LFCS

Silence is Golden:
Branching Bisimilarity is Decidable
for Context-Free Processes

by

Hans Hüttel

Silence is Golden: ....

Silence is Golden:
Branching Bisimilarity is Decidable for Context-Free Processes

Hans Hüttel *

Laboratory for Foundations of Computer Science
James Clerk Maxwell Building
University of Edinburgh
Edinburgh EH9 3JZ
SCOTLAND
hans@dcs.ed.ac.uk

Abstract

We show that the branching bisimulation equivalence introduced by Rob van Glabbeek is decidable for the class of normed, recursively defined BPA processes with silent actions, thus generalizing the decidability result for strong bisimilarity by Baeten, Bergstra, and Klop.

Note: This is the full, revised version of the paper with the same title that appeared at the 3rd International Workshop on Computer-Aided Verification (CAV91) in Aalborg, 1-4 July 1991.

1 Introduction

In their paper [BBK87a] Baeten, Bergstra, and Klop showed that strong bisimulation equivalence is decidable for normed recursively defined BPA processes [BK88], a class of processes corresponding to that of irredundant context-free grammars without empty productions. In this paper we generalize this result by showing that the branching bisimilation equivalence introduced by van Glabbeek and Weijland in [vGW89] is decidable for the class of normed recursively defined BPA processes

---

*Present address: Department of Mathematics and Computer Science, Aalborg University Centre, Fredrik Bajersvej 7E, 9220 Aalborg Ø, Denmark. E-mail: hans@iesd.auc.dk

with silent actions. The proof generalizes that of the decidability of strong bisimulation equivalence given by Colin Stirling and the present author in [HS91], relying as it does on a similar tableau-based decision method. This tableau method is related to the equivalence algorithms for certain classes of context-free grammars introduced by Korenjak and Hopcroft [KH66], and it directly reflects usual intuitions about determining the equivalence of processes by means of successive matchings of transitions.

Since the class of processes considered allows infinite state spaces, the present result further underpins the fact that the decidability of behavioural equivalences can extend beyond the finite-state case, and that decidability should be seen as a criterion for determining the relative merits and deficiencies of behavioural equivalences.

Section 2.1 introduces the notion of branching bisimilarity and the class of normed $\text{BPA}^\tau_{\text{rec}}$ processes. In Section 4 we describe the tableau system, prove its soundness and completeness, give a complexity measure and establish the decidability result of branching bisimilarity. Section 5 sums up conclusions and gives suggestions for further work.

2 Preliminaries

2.1 Branching bisimilarity

The processes that we will be looking at have their behavioural semantics given by transition graphs with silent actions. For comparison we first describe the notion of weak bisimulation equivalence, introduced by Milner [Mil80, Mil89]. This equivalence is essentially bisimulation equivalence defined on the derived *weak* transition relations that disregard silent actions.

Definition 2.1 *For a transition graph* $\mathcal{G} = (Pr, Act \cup \{\tau\}, \rightarrow)$ *with silent action* τ, *the* weak *transition relations* $\{ \overset{s}{\Longrightarrow} \mid s \in Act \cup \{\epsilon\}\}$ *are given by* $\overset{a}{\Longrightarrow} = \overset{\tau}{\rightarrow}{}^* \overset{a}{\rightarrow} \overset{\tau}{\rightarrow}{}^*$ *for* $a \in Act$ *and* $\overset{\epsilon}{\Longrightarrow} = \overset{\tau}{\rightarrow}{}^*$

In the definition below, we use the 'observational' mapping $\phi : (Act \cup \{\tau\})^* \rightarrow Act^*$ which is the homomorphic extension of the function defined by $\phi(a) = a$ for $a \in Act$ and $\phi(\tau) = \epsilon$.

Definition 2.2 [Mil89] *A* weak bisimulation *on* $\mathcal{G}$ *is a symmetric relation* $R \subseteq Pr \times Pr$ *such that whenever* pRq *for any* $a \in Act \cup \{\tau\}$ *we have that* $p \overset{a}{\rightarrow} p'$ *implies that there exists a* q' *such that* $q \overset{\phi(a)}{\Longrightarrow} q'$ *with* $p'Rq'$. *We define* $\approx$ *by*

$$\approx = \{(p,q) \mid pRq \text{ for some weak bisimulation } R\}$$

If $p \approx q$ *we say that* p *and* q *are* weakly bisimilar.

The notion of branching bisimilarity was put forward by van Glabbeek and Weijland in [vGW89] as an alternative to weak bisimulation.

Definition 2.3 [vGW89] *A branching bisimulation (bb) on $\mathcal{G}$ is a symmetric relation $R \subseteq Pr \times Pr$ such that whenever pRq for any $a \in Act \cup \{\tau\}$ we have that $p \xrightarrow{a} p'$ implies*

- *$a = \tau$ and $p'Rq$ or*

- *there exist q'_1, q' such that $q \stackrel{\epsilon}{\Longrightarrow} q'_1 \xrightarrow{a} q'$ with $pRq'_1, p'Rq'$*

We define $\approx_b$ by
$$\approx_b = \{(p,q) \mid pRq \text{ for some bb } R\}$$
If $p \approx_b q$ we say that p and q are branching bisimilar.

Unlike weak bisimulation equivalence, changes in branching properties caused by individual τ-transitions must always be taken into account in branching bisimulation. (Example 3.2 provides an example of the importance of this, namely two processes that are weakly bisimilar but not branching bisimilar). An equivalent definition which reflects this *stuttering property* better is the one below which we will be using in the tableau system presented in Section 4.

Proposition 2.1 *A branching bisimulation on $\mathcal{G}$ is a symmetric relation $R \subseteq Pr \times Pr$ such that whenever pRq for any $a \in Act \cup \{\tau\}$ we have that if $p \xrightarrow{a} p'$ then either*

- *$a = \tau$ and $p'Rq$ or*

- *there exist $q'_0, \ldots, q'_n, q'$ such that $q = q'_0 \xrightarrow{\tau} q'_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} q'_n \xrightarrow{a} q'$ with pRq'_i for $0 \le i \le n$ and $p'Rq'$.*

van Glabbeek proves that the above notion of branching bisimilarity is indeed equivalent to that of Definition 2.3 by introducing a notion of what he calls a semi-branching bisimulation where the conditions for matching $p \xrightarrow{\tau} p'$ have been relaxed to allow matches of the form $q \stackrel{\epsilon}{\Longrightarrow} q'$ with $p'Rq'$. He then proceeds to show that the maximal semi-bb satisfies the stuttering property of Definition 2.1. For the details of the proof, see [vG90].

3 Normed BPA$^\tau_{\text{rec}}$

Basic Process Algebra (BPA) was introduced by Bergstra and Klop in [BK84]. Extending BPA with silent actions gives us the class BPA$^\tau_{\text{rec}}$ [BK88] consisting of processes given by systems of defining equations $\Delta = \{X_i \stackrel{\text{def}}{=} E_i \mid 1 \le i \le m\}$. The process expressions E_i are given by the syntax

$$E ::= a \mid \tau \mid E_1 + E_2 \mid E_1 E_2 \mid X$$

where τ is a new, silent action not in Act. In the following, elements of Var^* will be denoted by Greek letters: $\alpha, \beta, \ldots$ and BPA expressions in general by $E, F \ldots$. The operational semantics given by the transition relations $\{ \xrightarrow{a} \mid a \in Act \cup \{\tau\}\}$ is as given below.

