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by

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Department of Computer Science
University of Edinburgh
The King's Buildings
Edinburgh EH9 3JZ

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Fixpoint and Loop Constructions as Colimits ^{*}

C. Barry Jay [†]

Department of Computer Science, University of Edinburgh
The King's Buildings, Mayfield Road, Edinburgh, U.K. EH9 3JZ
e-mail: cbj@lfc.ed.ac.uk

Abstract

The constructions of fixpoints and while-loops in a category of domains can be derived from the colimit, $\text{loop}(f)$ of a diagram which consists of a single endomorphism $f : D \rightarrow D$. If f is increasing then the colimiting map is the least-fixpoint function Y and

$$\text{loop}(f) = \text{fix}(f)$$

the subobject of fixpoints. If $f = \text{cond}(b, g, 1)$ is the conditional of a while-program then

$$\text{loop}(f) = (D_{\neg b} + \text{loop}_{\infty}(g))_{\perp}$$

the lifted sum of the terminating values (where b is false) and the infinite loops.

1 Fixpoints

Let $\mathbf{Pos}(\omega)$ the category of ω -complete partial orders and continuous functions (bottomless c.p.o.'s). A bottom element ($\perp$) can be adjoined to such a poset D by *lifting* it to $D_{\perp}$. Also, if $g : D \rightarrow D'$ is continuous then $g* : D_{\perp} \rightarrow D'_{\perp}$ is the strict extension of g i.e. agrees with g on D and preserves $\perp$ [4].

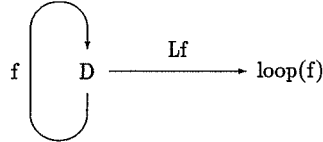
Let $f : D \rightarrow D$ be a continuous map in $\mathbf{Pos}(\omega)$. Its fixpoints are defined by $\text{fix}(f) = \{d \in D \mid fd = d\}$. Categorically, this definition asserts that the inclusion $i : \text{fix}(f) \rightarrow D$ is the limit of the loop:

$$\text{fix}(f) \xrightarrow{\quad} \begin{array}{c} \text{f} \\ \downarrow \\ \text{---} \rightarrow D \\ \uparrow \end{array}$$

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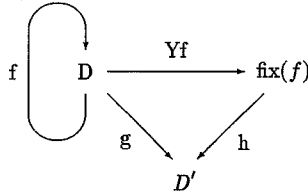
Equivalently, it is the equaliser of the pair $f, 1_D : D \rightarrow D$. (but not of the diagram $D \xrightarrow{f} D$ whose limit is just 1_D). Of course, this definition does not show how to construct any fixpoints. Of more use is the diagram's *colimit*, Lf



which will identify all of the elements of the sequence $\{f^n d\}$. The colimit exists since $\mathbf{Pos}(\omega)$ is cocomplete [3] (though the colimit constructions are non-trivial).

Theorem 1.1 *Let $f \geq 1 : D \rightarrow D$ be increasing, i.e. $fd \geq d$ for all $d \in D$. Then the colimit of the loop is $\text{fix}(f)$ and $Lf = Yf : D \rightarrow \text{fix}(f)$ is the least fixpoint function.*

Proof Yf is defined as usual [6] by $Yf(d) = \bigvee f^n d$ and is denoted by Y when there is no risk of ambiguity. Recall [2] that it is the colimit if whenever $g : D \rightarrow D'$ is a continuous function such that $gf = g$ then there is a unique continuous function $h : \text{fix}(f) \rightarrow D'$ satisfying $h.Y = g$.



More generally, we may consider an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ equipped with a natural transformation $\alpha : 1 \Rightarrow F$. Define $\text{fix}(\alpha)$ be its inverter [7] i.e. the full subcategory of objects for which α is an isomorphism, and $L(\alpha) : \mathcal{C} \rightarrow \text{loop}(\alpha)$ to be the coinverter, i.e. the universal functor whose composite with α is invertible.

