $s_P^{\sigma_0,\sigma_1,\sigma_2}(p_0)(p_1)(p_2) = (p_0(p_2))(p_1(p_2))$ for every $\sigma_0, \sigma_1, \sigma_2 \in \Sigma$
- $\top_P = 1, \;\; \bot_P = 0$

It is clear that $\mathcal{P}$ is an extensional $\lambda^\Sigma(C)$-model. Let **MON** be the set of equations between terms in $\Lambda^\Sigma(C)$ that hold in $\mathcal{P}$.

4.2 Programming computable functions PCF

The theory **PCF** was defined by Plotkin in [Plo77].

Let Σ be the set of types with two ground types, B and N. C has the following constants

- $\bot^\sigma : \sigma$ for every $\sigma \in \Sigma$
- $F^\sigma : (\sigma \to \sigma) \to \sigma$ for every $\sigma \in \Sigma$
- $\mathit{true}, \mathit{false} : B$
- $n : N$ for every $n \in \omega$
- $\mathit{if} - \mathit{then} - \mathit{else} - : B^3 \to B$

Here

$$B^3 \to B = B \to (B \to (B \to B))$$

Next, define a Σ-typed cppo **C** by letting

- $\mathbf{C}^B = \mathbf{2}_\bot$
- $\mathbf{C}^N = \omega_\bot$
- $\mathbf{C}^{\sigma_0 \to \sigma_1} = \langle C^{\sigma_0 \to \sigma_1}, \leq_C^{\sigma_0 \to \sigma_1}, \bot_C^{\sigma_0 \to \sigma_1} \rangle$ where
    - $C^{\sigma_0 \to \sigma_1} = \{ f : C^{\sigma_0} \to C^{\sigma_1} \mid f \text{ is continuous} \}$
    - $$f_0 \leq_C^{\sigma_0 \to \sigma_1} f_1 \leftrightarrow$$
      $$\forall c_0, c_1 \in C^{\sigma_0} \; (c_0 \leq_C^{\sigma_0} c_1 \to f_0(c_0) \leq_C^{\sigma_1} f_1(c_1))$$
    - $\bot_C^{\sigma_0 \to \sigma_1}(c) = \bot_C^{\sigma_1}$

As in the previous example define a structure

$$\mathcal{C} = \langle C, \cdot_C, k_C, s_C, C_C \rangle$$

by setting

- $f \cdot_C^{\sigma_0,\sigma_1} c = f(c)$ for every $\sigma_0, \sigma_1 \in \Sigma$
- $k_C^{\sigma_0,\sigma_1}(c_0)(c_1) = c_0$ for every $\sigma_0, \sigma_1 \in \Sigma$
- $s_C^{\sigma_0,\sigma_1,\sigma_2}(c_0)(c_1)(c_2) = (c_0(c_2))(c_1(c_2))$ for every $\sigma_0, \sigma_1, \sigma_2 \in \Sigma$
- $\perp_C^B = \perp_2$
- $\perp_C^N = \perp_\omega$
- $\perp_C^{\sigma_0 \to \sigma_1}(c) = \perp_C^{\sigma_1}$
- $true_C = 0$
- $false_C = 1$
- $F_C^\sigma(f) = \bigvee_{n\in\omega}\{f^n(\perp^\sigma)\}$
- $if - then - else -_C =$ the $\leq_C^{B^3 \to B}$-least function f such that

$$\begin{aligned} f(true_C, c_0, c_1) &= c_0 \\ f(false_C, c_0, c_1) &= c_1 \end{aligned}$$

Remark. $\perp$ should be thought of as representing 'undefined'. With this interpretation *if – then – else –* is *sequential* in the sense that its first argument has to be evaluated to obtain a result. Note that

$$(if \perp then\ true\ else\ true)_C = \perp_C^B$$

It could be argued that the value of this expression should be $true_C$ since the result does not depend on whether the antecedent is true or false. This, however, would require *parallel* evaluation of the arguments. □

PCF will now *not* be the set of equations that hold in $\mathcal{C}$. Rather we want two terms to be equivalent if they show the same behaviour in all program contexts, i.e. if one can be substituted for the other in all programs. We make this idea precise.

Definition 4.1 *A* program *is a closed* $\Lambda^{\Sigma}(C)$*-term of ground type.*

Definition 4.2 *For every* $\sigma \in \Sigma$ *the* Σ*-typed set of* σ-contexts $\Lambda^{\Sigma}(C)[-]^{\sigma}$ *is inductively defined by*

- $[-]^{\sigma} \in \Lambda^{\sigma}(C)[-]^{\sigma}$
- *if* $t_0 \in \Lambda^{\sigma_0 \to \sigma_1}(C)$ *and* $t_1 \in \Lambda^{\sigma_0}(C)[-]^{\sigma_2}$ *then* $t_0 t_1 \in \Lambda^{\sigma_1}[-]^{\sigma_2}$
- *if* $t_0 \in \Lambda^{\sigma_0 \to \sigma_1}(C)[-]^{\sigma_2}$ *and* $t_1 \in \Lambda^{\sigma_0}(C)$ *then* $t_0 t_1 \in \Lambda^{\sigma_0 \to \sigma_1}(C)[-]^{\sigma_2}$
- *if* $x \in \mathrm{Vars}^{\sigma_0}$ *and* $t \in \Lambda^{\sigma_1}[-]^{\sigma_2}$ *then* $\lambda x.t \in \Lambda^{\sigma_0 \to \sigma_1}[-]^{\sigma_2}$

Now let

$$|t_0 : \sigma = t_1 : \sigma| \in \mathbf{PCF}$$

if in all program σ-contexts $t[-]$

$$\mathcal{C} \models t[t_0] = t[t_1]$$

It is clear that

$$Th(\mathcal{C}) \subset \mathbf{PCF}$$

because if two terms are identical in the model, their behaviour is the same in all contexts. We now proceed to show that the above inclusion is *proper* and hence $\mathcal{C}$ is not a fully abstract model of **PCF**.

For this consider the two terms

$$portest_{true},\ portest_{false} : (B \to (B \to B)) \to B$$

Let v stand for either *true* or *false* and set

$$\begin{aligned}
portest_v = \lambda f.(&if(f\ true\ \bot)\\
&then(if(f\ \bot\ true)\\
&\quad then(if(f\ false\ false)\\
&\qquad then\ \bot\\
&\qquad else\ v)\\
&\quad else\ \bot)\\
&else\ \bot)
\end{aligned}$$

Define *parallel or* (*por*) to be the $\leq_C^{B\to(B\to B)}$- minimal function f such that

$$\begin{aligned} f(true_C, c) &= true_C \\ f(c, true_C) &= true_C \\ f(false_C, false_C) &= false_C \end{aligned}$$

Now,

$$(portest_{true})_C(por) = true_C$$

and

$$(portest_{false})_C(por) = false_C$$

hence

$$(portest_{true})_C \neq (portest_{false})_C$$

Therefore we have

$$|portest_{true} = portest_{false}| \notin Th(\mathcal{U})$$

On the other hand we do have

$$|portest_{true} = portest_{false}| \in \mathbf{PCF}$$

This is so because *por*, the only value in $C^{B\to(B\to B)}$ at which $(portest_{true})_C$ and $(portest_{false})_C$ differ is *not* denoted by any $\Lambda^\Sigma(C)$-term and the two show the same behaviour in all $B \to (B \to B)$-contexts. Which leaves us with the question why *por* is not denotable.

Before giving an answer, we show that $leftor_C$, the function which is minimal among those $f : (B \to (B \to B)$ for which

$$\begin{aligned} f(true_C, c) &= true_C \\ f(false_C, false_C) &= false_C \end{aligned}$$

is denotable. Simply put

$$leftor \triangleq \lambda xy.if\ x\ then\ true\ else\ y$$

$leftor_C$ can be thought of as evaluating its left argument first, and only if this shows a result does it look at its right argument.

The general situation is described in a theorem proved by Berry in [Ber78]. To state it we, introduce an order $\sqsubseteq$ on the $CL^\Sigma(C)$-terms.

Definition 4.3 *$\sqsubseteq$ is inductively defined on $\Lambda^{\Sigma}(C)^2$.*

- $\bot^{\sigma} \sqsubseteq^{\sigma} t$ *for all* $\sigma \in \Sigma$ *and* $t : \sigma$
- *if* $t_0 \sqsubseteq^{\sigma_0 \to \sigma_1} t_1$ *then* $t_0 t \sqsubseteq^{\sigma_1} t_1 t$ *for all* $\sigma_0, \sigma_1 \in \Sigma$
- *if* $t_0 \sqsubseteq^{\sigma_0} t_1$ *then* $t t_0 \sqsubseteq^{\sigma_1} t t_1$ *for all* $\sigma_0, \sigma_1 \in \Sigma$
- *if* $x \in \mathrm{Vars}^{\sigma_0}$ *and* $t_0 \sqsubseteq^{\sigma_1} t_1$ *then* $\lambda x.t_0 \sqsubseteq^{\sigma_0 \to \sigma_1} \lambda x.t_1$ *for all* $\sigma_0, \sigma_1 \in \Sigma$

We have

Theorem 4.4 *Let $t_0 : \sigma_0$ be a $CL^{\Sigma}(C)$-term such that*

$$\mathbf{CL}^{\Sigma}(C) + \mathbf{PCF} \vdash t_0 = \bot^{\sigma_0}$$

Then either

$$\mathbf{CL}^{\Sigma}(C) + \mathbf{PCF} \vdash t_1 = \bot^{\sigma_0}$$

for all t_1 with $t_0 \sqsubseteq^{\sigma_0} t_1$, or there is an occurrence of a $\bot^{\sigma_1}$ in t_0 (the sequentiality index of t_0*), such that for all t_1 with $t_0 \sqsubseteq^{\sigma_0} t_1$ the following statement holds: whenever*

$$\mathbf{CL}^{\Sigma}(C) + \mathbf{PCF} \nvdash t_1 = \bot^{\sigma_0}$$

then there exists a term $t_2 : \sigma_0$ such that $t_2 \sqsubseteq^{\sigma_0} t_1$ and t_2 has been obtained from t_0 by replacing $\bot^{\sigma_1}$ with another term.

There is then obviously no term t denoting *por*: it would have to satisfy

$$\mathbf{CL}^{\Sigma}(C) + \mathbf{PCF} \vdash t \bot^{B} \bot^{B} = \bot^{B}$$

and

$$\mathbf{CL}^{\Sigma}(C) + \mathbf{PCF} \vdash t\ \mathit{true}\ \bot^{B} = \mathit{true}$$

as well as

$$\mathbf{CL}^{\Sigma}(C) + \mathbf{PCF} \vdash t \bot^{B}\ \mathit{true} = \mathit{true}$$

which is ruled out by the above theorem.