Definition 3.1 *Any system of BPA process equations Δ defines a labelled transition graph. The transition relations are given as the least relations satisfying the following rules:*

$$\frac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'} \qquad\qquad \frac{F \xrightarrow{a} F'}{E + F \xrightarrow{a} F'}$$

$$\frac{E \xrightarrow{a} E'}{EF \xrightarrow{a} E'F} \qquad\qquad a \xrightarrow{a} \epsilon \quad a \in Act \cup \{\tau\}$$

$$\frac{E \xrightarrow{a} E'}{X \xrightarrow{a} E'} \qquad X \stackrel{\text{def}}{=} E \in \Delta$$

We restrict our attention to *weakly normed* systems of process equations.

Definition 3.2 *The* weak norm *of any $X \in Var$ is given by*

$$||X|| = \min\{length(w) \mid X \stackrel{w}{\Longrightarrow} \epsilon, w \in Act^*\}$$

A system of defining equations Δ is weakly normed *if for any $X \in Var$ $0 < ||X|| < \infty$. The maximal norm of any variable in Δ is defined by $M_\Delta = \max\{||X|| \mid X \in Var\}$.*

Since norms must be strictly positive, all variables must eventually perform an observable action and processes can therefore *not* terminate silently.

In Section 4 we shall also need the notion of a strong norm (cf. [HS91]).

Definition 3.3 *The* strong norm *of a BPA^τ_{rec} expression E is defined as*

$$|E| = \min\{length(w) \mid E \xrightarrow{w} \epsilon, w \in Act^+\}$$

A system of defining equations Δ is strongly normed *if for any variable $X \in Var$ $|X| < \infty$. The maximal strong norm of any variable in Δ is $m_\Delta = \max\{|X| \mid X \in Var\}$.*

4

Clearly, if Δ is weakly normed it is also strongly normed.

Finally, we restrict our attention to systems of defining equations given in 3-Greibach Normal Form (3-GNF).

Definition 3.4 *A system of BPA equations Δ is said to be in* Greibach Normal Form (GNF) *if all equations are of the form*

$$\{X_i \stackrel{\text{def}}{=} \sum_{j=1}^{n_i} a_{ij}\alpha_{ij} \mid 1 \le i \le m\}$$

If for each i,j the variable sequence α_{ij} has $length(\alpha_{ij}) < k$, Δ is said to be in k-GNF.

The normal form is called Greibach Normal Form by analogy with context-free grammars (without the empty production) in Greibach Normal Form (see e.g. [HU79]). There is an obvious correspondence with grammars in GNF: process variables correspond to non-terminals, the root is the start symbol, actions correspond to terminals, and each equation $X_i \stackrel{\text{def}}{=} \sum_{j=1}^{n_i} a_{ij}\alpha_{ij}$ can be viewed as the family of productions $\{X_i \to a_{ij}\alpha_{ij} \mid 1 \le j \le n_i\}$. The notion of normedness says that the grammar must not have useless productions. The requirement of norms being positive is in analogy with the requirement that a grammar has no empty productions.

It is well-known that any context-free language (without the empty string) is generated by a grammar in 3-GNF [HU79]. One can show that actually is not a real restriction, since any system of process equations Δ in BPA^τ_{rec} can effectively be rewritten to a Δ' which is *strongly* bisimilar to Δ and therefore weakly normed iff Δ is [BBK87b]. This leaves us with transition graphs whose states are strings of process variables; the further restriction to variable sequences of length at most 2 guarantees limited growth when determining single transitions:

Proposition 3.1 *Suppose Δ is in 3-GNF. Then, for any $\alpha \in Var^*$, whenever $\alpha \stackrel{a}{\to} \alpha'$ we have $length(\alpha') \le length(\alpha) + 1$.*

Because weak norms are assumed strictly positive, we have a simple relationship between lengths and norms:

Proposition 3.2 *For $\alpha \in Var^*$ $length(\alpha) \le ||\alpha||$ and $||\alpha|| \le M_\Delta length(\alpha)$.*

The weak norm is additive under sequential composition:

Proposition 3.3 *For $\alpha, \beta \in Var^*$ $||\alpha\beta|| = ||\alpha|| + ||\beta||$.*

PROOF: Let w_α be any shortest string with the property that $\alpha \stackrel{w_\alpha}{\Longrightarrow} \epsilon$ and let w_β be any shortest string with the property that $\beta \stackrel{w_\beta}{\Longrightarrow} \epsilon$. Clearly, $w_\alpha w_\beta$ is a shortest string u such that $\alpha\beta \stackrel{u}{\Longrightarrow} \epsilon$. $\square$

$$A \xrightarrow{b} BC \xrightarrow{\tau} AC \xrightarrow{b} BCC \cdots \xrightarrow{b} AC^n \xrightarrow{b} \cdots$$

$$a \downarrow \qquad a \downarrow \qquad \qquad a \downarrow$$

$$\epsilon \xleftarrow{\quad c \quad} C \xleftarrow{\quad c \quad} \cdots \xleftarrow{c} C^n \xleftarrow{\quad c \quad} \cdots$$
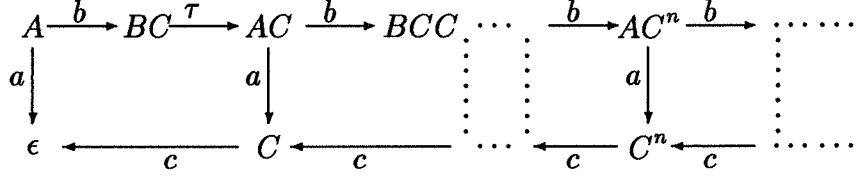
Figure 1: Transition graph for $A \stackrel{\text{def}}{=} a + bBC$; $B \stackrel{\text{def}}{=} \tau A$; $C \stackrel{\text{def}}{=} c$

Definition 3.5 *The observational language $L_{obs}(X)$ accepted by a variable X is defined by $L_{obs}(X) = \{w \mid X \stackrel{obs}{\Longrightarrow} \epsilon\}$.*

We have

Proposition 3.4 [vG90] *If $\alpha \approx_b \beta$ then $L_{obs}(\alpha) = L_{obs}(\beta)$.*

Note that for weakly normed systems this implies

Proposition 3.5 $\alpha \approx_b \beta$ *implies that* $||\alpha|| = ||\beta||$.

Example 3.1 Consider $\Delta_1 = \{A \stackrel{\text{def}}{=} a + bBC$; $B \stackrel{\text{def}}{=} \tau A$; $C \stackrel{\text{def}}{=} c\}$ and $\Delta_2 = \{X \stackrel{\text{def}}{=} a + bXY$; $Y \stackrel{\text{def}}{=} c\}$. The transition graphs are shown in Figure 1. For Δ_1 we have $L_{obs}(A) = \{b^n ac^n \mid n \geq 0\}$, $L_{obs}(B) = L_{obs}(A)$ and $L_{obs}(C) = \{c\}$. $X \approx_b A$ because of the branching bisimulation

$$\{(XY^n, AC^n) \mid n \geq 0\} \cup \{(XY^n, BC^n) \mid n \geq 1\} \cup \{(Y^n, C^n) \mid n \geq 1\} \cup \{(\epsilon, \epsilon)\}$$

□

For the tableau system we need the following results - firstly, $\approx_b$ is a congruence w.r.t. sequential composition:

Proposition 3.6 *If $\alpha_1 \approx_b \beta_1$ and $\alpha_2 \approx_b \beta_2$ then $\alpha_1 \alpha_2 \approx_b \beta_1 \beta_2$*

