LFCS

Laboratory for Foundations of Computer Science
Department of Computer Science - The University of Edinburgh

An n-Categorical Pasting Theorem

by

A. J. Power

An n-Categorical Pasting Theorem

LFCS Report Series

ECS-LFCS-91-190

LFCS
Department of Computer Science
University of Edinburgh
The King's Buildings
Edinburgh EH9 3JZ

November 1991

Copyright © 1991, LFCS

Copyright © 1991, Laboratory for Foundations of Computer Science,
University of Edinburgh. All rights reserved.

Reproduction of all or part of this work
is permitted for educational or research use
on condition that this copyright notice is
included in any copy.

AN n-CATEGORICAL PASTING THEOREM

A.J.Power*
Department of Computer Science
University of Edinburgh
King's Buildings
Edinburgh EH9 3JZ
Scotland.

dedicated to Max Kelly on the occasion of his 60th birthday

Abstract

In order to facilitate the study of 2-categories with structure, we state and prove an n-categorical pasting theorem. This is based upon a new definition of n-pasting scheme that generalises Johnson's definition of a well-formed loop-free pasting scheme by weakening his no direct loops condition. We define n-pasting, prove the theorem, and show that for n=3, it incorporates all possible composites of n-cells. We show that that is not true for higher n. We define the horizontal n-category of an (n+1)-category to generalise that of a 2-category, we define horizontal and vertical composition for an (n+1)-category and we state and prove an interchange law. We also study further conditions on a pasting diagram and their impact upon how one may evaluate a composite, and we express Street's free n-categories in terms of left adjoints.

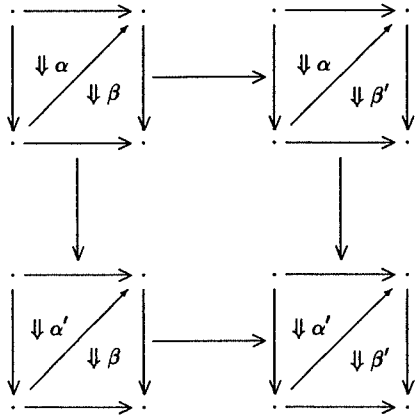
1 Introduction

"Pasting" has become a well known and valuable tool in the study of 2-categories. In [8], a "2-categorical pasting theorem" was proved in order to make precise the notion of

*This research was supported by the Australian Research Council and by ESPRIT Basic Research Action 3245: Logical Frameworks Design, Implementation and Experiment.

pasting as outlined in [7]. Now, in various articles such as [2], [9], [10] and implicitly in [11], 3-categories or mild generalisations of them are used to study 2-categories with extra structure. So we seek a 3-categorical pasting theorem in order to facilitate that. In fact, an n-categorical pasting theorem for each positive natural number n is only a little more difficult, and it seems a more natural level of generality; so that is the aim of this paper.

The idea of pasting is to unify the n different sorts of composite in the definition of n-category into one general notion of "pasting composite". For instance, for n=3, given 3-cells $\gamma : \alpha \to \alpha'$ and $\gamma' : \beta \to \beta'$, we should like to know directly from the definition of pasting composite that a diagram such as



commutes, where the 3-cells are the evident ones generated by γ and γ'.

There has been and there continues to be much work on this problem. Michael Johnson produced an n-categorical pasting theorem in his thesis [3] and published the result in [4]. Johnson investigated two classes of schemes. The first class, loop-free schemes, are of interest because the free n-category on an n-dimensional such admits a particularly easy construction. We do not handle these: a counterexample appears in Section 2. The second class, well-formed loop-free schemes, are the pasting schemes of interest to us. These parametrize composable diagrams in n-categories. Johnson realised that his definition was not broad enough to include all possible composites of n-cells. In particular,

it was not broad enough to allow all of the constructions he sought, specifically that of a certain tensor product of pasting schemes, as outlined in his lecture [5]. The difficulty was that his "no direct loops" condition was too strong. Here, we generalise Johnson's definition. Specifically, we drop most of the no direct loops condition, and what remains of it is true of all composites of n-cells. This allows us to capture all possible composites for n=3; we also capture a large class of composites for higher n. Alas, we do not capture all: an example appears in Section 3. That example is interesting as it is not merely a technical example but acts as a counterexample to a large class of conjectures. For instance, a variant of that example exhibits a composite of 4-cells that fails to satisfy the fourth (and most substantial other than no direct loops) requirement of Johnson's "loop-freeness" condition. It is not immediately evident from the two definitions that the definition here generalises that of Johnson, but it does follow from an analysis of the early results of Johnson's paper [4] as he has indicated in a private communication for general n and shown in detail for n=2 in a Sydney Category seminar in 1988.