Milner ([Mil77]) proved that there is a unique (up to isomorphism) fully abstract extensional model of **PCF**, with ground types $\mathbf{2}_{\bot}$ and $\omega_{\bot}$. The representable functions in this fully abstract extensional model are called *sequential.* The model in [Mil77] was obtained by syntactic means; a *semantic* definition of the sequential functions has yet to be found. Some progress has been made and is recorded in [BCL85].

4.3 The main theorem (typed version)

In this section we shall prove the most important result of this thesis. Before we state it as a theorem, we look again at the two ways of deriving new combinatory algebras from given ones.

We first demonstrated that given a $\neg\neg$-separated ca $\mathcal{U}$, we can turn the family of stable $\neg\neg$-singletons of U into a ca $\mathcal{U}^s$. All sub-ca's $\mathcal{V}$ of $\mathcal{U}^s$ for which $\mathcal{U} \tilde{\subset} \mathcal{V}$ are equationally equivalent. They are either all extensional, or none of them is.

Further we saw that, given a full function space model $\mathcal{V}$ and a Γ-indexed family $\{R^\gamma\}_{\gamma\in\Gamma}$ of inhabited sets, such that $R^\gamma \subset V^\gamma$ for all $\gamma \in \Gamma$, we can define a sub-ca of $\mathcal{V}$ whose universe consists of those points $v \in V$ that satisfy Rv, where R is the logical relation induced by $\{R^\gamma\}_{\gamma\in\Gamma}$. This subalgebra is not necessarily weakly extensional.

We now start at the other end: Take a ca $\mathcal{U}$. Is it consistent with **IZF** that there is a full function space model $\mathcal{V}$ such that for all $\gamma \in \Gamma$

$$U^\gamma \subset V^\gamma \subset (U^s)^\gamma$$

and $\mathcal{U}$ is isomorphic to $\mathcal{E}(\{U^\gamma\}_{\gamma\in\Gamma}, \mathcal{V})$? I do not know the answer to this question in full generality. I can however give a result that solves an instance of the problem.

Before making this claim precise we should take note of the following fact.

Lemma 4.5 *Let $\mathcal{M} \in \mathbf{V}$ be a combinatory algebra over C, and $\mathbf{C}$ a category. Then*

$$\mathbf{V}^{\mathbf{C}} \models \mathcal{M}^* \text{ is a combinatory algebra over } C$$

Moreover $\mathcal{M}$ is equationally equivalent to $\mathcal{M}^*$. The same applies to (weak) extensionality. Therefore $\mathcal{M}$ is a $CL^\Sigma(C)$-model, respectively an extensional $\lambda^\Sigma(C)$-model, iff $\mathcal{M}^*$ is.

Proof. All concepts involved are defined by restricted formulas. The result follows therefore from Theorem 2.32. □

So let us now state the main theorem.

Theorem 4.6 *Let $\mathcal{M} \in \mathbf{V}$ be an extensional $\lambda^\Sigma(C)$-model. Then there exists a tree $\mathbf{P}$ dependent only on the cardinality of $\mathcal{M}$ such that $\mathbf{V}^{\mathbf{P}^{op}}$ satisfies the following statement.*

There is a full function space model $\mathcal{U}$ such that

$$(M^*)^\gamma \subset U^\gamma \subset ((M^*)^s)^\gamma$$

for all $\gamma \in \Gamma$ and

$$\mathcal{M}^* \cong \mathcal{E}(\{(M^*)^\gamma\}_{\gamma\in\Gamma}, \mathcal{U}) \tilde{\subset} \mathcal{U} \tilde{\subset} (\mathcal{M}^*)^s$$

Before giving a proof of Theorem 4.6, we shall go back to the two examples at the beginning of this chapter and put them in perspective.

4.3.1 MON revisited

We get two results from Theorem 4.6.

- There is a tree $\mathbf{P}$ such that in $\mathbf{V}^{\mathbf{P}^{op}}$ the monotone function model is *isomorphic* to an extension model. Exactly the monotone functions have extensions in the full function space model. All functions in the full function space model are monotone with respect to the extension of the partial order in the monotone function model.
- The monotone function model is *equationally equivalent* to a full function space model.

4.3.2 PCF revisited

There is a tree $\mathbf{P}$ such that in $\mathbf{V}^{\mathbf{P}^{op}}$ (the image of) the fully abstract classical model of **PCF** is *isomorphic* to an extension model. Exactly the sequential functions have extensions in the full function space model. The classical model is *equationally equivalent* to the full function space model. Therefore the full function space model is also fully abstract. Since sequentiality was defined in terms of full abstraction, all the functions in the full function space model can be said to be sequential. This of course still does not give us a semantic description of 'sequentiality'.

4.3.3 Another thought

We have not made use of the fact that the tree used in building our (constructive) models only depends on the cardinality of the original (classical) models. Every theory that has a fully abstract model, has a *countable* fully abstract model.

So, if we are only interested in theories, we can find a set theoretic universe in which all theories with a fully abstract (classical) model have a fully abstract full function space model.

4.3.4 The proof of the main theorem

Let

$$\mathcal{M} = \langle M, \cdot_M, k_M, s_M, C_M \rangle$$

be an extensional $\lambda^{\Sigma}(C)$-model.

Add M as constants and $|M|^+$ variables for every type $\sigma \in \Sigma$ to $\Lambda^{\Sigma}(C)$.

Definition 4.7 *An* environment *E is a sequence in $\bigcup_{\sigma\in\Sigma}(\mathrm{Vars}^{\sigma} \times M^{\sigma})$, in which every variable appears at most once. An environment E is* finite *if it is a finite set. E is* total *if every variable appears in it.*

The obvious interpretation of an environment is that of a list of variables and the values assigned to them. If E_0 and E_1 are environments let E_0E_1 denote the environment that has been obtained by

- *adding* $\langle x, v\rangle$ to E_0 if x does not appear in E_0
- *replacing* in E_0 the value of x by v if x appears in E_0

for all pairs $\langle x, v\rangle$ in E_1. If we want to give a finite environment explicitly, we shall write

$$\{v_0/x_0; \ldots ; v_{n-1}/x_{n-1}\}$$

or

$$\{x_0 := v_0; \ldots ; x_{n-1} := v_{n-1}\}$$

for

$$\langle\langle x_0, v_0\rangle, \ldots , \langle x_{n-1}, v_{n-1}\rangle\rangle$$

Finally, let $E(x)$ denote the value associated with x in E, and write $x \in \mathrm{dom}(E)$ if E assigns a value to x.

Now let $\mathcal{E}$ designate the set of all environments, and $\mathcal{E}_{\mathrm{fin}}$ and $\mathcal{E}_{\mathrm{tot}}$ the sets of finite and total environments. $\mathcal{E}$ will be ordered by extension $\subset$. Evidently, in this order the empty environment is the least point in $\mathcal{E}$. Set

$$\boldsymbol{\mathcal{E}}_{\mathbf{fin}} = \langle \mathcal{E}_{\mathrm{fin}}, \subset, \emptyset \rangle$$

Remark. Environments are usually defined as *sets* rather than *lists* of variable-value-pairs. We prefer lists because $\mathcal{E}_{\text{fin}}$ ordered by extension is a tree. □

Note that $\mathcal{E}_{\text{fin}}$ modulo isomorphism only depends on the *cardinality* of M.

Notation. Terms t of $\lambda^{\Sigma}(C)$ are evaluated with respect to a total environment E. So let t_E stand for $[\![t]\!]^M_E$ and write

$$\mathcal{M}, E \models t_0 = t_1$$

if $(t_0)_E = (t_1)_E$. □

We shall prove the claims of the main theorem in three steps.

1. We construct $\mathcal{U} \in \mathbf{V}^{\mathcal{E}_{\text{fin}}{}^{\text{op}}}$.

2. We prove that $\mathcal{U}$ is a full function space model. This we do by defining a Σ-typed bijection $\phi : U \hookrightarrow\!\!\!\!\!\rightarrow \text{Fsp}(\{U^\gamma\}_{\gamma\in\Gamma})$.

3. We prove that $\phi|M^*$ is a surjection onto $|\mathcal{E}(\{(M^*)^\gamma\}_{\gamma\in\Gamma},\mathcal{U})|$ and therefore a ca-isomorphism between $\mathcal{M}^*$ and $\mathcal{E}(\{(M^*)^\gamma\}_{\gamma\in\Gamma},\mathcal{U})$.

1. We describe $\mathcal{U}$. For every $\sigma \in \Sigma$ let $U^\sigma \in \mathbf{V}^{\mathcal{E}_{\text{fin}}{}^{\text{op}}}$ be the set of all subsets $U_t \in \mathbf{V}^{\mathcal{E}_{\text{fin}}{}^{\text{op}}}$ of M^* where $t \in \Lambda^{\Sigma}(C)$ such that for all $m \in M^\sigma$ and all $E_0 \in \mathcal{E}_{\text{fin}}$

$$\begin{aligned} E_0 \Vdash_{\emptyset} m^* \in U_t \leftrightarrow \\ \forall E_1 \in \mathcal{E}_{\text{tot}}\,(E_0 \subset E_1 \to t_{E_1} = m) \end{aligned}$$

It is clear from this definition that for all $t \in \Lambda^\sigma(C)$

$$\mathbf{V}^{\mathcal{E}_{\text{fin}}{}^{\text{op}}} \models U_t \text{ is a stable } \neg\neg\text{-singleton.}$$

Further we have

$$\begin{aligned} E_0 \Vdash_{\emptyset} U_{t_0} = U_{t_1} \leftrightarrow \\ \forall E_1 \in \mathcal{E}_{\text{tot}}\,(E_0 \subset E_1 \to (t_0)_{E_1} = (t_1)_{E_1}) \end{aligned} \tag{4.1}$$

We define application.

$$\begin{aligned} \mathbf{V}^{\mathcal{E}_{\text{fin}}{}^{\text{op}}} \models\ & U_{t_0} \cdot_U U_{t_1} \triangleq \\ & U_{t_0} \overset{s}{\cdot}_{M^*} U_{t_1} = \\ & \{\, m \in M^* \mid \forall m_0 \in U_{t_0}\, \forall m_1 \in U_{t_1}\, (m = m_0 \cdot_{M^*} m_1) \,\} = \\ & U_{t_0 t_1} \end{aligned}$$

The second equation holds by Lemma 3.39; we justify the third. Pick $E_0, E_1 \in \mathcal{E}_{\mathrm{fin}}$, such that $E_0 \subset E_1$. Take $m \in M$. We have

$$E_1 \Vdash_\emptyset m^* \in \{ m \in M^* \mid \forall m_0 \in U_{t_0}\ \forall m_1 \in U_{t_1}\ (m = m_0 \cdot_{M^*} m_1) \}$$

iff for all $E \in \mathcal{E}_{\mathrm{tot}}$ with $E_1 \subset E$

$$m = [\![t_0]\!]_E^M \cdot_M [\![t_1]\!]_E^M = [\![t_0 t_1]\!]_E^M$$

iff

$$E_1 \Vdash_\emptyset m^* \in U_{t_0 t_1}$$

As for constants $c \in C$, we define

$$\mathbf{V}^{\mathcal{E}_{\mathrm{fin}}^{\mathrm{op}}} \models c_U \triangleq (c_M^*)^s = \{c_M^*\} = U_{c_M}$$

We have by Lemma 3.19 that $\mathcal{U}$ is a subalgebra of $(\mathcal{M}^*)^s$ in $\mathbf{V}^{\mathcal{E}_{\mathrm{fin}}}$.