PROOF: $R = \{(\alpha_1 \alpha_2, \beta_1 \beta_2) \mid \alpha_1 \approx_b \beta_1, \alpha_2 \approx_b \beta_2\}$ is a bb. It is obvious that R is symmetric. Suppose $(\alpha_1 \alpha_2, \beta_1 \beta_2) \in R$. If $\alpha_1 \alpha_2 \xrightarrow{a} \alpha'$ this must be due to either $\alpha_1 \xrightarrow{a} \alpha'_1$ or $\alpha_1 = \epsilon$ and $\alpha_2 \xrightarrow{a} \alpha'_2$. In the former case $\alpha_1 \alpha_2 \xrightarrow{a} \alpha'_1 \alpha_2$, in the latter $\alpha_1 \alpha_2 \xrightarrow{a} \alpha'_2$. If $\alpha_1 \xrightarrow{a} \alpha'_1$ there are two possibilities. Either $a = \tau$ and $\alpha'_1 \approx_b \beta_1$, but then $(\alpha'_1 \alpha_2, \beta_1 \beta_2) \in R$. Or there exist β'_1, β''_1 with $\beta \stackrel{\epsilon}{\Longrightarrow} \beta'_1 \xrightarrow{a} \beta''_1$ such that $\alpha_1 \approx_b \beta'_1$ and $\alpha'_1 \approx_b \beta''_1$. Then $\beta_1 \beta_2 \stackrel{\epsilon}{\Longrightarrow} \beta'_1 \beta_2 \xrightarrow{a} \beta''_1 \beta_2$ and $(\alpha_1 \alpha_2, \beta'_1 \beta_2) \in R$, $(\alpha'_1 \alpha_2, \beta''_1 \beta_2) \in R$. If $\alpha_1 = \epsilon$ we must have $\beta_1 = \epsilon$ and thus $\beta_2 \stackrel{\epsilon}{\Longrightarrow} \beta'_2 \xrightarrow{a} \beta''_2$ with $\alpha_2 \approx_b \beta'_2$ and $\alpha'_2 \approx_b \beta''_2$, so $(\alpha_1 \alpha_2, \beta_1 \beta'_2) \in R$ and $(\alpha_1 \alpha'_2, \beta_1 \beta''_2) \in R$. □

The other result is a 'split' lemma that allows us to discard identical tails:

Lemma 3.1 *If $\alpha_1\alpha \approx_b \alpha_2\alpha$ then $\alpha_1 \approx_b \alpha_2$.*

PROOF: By Proposition 3.3 $||\alpha_1|| = ||\alpha_2||$, and since all variables are normed, $\alpha_1 = \epsilon$ iff $\alpha_2 = \epsilon$. Now $R = \{(\alpha_1,\alpha_2) \mid \alpha_1\alpha \approx_b \alpha_2\alpha\}$ is a bb. It is obvious that R is symmetric. Suppose $(\alpha_1,\alpha_2) \in R$ and $\alpha_1 \neq \epsilon$. Then also $\alpha_2 \neq \epsilon$, since $||\alpha_1|| = ||\alpha_2||$. So if $\alpha_1 \xrightarrow{a} \alpha_1'$ then $\alpha_1\alpha \xrightarrow{a} \alpha_1'\alpha$ can either be matched by $a = \tau$ and $\alpha_1'\alpha \approx_b \alpha_2\alpha$, implying $\alpha_1' R\alpha_2$, or by $\alpha_2\alpha \xRightarrow{\epsilon} \alpha_2'\alpha \xrightarrow{a} \alpha_2''\alpha$ with $\alpha_2'\alpha \approx_b \alpha_1\alpha$ and $\alpha_2''\alpha \approx_b \alpha_1'\alpha$. The latter holds, since weak norms are positive, so $\alpha_2\alpha \xRightarrow{\epsilon} \xrightarrow{a}$ because $\alpha_2 \xRightarrow{\epsilon} \xrightarrow{a}$. $\qquad\square$

It is important to note that this does *not* hold for weak bisimulation. The following counterexample arose in a discussion with Kim Larsen and is due to him.

Example 3.2 Consider $\Delta = \{X = aY, Y = a + \tau X, A = a + aB, B = a\}$. As $||X|| = 2, ||Y|| = 1, ||A|| = 1$ and $||B|| = 1$, Δ clearly obeys all requirements stated above. It is easily seen that $X \approx BY$ and that $A \not\approx B$. However, we have $AY \approx BY$, since $\{(AY, BY), (BY, X), (Y, Y), (\epsilon, \epsilon), (X, X)\}$ is a weak bisimulation. The problem lies in the fact that weak bisimilarity does not require the results of intermediate steps in weak transitions to be related. In particular, $AY \xrightarrow{a} Y$ is matched by $BY \xRightarrow{a} X$. The latter is due to $BY \xrightarrow{a} Y \xrightarrow{\tau} X$, where we clearly have that $AY \not\approx Y$. $\qquad\square$

4 A tableau system for branching bisimulation

4.1 Building tableaux

A tableau for determining branching bisimilarity is a maximal proof tree built using the proof rules in Table 1. Tableaux consist of a number of subtableaux. These are built from successive applications of the STEP rule.

STEP is applicable iff there is a possibility of matching transitions. A *possible match* is any set of equations whose sides are the results of successful matching transitions according to the definition of branching bisimilarity in Proposition 2.1:

Definition 4.1 *A set of equations M is a possible match for $\alpha = \beta$ if for any $a \in Act$ we have that if $\alpha \xrightarrow{a} \alpha'$ then either*

- *$a = \tau$ and $\alpha' = \beta \in M$ or*

- *there exist $\beta_0' = \beta, \ldots, \beta_n', \beta'$ such that $\beta_0 \xrightarrow{\tau} \beta_1' \xrightarrow{\tau} \cdots \xrightarrow{\tau} \beta_n' \xrightarrow{a} \beta'$ with $\alpha = \beta_i' \in M$ for $0 \le i \le n$ and $\alpha' = \beta' \in M$.*

and similarly for any $\beta \xrightarrow{a} \beta'$.

This definition appears to allow *infinitely* many possible matches, since there seems to be no bound on the length n of a matching transition sequence. However, this is not the case. Firstly, we have

Proposition 4.1 *If $\alpha \approx_b \beta$ we can find a possible match M for $\alpha = \beta$ such that whenever $\alpha \xrightarrow{a} \alpha'$ is matched by $\beta_0' = \beta, \ldots, \beta_{n'}', \beta'$ such that $\beta_0 \xrightarrow{\tau} \beta_1' \xrightarrow{\tau} \cdots \xrightarrow{\tau} \beta_n' \xrightarrow{a} \beta'$ with $\alpha = \beta_i' \in M$ for $0 \le i \le n$ and $\alpha' = \beta' \in M$ all β_i' $(0 \le i \le n)$ are distinct.*

PROOF: If some state β_j occurred twice in $\beta_0 \xrightarrow{\tau} \beta_1' \xrightarrow{\tau} \cdots \xrightarrow{\tau} \beta_n' \xrightarrow{a} \beta'$ we could remove any states between the two occurrences of β_j and still have a matching sequence. □

Secondly, we have

Proposition 4.2 *If $X\alpha \approx_b Y\beta$ and $X\alpha \xrightarrow{a} \alpha'\alpha$ is matched by $Y\beta \stackrel{\epsilon}{\Longrightarrow} \beta_1'\beta \xrightarrow{a} \beta'\beta$ any intermediate state β'' in $Y\beta \stackrel{\epsilon}{\Longrightarrow} \beta_1'\beta$ has $length(\beta'') \le M_\Delta + length(\beta)$. Furthermore, $length(\beta') \le M_\Delta + length(\beta) + 1$*

PROOF: For any single transition step $\beta_1 \xrightarrow{\tau} \beta_2$ in $Y\beta \stackrel{\epsilon}{\Longrightarrow} \beta_1'\beta$ Proposition 3.1 applies, so $length(\beta_2) \le 1 + length(\beta_1)$. Moreover, we must have $\beta_2 \approx_b X\alpha$ so by Proposition 3.5 we have $||\beta_2|| = ||X\alpha||$. Thus, in the worst case, where $||Y|| = M_\Delta$, we would have replaced Y by a sequence of M_Δ variables each having weak norm 1. □

The outbranching is a multiple of the bound $B_{X,Y}$ on the number of single transition steps for $X\alpha = Y\beta$; this factor only depends on the leftmost variables.