It should be noted that greater generality for pasting schemes does not necessarily imply a more general pasting theorem. For instance, our Example 3.12 is not one of Johnson's well-formed loop-free pasting schemes, but diagrams of that form can be handled by Johnson's pasting theorem by parametrizing them by loop-free schemes.

Our approach is to give a topological foundation and to proceed by induction. Typically, one does not try to use 3-pasting schemes until one already has 2-pasting schemes to act as domains and codomains of the 3-cells; similarly for higher dimensions. So it seems reasonable to use the definition and results of n-pasting schemes in the definition of an (n+1)-pasting scheme. This is a convenient way to order the information so that it is less overwhelming in its detail. Just as a 2-cell can be seen as two parallel directed paths in a graph, together with a homotopy in 2-space from one to the other [8], we wish to regard a 3-cell as a homotopy in 3-space, and similarly for higher n. Using this paradigm, one may regard Ross Street's pictures [13] as n-pasting schemes together with a choice of the order in which some n-cells are composed.

Central to our approach is our "interchange law". Just as one speaks of horizontal and vertical composition for a 2-category, the various n+1 sorts of composite in the definition of an (n+1)-category may also be viewed naturally in terms of two sorts of composition:

vertical composition, which is the top level composition of the (n+1)-category; and horizontal composition, which is the n-pasting composition of the (n+1)-category. These two sorts of composition satisfy an appropriately generalised interchange law given in Section 6. It is from this result that we deduce the unicity of a pasting composite.

There is other current work on giving a topological analysis of n-categories. Specifically, Kapranov and Voevodsky [6] are also working on pasting results that are topological in flavour, and Al-Agl and Steiner [1] have characterised ∞-categories in terms of simplexes. However, in both cases, the work is somewhat different in direction to ours. Of course, much of the work here was ultimately inspired by Street's fundamental paper [13], as explained in Section 8. It is possible to exhibit in terms of left adjoints the free n-categories described internally by Street [13]. A consequence of our result is that one can do so for a large class of them by means of pasting diagrams, hence in topological terms.

In Section 2, we commence the induction with the definitions for ordinary categories, i.e. in the case n=1, and we recall the definitions for 2-categories as in [8]. The definitions for 2-categories are not needed for the induction, but they illustrate the first nontrivial case. In Section 3, we give the inductive definitions needed to define pasting diagrams. In Section 4, we give the simple topological results that we require. In Section 5, we define horizontal pasting. In Section 6, we define horizontal and vertical composition in an (n+1)-category and give our "interchange law" and our "n-categorical pasting theorem". In Section 7, we show that we have captured all possible composites in a 3-category, we discuss our failure for higher n, and we illustrate a strengthening of the conditions that allow us greater flexibility in describing a composite. Finally, in Section 8, we explain limiting behaviour as n approaches infinity, and show how the work relates to that of Street [13], specifically to his free ω-categories.

Throughout, we shall assume that n-space is oriented for every n, and all embeddings into n-space are piecewise linear. This avoids irrelevant topological detail.

I should like to offer warm thanks to Michael Johnson and to Ross Street for valuable discussions on this work. This paper is in final form and no similar paper has been or is being submitted elsewhere.

2 Commencing the induction

A *0-pasting scheme* is the point of 0-space.

A *1-computad* is a finite directed graph; a *1-computad morphism* is a map of directed graphs.

A *1-pasting scheme* is a finite non-empty set G together with an embedding g of G into the oriented line. We identify G with its image, and we call the elements of G *0-cells.* G partitions the rest of the line into a finite number of disconnected open intervals and two rays. We call the open intervals *1-cells* and we define the *domain* and *codomain* of a 1-cell f to be its endpoints, written $domf$ and $cod\,f$, with $domf < cod\,f$ in the order determined by the orientation of the line. The *domain* of the pasting scheme, denoted $domG$, is the least element of G in the order determined by the orientation of the line; dually, the *codomain* of (G,g), denoted $codG$, is the greatest element. The 0-cells and 1-cells of (G,g) together with the domain and codomain functions form a directed graph, which we call the *underlying* 1-computad of (G,g), and which we denote by $C(G)$.
A *1-pasting diagram* in a 1-computad H consists of a 1-pasting scheme (G,g) together with a 1-computad morphism $h : C(G) \to H$. The *domain* of a 1-pasting diagram $((G,g),h)$ is $domG$ together with the restriction of h to $C(domG)$; dually for *codomain.* A 1-pasting diagram $((G,g),h)$ is called simply a *1-pasting scheme in* H if h is a monomorphism in the category of directed graphs. Every 1-pasting scheme with at least two 0-cells is called *spherical.* A 1-computad is *locally spherical* if each 1-cell together with its domain and codomain is a spherical 1-pasting scheme, i.e. if every 1-cell has its domain and codomain different 0-cells. A *1-subpasting scheme* of a 1-pasting scheme (G,g) is a subset J of G for which given $j < g < j'$ with j and j' in J and g in G, it follows that g lies in J.