2. We prove that $\mathcal{U}$ is a full function space model. We construct the Σ-typed bijection

$$\phi\ :\ U \hookrightarrow\!\!\!\!\rightarrow \mathrm{Fsp}(\{U^\gamma\}_{\gamma\in\Gamma})$$

At ground types we put

$$\phi^\gamma = \mathrm{id}_{U^\gamma}$$

For higher types assume that ϕ^{σ_0} and ϕ^{σ_1} are bijections. Define

$$\phi^{\sigma_0\to\sigma_1}\ :\ U^{\sigma_0\to\sigma_1} \to (U^{\sigma_1})^{U^{\sigma_0}}$$

by setting for $t_0 : \sigma_0 \to \sigma_1,\ t_1 : \sigma_0$

$$\phi^{\sigma_0\to\sigma_1}(U_{t_0})(U_{t_1}) = U_{t_0} \cdot_U U_{t_1}$$

We must show that $\phi^{\sigma_0\to\sigma_1}$ is a bijection.

Hence

(a) $\phi^{\sigma_0\to\sigma_1}$ is surjective.

We must convince ourselves that

$$\mathbf{V}^{\mathcal{E}_{\mathrm{fin}}^{\mathrm{op}}} \models \forall f\ :\ U^{\sigma_0} \to U^{\sigma_1}\ \exists u \in U^{\sigma_0\to\sigma_1}\ (\phi^{\sigma_0\to\sigma_1}(u) = f)$$

So pick $E_0, E_1 \in \mathcal{E}_{\text{fin}}$, such that $E_0 \subset E_1$, and $f' \in (U^{\sigma_1})^{U^{\sigma_0}}(E_1)$. We check that

$$E_1 \Vdash_{\{f'/f\}} \exists u \in U^{\sigma_0 \to \sigma_1} \, (\phi^{\sigma_0 \to \sigma_1}(u) = f)$$

Take

$$f^{\Lambda} \; : \; \Lambda^{\sigma_0}(C) \to \Lambda^{\sigma_1}(C)$$

such that

$$f^{\Lambda}(t_0) = t_1 \to E_1 \; \Vdash_{\{f'/f\}} \; f(U_{t_0}) = U_{t_1}$$

Pick $x \in \text{Vars}^{\sigma_0} \setminus (\bigcup_{m \in M} \text{FV}(f^{\Lambda}m) \cup \text{dom}E_1)$. This can be done since the right hand side is of cardinality $|M|$.

$$E_1 \; \Vdash_{\{f'/f\}} \; U_{f^{\Lambda}x} = U_{(\lambda x.t_1)x}$$

for some $t_1 \in \Lambda^{\sigma_1}(C)$, so

$$E_1 \; \Vdash_{\{f'/f\}} \; f(U_x) = U_{f^{\Lambda}x} = U_{(\lambda x.t_1)x} = U_{\lambda x.t_1} \cdot_U^{\sigma_0,\sigma_1} U_x$$

We wish to prove that then for every $t_0 \in \Lambda^{\sigma_0}(C)$

$$E_1 \; \Vdash_{\{f'/f\}} \; f(U_{t_0}) = U_{f^{\Lambda}t_0} = U_{(\lambda x.t_1)t_0} = U_{\lambda x.t_1} \cdot_U^{\sigma_0,\sigma_1} U_{t_0}$$

i.e. that $U_{\lambda x.t_1}$ represents f'. In order to prove equality in the middle recall (4.1). Pick $E_2 \in \mathcal{E}_{\text{tot}}$, such that $E_1 \subset E_2$.
We show that

$$\begin{aligned}
(f^{\Lambda}t_0)_{E_2} &= (f^{\Lambda}(t_0)_{E_2})_{E_2} \\
&= (f^{\Lambda}(t_0)_{E_2})_{E_2\{x := (t_0)_{E_2}\}} \\
&= (f^{\Lambda}x)_{E_2\{x := (t_0)_{E_2}\}} \\
&= ((\lambda x.t_1)x)_{E_2\{x := (t_0)_{E_2}\}} \\
&= ((\lambda x.t_1)t_0)_{E_2}
\end{aligned}$$

To justify

$$(f^{\Lambda}t_0)_{E_2} = (f^{\Lambda}(t_0)_{E_2})_{E_2}$$

let $\text{FV}(t_0) \subset \vec{y}$.

$$E_1\{\vec{y} := E_2(\vec{y})\} \; \Vdash_{\{f'/f\}} \; U_{t_0} = U_{(t_0)_{E_2}}$$

so

$$E_1\{\vec{y} := E_2(\vec{y})\} \Vdash_{\{f'/f\}} U_{f^\Lambda t_0} = U_{f^\Lambda(t_0)_{E_2}}$$

Further since $x \notin \mathrm{Vars}(f^\Lambda(t_0)_{E_2})$

$$(f^\Lambda(t_0)_{E_2})_{E_2} = (f^\Lambda(t_0)_{E_2})_{E_2\{x := (t_0)_{E_2}\}} \tag{4.2}$$

Then

$$E_1\{x := (t_0)_{E_2}\} \Vdash_{\{f'/f\}} U_{(t_0)_{E_2}} = U_x$$

implies

$$E_1\{x := (t_0)_{E_2}\} \Vdash_{\{f'/f\}} U_{f^\Lambda(t_0)_{E_2}} = U_{f^\Lambda x}$$

hence

$$(f^\Lambda(t_0)_{E_2})_{E_2\{x := (t_0)_{E_2}\}} = (f^\Lambda x)_{E_2\{x := (t_0)_{E_2}\}}$$

Finally

$$(f^\Lambda x)_{E_2\{x := (t_0)_{E_2}\}} = ((\lambda x.t_1)x)_{E_2\{x := (t_0)_{E_2}\}} = ((\lambda x.t_1)t_0)_{E_2}$$

(b) $\phi^{\sigma_0 \to \sigma_1}$ is injective.
We have in $\mathbf{V}^{\mathcal{E}_{\mathrm{fin}}^{\mathrm{op}}}$: $\mathcal{M}^* \tilde{\subset} \mathcal{U} \tilde{\subset} (\mathcal{M}^*)^s$ and $\mathcal{M}^*$ is extensional, hence by Lemma 3.51 so is $\mathcal{U}$. This implies that ϕ is injective.

3. $\phi|M^* : \mathcal{M}^* \to \mathcal{E}(\{(M^*)^\gamma\}_{\gamma\in\Gamma}, \mathcal{U})$ is a ca-isomorphism.
Identify M^* with its image in $|\mathcal{E}(\{(M^*)^\gamma\}_{\gamma\in\Gamma}, \mathcal{U})|$ under ϕ. We must check that $\phi^{\sigma_0\to\sigma_1}$ is surjective. So pick $E_0 \in \mathcal{E}_{\mathrm{fin}}$ and $f' \in (U^{\sigma_1})^{U^{\sigma_0}}(E_0)$ such that

$$E_0 \Vdash_{\{f'/f\}} f((M^*)^{\sigma_0}) \subset (M^*)^{\sigma_1}$$

By 2. there is a term $t \in \Lambda^{\sigma_0\to\sigma_1}$ such that U_t represents f'. We wish to prove that

$$E_0 \Vdash_{\{f'/f\}} U_t \in (M^*)^{\sigma_0\to\sigma_1}$$

So assume

$$E_0 \not\Vdash_{\{f'/f\}} U_t \in (M^*)^{\sigma_0\to\sigma_1} \tag{4.3}$$

We shall derive a contradiction. (4.3) means that for all $m \in M^{\sigma_0\to\sigma_1}$

$$E_0 \not\Vdash_{\{f'/f\}} m^* \in U_t$$

hence there are $E_1, E_2 \in \mathcal{E}_{\mathrm{tot}}$ with $E_0 \subset E_1$ and $E_0 \subset E_2$, such that

$$t_{E_1} \neq t_{E_2}$$

By extensionality there is an $m_0 \in M^{\sigma_0}$ such that

$$t_{E_1} \cdot_M m_0 \neq t_{E_2} \cdot_M m_0$$

Therefore for all $m \in M^{\sigma_0}$

$$E_0 \not\Vdash_{\{f'/f\}} m^* \in U_t \cdot_U U_{m_0}$$

and

$$E_0 \not\Vdash_{\{f'/f\}} U_t \cdot_U U_{m_0} \in (M^*)^{\sigma_1}$$

We deduce by <RAA>

$$E_0 \Vdash_{\{f'/f\}} U_t \in (M^*)^{\sigma_0 \to \sigma_1}$$

This concludes the proof of the main theorem.

4.3.5 Related results and discussion

Evidently in the proof of the main theorem

$$\mathbf{P} = \mathcal{E}_{\mathbf{fin}} \cong (|\mathcal{M}|^+)^{<\omega_0}$$

If we now remember that for every extensional $\lambda^{\Sigma}(C)$-model there is an equationally equivalent *countable* extensional $\lambda^{\Sigma}(C)$-model we can combine Lemma 3.50 and Theorem 4.6 and draw the following conclusion.

Theorem 4.8 *Let $\mathcal{M}$ be an extensional $\lambda^{\Sigma}(C)$-model and $\mathbf{P} = \omega_1^{<\omega_0}$. Then $\mathbf{V}^{\mathbf{P}^{\mathrm{op}}}$ satisfies*

> *There is a full function space model which is equationally equivalent to $\mathcal{M}^*$.*

And finally we add

Theorem 4.9 *Let T be a $\lambda^{\Sigma}(C)$-theory that has a fully abstract extensional model and $\mathbf{P} = \omega_1^{<\omega_0}$. Then $\mathbf{V}^{\mathbf{P}^{\mathrm{op}}}$ satisfies*

> *T has a fully abstract full function space model.*

We add a few comments about the proof of the main theorem.

First we note that the condition that $\mathbf{P}$ be $|M|^+$-branching is unnecessarily strict, albeit convenient. All we have to require of the underlying tree is that it be hereditarily of width $|M|^+$.