Definition 4.2 $B_{X,Y}$ *is the cardinality of*

$$\{\alpha' \mid X \xrightarrow{a} \alpha', a \in Act \cup \{\tau\}\} \cup \{\beta' \mid Y\beta \xrightarrow{a} \beta', a \in Act \cup \{\tau\}\}$$

Proposition 4.3 *Let v be the cardinality of Var. If $X\alpha \approx_b Y\beta$, there is a possible match for $X\alpha = Y\beta$ with at most $B_{X,Y} \sum_{j=2}^{2K}(j-1)v^j$ equations in a match, where $K = M_\Delta + 1 + \max(length(\alpha), length(\beta))$.*

PROOF: By Proposition 4.2, we see that there is a possible match for $X\alpha = Y\beta$ with expressions with lengths bounded by $K = M_\Delta + 1 + \max(length(\alpha), length(\beta))$. So the total length on an expression is at most $2K$, meaning that there are at most $\sum_{j=2}^{2K}(j-1)v^j$ different equations for any $X\alpha \xrightarrow{a} \alpha'\alpha$ in a possible match. □

Clearly, STEP is forwards sound in the following sense:

Proposition 4.4 (Forwards soundness of STEP) *If $\alpha \approx_b \beta$, then there is a possible match M such that whenever $\alpha' = \beta' \in M$ we have $\alpha' \approx_b \beta'$.*

An eliminating subtableau for $X\alpha = Y\beta$ consists of attempted matches to the depth where an equation of the form $\alpha = \gamma\beta$ is reached. When $|X| \leq |Y|$ each non-residual leaf of an eliminating subtableau for $X\alpha = Y\beta$ is either labelled $\alpha = \gamma\beta$ (a *residual* of the subtableau), or $\alpha_i\alpha = \beta_i\beta$. Because the number of successive attempted matches is $|X|$ there is at least one residual and since all norms are strictly positive, α and β must persist as suffixes throughout the subtableau. For any such subtableau we pick one residual node and call it *the* residual. If instead $|Y| < |X|$ the same holds, only now the residual is $\gamma\alpha = \beta$. Unless a subtableau leaf is a successful terminal (Definition 4.4 below) it is used as the basis of a new subtableau. However, before a new subtableau is constructed, for every leaf one of the SUB rules is used to trim the length of the expressions in the new subtableau root. The SUB rules are forwards sound in the following sense:

Proposition 4.5 (Soundness of SUBL and SUBR) *If $\alpha_i\alpha \approx_b \beta_i\beta$ and $\alpha \approx_b \gamma\beta$ then $\alpha_i\gamma \approx_b \beta_i$. If $\gamma\alpha_i \approx_b \beta_i$ then $\alpha_i \approx_b \beta_i\gamma$*

PROOF: From Propositions 3.6 and 3.1.   □

Rule within a subtableau

STEP $$\dfrac{\alpha = \beta}{\alpha_1 = \beta_1 \ldots \alpha_k = \beta_k}$$ where $\{\alpha_1 = \beta_1 \ldots \alpha_k = \beta_k\}$

is a possible match for $\alpha = \beta$

Rules for new subtableaux

SUBL $$\dfrac{\alpha_i\alpha = \beta_i\beta}{\alpha_i\gamma = \beta_i}$$ where $\alpha = \gamma\beta$ is the residual

SUBR $$\dfrac{\alpha_i\alpha = \beta_i\beta}{\alpha_i = \beta_i\gamma}$$ where $\gamma\alpha = \beta$ is the residual

---

Table 1: The tableau rules

The rules are only applied to nodes that are not *terminal*. Terminal nodes can either be *successful* or *unsuccessful*.

9

Definition 4.3 *A tableau node is an* unsuccessful terminal *if it has one of the forms*

1. $\alpha = \beta$ *with* $\|\alpha\| \neq \|\beta\|$

2. $\alpha = \beta$ *with* $\alpha \neq \epsilon, \beta \neq \epsilon$ *and no possible match exists (i.e.* STEP *is inapplicable).*

In both of these cases it is obvious that the expressions compared are not branching bisimilar. Thus, whenever we see an unsuccessful terminal the whole tableau construction aborts.

The nodes that can be successful terminals are those that are potential roots of eliminating subtableaux:

Definition 4.4 *A* residual or consequent *of an application of a* SUB *rule is a* successful terminal *if it has one of the forms*

1. $\alpha = \beta$ *where there is another subtableau root above it on the path from the root also labelled* $\alpha = \beta$

2. $\alpha = \alpha$

$$
\cfrac{\cfrac{X = A}{\cfrac{\epsilon = \epsilon \qquad \cfrac{XY = BC}{XY = BC}\; \text{SUBL}}{\cfrac{XY = AC}{X = A}\; \text{SUBL} \quad \cfrac{XYY = BCC}{XY = BC}\; \text{SUBL} \quad \cfrac{Y = C}{\epsilon = \epsilon}\; \text{STEP}}}\; \text{STEP}}{}\; \text{STEP}
$$

Figure 2: A successful tableau for $X = A$

Example 4.1 (Example 3.1 cont.) The tableau in Figure 2 is a successful tableau for $X = A$. □

4.2 Termination, completeness, and soundness

It is important for our decidability result that all tableaux are finite. This follows from reasoning entirely similar to that for the tableau system for strong bisimulation.

Theorem 4.1 *For any equation* $\alpha = \beta$ *all tableaux are finite.*

10

PROOF: Since our tableaux are finitely branching by Proposition 4.3, by König's Lemma an infinite tableau would have an infinite path. This would then be caused by the combined absence of unsuccessful termination and the successful termination condition 1 along that path. Since we have assumed 3-GNF and normedness, there is a uniform bound on the total length of the consequent of a SUB rule. Assume wlog that we have a subtableau with root $X\alpha = Y\beta$ and that a SUBL rule was applied to a subtableau leaf:

$$\frac{\alpha_1\alpha = \beta_1\beta}{\alpha_1\gamma = \beta_1} \quad \text{SUBL}$$

Since the depth of the subtableau is at most m_Δ, repeated applications of Proposition 4.2 tell us that $length(\alpha_1) \leq m_\Delta(M_\Delta + 1)$, $length(\beta_1) \leq m_\Delta(M_\Delta + 1)$ and $length(\gamma) \leq m_\Delta(M_\Delta + 1)$. This implies a uniform bound on the length of SUB consequents of $3m_\Delta(M_\Delta + 1)$, so there can be no infinite path through infinitely many SUB applications since there are of course only finitely many different equations of any given length. Nor can an infinite path pass through infinitely many residuals. For if a residual $\alpha_0 = \beta_0$ is above the residual $\alpha_1 = \beta_1$ we have that $||\alpha_0|| = ||\beta_0|| < ||\alpha_1|| = ||\beta_1||$. By Proposition 3.2, any subsequence of residuals therefore has a uniform bound on the total lengths of expressions compared, again ensuring termination. $\quad\square$

It is easily seen that the tableau system is complete:

Theorem 4.2 *If $\alpha \approx_b \beta$, $\alpha = \beta$ has a successful tableau.*

PROOF: By the forwards soundness of the STEP and SUB rules (Propositions 4.4 and 4.5) we can use the tableau rules in such a way that only valid consequents arise. Clearly this must give rise to a finite, successful tableau. $\quad\square$