Theorem 1.2 *If $\mathcal{C}$ has, and F preserves, colimits of countable chains then we can take $\text{loop}(\alpha) = \text{fix}(\alpha)$ with $L(\alpha)C = \text{colim}\{F^n \alpha_C : F^n C \rightarrow F^{n+1} C\}$.*

Proof Just generalise the previous proof in the style of [5]. $\square$

A similar result holds in the category of complete metric spaces and distance-decreasing functions: if $f : X \rightarrow X$ is Lipschitz, i.e. there is a constant $\lambda < 1$ such that for all $x, y \in X$

$$d(fx, fy) \leq \lambda d(x, y)$$

then $\text{loop}(f) = \text{fix}(f)$. Lawvere [1] observed that metric spaces can be thought of as *enriched* categories with distance-decreasing functions as functors. Also, in any category, if $f^2 = f : D \rightarrow D$ then $\text{loop}(f) = \text{fix}(f)$ whenever they exist. Is there a common generalisation of these results?

Of course, we may want to compute fixpoints of a continuous function f even when it is not always increasing. One solution is to cut the domain down to those points where it is. The *postfix* points of f (pace *prefix points* [5]) are defined by

$$D_0 = \text{postfix}(f) = \{d \in D \mid d \leq fd\}$$

(Taking f as a functor, this is the *inserter* [7] from 1_D to f .) Now the postfix points are closed under the application of f and include the fixpoints. Thus the restriction of f to D_0 is an increasing endomorphism f_0 which has the same fixpoints as f , whence $Yf_0(d)$ is the least fixpoint over d .

For example, the factorial function $\text{fact} : N_{\perp} \rightarrow N_{\perp}$ on the lifted natural numbers is the least fixpoint of $h : (N_{\perp} \rightarrow N_{\perp}) \rightarrow (N_{\perp} \rightarrow N_{\perp})$ given by

$$\begin{aligned} h(g)(0) &= 1 \\ h(g)(n+1) &= (n+1) * g(n) \end{aligned}$$

Here $\text{postfix}(h)$ consists of fact and its initial segments fact_n where

$$\begin{aligned} \text{fact}_n(k) &= k! & \text{if } 0 \leq k \leq n \\ &= \perp & \text{otherwise} \end{aligned}$$

Further $h_0(\text{fact}_n) = \text{fact}_{n+1}$ and so we may identify $\text{postfix}(h)$ with $\omega + 1$ (the free ω -chain $0 \leq 1 \leq \dots \leq \omega$) and h_0 with the successor function. Clearly then Yh_0 is the unique map $\omega + 1 \rightarrow \{\omega\}$ where ω corresponds to fact .

Although this method recovers the fixpoints, as a method of assigning fixpoints it is quite limited. For example, if some tail of the sequence $\{f^n d\}$ is a chain then its supremum is a fixpoint naturally associated to d , even if $d \not\leq fd$.

In more detail, consider $f : N \times N \rightarrow N \times N$ defined by

$$\begin{aligned} f(0, n) &= (0, n) \\ f(m+1, n) &= (m, (m+1)n) \end{aligned}$$

Clearly $\text{fix}(f) = \text{postfix}(f) = \{(0, n) \mid n \in N\}$. The interesting question is how to map the point (m, n) to the ‘limit’ of the sequence $\{f^k(m, n)\}$ which becomes stationary at $(0, m!n)$.

Lemma 1.3 $Lf = \text{fact}^* : N \times N \rightarrow N$ where $\text{fact}^*(m, n) = m!n$.

Proof A simple induction shows that if $gf = g : N \times N \rightarrow D$ then $g(m, n) = g(0, m!n)$. Define $h : N \rightarrow D$ by $h(n) = g(0, n)$. $\square$

Now fact is

$$N \xrightarrow{\langle 1_N, \Delta(1) \rangle} N \times N \xrightarrow{\text{fact}^*} N$$

where $\Delta(k)$ is the function which is constantly k . Note that, unlike the usual fixpoint construction of fact , this doesn’t require function spaces or $\perp$.

2 While-loops

The last example above has the pleasant property that every sequence $\{f^n d\}$ becomes stationary at a desired value (i.e. a pair (m, n) where $m = 0$) after finitely many steps. In general when evaluating a loop there will be other sequences which are either not stationary, or become so at a value where some undesirable condition holds. When defining the semantics of a while-loop, however, those undesirable sequences represent infinite loops which should perhaps be discarded (mapped to $\perp$). This proves to be easy as the loop object is a lifted sum of the finite and infinite loops.

Taylor [8] has also tried to give a semantics for while loops using colimits, specifically, via coequalisers in a topos.