A *labelling* of a 1-pasting scheme (G,g) in a category A is a 1-pasting diagram $((G,g),a)$ in A. A *1-pasting composite*, or equally a *strong composite*, of a labelled 1-pasting scheme in a category A is the composite in A of the sequence of arrows determined by the diagram. We call any composite in a category A a *vertical composite.*

We now recall the definitions and a few results for n=2, as appear in [8].

Definition 2.1 A *2-computad* consists of a directed graph G, together with a set G_2 and two functions $dom, cod : G_2 \to diags(G)$, where $diags(G)$ is the set of directed paths in G, such that $dom\, dom = dom\, cod$ and $cod\, dom = cod\, cod$. A *2-computad morphism* from (G_2, G, dom, cod) to (H_2, H, dom, cod) consists of a morphism of directed graphs $f : G \to H$ and a function $f_2 : G_2 \to H_2$ such that $dom\, f_2 = f\, dom$ and $cod\, f_2 = f\, cod$. The category *2-Computad* is the category of 2-computads and 2-computad morphisms, with the evident composition.

Definition 2.2 A *2-pasting scheme* consists of a (non-empty) connected finite directed graph G together with an embedding g of G into oriented 2-space subject to the following conditions. The image of G divides the rest of 2-space into finitely many disconnected regions, called faces. The boundary of each interior face F must be of the form $\sigma(F)(\tau(F))^{-1}$ for directed paths $\sigma(F)$ and $\tau(F)$ in G of non-zero length, if moving clockwise around the boundary; and there must exist vertices $s(G)$ and $t(G)$ in the exterior face for which given any vertex v, there exist directed paths from $s(G)$ to v and from v to $t(G)$.

Remark 2.3 In [8], it was assumed in the definition of 2-pasting scheme that $s(G)$ and $t(G)$ are distinct, but that assumption was inessential to the argument. If they are the same, then the 2-pasting scheme must be trivial.

With the above notation, the interior faces F of (G, g) are called *2-cells*, and $domF$ and $codF$ are defined to be $\sigma(F)$ and $\tau(F)$ respectively. So, (G, g) has an underlying 2-computad given by the graph G and the 2-cells of (G, g); we denote this 2-computad by $C(G)$. It also follows that the exterior face E also has boundary of the form $\sigma(E)(\tau(E))^{-1}$ for directed paths $\sigma(E)$ and $\tau(E)$ of non-zero length, if moving around the boundary with E always on one's right side; these are called $codG$ and $domG$ respectively.

Proposition 2.4 If (G, g) is a 2-pasting scheme, then G has no directed cycle.

Proposition 2.5 If a 2-pasting scheme (G, g) has at least one interior face, then there exists an interior face F such that $domF$ lies entirely on $domG$.

Definition 2.6 A *2-pasting diagram* in a 2-computad H consists of a 2-pasting scheme (G, g) together with a 2-computad morphism $h : C(G) \to H$. A *labelling* of a 2-pasting scheme (G, g) in a 2-category A is a 2-pasting diagram $((G, g), a)$ in A.

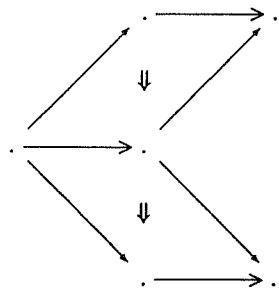
Theorem 2.7 (A 2-categorical pasting theorem) Every labelling of a 2-pasting scheme has a unique composite.

Proof: The existence of a composite follows from Proposition 2.5 and induction on the number of interior faces; unicity also follows directly by induction. □

Remark 2.8 For n=2, pasting composition describes all possible composites of a set of 2-cells. This is evident because a horizontal composite of labelled 2-pasting schemes is a labelled 2-pasting scheme, and a vertical composite of labelled 2-pasting schemes is a labelled 2-pasting scheme.