Finally we must show soundness of the tableau system, namely that the existence of a successful tableau for $\alpha = \beta$ indicates that $\alpha \approx_b \beta$. This follows from the fact that the tableau system tries to construct a 'bisimulation up to a sequential congruence', which, if a successful tableau is reached, consists of the symmetric closure of the set of nodes in the successful tableau. This notion is the counterpart of the notion of a self-bisimulation used in the tableau system of [HS91] and in [Cau90]. In order to define the corresponding notion for branching bisimulation, we need a simple rephrasing of Proposition 2.1:

Proposition 4.6 *A branching bisimulation on a transition graph $\mathcal{G}$ is a symmetric relation $R \subseteq Pr \times Pr$ such that whenever pRq for any $a \in Act \cup \{\tau\}$ we have that if $p = p_0 \xrightarrow{\tau} p_1 \xrightarrow{\tau} \cdots p_m \xrightarrow{a} p'$ then there exist $q_0, q_1, \ldots, q_m, q'$ such that $q_0 = q$ and $p_i R q_i$ for $1 \leq i \leq m$, $p'Rq'$ and for $i < m$*

- $- q_i = q_{i+1}$ *or*

– there exist $q_{i_1}, \ldots, q_{i_{n(i)}}$ such that $q_i \xrightarrow{\tau} q_{i_1} \xrightarrow{\tau} \cdots q_{i_{n(i)}} \xrightarrow{\tau} q_{i+1}$ with $p_i R q_{ij}$
  for $1 \le j \le n(i)$

and either

- $a = \tau$ and $q_m = q'$ or

- there exist $q_{m_1}, \ldots, q_{m_{n(m)}}$ such that $q_m \xrightarrow{\tau} q_{m_1} \xrightarrow{\tau} \cdots q_{m_{n(m)}} \xrightarrow{a} q'$

PROOF: Clearly, any relation that satisfies the conditions of the proposition is a bb (let $m = 0$). For the reverse direction, suppose $p = p_0 \xrightarrow{\tau} p_1 \xrightarrow{\tau} \cdots p_m \xrightarrow{a} p'$. One then uses a straightforward induction in m.

Base case - $m = 0$: This is immediate, since here the definitions coincide.

Step - assuming for $m = k$: Here we know by induction hypothesis that the transition sequence $p = p_0 \xrightarrow{\tau} p_1 \xrightarrow{\tau} \cdots p_m$ can be matched according to the conditions in the proposition. We then extend the match to cover $p_m \xrightarrow{a} p'$ by appealing to Proposition 2.1.

$\square$

Definition 4.5 *For any binary relation R on Var^*, $\xrightarrow[R]{}$ is the least precongruence w.r.t. sequential composition that contains R, $\xleftrightarrow[R]{}$ the symmetric congruence of $\xrightarrow[R]{}$ and $\xleftrightarrow[R]{}^*$ the transitive closure of $\xleftrightarrow[R]{}$ and thus the least congruence w.r.t. sequential composition containing R.*

Definition 4.6 *A branching bisimulation up to sequential congruence (sbb) is a symmetric relation $R \subseteq Var \times Var$ such that whenever $\alpha R \beta$ $\alpha = \epsilon$ iff $\beta = \epsilon$ and for any $a \in Act \cup \{\tau\}$ we have that if $\alpha = \alpha_0 \xrightarrow{\tau} \alpha_1 \xrightarrow{\tau} \cdots \alpha_m \xrightarrow{a} \alpha'$ then there exist $\beta_0, \beta_1, \ldots, \beta_m, \beta'$ such that $\beta_0 = \beta$ and $\alpha_i \xleftrightarrow[R]{}^* \beta_i$ for $1 \le i \le m$, $\alpha' \xleftrightarrow[R]{}^* \beta'$ and for $i < m$*

- $\beta_i = \beta_{i+1}$ or
  - there exist $\beta_{i_1}, \ldots, \beta_{i_{n(i)}}$ s.t. $\beta_i \xrightarrow{\tau} \beta_{i_1} \xrightarrow{\tau} \cdots \beta_{i_{n(i)}} \xrightarrow{\tau} \beta_{i+1}$ with $\alpha_i \xleftrightarrow[R]{}^* \beta_{ij}$
    for $1 \le j \le n(i)$

and either

- $a = \tau$ and $\beta_m = \beta'$ or

- there exist $\beta_{m_1}, \ldots, \beta_{m_{n(i)}}$ s.t. $\beta_m \xrightarrow{\tau} \beta_{m_1} \xrightarrow{\tau} \cdots \beta_{m_{n(m)}} \xrightarrow{a} \beta'$ with $\alpha_m \xleftrightarrow[R]{}^* \beta_{mj}$
  for $1 \le j \le n(m)$.

The reason why a bisimulation up to sequential congruence can be said to be an essential part of a bisimulation lies in the following result.

Lemma 4.1 *If R is an sbb then $\overset{\longleftrightarrow}{R}{}^{*}$ is a bb.*

PROOF: It is clear that $\overset{\longleftrightarrow}{R}{}^{*}$ is symmetric. Now suppose $\alpha \overset{\longleftrightarrow}{R}{}^{*} \beta$; we must show that we can match transitions within $\overset{\longleftrightarrow}{R}{}^{*}$ as required by Definition 4.6. $\alpha \overset{\longleftrightarrow}{R}{}^{*} \beta$ must be due to $\alpha \overset{\longleftrightarrow}{R}{}^{n} \beta$ for some n, and the proof now proceeds by induction in n:

Base case 1, $n = 0$ is obvious, since $\alpha = \beta$ in this case.

Base case 2, $n = 1$ must have $\alpha = \sigma\alpha_0\chi, \beta = \sigma\beta_0\chi$ with $\alpha_0 R \beta_0$. If $\alpha_0 = \beta_0 = \epsilon$, we are done. Otherwise, if $\sigma \neq \epsilon$ any transition sequence

$$\alpha \overset{\tau}{\to} \alpha_1 \overset{\tau}{\to} \cdots \alpha_m \overset{a}{\to} \alpha' \tag{1}$$

is due to

$$\sigma \overset{\tau}{\to} \sigma_1 \overset{\tau}{\to} \cdots \sigma_m \overset{a}{\to} \sigma'$$

with $\alpha_i = \sigma_i\alpha_0\chi$ for $1 \leq i \leq m$ and $\alpha' = \sigma'\alpha_0\chi$. It is easy to see that this transition sequence can be matched by

$$\sigma\beta_0\chi \overset{\tau}{\to} \sigma_1\beta_0\chi \overset{\tau}{\to} \cdots \sigma_m\beta_0\chi \overset{a}{\to} \sigma'\beta_0\chi$$

with $\sigma_i\alpha_i\chi \overset{\longleftrightarrow}{R}{}^{*} \sigma_i\beta_i\chi$ for all $1 \leq i \leq m$ and $\sigma'\alpha_0\chi \overset{\longleftrightarrow}{R}{}^{*} \sigma'\beta_0\chi$.