Secondly, it is not certain whether the underlying tree cannot be trimmed even further, i.e. whether it could be hereditarily of width $|M|$ or even smaller. Going over the proof, we see that we had to make sure that there existed a *generic* point U_x for every function f at $\emptyset$, i.e. a point that determines the behaviour of f at every U_m for $m \in M$ and hence f itself. The point where we needed the assumption that there was a variable x *not free* in $f^\Lambda(m)$ for all $m \in M$ is (4.2). It would suffice that for every f there is an x which is *irrelevant* in $f^\Lambda(m_0)$ for all $m_0 \in M$ in the sense that

$$(f^\Lambda m_0)_E = (f^\Lambda m_0)_{E\{x := m_1\}}$$

for all total environments E and $m_1 \in M$. $(\lambda x.y)z$ is an example of an expression in which z is free but irrelevant.

It is my guess that an x with the property just described indeed exists for every f. What consequences would a proof of this conjecture have?

In the case of M being countable we obtain the main theorem with the weaker assumption of $\mathbf{P}$ being a tree of hereditary width ω_0 (e.g. the binary tree). On the other hand we shall see in Chapter 6 that we cannot make $\mathbf{P}$ any smaller, i.e. somewhere of only finite width. So we would know that we have achieved the best result possible in this area.

It is moreover quite probable that a model $\mathbf{V}^{\mathbf{P}^{\mathrm{op}}}$ with $\mathbf{P}$ countable is a very different world from a $\mathbf{P}$ containing an uncountable antichain. This is certainly the case in the classical models $\mathbf{V}^{\mathbf{B}}$ (c.f. [Bel77]) where

$$-^* : \mathbf{V} \to \mathbf{V}^{\mathbf{B}}$$

preserves ordinals iff there is no uncountable antichain in $\mathbf{B}$.

As a third point we might note that in the proof of the main theorem $\mathcal{U}$ is the subalgebra of $(\mathcal{M}^*)^s$ generated by the Σ-typed set $V \in \mathbf{V}^{\mathbf{P}^{\mathrm{op}}}$, $V \subset (M^*)^s$ that contains all U_m for all $m \in M$ and all U_x for $x \in$ Vars. We have

Lemma 4.10 *For all $\sigma \in \Sigma$*

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models \neg\, \forall U_0 \in V^\sigma\ \exists U_1 \in (M^*)^\sigma\ (U_0 \neq U_1)$$

Proof. Pick $E_0, E_1 \in \mathcal{E}_{\text{fin}}$, $E_0 \subset E_1$, $x : \sigma \notin \text{dom} E_1$. Then for any $m \in M^\sigma$

$$E_1\{x := m\} \Vdash_\emptyset U_x = U_m$$

hence

$$E_1 \not\Vdash_\emptyset \exists U_1 \in (M^*)^\sigma \, (U_x \neq U_1)$$

therefore

$$E_0 \not\Vdash_\emptyset \forall U_0 \in V^\sigma \, \exists U_1 \in (M^*)^\sigma \, (U_0 \neq U_1)$$

and the result follows. □

We add the obvious corollary

Lemma 4.11 *For all $\sigma \in \Sigma$*

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models \neg \, \forall U_0 \in U^\sigma \, \exists U_1 \in (M^*)^\sigma \, (U_0 \neq U_1) \tag{4.4}$$

It is worth asking whether Lemma 4.11 can be strengthened by substituting U for M^* in (4.4). This is not the case.

Lemma 4.12 *Consider the theory* $\mathbf{PCF}^-$, *which consists of all equations in* $\mathbf{PCF}$ *that contain neither* $\perp$ *nor* F. *Let* $\mathcal{M}$ *be a fully abstract model of* $\mathbf{PCF}^-$, $\mathbf{P}$ *and* U *as in the proof of the main theorem. Then*

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models \forall U_0 \in U^B \, \exists U_1 \in U^B \, (U_0 \neq U_1)$$

Proof. Pick $U_0' \in U^B$. Set

$$U_1' \triangleq \{ \, m \in (M^*)^B \mid \forall m_0 \in U_0' \, (m = \textit{if } m_0 \textit{ then false else true}) \, \}$$

Evidently

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models U_1' \in U^B$$

and yet

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models U_0' \neq U_1'$$

□

Lemma 4.10 and the fact that $\mathcal{U}$ is generated by V raises the question whether we might not be able to give a first order 'recipe' for the construction of $\mathcal{U}$. What do we have to require of a Σ-typed set V, such that

$$M \subset V \subset M^s$$

and the theorem holds with $\mathcal{U}$ replaced by the model generated by V? Condition (4.4) is evidently not enough; it holds of $\mathcal{M}^s$ itself, for instance, and not all functions from M^s into itself are representable in $\mathcal{M}^s$. We shall come back to this point in Chapter 6 and attempt an axiomatic description of the situation.

Now we turn our attention to the untyped models.

Chapter 5

Untyped extensional λ-models

As in the previous chapter we shall start with a classical example. Then we proceed to prove for the untyped case an equivalent result to Theorem 4.6.

5.1 Reflexive sets

In Chapter 3, Section 3.6, we saw that there is an obvious bijection

$$F : U \hookrightarrow\!\!\!\!\rightarrow [U \to U]$$

from an extensional λ-model $\mathcal{U}$ to its space of representable functions. F was defined as

$$(F(u_0))(u_1) \triangleq u_0 \cdot_U u_1$$

This suggests the following question: Under which conditions can a subset $\mathcal{F}$ of the space of total endofunctions on a set U be regarded as the space of representable functions of an extensional model with U as its universe? In other words: When can we define a bijection

$$F : U \hookrightarrow\!\!\!\!\rightarrow \mathcal{F}$$

such that U equipped with $\cdot_F$ now *defined* as

$$u_0 \cdot_F u_1 \triangleq (F(u_0))(u_1) \tag{5.1}$$

and

$$k_F \triangleq F^{-1}(\lambda x.F^{-1}(\lambda y.x)) \tag{5.2}$$

$$s_F \triangleq F^{-1}(\lambda x.F^{-1}(\lambda y.F^{-1}(\lambda z.x \cdot_F z \cdot_F (y \cdot_F z)))) \tag{5.3}$$

is a λ-model that has $\mathcal{F}$ as its space of representable functions?

The easiest case is where $\mathcal{F} = U^U$, i.e. when $\mathcal{F}$ is the full function space. Then *every* bijection gives rise to a λ-model. However, classically we have the following lemma.

Lemma 5.1 (ZF+ <AC(X,X)>) *Let* $X \approx X^X$. *Then* X *is a singleton.*

Proof. Let

$$F : X \hookrightarrow\!\!\!\!\rightarrow X^X$$

and assume

$$\forall x_0 \in X \; \exists x_1 \in X \; (x_0 \neq x_1)$$

Define, using <AC(X,X)>,

$$f : X \to X$$

by picking for every $x'_0 \in X$ as $f(x'_0)$ a value x'_1 distinct from $(F(x'_0))(x'_0)$. Then f is not in the range of F, therefore

$$\neg \, \forall x_0 \in X \; \exists x_1 \in X \; (x_0 \neq x_1)$$

which is equivalent to

$$\exists x_0 \in X \; \forall x_1 \in X \; (x_0 = x_1)$$

□

Constructively, we can repeat the the above argument only part of the way.

Lemma 5.2 (IZF+ <AC(X,X)>) *Let* $X \approx X^X$. *Then*

$$\neg \, \forall x_0 \in X \; \exists x_1 \in X \; (x_0 \neq x_1)$$

Proof. Assume

$$F : X \hookrightarrow\!\!\!\!\rightarrow X^X$$

and

$$\forall x_0 \in X \; \exists x_1 \in X \; (x_0 \neq x_1)$$

Define

$$f : X \to X$$

by picking for every $x'_0 \in X$ as $f(x'_0)$ a value x'_1 distinct from $(F(x'_0))(x'_0)$. Then f is not in the range of F, therefore

$$\neg \, \forall x_0 \in X \; \exists x_1 \in X \; (x_0 \neq x_1)$$

□

This, of course, is constructively *not* equivalent to X being a singleton.

We need the following definition for a description of the more general situation.

Definition 5.3 *Let* **C** *be a cartesian closed subcategory of* **Set** *and* $U \in \mathbf{C}$. U *is a* reflexive object *if there exist morphisms*

$$F : U \to U^U$$

and

$$G : U^U \to U$$

such that

$$F \circ G = \mathrm{id}_{U^U}$$

If U is a reflexive object, the structure $\mathcal{U}$ defined by 5.1–5.3 is a *CL*-algebra. If F is bijective, $\mathcal{U}$ is an extensional λ-model, c.f. [Koy84].

As a nontrivial example, we shall give the construction of a reflexive object in **CPPO**. It made its appearance in [Sco72]. For complete details and related models see [Koy84] and [Bet88].

5.2 The inverse limit construction

Let

$$\mathbf{D} = \langle D, \leq, \perp \rangle$$

be a cppo. Define an ω-indexed family $\{\mathbf{D}_n\}_{n\in\omega}$ of cppo's by setting

- $\mathbf{D}_0 \triangleq \mathbf{D}$
- $\mathbf{D}_{n+1} = \langle D_{n+1}, \leq_{n+1}, \perp_{n+1} \rangle$ where
  - $D_{n+1} = \{ f : D_n \to D_n \mid f \text{ is continuous} \}$
  - $\leq_{n+1}$ is the pointwise ordering
  - $\perp_{n+1}(x) = \perp_n$ for $x \in D_n$

Furthermore define an ω-indexed family $\{\langle \phi_n, \psi_n \rangle\}_{n\in\omega}$ of pairs of morphisms where

$$\begin{aligned} \phi_n \ : \ & D_n \to D_{n+1} \\ & x \mapsto \lambda y.x \end{aligned}$$

$$\begin{aligned} \psi_n \ : \ & D_{n+1} \to D_n \\ & f \mapsto f(\bot_n) \end{aligned}$$

$\langle \phi_n, \psi_n \rangle$ are *retractions*, i.e.

$$\psi_n \circ \phi_n = \mathrm{id}_{D_n}$$

and

$$\phi_n \circ \psi_n \leq_{n+2} \mathrm{id}_{D_{n+1}}$$

Now let Δ be the diagram in **CPPO** having $\mathbf{D}_n$ as vertices and ψ_n as edges. Define

$$\mathbf{D}_\infty \triangleq \lim \Delta$$

Let

$$\pi_n \ : \ \mathbf{D}_\infty \to \mathbf{D}_n$$

be the edges of the limiting cone.

D_∞ is the set of all sequences $x = \langle x_n \rangle_{n \in \omega}$ such that

$$x_n \in D_n \ \text{ and } \ x_n = \psi_n(x_{n+1}) \ \text{ for all } n \in \omega.$$

π_n maps $x \in D_\infty$ to x_n.