First we need some notation. For any full sub-object D_1 of D (i.e. sub-c.p.o. whose order is the restriction of that of D) define the full sub-objects

$$\begin{aligned} f^\forall(D_1) &= \{d \in D \mid (\forall n \geq 0) f^n d \in D_1\} \\ f^\exists(D_1) &= \{d \in D \mid (\exists n \geq 0) f^n d \in D_1\} \end{aligned}$$

The set $f^\forall(D_1)$ is the greatest fixpoint of $f^{-1} : \text{sub}(D) \rightarrow \text{sub}(D)$ contained in D_1 while $f^\exists(D_1)$ is the least fixpoint containing it if $D_1 \subseteq \text{fix}(f)$.

Let $B = \{tt, ff\}$ be the c.p.o. of booleans and consider a continuous function $b : D \rightarrow B_\perp$ which is to be thought of as a condition on D . Define

$$\begin{aligned} D_b &= b^{-1}\{tt\} \\ D_{\neg b} &= b^{-1}\{\perp\} \end{aligned}$$

which are open and closed, respectively, in D . Note that $D_{\neg b}$ is defined using $\neg : B \rightarrow B$ the negation.

The conditional $\text{cond}(b, g, g') : D \rightarrow D'_\perp$ for continuous functions $g, g' : D \rightarrow D'$ is given by

$$\text{cond}(b, g, g')(d) = \begin{cases} gd & \text{if } d \in D_b \\ g'd & \text{if } d \in D_{\neg b} \\ \perp & \text{if } d \in D_{\neg b} \end{cases}$$

The loop ‘while b do g ’ should be interpreted by a function $\text{while}(b, g) : D \rightarrow D_\perp$ which satisfies

$$\text{while}(b, g) = \text{cond}(b, \text{while}(b, g)^* g, 1_D)$$

Its finite and infinite loops are those of $f = \text{cond}(b, g, 1_D)^*$.

Theorem 2.1 *Let*

$$\begin{aligned} D_1 &= f^\exists(D_{\neg b})_\perp \\ D_2 &= f^\exists(b^{-1}\{ff, \perp\}) \\ D_3 &= f^\forall(b^{-1}\{tt, \perp\}) \end{aligned}$$

Then the following diagram of inclusions is a pushout

$$\begin{array}{ccc} D_1 & \longrightarrow & D_2 \\ \downarrow & & \downarrow \\ D_3 & \longrightarrow & D_\perp \end{array}$$

Further, f restricts to endomorphisms $f_i : D_i \rightarrow D_i$ for $i = 1, 2, 3$. Thus

$$\begin{array}{ccc} \text{loop}(f_1) & \longrightarrow & \text{loop}(f_2) \\ \downarrow & & \downarrow \\ \text{loop}(f_3) & \longrightarrow & \text{loop}(f) \end{array}$$

is also a pushout which implies (writing $\text{loop}_\infty(f)$ for $\text{loop}(f_3)$)

$$\text{loop}(f) = (D_{\neg b} + \text{loop}_\infty(f))_\perp$$

and

$$(Lf)d = \begin{cases} g^n d & \text{if } g^n d \in D_{\neg b} \\ L(f_3)d & \text{if } d \in D_3 \\ \perp & \text{if } d \in D_1 \end{cases}$$

Proof Let $h_i : D_i \rightarrow D'$ for $i = 2, 3$ be continuous functions whose restrictions to D_1 are equal. It suffices to show that they determine a unique continuous function $h : D \rightarrow D'$ whose restriction to D_i is h_i for $i = 2, 3$. Clearly there is a unique such function. Its continuity follows since the set-theoretic differences $D_2 - D_1$ and $D_3 - D_1$ are open subsets.

Now the colimit of a pushout is just the pushout of the colimits. Thus $\text{loop}(f)$ is the pushout of the $\text{loop}(f_i)$. $\square$

Thus we can interpret the while-loop by

$$D \xrightarrow{j} D_\perp \xrightarrow{Lf} \text{loop}(f) \xrightarrow{p} (D_{\neg b})_\perp \xrightarrow{k_\perp} D_\perp$$

where j and k are the obvious inclusions, and p is the projection from the lifted sum (that maps all elements of $\text{loop}_\infty(f)$ to $\perp$).

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