If $\sigma = \epsilon$, (1) becomes

$$\alpha_0\chi \overset{\tau}{\to} \alpha_{01}\chi \overset{\tau}{\to} \cdots \alpha_{0m}\chi \overset{a}{\to} \alpha_0'\chi \tag{2}$$

due to

$$\alpha_0 \overset{\tau}{\to} \alpha_{01} \overset{\tau}{\to} \cdots \alpha_{0m} \overset{a}{\to} \alpha_0'$$

There exist $\beta_{00}, \ldots, \beta_{0m}, \beta_0'$ where either $\beta_{0i} = \beta_{0i+1}$ for $i \geq 1$ or there exist $\beta_{0i_1}', \ldots, \beta_{0i_{n(1)}}'$ such that $\beta_0 = \beta_{00} \overset{\tau}{\to} \cdots \beta_{01} \overset{\tau}{\to} \cdots \beta_{0m} \overset{\tau}{\to} \cdots \overset{a}{\to} \beta_0'$ is a matching transition sequence with $\alpha_{0i} \overset{\longleftrightarrow}{R}{}^{*} \beta_{0i}$ and $\alpha_0' \overset{\longleftrightarrow}{R}{}^{*} \beta_0'$. If $a = \tau$, possibly $\beta_{0m}' = \beta_0'$. Here (2) is matched by

$$\beta_0\chi \overset{\tau}{\to} \cdots \beta_{01}\chi \overset{\tau}{\to} \cdots \beta_{0m}\chi \overset{\tau}{\to} \cdots \overset{a}{\to} \beta_0'\chi$$

where possibly $\beta_{0m}'\chi = \beta_0'\chi$.

Step, $n = k + 1$ *assuming for k, where $k \geq 1$:* Here there is an ρ such that $\alpha \overset{\longleftrightarrow}{R}{}^{k} \rho \overset{\longleftrightarrow}{R} \beta$. By induction hypothesis, we must have that (1) is matched using $\rho_0 = \rho, \rho_1, \ldots, \rho_m, \rho'$ where $\alpha_i \overset{\longleftrightarrow}{R}{}^{*} \rho_i$ and either $\rho_i = \rho_{i+1}$ or there exist $\rho_{i1}, \ldots, \rho_{in(i)}$ such that

$$\rho_i \overset{\tau}{\to} \rho_{i1} \overset{\tau}{\to} \cdots \overset{\tau}{\to} \rho_{in(i)} \overset{\tau}{\to} \rho_{i+1} \tag{3}$$

13

with $\alpha_i \xrightarrow[R]{}{}^* \rho_{ij}$ for $1 \le j \le n(i)$ and either $\rho_m = \rho'$ with $a = \tau$ or there exist $\rho_{m1}, \ldots, \rho_{mn(m)}$ such that

$$\rho_m \xrightarrow{\tau} \rho_{m1} \xrightarrow{\tau} \cdots \xrightarrow{\tau} \rho_{mn(n)} \xrightarrow{\tau} \rho'$$

Each transition sequence of the form (3) can be matched by β_i, β_{i+1} such that either $\beta_i = \beta_{i+1}$ or there exist $\beta_{i1}, \ldots, \beta_{ik(i)}$ such that

$$\beta_i \xrightarrow{\tau} \beta_{i1} \xrightarrow{\tau} \beta_{i2} \cdots \beta_{ik_i} \xrightarrow{\tau} \beta_{i+1}$$

such that for all $1 \le i \le m$ we have $\rho_i \xrightarrow[R]{}{}^* \beta_i$ and $\rho_i \xrightarrow[R]{}{}^* \beta_{ij}$ for $1 \le j \le n(i)$. And if $\rho_m \ne \rho'$ we can match similarly with $\beta_m \xrightarrow{\tau} \cdots \xrightarrow{a} \beta'$. By transitivity of $\xrightarrow[R]{}{}^*$ we see that the concatenation of the matching sequences

$$\beta_0 \xrightarrow{\tau} \cdots \beta_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} \beta_m \xrightarrow{\tau} \cdots \xrightarrow{a} \beta'$$

constitutes the desired matching sequence of transitions for (1). $\qquad \square$

Corollary 4.1 $\alpha \approx_b \beta$ *iff there is an sbb R such that $\alpha R \beta$.*

PROOF: From the above and from the fact that any bb is an sbb. $\qquad \square$

We now have

Theorem 4.3 *If $\alpha = \beta$ has a successful tableau $\mathbf{T}$ then*

$$R_{\mathbf{T}} = \{(\alpha', \beta') \mid \alpha' = \beta' \text{ or } \beta' = \alpha' \text{ is an equation in } \mathbf{T}\}$$

is an sbb.

PROOF: It is obvious that $R_{\mathbf{T}}$ is symmetric. Since all variables have positive norms, we see that $\alpha' = \epsilon$ iff $\beta' = \epsilon$. We must now show that for any $(\alpha', \beta') \in R_{\mathbf{T}}$ any transition sequence

$$\alpha' \xrightarrow{\tau} \alpha'_1 \xrightarrow{\tau} \cdots \alpha'_m \xrightarrow{a} \alpha'' \tag{4}$$

can match within $\xrightarrow[R_{\mathbf{T}}]{}{}^*$. We now proceed by induction in the length m of the τ-transition.

Base case, $m = 0$: Here (4) is $\alpha' \xrightarrow{a} \alpha'_1$, and there are now four cases to be considered, depending on where in $\mathbf{T}$ $\alpha' = \beta'$ is found.

- If $\alpha' = \beta'$ is a successful terminal due to condition 2, it is obvious that we can match within $\xrightarrow[R_{\mathbf{T}}]{}{}^*$, since any least congruence contains the identity.

14

- If $\alpha' = \beta'$ is a successful terminal due to condition 1 this means that there is a subtableau root above it also labelled $\alpha' = \beta'$. This node must be the premise of a STEP application, and because T is successful, a possible match exists so we can find a matching sequence

$$\beta' \xrightarrow{\tau} \beta''_{11} \xrightarrow{\tau} \cdots \xrightarrow{\tau} \beta''_{1n(1)} \xrightarrow{a} \beta''$$

where $(\alpha', \beta''_{1j}) \in R_T$ for $1 \leq j \leq n(1)$ and $(\alpha'', \beta'') \in R_T$.

- If $\alpha' = \beta'$ is an internal node and the premise of a STEP, we proceed exactly as in the previous case for termination condition 1.

- If $\alpha' = \beta'$ is an internal node and the premise of a SUB, assume wlog that the rule applied was SUBL. Further suppose $\alpha' = \beta'$ is $X_1\alpha_1\alpha = Y_1\beta_1\beta$ and that $\alpha = \gamma\beta$ is the residual. Then we have

$$\frac{X_1\alpha_1\alpha = Y_1\beta_1\beta}{X_1\alpha_1\gamma = Y_1\beta_1} \quad \text{SUBL}$$

If $X_1\alpha_1\gamma = Y_1\beta$ is a terminal node it can either be an identity (condition 2) or a repeated occurrence (condition 1). In the former case we know that the transition $X_1\alpha_1\gamma \xrightarrow{a} \alpha''_a\alpha_1\gamma$ can be matched by $Y_1\beta \xrightarrow{a} \beta''_a\beta_1$ where $\alpha''_a\alpha_1\gamma = \beta''_a\beta_1$, so clearly $\alpha''_a\alpha_1\gamma \xleftrightarrow[R_T]{}^* \beta''_a\beta_1$.