In the introduction, it was mentioned that our pasting schemes generalise the well-formed loop-free pasting schemes of Johnson, but not his loop-free schemes in general. For n=2, our pasting schemes are precisely his well-formed loop-free pasting schemes as he showed in a lecture in Sydney in 1988, but even then, we do not include all of his loop-free schemes, for instance the following.

Example 2.9 Consider the following diagram:



(2.1)

Observe that this is not one of our 2-pasting schemes since there is no candidate for $t(G)$. It is however one of Johnson's loop-free schemes. It is not well-formed since its domain is not even connected.

This generality allows Johnson to give an explicit description of the free 2-category on some data, but it evidently does not parametrize a composable diagram in general since the two 2-cells are not composable.

In order to study higher dimensions, we add the following definitions. The *domain* of a 2-pasting diagram $((G,g),h)$ in a 2-computad H is $domG$ together with the restriction of h to $C(domG)$; dually for *codomain*. A 2-pasting diagram $((G,g),h)$ is called simply a *2-pasting scheme in* H if h is a monomorphism in 2-*Computad*, i.e. if h is injective on k-cells for k=0 to 2. A 2-pasting scheme is called *spherical* if $domG \cup codG$ is homeomorphic to the 1-sphere. A 2-computad G is called *locally spherical* if each 2-cell in it is a spherical 2-pasting scheme containing just one 2-cell. A *2-pasting inclusion* into a 2-pasting scheme (G,g) is a directed graph inclusion j into G such that the composite gj is a 2-pasting scheme for which every 2-cell is a 2-cell of (G,g). A *2-subpasting scheme* is an equivalence class of 2-pasting inclusions under the evident equivalence.

3 Continuing the induction

This section is devoted to defining the terms n-computad and n-pasting scheme. In order to simplify the exposition, it is convenient to define several subsidiary terms. We have already commenced with n=1 and exhibited the ideas to follow by recalling the definitions for n=2.

Generalising the situation for n=2, an (n+1)-computad will be an (n+1)-dimensional cellular complex, together with an orientation assigned to each k-cell for k=0 to n+1, subject to some axioms. Each (n+1)-cell will have a domain and a codomain, each of which will be an n-pasting scheme, hence the induction. An (n+1)-pasting scheme will consist of an n-computad embedded in (n+1)-space: this is why we have a mutual induction on pasting schemes and computads. An (n+1)-pasting scheme (G,g) will be called spherical if its domain and codomain, which will be subcomputads of G, are spherical n-pasting

schemes and if $domG \cup codG$ is homeomorphic to the n-sphere. An (n+1)-computad will be called locally spherical if for each (n+1)-cell, the subcomputad generated by taking its domain and codomain is a spherical (n+1)-pasting scheme (with only one (n+1)-cell in it). Given an (n+1)-pasting scheme (G, g), it will generate an (n+1)-computad by taking the embedding and the region enclosed by it; this (n+1)-computad will be denoted by $C(G)$. An (n+1)-pasting diagram in an (n+1)-computad H will be an (n+1)-pasting scheme (G, g) together with an (n+1)-computad morphism $h : C(G) \to H$.

Definition 3.1 An *(n+1)-computad* consists of an n-computad G_n, a set G_{n+1}, and two functions $dom, cod : G_{n+1} \to diags(G_n)$, where $diags(G_n)$ is the set of n-pasting diagrams (See Definition 3.13) in G_n, such that $dom\, dom = dom\, cod$ and $cod\, dom = cod\, cod$. An *(n+1)-computad morphism* from (G_{n+1}, G_n, dom, cod) to (H_{n+1}, H_n, dom, cod) consists of an n-computad morphism $f_n : G_n \to H_n$ and a function $f_{n+1} : G_{n+1} \to H_{n+1}$ such that $dom\, f_{n+1} = f_n\, dom$ and $cod\, f_{n+1} = f_n\, cod$. The category $(n+1)$-*Computad* is the category of (n+1)-computads and (n+1)-computad morphisms, with the evident composition.

Proposition 3.2 Let (G, g) be a locally spherical (See Definition 3.15) n-computad G together with an embedding g of G into oriented (n+1)-space. Then the orientation of (n+1)-space determines an orientation for each n-cell of G, giving it a "left hand side" and a "right hand side".