$\mathbf{D}_\infty$ is also the colimit of the diagram with vertices $\mathbf{D}_n$ and edges ϕ_n. Let ι_n be the edges of the colimiting cone. ι_n maps $x \in D_n$ to

$$\langle \ldots, \psi_{n-1}(x), x, \phi_n(x), \ldots \rangle$$

Note that the $\langle \iota_n, \pi_n \rangle$ are themselves retractions.

Now define application $\cdot_\infty$ on D_∞ by

$$x \cdot_\infty y \triangleq \bigvee_{n \in \omega} \{ \iota_n((\pi_{n+1}(x))(\pi_n(y))) \}$$

$\langle \iota_n((\pi_{n+1}(x))(\pi_n(y))) \rangle_{n \in \omega}$ is an ω-chain in $\mathbf{D}_\infty$, so the supremum exists.

The morphism

$$F \ : \ \mathbf{D}_\infty \to \mathbf{D}_\infty^{\mathbf{D}_\infty}$$

is a bijection and with k_∞, s_∞ defined as above the structure $\mathcal{D}_\infty$ is an extensional λ-model.

Scott (see [Sco80]) hat the idea to use the Yoneda functor to embed **CPPO** in $\mathbf{Set}^{\mathbf{CPPO}^{\mathrm{op}}}$. The reflexive *object* $\mathbf{D}_\infty$ will be mapped to a presheaf $Y(\mathbf{D}_\infty)$, which under the right interpretation—which is very similar to the one given in Chapter 2—is a reflexive *set* in the (constructive) universe:

$$\mathbf{Set}^{\mathbf{CPPO}^{\mathrm{op}}} \models Y(F) \; : \; Y(D_\infty) \to Y(D_\infty^{D_\infty}) \text{ is a bijection.}$$

The resulting extensional (full functionspace) λ-model has the same theory as $\mathcal{D}_\infty$. In fact one might argue that the two models are the same thing, once seen externally, once internally.

It lies in the nature of presheaf models constructed from categories such as **CPPO** that they are fairly awkward to handle. It will be difficult to obtain *relative consistency results*, i.e. to find out, for instance, which classical or constructive statements are consistent with the existence of a nontrivial set having the same number of points as its function space.

More importantly, not every *CL*-theory (nor λ-theory for that matter) is the theory of a model obtained from a reflexive object in **CPPO**, and certain theories may not have a full functionspace model in the universe.

5.3 The main theorem (untyped)

We shall now proceed to state and prove the untyped version of our main theorem.

Theorem 5.4 *Let $\mathcal{M}$ be an extensional $\lambda(C)$-model. Then there exists a tree* $\mathbf{P}$ *which only depends on the cardinality of M such that* $\mathbf{V}^{\mathbf{P}^{\mathrm{op}}}$ *satisfies the following statement.*

There is a full function space model $\mathcal{U}$ such that

$$M^* \subset U \subset (M^*)^s$$

and

$$\mathcal{M}^* \cong \mathcal{E}(\{M\},\mathcal{U}) \mathrel{\tilde{\subset}} \mathcal{U} \mathrel{\tilde{\subset}} (\mathcal{M}^*)^s$$

Moreover there is a bijection

$$F \; : \; U \hookrightarrow\!\!\!\!\to U^U$$

such that

$$U_0 \cdot_U U_1 = (F(U_0))(U_1)$$

and therefore

- $k_U = F^{-1}(\lambda x.F^{-1}(\lambda y.x))$
- $s_U = F^{-1}(\lambda x.F^{-1}(\lambda y.F^{-1}(\lambda z.x \cdot_U z \cdot_U (y \cdot_U z))))$

The proof is almost the same as the one of the typed theorem.

Proof. The definition of 'environment' in the proof of Theorem 4.6 applies of course also to the untyped language. Add M as constants and $|M|^+$ variables to $\Lambda(C)$. Again we work with the tree $\langle \mathcal{E}_{\text{fin}}, \subset, \emptyset \rangle$. We describe U first.

Let $U \in \mathbf{V}^{\mathcal{E}_{\text{fin}}^{\text{op}}}$ be the set of all stable singletons U_t of M^* where $t \in \Lambda(C)$, such that for all $m \in M$ and $E_0 \in \mathcal{E}_{\text{fin}}$

$$\begin{aligned} E_0 \Vdash\ & m^* \in U_t \leftrightarrow \\ & \forall E_1 \in \mathcal{E}_{\text{tot}}\ (E_0 \subset E_1 \rightarrow t_{E_1} = m) \end{aligned}$$

Again

$$\begin{aligned} E_0 \Vdash\ & U_{t_0} = U_{t_1} \leftrightarrow \\ & \forall E_1 \in \mathcal{E}_{\text{tot}}\ (E_0 \subset E_1 \rightarrow (t_0)_{E_1} = (t_1)_{E_1}) \end{aligned} \tag{5.4}$$

Define the function $F \in \mathbf{V}^{\mathcal{E}_{\text{fin}}^{\text{op}}}$ by

$$\mathbf{V}^{\mathcal{E}_{\text{fin}}^{\text{op}}} \models (F(U_{t_0}))(U_{t_1}) = U_{t_0 t_1}$$

We have indeed

$$\mathbf{V}^{\mathcal{E}_{\text{fin}}^{\text{op}}} \models U_{t_0} \cdot_U U_{t_1} = U_{t_0 t_1} = (F(U_0))(U_1)$$

We prove that F is bijective.

1. F is surjective.

   We must convince ourselves that

   $$\mathbf{V}^{\mathcal{E}_{\text{fin}}^{\text{op}}} \models \forall f \in U^U\ \exists U_0 \in U\ (F(U_0) = f)$$

   So pick $E_0, E_1 \in \mathcal{E}_{\text{fin}}, E_0 \subset E_1$ and $f' \in U^U(E_1)$. We must prove that

   $$E_1 \Vdash_{\{f'/f\}} \exists U_0 \in U\ (F(U_0) = f)$$

   Take

   $$f^\Lambda\ :\ \Lambda(C) \rightarrow \Lambda(C)$$

such that

$$f^{\Lambda}(t_0) = t_1 \to E_1 \Vdash_{\{f'/f\}} f(U_{t_0}) = U_{t_1}$$

Pick $x \notin \bigcup_{m \in M} \mathrm{FV}(f^{\Lambda}m) \cup \mathrm{dom} E_1$. This can be done since the right hand side is of cardinality $|M|$.

$$E_1 \Vdash_{\{f'/f\}} U_{f^{\Lambda}x} = U_{(\lambda x.t_1)x}$$

for some $t_1 \in \Lambda(C)$, so

$$E_1 \Vdash_{\{f'/f\}} f(U_x) = U_{f^{\Lambda}x} = U_{(\lambda x.t_1)x} = U_{\lambda x.t_1} \cdot_U U_x$$

We wish to prove that then for every $t_0 \in \Lambda(C)$

$$E_1 \Vdash_{\{f'/f\}} f(U_{t_0}) = U_{f^{\Lambda}t_0} = U_{(\lambda x.t_1)t_0} = U_{\lambda x.t_1} \cdot_U U_{t_0}$$

i.e. that $U_{\lambda x.t_1}$ represents f'. In order to prove equality in the middle recall (5.4). Pick $E_2 \in \mathcal{E}_{\mathrm{tot}}$, such that $E_1 \subset E_2$. To justify

$$(f^{\Lambda}t_0)_{E_2} = (f^{\Lambda}(t_0)_{E_2})_{E_2}$$

let $\mathrm{FV}(t_0) \subset \vec{y}$.

$$E_1\{\vec{y} := E_2(\vec{y})\} \Vdash_{\{f'/f\}} U_{t_0} = U_{(t_0)_{E_2}}$$

so

$$E_1\{\vec{y} := E_2(\vec{y})\} \Vdash_{\{f'/f\}} U_{f^{\Lambda}t_0} = U_{f^{\Lambda}(t_0)_{E_2}}$$

Further

$$(f^{\Lambda}(t_0)_{E_2})_{E_2} = (f^{\Lambda}(t_0)_{E_2})_{E_2\{x := (t_0)_{E_2}\}} \text{ since } x \notin \mathrm{Vars}(f^{\Lambda}(t_0)_{E_2})$$

Then

$$E_1\{x := (t_0)_{E_2}\} \Vdash_{\{f'/f\}} U_{(t_0)_{E_2}} = U_x$$

implies

$$E_1\{x := (t_0)_{E_2}\} \Vdash_{\{f'/f\}} U_{f^{\Lambda}(t_0)_{E_2}} = U_{f^{\Lambda}x}$$

hence

$$(f^{\Lambda}(t_0)_{E_2})_{E_2\{x := (t_0)_{E_2}\}} = (f^{\Lambda}x)_{E_2\{x := (t_0)_{E_2}\}}$$

Finally

$$(f^{\Lambda}x)_{E_2\{x := (t_0)_{E_2}\}} = ((\lambda x.t_1)x)_{E_2\{x := (t_0)_{E_2}\}} = ((\lambda x.t_1)t_0)_{E_2}$$

2. F is injective.
   This follows from Lemma 3.51 and the fact that $\mathcal{M}^* \tilde{\subset} \mathcal{U} \tilde{\subset} (\mathcal{M}^*)^s$ and $\mathcal{M}$ is extensional.

Next we prove that $F|M^* : M^* \to |\mathcal{E}(\{(M^*)\},\mathcal{U})|$ is a bijection. Injectivity is clear. For surjectivity of F pick $E_0 \in \mathcal{E}_{\text{fin}}$, and $f' \in U^U(E_0)$ such that

$$E_0 \Vdash_{\{f'/f\}} f(M^*) \subset M^*$$

By 2. f' is represented by U_{t_0} for some $t_0 \in \Lambda(C)$. To prove that

$$E_0 \Vdash_{\{f'/f\}} U_{t_0} \in M^*$$

assume

$$E_0 \nVdash_{\{f'/f\}} U_{t_0} \in M^* \tag{5.5}$$

and derive a contradiction. (5.5) means that for all $m \in M$

$$E_0 \nVdash_{\{f'/f\}} m^* \in U_{t_0}$$

hence there are $E_1, E_2 \in \mathcal{E}_{\text{tot}}$ with $E_0 \subset E_1$ and $E_0 \subset E_2$, such that

$$(t_0)_{E_1} \neq (t_0)_{E_2}$$

By extensionality there is an $m_0 \in M$ such that

$$(t_0)_{E_1} \cdot_M m_0 \neq (t_0)_{E_2} \cdot_M m_0$$

Therefore for all $m \in M$

$$E_0 \nVdash_{\{f'/f\}} m^* \in (U_{t_0} \cdot_U U_{m_0})$$

and

$$E_0 \nVdash_{\{f'/f\}} U_{t_0} \cdot_U U_{m_0} \in (M^*) \lightning$$

We deduce by <RAA>

$$E_0 \Vdash_{\{f'/f\}} U_{t_0} \in M^*$$

This concludes the proof of the main theorem. □

Again we list some corollaries.