In the latter case the situation is the same as when STEP is applied to the SUB consequent. Here the transition $X_1\alpha_1\gamma \xrightarrow{a} \alpha''_a\alpha_1\gamma$ will be matched either by $Y_1\beta_1$ (if $a = \tau$) with $\alpha''_a\alpha_1\gamma R_T Y_1\beta_1$ or by some

$$Y_1\beta_1 \xrightarrow{\tau} \beta''_{11}\beta_1 \xrightarrow{\tau} \cdots \xrightarrow{a} \beta''_{1n(1)}\beta_1 \xrightarrow{a} \beta''_a\beta_1$$

with $(\alpha''_a\alpha_1\gamma, \beta''_{1j}\beta_1) \in R_T$ for $1 \leq j \leq n(1)$ and $(\alpha''_a\alpha_1\gamma, \beta''_a\beta_1) \in R_T$. Clearly $X_1\alpha_1\alpha \xrightarrow{a}$ iff $X_1\alpha_1\gamma \xrightarrow{a}$ and thus (4) is of the form $X_1\alpha_1\alpha \xrightarrow{a} \alpha''_a\alpha_1\alpha$ which can be matched either by $Y_1\beta_1\beta$ if $a = \tau$ or by

$$Y_1\beta_1\beta \xrightarrow{\tau} \beta''_{11}\beta_1\beta \cdots \xrightarrow{a} \beta_{1n}\beta_1\beta \xrightarrow{a} \beta''_a\beta_1\beta$$

For then $X_1\alpha_1\alpha \xleftrightarrow[R_T]{}^* \beta''_{1j}\beta_1\beta$ for $1 \leq j \leq n(1)$ and $\alpha''_a\alpha_1\alpha \xleftrightarrow[R_T]{}^* \beta''_a\beta_1\beta$. This holds, since $\alpha \xleftrightarrow[R_T]{}^* \gamma\beta$ and $X_1\alpha_1\gamma R_T\beta''_{1j}\beta_1$ implies $X_1\alpha_1\gamma\beta \xleftrightarrow[R_T]{}^* \beta''_{1j}\beta_1\beta$ giving $X_1\alpha_1\alpha \xleftrightarrow[R_T]{}^* \beta''_{1j}\beta_1\beta$ and $\alpha''_a\alpha_1\gamma\beta R_T\beta''_a\beta_1\beta$, implying $\alpha''_a\alpha_1\alpha \xleftrightarrow[R_T]{}^* \beta''_a\beta_1\beta$. Similarly, $\alpha'_a\alpha_1\alpha \xleftrightarrow[R_T]{}^* Y_1\beta_1\beta$.

15

Step: Suppose $\alpha' \xrightarrow{\tau} \alpha'_1 \xrightarrow{\tau} \cdots \alpha'_m \xrightarrow{a} \alpha''$. If $\alpha' = \beta'$ is a terminal because of condition 2, it is obvious that we can match within $\xrightarrow[R_T]{}{}^*$. Otherwise, by the base case we know that $\alpha' \xrightarrow{\tau} \alpha'_1$ can be matched either by $\alpha'_1 \xrightarrow[R_T]{}{}^* \beta'$ or by

$$\beta' \xrightarrow{\tau} \beta'_{11} \xrightarrow{\tau} \cdots \xrightarrow{\tau} \beta''_{1n(1)} \xrightarrow{a} \beta'_1$$

where $\alpha' \xrightarrow[R_T]{}{}^* \beta''_{1j}$ for $1 \leq j \leq n(1)$ and $\alpha'_1 \xrightarrow[R_T]{}{}^* \beta'_1$. Now if $(\alpha'_1, \beta'_1) \in R_T$ (or $(\alpha'_1, \beta') \in R_T$) we know by induction hypothesis that we can match the transition sequence $\alpha'_1 \xrightarrow{\tau} \cdots \alpha'_m \xrightarrow{a} \alpha''$ within $\xrightarrow[R_T]{}{}^*$ so our match for (4) simply consists in combining these two matches. From the base case we see that the only case where it might *not* be the case that $(\alpha'_1, \beta'_1) \in R_T$ or $(\alpha'_1, \beta') \in R_T$ is when $\alpha' = \beta'$ is the premise of a SUB rule. So assume wlog that the rule applied was SUBL and that $\alpha' = \beta'$ is $X_1\alpha_1\alpha = Y_1\beta_1\beta$ with $\alpha = \gamma\beta$ as the residual. Then we have

$$\frac{X_1\alpha_1\alpha = Y_1\beta_1\beta}{X_1\alpha_1\gamma = Y_1\beta_1} \quad \text{SUBL}$$

So here $X_1\alpha_1\alpha \xrightarrow{\tau} X_2\alpha_2\alpha_1\alpha$, so consequently $\alpha'_1 = \beta'_1$ is of the form $X_2\alpha_2\alpha_1\alpha = Y_2\beta_2\beta_1\beta$ with either $(X_2\alpha_2\alpha_1\gamma, Y_2\beta_2\beta_1) \in R_T$ or $X_2\alpha_2\alpha_1\gamma = Y_2\beta_2\beta_1$, if we are dealing with a terminal of type 2. A closer look at the transition sequence $\alpha'_1 \xrightarrow{\tau} \cdots \alpha'_m \xrightarrow{a} \alpha''$ reveals that it is of the form

$$X_2\alpha_2\alpha_1\alpha \xrightarrow{\tau} \alpha_3\alpha_1\alpha \cdots \xrightarrow{\tau} \alpha_m\alpha_1\alpha \xrightarrow{a} \alpha''_1\alpha_1\alpha \tag{5}$$

which is due to

$$X_2\alpha_2\alpha_1\gamma \xrightarrow{\tau} \alpha_3\alpha_1\gamma \cdots \xrightarrow{\tau} \alpha_m\alpha_1\gamma \xrightarrow{a} \alpha''_1\alpha_1\gamma$$

Both in the cases when $X_2\alpha_2\alpha_1\gamma = Y_2\beta_2\beta_1$ is a terminal of type 2 and when we have $(X_2\alpha_2\alpha_1\gamma, Y_2\beta_2\beta_1) \in R_T$, we know (in the latter case by induction hypothesis) that there must exist $\beta_i\beta_1$ for $3 \leq i \leq m$ and $\beta''_1\beta_1$ such that either

$$\beta_2\beta_1 \xrightarrow{\tau} \beta_3\beta_1 \cdots \beta_m\beta_1 \cdots \xrightarrow{a} \beta''_1\beta_1$$

or, if $a = \tau$,

$$\beta_2\beta_1 \xrightarrow{\tau} \beta_3\beta_1 \cdots \beta_m\beta_1 \cdots = \beta''_1\beta_1$$

with $\alpha_i\alpha_1\gamma \xrightarrow[R_T]{}{}^* \beta_i\beta_1$ and either $\beta_i\beta_1 = \beta_{i+1}\beta_1$ or there are $\beta_{ij}\beta_1$ for $1 \leq j \leq n(i)$ such that $\beta_i\beta_1 \xrightarrow{\tau} \beta_{i1}\beta_1 \xrightarrow{\tau} \cdots \beta_{i+1}\beta_1$ with $\alpha_i\alpha_1\gamma \xrightarrow[R_T]{}{}^* \beta_{ij}\beta_1$ and $\alpha''_1\alpha_1\gamma \xrightarrow[R_T]{}{}^* \beta''_1\beta_1$. But then our match for $\alpha' \xrightarrow{\tau} \alpha'_1 \xrightarrow{\tau} \cdots \alpha'_m \xrightarrow{a} \alpha''$ consists of the concatention of the match for $\alpha' \xrightarrow{\tau} \alpha'_1$ and that for (5), which is

16

$$\beta_2\beta_1\beta \xrightarrow{\tau} \beta_3\beta_1\beta \cdots \beta_m\beta_1\beta \cdots \xrightarrow{a} \beta_1''\beta_1\beta$$

with $\beta_i\beta_1\beta$ for $3 \leq i \leq m$ and if $\beta_i\beta_1\beta \neq \beta_{i+1}\beta_1\beta$ the additional $\beta_{ij}\beta_1\beta$ for $1 \leq j \leq n(i)$. Clearly we now have that $\alpha_i\alpha_1\alpha \underset{R_T}{\longrightarrow}{}^* \beta_i\beta_1\beta$, $\alpha_i\alpha_1\alpha \underset{R_T}{\longrightarrow}{}^* \beta_{ij}\beta_1\beta$ and $\alpha_1''\alpha_1\alpha \underset{R_T}{\longrightarrow}{}^* \beta_1''\beta_1\beta$. $\qquad\square$

So we now get the soundness of the tableau system as

Corollary 4.2 *If $\alpha = \beta$ has a successful tableau then $\alpha \approx_b \beta$.*

4.3 Complexity of the tableau system and decidability

The complexity of the tableau system can be measured in terms of the maximal depth of a tableau, i.e. the length w.r.t. STEP applications of the longest possible path in a successful tableau for an equation $X\alpha = Y\beta$. Let v be the cardinality of Var. By the discussion in the proof of Theorem 4.1 we have that any SUB consequent has a length of at most $3m_\Delta(M_\Delta + 1)$, so an upper bound on the number of distinct SUB consequents along any tableau path is $\sum_{j=2}^{3m_\Delta(M_\Delta+1)}(j-1)v^j$. Between any two SUB consequents there are at most $\lceil\frac{3m_\Delta(M_\Delta+1)}{2}\rceil$ residuals, since the worst that happen is that the total norm decreases by 2 in every subtableau along the way. Thus, any path that contains SUB consequents can have at most $\lceil\frac{3m_\Delta(M_\Delta+1)}{2}\rceil \sum_{j=2}^{3m_\Delta M_\Delta+3}(j-1)v^j$ subtableau roots. As for the leftmost path, all of whose subtableau roots are residuals, there can be at most $\max(||\alpha||, ||\beta||)$ residuals, since the norm of the residuals is strictly decreasing. So, since a subtableau can have a depth w.r.t. STEP applications of at most m_Δ, any path can have a length of at most

$$\max(||\alpha||, ||\beta||, \lceil\frac{3m_\Delta(M_\Delta + 1)}{2}\rceil \sum_{j=2}^{3m_\Delta(M_\Delta+1)}(j-1)v^j) \tag{6}$$

STEPs.

We also have an upper bound on the outbranching of any tableau for $X\alpha = Y\beta$. This follows from the fact that there is a uniform upper bound on the total length of any subtableau root in any tableau for $X\alpha = Y\beta$. Any subtableau root along the leftmost path is a residual and has its total length bounded by $2\max(||\alpha||, ||\beta||)$. Since, as we saw, any SUB consequent has its length bounded by $3m_\Delta(M_\Delta + 1)$, and thus also has a norm of at most $3m_\Delta(M_\Delta + 1)$, any of its following residuals must also have a length of at most $3m_\Delta(M_\Delta + 1)$. So the length of any subtableau root is bounded by

$$L = \max(2\max(||\alpha||, ||\beta||), 3m_\Delta(M_\Delta + 1))$$

By repeated applications of Proposition 4.2 we see that any node in a sub-tableau has a length of at most $2m_\Delta(M_\Delta + 1) + L$. By Proposition 4.3 this means that there is a uniform upper bound on the number of STEP consequents at any point in any tableau for $X\alpha = Y\beta$ of

$$\max\{B_{X,Y} \mid X, Y \in Var\} \sum_{j=2}^{2m_\Delta(M_\Delta+1)+L} (j-1)v^j \tag{7}$$

Theorem 4.4 *For any $X\alpha = Y\beta$ there are finitely many possible tableaux.*

PROOF: From (6) and (7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 4.5 *For any weakly normed Δ it is decidable whether or not $\alpha \approx_b \beta$ for $\alpha, \beta \in Var^*$.*

PROOF: A naive decision procedure for $\approx_b$ constructs all the finitely many tableaux for $\alpha = \beta$, answering 'yes' if a successful tableau occurs and 'no' otherwise. □

5 Conclusions and directions for further work

We have here shown that the branching bisimilarity of [vGW89] is decidable for the class of normed BPA processes with silent actions by giving a tableau system. This system has exponential complexity in terms of the longest possible path of a generated tableau; however, in the case of a successful tableau we get additional information in the form of a finite relation whose congruence closure w.r.t. sequential composition is a bisimulation containing the initial equation.

The results of [GH91] show that all known strong equivalences except bisimilarity are undecidable for normed BPA. This means that their weak counterparts also are undecidable, but there are still several open questions for the weak versions of bisimulation equivalence. For branching bisimulation, the restriction to processes with strictly positive norms is rather strong, as it rules out the possibility of a process terminating silently. A problem with having nullary norms in the tableau system is that we no longer are guaranteed that α and β persist throughout an eliminating subtableau for $X\alpha = Y\beta$, since a match for $X\alpha \xrightarrow{a}$ may require access to observable actions inside β. So the natural question is whether there is a way of introducing nullary norms. Moreover, we would of course also want to get rid of the restriction of normedness altogether. However, since this problem also needs to be tackled for strong bisimulation equivalence, it seems that progress must first be made here before we can give any answer for the branching bisimulation case. Last, but not least the questions for weak bisimilarity all remain open. As we saw, Lemma 3.1 does not hold for this equivalence so a different approach must be used in that case.

Finally, it would be interesting if we could give a syntax-directed version of our tableau system for branching bisimulation since this could give us an equational theory of $\approx_b$ over normed BPA^τ_{rec} along the lines of [HS91]. A naive approach would be to add the τ-laws for branching bisimulation to the equational theory for strong bisimilarity of [HS91], where a proof system is given that consists of rules that can simulate the tableau construction. However, this theory would *not* be powerful enough for this; the problem lies in simulating the STEP rule.

References

[BBK87a] J.C.M. Baeten, J.A. Bergstra, and J.W. Klop. Decidability of bisimulation equivalence for processes generating context-free languages. In *LNCS 259*, pages 93–114. Springer-Verlag, 1987.

[BBK87b] J.C.M. Baeten, J.A. Bergstra, and J.W. Klop. Decidability of bisimulation equivalence for processes generating context-free languages. Technical Report CS-R8632, CWI, September 1987.

[BK84] J.A. Bergstra and J.W. Klop. Process algebra for synchronous communication. *Information and Control*, 60:109–137, 1984.

[BK88] J.A. Bergstra and J.W. Klop. Process theory based on bisimulation semantics. In J.W. de Bakker, W.P de Roever, and G. Rozenberg, editors, *LNCS 354*, pages 50–122. Springer-Verlag, 1988.

[Cau90] D. Caucal. Graphes canoniques de graphes algébriques. *Informatique théorique et Applications (RAIRO)*, 24(4):339–352, 1990.

[GH91] J.F. Groote and Hüttel. Undecidable equivalences for basic process algebra. Technical Report ECS-LFCS-91-169, Department of Computer Science, University of Edinburgh, August 1991.

[HS91] H. Hüttel and C. Stirling. Actions speak louder than words: Proving bisimilarity for context-free processes. In *Proceedings of 6th Annual Symposium on Logic in Computer Science (LICS 91)*, pages 376–386. IEEE Computer Society Press, 1991.

[HU79] J. Hopcroft and J.D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley, 1979.

[KH66] A.J. Korenjak and J.E. Hopcroft. Simple deterministic languages. In *Proceedings of Seventh Annual IEEE Symposium on Switching and Automata Theory*, pages 36–46, 1966.

[Mil80] Robin Milner. *A Calculus of Communicating Systems, LNCS 92*. Springer-Verlag, 1980.

[Mil89]   R. Milner. *Communication and Concurrency*. Prentice-Hall International, 1989.

[vG90]   R.J. van Glabbeek. *Comparative Concurrency Semantics and Refinement of Actions*. PhD thesis, CWI/Vrije Universiteit, 1990.

[vGW89]   R.J. van Glabbeek and W.P. Weijland. Branching time and abstraction in bisimulation semantics (extended abstract). In G.X. Ritter, editor, *Information Processing 89*, pages 613–618. North-Holland, 1989.