Theorem 5.5 *Let $\mathcal{M}$ be an extensional $\lambda(C)$-model and $\mathbf{P} = \omega_1^{<\omega_0}$. Then $\mathbf{V}^{\mathbf{P}^{op}}$ satisfies*

There is a full function space model which is equationally equivalent to $\mathcal{M}^*$.

And finally we add

Theorem 5.6 *Let T be a $\lambda(C)$-theory that has a fully abstract extensional model and $\mathbf{P} = \omega_1^{<\omega_0}$. Then $\mathbf{V}^{\mathbf{P}^{\mathrm{op}}}$ satisfies*

T has a fully abstract full function space model.

As in the chapter on the typed models we investigate some of the properties of models and sets concerned.

First we note that Lemma 4.10 has an obvious equivalent.

Lemma 5.7 *Let $\mathbf{P}, \mathcal{M}, \mathcal{U}$ be as in Theorem 5.4. Then*

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models \neg\, \forall U_0 \in U\ \exists U_1 \in M^*\ (U_0 \neq U_1)$$

Indeed and perhaps surprisingly in view of Lemma 4.12 we have

Lemma 5.8 *Let $\mathbf{P}, \mathcal{M}, \mathcal{U}$ be as in Theorem 5.4. Then*

$$\mathbf{V}^{\mathbf{P}^{\mathrm{op}}} \models \neg\, \forall U_0 \in U\ \exists U_1 \in U\ (U_0 \neq U_1)$$

Proof. Pick $E_0, E_1 \in \mathcal{E}_{\mathrm{fin}}$, $E_0 \subset E_1$, and an x such that $x \notin \mathrm{dom}(E_1)$. Pick $t_0 \in \Lambda(C)$. We shall show that

$$E_1 \ \Vdash\!\!\!\!/_{\emptyset}\ U_x \neq U_{t_0} \tag{5.6}$$

We do this by case analysis

1. Assume $x \notin \mathrm{FV}(t_0)$. Take any $E_2 \in \mathcal{E}_{\mathrm{tot}}$ such that $E_1 \subset E_2$. Then

$$\mathcal{M}, E_2\{x := (t_0)_{E_2}\} \models x = (t_0)_{E_2} = t_0$$

   But

$$E_1 \subset E_2\{x := (t_0)_{E_2}\}$$

   therefore (5.6).

2. Assume $x \in \mathrm{FV}(t_0)$. $(\lambda x.t_0)$ has a fixpoint t_1. Take $E_2 \in \mathcal{E}_{\mathrm{tot}}$ such that $E_1 \subset E_2$. Then

$$\mathcal{M}, E_2\{x := (t_1)_{E_2}\} \models x = (t_1)_{E_2} =$$
$$((\lambda x.t_0)t_1)_{E_2} = (t_0(t_1/x))_{E_2} = t_0((t_1)_{E_2}/x) = t_0$$

   Now, $E_1 \subset E_2\{x := (t_1)_{E_2}\}$, therefore again (5.6).

□

We know from Lemmas 4.12 and 5.2 that the Axiom of Choice does not hold at least at type B for U, i.e. that

$$\mathbf{V}^{\mathcal{E}_{\mathrm{fin}}{}^{\mathrm{op}}} \not\models \,<\!\mathrm{AC}(U^B,U^B)\!>$$

For the untyped case, Lemma 5.8 gives us hope. We have the

Conjecture. The existence of a set U with two distinct points, $U \approx U^U$ and $<\!\mathrm{AC(U,U)}\!>$ is consistent with **IZF**. □

I have neither been able to prove nor disprove that $<\!\mathrm{AC(U,U)}\!>$ for U in the proof of Theorem 5.4.

Chapter 6

Related results and open problems

6.1 Extensions galore

Clearly when $\mathcal{M}$ is a nontrivial classical $\lambda(C)$-model, $\mathcal{U}$ a full function space model and

$$\mathcal{M} \approx \mathcal{E}(\{M\},\mathcal{U})$$

then not all functions $f : M \to M$ have extensions in U^U. This has serious consequences for the world where M and U live. First we prove that U is rather 'fuzzy'.

Lemma 6.1 *Let X_0, X_1 be sets, X_0 classical and inhabited, and $X_0 \subset X_1$. Assume that not all functions $f : X_0 \to X_0$ have an extension in $X_1 \Rightarrow X_1$. Then*

$$\neg\, \forall x_0, x_1 \in X_1\, (x_0 \neq x_1 \vee \neg\neg\, x_0 = x_1)$$

Proof. Assume

$$\forall x_0, x_1 \in X_1\, (x_0 \neq x_1 \vee \neg\neg\, x_0 = x_1) \tag{6.1}$$

Pick $f'_0 : X_0 \to X_0$. We shall construct an extension $f_1 : X_1 \to X_1$. Take $y' \in X_0$ and $x'_1 \in X_1$. We have

$$\forall x_0 \in X_0\, (x'_1 \neq x_0 \vee \neg\neg\, x'_1 = x_0)$$

hence $\phi(x_0) \equiv \neg\neg\, x'_1 = x_0$ is a decidable relation on X_0. By classicality of X_0 we get

$$\exists x_0 \in X_0\, (\neg\neg\, x'_1 = x_0) \vee \neg\, \exists x_0 \in X_0\, (\neg\neg\, x'_1 = x_0)$$

Consider the first option. We prove that in fact

$$\exists! x_0 \in X_0\, (\neg\neg\, x'_1 = x_0)$$

Pick $x_0', x_0'' \in X_0$ such that

$$\neg\neg\, x_1' = x_0' \wedge \neg\neg\, x_1' = x_0''$$

It follows that

$$\neg\neg\, x_0' = x_0''$$

and by discreteness of X_0

$$x_0' = x_0''$$

This allows us to define $f_1(x_1')$ as $f_1(x_0')$ where x_0' is the unique point in X_0 such that $\neg\neg\, x_1' = x_0'$.

In the second case define

$$f_1(x_1') \triangleq y'$$

f_1 is well defined and total. The result follows. □

Remark. In the above proof X_0 really does have to be classical, discreteness alone is not enough. To see this consider the model $\mathbf{V}^{\mathbf{2}^{op}}$. Define

$$X_0 \triangleq \{\, 0 \mid \{1\} \,\} \cup \{\, 1 \mid \{1\} \,\}$$

and

$$X_1 \triangleq \{\, 0 \mid \{0,1\} \,\} \cup \{\, 1 \mid \{1\} \,\}$$

Then the function $f : X_0 \to X_0$ for which

$$1 \Vdash_{\emptyset} f(0) = 1$$

has no extension in $X_1 \Rightarrow X_1$, but of course

$$\mathbf{V}^{\mathbf{2}^{op}} \models \forall x_0, x_1 \in X_1\, (x_0 \neq x_1 \vee \neg\neg\, x_0 = x_1)$$

□

As a corollary to Lemma 6.1 we note

Lemma 6.2 *Let X_0 be a classical inhabited set and X_1 such that $X_0 \subset X_1$ and not all functions in $X_0 \Rightarrow X_0$ have extensions in $X_1 \Rightarrow X_1$. Then*

$$\neg\, \forall \phi\, (\neg\, \phi \vee \neg\neg\, \phi)$$

Finally, Lemma 6.1 taken together with Lemma 2.52 and Lemma 2.45 yields

Lemma 6.3 *Let* $\mathbf{C}$ *be a category,* $1 \hookrightarrow X_0 \in \mathbf{V}$ *and* $X_1 \in \mathbf{V}^{\mathbf{C}}$. *Assume* $\mathbf{V}^{\mathbf{C}}$ *satisfies*

> $X_0^* \subset X_1$ *and not all functions in* $X_0^* \Rightarrow X_0^*$ *have extensions in* $X_1 \Rightarrow X_1$.

Then for all $C \in \mathbf{C}$ *and for all* $f_0 : C_0 \to C$ *there exist* $f_1 : C_1 \to C$ *and* $f_2 : C_2 \to C$ *such that* $f_0 \preceq_C f_1$, $f_0 \preceq_C f_2$ *and* f_1 *and* f_2 *are incompatible.*

If $\mathbf{C}$ happens to be a tree, it has hereditary width at least ω.

Now, if not all functions $f : X_0 \to X_0$ have extensions in $X_1 \Rightarrow X_1$, what can be said about those that do have one? It is easy to see that the identity and all constant functions have extensions. Indeed we shall now give two conditions on X_1 such that these are the only ones.

Lemma 6.4 *Let* X_0 *be inhabited and classical and* X_1 *a superset of* X_0 *such that*

1. $\neg \forall x_1 \in X_1 \, \exists x_0 \in X_0 \, (x_1 \neq x_0)$
2. $\forall x_0, x_1 \in X_1 \, ((x_0 \in X_0 \to x_1 \in X_0) \to x_1 \in X_0 \vee x_0 = x_1)$

Then

$$\forall f_0 \in X_0^{X_0} \, (\exists f_1 \in X_1^{X_1} \, (f_0 \subset f_1) \to f_0 = \mathrm{id}_{X_0} \vee \exists y \in X_0 \, (\forall x \in X_0 \, (f(x) = y)))$$

Proof. Pick $f_0' : X_0 \to X_0$. Assume f_0' has an extension and is neither constant nor the identity. We shall prove that then

$$\forall x_1 \in X_1 \, \exists x_0 \in X_0 \, (x_0 \neq x_1)$$

There are points $x_0', x_0'' \in X_0$ such that

$$\begin{array}{ccc} x_0' & \neq & x_0'' \\ & & \neq \\ f_0'(x_0') & \neq & f_0'(x_0'') \end{array}$$

Let $f_1 \in X_1 \Rightarrow X_1$ be the extension of f_0'. We have

$$\forall x_1 \in X_1 \, (x_1 \in X_0 \to f_1(x_1) \in X_0)$$

hence by 2.

$$\forall x_1 \in X_1 \, (f_1(x_1) \in X_0 \vee x_1 = f_1(x_1))$$

Now pick $x_1' \in X_1$. We have two cases.

1. If $f_1(x_1') \in X_0$, then either

$$f_1(x_1') \neq f_0'(x_0')$$

or

$$f_1(x_1') \neq f_0'(x_0'')$$

In the first case $x_1' \neq x_0'$, in the second $x_1' \neq x_0''$.

2. If $f_1(x_1') = x_1'$, then $x_1' \neq x_0''$.

Therefore always

$$\exists x_0 \in X_0 \, (x_1' \neq x_0)$$

Hence our assumption that f_0' is neither constant nor the identity is false, and by Lemma 2.41 the result follows. □

The sets $M^* \subset V$ of Lemma 4.10 fit description 1. and 2. of Lemma 6.4.

The same example shows that condition 2. is essential. $M^* \subset (M^*)^s$ fit 1., but as we saw in Lemma 3.45, every function in $M^* \Rightarrow M^*$ has an extension in $(M^*)^s \Rightarrow (M^*)^s$.

We have seen that if X_0 is a nontrivial classical set, and X_1 is a superset of X_0 such that not all functions $f : X_0 \to X_0$ have extensions in $X_1 \Rightarrow X_1$, then X_1 is fairly non classical, indeed we proved that it is strongly non discrete. We shall postpone a further investigation of functions and try to capture this 'fuzziness' of sets.

6.2 Degrees of 'fuzziness'

We introduce a new concept.

Definition 6.5 *A set X is* unzerlegbar[1] *if for every subset X_0 of X*

$$X_0 \cup X_0^c = X \to X_0 = \emptyset \vee X_0 = X$$

In other words, the only decidable relations on X are the trivial ones.

Theorem 6.6 *Let X_0, X_1 be sets, X_0 classical and $2 \hookrightarrow X_0 \subset X_1$ Consider the following sentences.*

1. $\neg \forall x_1 \in X_1 \, \exists x_0 \in X_1 \, (x_0 \neq x_1)$

[1] 'cannot be split'

2. $\neg\, \forall x_1 \in X_1\, \exists x_0 \in X_0\, (x_0 \neq x_1)$

3. $\exists x_0, x_1 \in X_0\, (x_0 \neq x_1 \wedge \neg\, \forall x \in X_1\, (x_0 \neq x \vee x_1 \neq x))$

4. $\neg\, \forall x_0, x_1 \in X_0\, (x_0 \neq x_1 \rightarrow \forall x \in X_1\, (x_0 \neq x \vee x_1 \neq x))$

5. X_1 *is* unzerlegbar

6. $\neg\, \forall x_0, x_1 \in X_1\, (x_0 \neq x_1 \vee \neg\neg\, x_0 = x_1)$

7. $\neg\, \forall x_0, x_1 \in X_1\, (x_0 = x_1 \vee x_0 \neq x_1)$

Exactly the following implications hold

$$1. \rightarrow 2. \rightarrow 3. \rightarrow 4. \rightarrow 6. \rightarrow 7.$$

$$1. \rightarrow 5. \rightarrow 6.$$

Proof.

- 1. → 2.
  This is clear. For an example where 2. does not imply 1. see Lemmas 4.11 and 4.12.

- 2. → 3.
  is also clear. An example where the converse fails can be constructed from M and U in the proof of Theorem 4.6. Pick $m_0', m_1' \in M$, such that $m_0' \neq m_1'$. Define
  $$U^+ \triangleq U + 1$$
  and
  $$M^+ \triangleq M + 1$$
  Then
  $$\mathbf{V}^{\mathbf{P}^{\text{op}}} \models \forall u \in U^+\, \exists m \in M^+\, (u \neq m)$$
  but also
  $$\mathbf{V}^{\mathbf{P}^{\text{op}}} \models \neg\, \forall u \in U^+\, (u \neq m_0' \vee u \neq m_1')$$

- 3. → 4.
  Assume 3. and
  $$y_0 \in X_0 \wedge y_1 \in X_0 \wedge y_0 \neq y_1 \wedge \neg\, \forall x \in X_1\, (y_0 \neq x \vee y_1 \neq x)$$

Assume

$$\forall x_0, x_1 \in X_0\, (x_0 \neq x_1 \rightarrow \forall x \in X_1\, (x_0 \neq x \vee x_1 \neq x))$$

Then in particular

$$\forall x_0 \in X_1\, (y_0 \neq x \vee y_1 \neq x)$$

This leads to the required contradiction.

For a counterexample to 4. → 3. consider the model $\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}}$. Define $x_n \in \mathbf{V}^{(2^{<\omega})^{\mathrm{op}}}$ for every $n \in \omega$ by

$$x_n \triangleq \{\, n \mid \{\, s \mid s(n) = 0 \,\}\,\} \cup \{\, n+1 \mid \{\, s \mid s(n) = 1 \,\}\,\}$$

and

$$X \triangleq \{\, x_n \mid n \in \omega \,\} \cup \{\, \{n\} \mid n \in \omega \,\}$$

Pick $s_0, s_1 \in 2^{<\omega}$ such that $s_0 \subset s_1$. Then

$$s_1 \Vdash_\emptyset \{|s_1|\} \neq \{|s_1| + 1\}$$

but

$$\begin{array}{ll} s_1 0 & \Vdash_\emptyset x_{|s_1|} = \{|s_1|\} \\ s_1 1 & \Vdash_\emptyset x_{|s_1|} = \{|s_1| + 1\} \end{array}$$

hence

$$s_1 \not\Vdash_\emptyset x_{|s_1|} \neq \{|s_1|\} \vee x_{|s_1|} \neq \{|s_1| + 1\}$$

therefore

$$s_1 \not\Vdash_\emptyset \forall n_0, n_1 \in \omega\, (n_0 \neq n_1 \rightarrow \forall x \in X\, (\{n_0\} \neq x \vee \{n_1\} \neq x))$$

and

$$s_0 \Vdash_\emptyset \neg \forall n_0, n_1 \in \omega\, (n_0 \neq n_1 \rightarrow \forall x \in X\, (\{n_0\} \neq x \vee \{n_1\} \neq x))$$

so

$$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \models \neg \forall n_0, n_1 \in \omega\, (n_0 \neq n_1 \rightarrow \forall x \in X\, (\{n_0\} \neq x \vee \{n_1\} \neq x))$$

To prove that

$$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \not\models \exists n_0, n_1 \in \omega\, (n_0 \neq n_1 \wedge \neg \forall x \in X\, (\{n_0\} \neq x \vee \{n_1\} \neq x))$$

pick $n_0', n_1' \in \omega$, such that $n_0' \neq n_1'$ and $s \in 2^{<\omega}$ such that

$$|s| \geq (n_0 \vee n_1) + 2$$

We prove that

$$s \Vdash_\emptyset \forall x \in X\ (\{n_0\} \neq x \vee \{n_1\} \neq x)$$

Pick $n \in \omega$. If n is neither of $n_0, n_0 - 1, n_1, n_1 - 1$ then

$$s \Vdash_\emptyset x_n \neq \{n_0\} \vee x_n \neq \{n_1\}$$

If n is one of the above, then

$$s \Vdash_\emptyset x_n = \{n_2\}$$

where n_2 is one of $n_0, n_0 - 1, n_1, n_1 - 1$ and therefore

$$s \Vdash_\emptyset x_n \neq \{n_0\} \vee x_n \neq \{n_1\}$$

Also,

$$s \Vdash_\emptyset \{n\} \neq \{n_0\} \vee \{n\} \neq \{n_1\}$$

We get

$$\emptyset \not\Vdash_\emptyset \exists n_0, n_1 \in \omega\ (n_0 \neq n_1 \wedge \neg \forall x \in X\ (\{n_0\} \neq x \vee \{n_1\} \neq x))$$

and the required result.

- 4. $\rightarrow$ 6.

  Assume 4. and

$$\forall x_0, x_1 \in X_1\ (x_0 \neq x_1 \vee \neg\neg\, x_0 = x_1)$$

Pick $x_0', x_1' \in X_0$ such that $x_0' \neq x_1'$. Then

$$\forall x_1 \in X_1\ (x_0' \neq x_1 \vee \neg\neg\, x_0' = x_1)$$

therefore

$$\forall x_1 \in X_1\ (x_0' \neq x_1 \vee x_1' \neq x_1)$$

The result follows.

To disprove the reverse implication consider the model $\mathbf{V}^{(3^{<\omega})^{op}}$. For all $n \in \omega$ and $i \in 2$ define $x_n^i \in \mathbf{V}^{(3^{<\omega})^{op}}$ by

$$x_n^i \triangleq \{\, i \mid \{\, s \mid s(n) = i \,\}\,\} \cup \{\, 2 \mid \{\, s \mid s(n) = 2 \,\}\,\}$$

Let

$$X \triangleq \{x^0_n\}_{n\in\omega} \cup \{x^1_n\}_{n\in\omega} \cup \{\{0\},\{1\}\}$$

Then clearly

$$\mathbf{V}^{(3^{<\omega})^{\mathrm{op}}} \models \forall x \in X\ (\{0\} \neq x \vee \{1\} \neq x)$$

therefore

$$\mathbf{V}^{(3^{<\omega})^{\mathrm{op}}} \not\models \neg\,\forall n_0, n_1 \in 2\ (n_0 \neq n_1 \rightarrow \forall x \in X\ (\{n_0\} \neq x \vee \{n_1\} \neq x))$$

On the other hand, take $s \in 3^{<\omega}$. Then

$$s0 \Vdash_\emptyset x^0_{|s|} = \{0\}$$
$$s1 \Vdash_\emptyset x^0_{|s|} \neq \{0\}$$

therefore

$$s \not\Vdash_\emptyset x^0_{|s|} \neq \{0\} \vee \neg\neg\, x^0_{|s|} = \{0\}$$

and

$$\mathbf{V}^{(3^{<\omega})^{\mathrm{op}}} \models \neg\,\forall x_0, x_1 \in X\ (x_0 \neq x_1 \vee \neg\neg\, x_0 = x_1)$$

- 6. → 7. is easy.

  A counterexample to 7. → 6. appears in Lemma 2.50.

- 1. → 5.

  Assume 1. Assume $X_0 \subset X_1$ and $X_0 \cup X_0^c = X_1$. Pick $x_1' \in X_1$. We may assume that $x_1' \in X_0$. Pick $x_1'' \in X_1$. Assume $x_1'' \in X_0^c$. Take $x' \in X_1$. Then either

  $$x' \in X_0 \text{ and } x' \neq x_1''$$

  or

  $$x' \in X_0^c \text{ and } x' \neq x_1'$$

  Therefore

  $$\forall x_0 \in X_1\ \exists x_1 \in X_1\ (x_0 \neq x_1)\ \lightning$$

  Hence

  $$x_1'' \notin X_0^c$$

  and

  $$x_1'' \in X_0$$

  Since x_1'' was general we have

  $$\forall x \in X\ (x \in X_0)$$

  or $X = X_0$.

- 5. → 6.
  Assume 5. and
  $$\forall x_0, x_1 \in X_1 \, (x_0 \neq x_1 \vee \neg\neg \, x_0 = x_1) \tag{6.2}$$
  Pick $x'_0, x'_1 \in X_0$ such that $x'_0 \neq x'_1$. Define
  $$X \triangleq \{ \, x \in X_1 \mid x \neq x'_0 \, \}$$
  Then both X and X^c is inhabited (by x'_1 and x'_0, respectively), yet by (6.2)
  $$X \cup X^c = X_1$$

- To see that 6. does not in general imply 5., take the model $\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}}$. Define $x_n \in \mathbf{V}^{(2^{<\omega})^{\mathrm{op}}}$ for all $n \in \omega$ by
  $$x_n \triangleq \{ \, 0 \mid \{ \, s \mid s(n) = 0 \, \} \, \} \cup \{ \, 1 \mid \{ \, s \mid s(n) = 1 \, \} \, \}$$
  Define
  $$X_0 \triangleq \{x_n\}_{n \in \omega} \cup \{\{0\}, \{1\}\}$$
  and
  $$X_1 \triangleq X_0 \cup \{\{2\}\}$$
  Pick $s \in 2^{<\omega}$. Then
  $$s \not\Vdash_\emptyset \; x_n \neq \{0\} \vee \neg\neg \, x_n = \{0\}$$
  for $n > |s|$, therefore
  $$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \models \neg \, \forall x_0, x_1 \in X_1 \, (x_0 \neq x_1 \vee \neg\neg \, x_0 = x_1)$$
  but X_1 can be split ('zerlegt').

It remains to be proved that neither 4. → 5. nor 5. → 4. The last example shows that 4. → 5. need not hold. We have
$$s \Vdash_\emptyset \; \{0\} \neq \{1\}$$
but
$$s \not\Vdash_\emptyset \; \forall x \in X_1 \, (x \neq \{0\} \vee x \neq \{1\})$$
hence
$$\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}} \models \neg \, \forall x_0, x_1 \in X_0 \, (x_0 \neq x_1 \to (\forall x \in X_1 \, (x_0 \neq x \vee x_1 \neq x)))$$

but as we have already seen, X_1 splits into X_0 and $\{\{2\}\}$.

To disprove 5. → 4., consider again the model $\mathbf{V}^{(3^{<\omega})^{op}}$. Define $x_n^i \in \mathbf{V}^{(3^{<\omega})^{op}}$ for all $n \in \omega$, $i \in 2$ by

$$x_n^0 \triangleq \{\, 0 \mid \{\, s \mid s(n) = 0 \,\} \,\} \cup \{\, 1 \mid \{\, s \mid s(n) = 1 \,\} \,\} \cup \{\, 2 \mid \{\, s \mid s(n) = 2 \,\} \,\}$$

and

$$x_n^1 \triangleq \{\, 1 \mid \{\, s \mid s(n) = 0 \,\} \,\} \cup \{\, 0 \mid \{\, s \mid s(n) = 1 \,\} \,\} \cup \{\, 2 \mid \{\, s \mid s(n) = 2 \,\} \,\}$$

Set

$$X \triangleq \{x_n^0\}_{n\in\omega} \cup \{x_n^1\}_{n\in\omega} \cup \{\{0\},\{1\},\{2\}\}$$

X is *unzerlegbar*. For pick $s \in 3^{<\omega}$ and assume that for some $X_0 \subset X$

$$s \Vdash_\emptyset X_0 \cup X_0^c = X \tag{6.3}$$

Further assume

$$s \Vdash_\emptyset \{2\} \in X_0^c$$

But we have

$$s0 \Vdash_\emptyset x_{|s|}^0 = \{0\} \in X_0$$

and

$$s2 \Vdash_\emptyset x_{|s|}^0 = \{2\} \in X_0^c \lightning$$

hence

$$s \not\Vdash_\emptyset \{2\} \in X_0^c$$

and by (6.3)

$$s \Vdash_\emptyset \{2\} \in X_0$$

With a similar argument we prove

$$s \Vdash_\emptyset \{1\} \in X_0$$

and finally for all $n \in \omega$

$$s \Vdash_\emptyset x_n \in X_0$$

On the other hand clearly

$$\mathbf{V}^{(3^{<\omega})^{op}} \models \forall n_0, n_1 \in 2\ (n_0 \neq n_1 \to \forall x \in X\ (\{n_0\} \neq x \vee \{n_1\} \neq x))$$

□

We are interested in one more and by now well known statement.

8. Not all functions $f : X_0 \to X_0$ have extensions in $X_1 \Rightarrow X_1$.

We have already seen that 8. $\to$ 6. Now we prove that neither 8. $\to$ 5. nor 8. $\to$ 4.

Proof.

- 8. does not entail 5.
  This is easily disproved by considering U in the proof of Theorem 5.4. Clearly not all functions $f : M + M \to M + M$ have extensions in $U + U \Rightarrow U + U$, yet $U + U$ is evidently *zerlegbar.*

- 8. does not entail 4.
  We can again consider $\mathbf{V}^{(3^{<\omega})^{\mathrm{op}}}$ and X constructed to disprove 4. $\to$ 6. The function $f : \{\{0\},\{1\}\} \to \{\{0\},\{1\}\}$ with

  $$f(\{0\}) = \{1\} \text{ and } f(\{1\}) = \{0\}$$

  does not have an extension in $X \Rightarrow X$.

  At $s \in 3^{<\omega}$ try to assign a value to $f(x^0_{|s|+1})$. We cannot have

  - $$s \Vdash_\emptyset f(x^0_{|s|}) = \{0\}$$
    because
    $$s0 \Vdash_\emptyset x^0_{|s|} = \{0\}$$
    hence
    $$s0 \Vdash_\emptyset f(x^0_{|s|}) = \{1\} \;↯$$

  - $$s \Vdash_\emptyset f(x^0_{|s|}) = \{2\}$$
    similar.

  - $$s \Vdash_\emptyset f(x^0_{|s|}) = \{1\}$$
    because
    $$s2 \Vdash_\emptyset x^0_{|s|} = x^1_{|s|}$$
    hence
    $$s2 \Vdash_\emptyset f(x^1_{|s|}) = \{1\}$$

but also

$$s1 \Vdash_{\emptyset} x^1_{|s|} = \{1\}$$

therefore

$$s1 \Vdash_{\emptyset} f(x^1_{|s|}) = \{0\}$$

But there is no point x in X such that

$$s1 \Vdash_{\emptyset} x = \{1\} \text{ and } s2 \Vdash_{\emptyset} x = \{0\}$$

- $$s \Vdash_{\emptyset} f(x^0_{|s|}) = x^i_n \text{ for } n \geq |s|$$

  because

  $$s0 \Vdash_{\emptyset} f(x^0_{|s|}) = \{1\}$$

  but

  $$s0 \not\Vdash_{\emptyset} x^i_n = \{1\}$$

□

We sum up: sets can have various degrees of 'fuzziness'. This can be thought of as a measure of our being able to distinguish between two points. Our list is admittedly fairly *ad hoc.* It would be interesting to find a general method of generating a 'fuzzy' hierarchy.

In classical mathematics, where all sets are discrete anyway, a measure of the distinctness of points can be introduced via *topologies.* It would be interesting to know whether the above results could be expressed and indeed refined in those terms.

6.3 Towards an axiomatic approach

Assume we have a classical nontrivial and extensional λ-model $\mathcal{M}$. What do we have to require (in first order terms) of a set U with $M \subset U \subset M^s$ such that there is a bijection between U and U^U and the induced λ-model $\mathcal{U}$ satisfies

$$\mathcal{M} \mathrel{\tilde{\subset}} \mathcal{U} \mathrel{\tilde{\subset}} \mathcal{M}^s?$$

Try this idea: we *construct* U by first taking a set U_0, $M \subset U_0 \subset M^s$ such that only the constant functions and the identity in M^M have extensions in $U_0^{U_0}$. We then add exactly enough points to U_0, obtaining a set U_1, such that all functions in M^M represented by a point in M have an extension in $U_1^{U_0}$. Then we add enough points to U_1, obtaining a set U_2 such that all the functions in $U_0^{U_0}$ have an extension in $U_2^{U_1}$ and continue in that manner defining an ω-chain of sets $\{U_n\}_{n\in\omega}$ whose union with some luck will be the required set U.

Now the details: Let U_0 be such that $M \subset U_0 \subset M^s$ and

1. $\neg\, \forall u_0 \in U_0\, \exists u_1 \in U_0\, (u_0 \neq u_1)$
2. $\forall u_0, u_1 \in U_0\, ((u_0 \in M \to u_1 \in M) \to u_1 \in M \vee u_0 = u_1)$

We know that exactly the constant functions and the identity in M^M have extensions in $U_0 \Rightarrow U_0$. We need the smallest set U_1 such that all functions in M^M represented by a point $m \in M$ have an extension in $U_0 \Rightarrow U_1$. We define

$$U_1 \triangleq M \cdot_{M^s} U_0$$

Now not necessarily all functions in $M \Rightarrow M$ represented by an $m \in M$ have an extension in $U_1 \Rightarrow U_1$. So again we have to add something. Moreover since we eventually need

$$U \approx U^U$$

we now want a U_2 such that all functions in $U_0 \Rightarrow U_0$ represented by a point $u_0 \in U_0$ have an extension in $U_1 \Rightarrow U_2$. We define

$$U_2 \triangleq U_0 \cdot_{M^s} U_1$$

And so on. Finally we set

$$U \triangleq \bigcup_{n\in\omega} U_n$$

and with any luck

$$U \approx U^U \tag{6.4}$$

I do not know whether (6.4) is provable. An indication that it might be is the fact that the set U appearing in the proof of the (untyped version) of the main theorem can be constructed in the way described above. On the other hand it is easy to see that sets U_0 such that $M \subset U_0 \subset M^s$ and conditions 1. and 2. hold exist in the model $\mathbf{V}^{(2^{<\omega})^{\mathrm{op}}}$. This would give an answer to the second point raised in Section 4.3.5.

6.4 Problems

I state again some of the problems worth investigating.

First, the main results of Sections 4. and 5. should be improved by pruning the underlying tree. Are Theorems 4.8 and 4.9 valid for $\mathbf{P} = \omega_0^{<\omega_0}$?

Then, we can ask again the question raised at the beginning of Section 4.3. Is it consistent with **IZF** that there is a full function space model $\mathcal{V}$ for any Σ-typed ca $\mathcal{U}$ such that for all $\gamma \in \Gamma$

$$U^\gamma \subset V^\gamma \subset (U^s)^\gamma$$

and $\mathcal{U}$ is isomorphic to $\mathcal{E}(\{U^\gamma\}_{\gamma\in\Gamma}, \mathcal{V})$? Related questions are: what can be said about the space of singletons of a set which is not $\neg\neg$-separated? Can the family of singletons of any ca be made into a combinatory algebra, and if so, which other properties are preserved?

This brings us to the last question. Is there a way to describe everything axiomatically as we tried to do in the previous section?

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