Proof: Since G is locally spherical, every n-cell in G is given by an inclusion of an n-pasting scheme with just one n-cell into G, and so composing this inclusion with g gives a piecewise linear embedding of an n-ball into oriented (n+1)-space. So the orientation of n-space specifies an orientation for the embedded n-cell. □

Notation 3.3 Given a finite locally spherical n-computad G together with an embedding g of G into (n+1)-space, the image of G divides the rest of (n+1)-space into finitely many disconnected regions. For each such region F, we write inF for the set of n-cells of G that are incident to F and with orientation into F, as given by Proposition 3.2. Dually, we write $outF$ for the set of n-cells of G that are incident to F and with orientation out of F. We shall call the interior regions *(n+1)-cells* of (G, g). Let the exterior region determined by such an embedding be denoted by E.

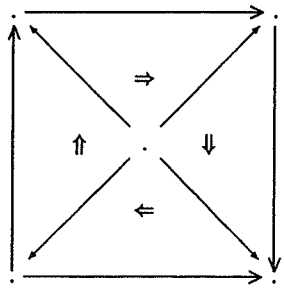
Definition 3.4 An *(n+1)-embedding* is a (non-empty) connected, finite, locally spherical n-computad G together with an embedding g of G into (n+1)-space subject to the following conditions:

1. if F is an interior region, then F is homeomorphic to the (n+1)-ball, and inF and $outF$, together with their domains and codomains, each forms a spherical n-pasting scheme, denoted $domF$ and $codF$ respectively, such that $domF \cup codF$ is the boundary of F; $dom(domF) = dom(codF)$; and $cod(domF) = cod(codF)$.

2. there exist n-pasting schemes $domG$ and $codG$ in G such that the n-cells of $domG$ are precisely the elements of $outE$; the n-cells of $codG$ are precisely the elements of inE; $dom(domG) = dom(codG)$; $cod(domG) = cod(codG)$; and $domG \cup codG$ is the boundary of E. Moreover, $domG$ extends to an n-sphere $S(domG)$ such that $G - domG$ lies strictly inside $S(domG)$.

Notation 3.5 Given an (n+1)-embedding (G, g), we shall denote $dom(domG)$ by S and $cod(domG)$ by T, and we shall denote the region enclosed by $S(domG)$ by $B(domG)$. Observe, more generally, that if X is any n-pasting scheme in G with domain S and codomain T, then X together with $S(domG) - domG$ forms an n-sphere. We shall denote this n-sphere by $S(X)$ and we shall denote the region it encloses by $B(X)$. Note that every n-cell of X is an element of $in(B(X))$. If F is an (n+1)-cell of (G, g), we shall denote by $d(F)$ the union over all k$\leq$n of the set of those k-cells of G that lie on the boundary of F but not on $codF$. Johnson called this the set of "beginnings" of F: it is equally the set of those k-cells in the interior of $domF$. Dually, we shall denote by $c(F)$ the union over all k$\leq$n of the set of those k-cells of G that lie on the boundary of F but not on $domF$. Johnson called this the set of "ends" of F. The n-computad G, the set of regions it encloses, and dom and cod as defined above, yield an (n+1)-computad, which we shall denote by $C(G)$.

Example 3.6 The following is an example of a 2-embedding that is not a 2-pasting scheme as defined in Section 2.



(3.1)

Observe that this 2-embedding has no candidate for $s(G)$, which by definition must lie in the exterior face.

Owing to this example, we evidently need an extra condition in order to obtain pasting schemes. Thus, we introduce the following.

Definition 3.7 An (n+1)-embedding (G, g) satisfies the *n-cell extension* condition if every n-cell x extends to an n-pasting scheme X in G with $domX = S$ and $codX = T$.

Remark 3.8 A 2-embedding that satisfies the 1-cell extension condition is exactly a 2-pasting scheme as defined in Section 2: that a 2-pasting scheme is a 2-embedding that satisfies 1-cell extension follows from Proposition 2.4; the converse is trivial. However, even for n=3, an n-embedding that satisfies (n-1)-extension does seem to suffice to determine what we require of an n-pasting scheme. The reason is that such a 3-embedding will have a "topmost" 3-cell F, and for any such, if $d(F)$ is removed from G, we want the domain of the new 3-embedding to be a 2-pasting scheme embedded in 3-space. That does not seem to be the case in general for n=3, although we do not have a counterexample. It is certainly false for higher n, as Example 3.11 shows. So we require a further definition. In fact, the following definition can be simplified substantially for n=3, as shown by Theorem 4.14.

Definition 3.9 An (n+1)-embedding (G, g) satisfies the *domain replacement* condition if for every n-pasting scheme X in G with domain S and codomain T,

1. if F is an (n+1)-cell with $domF$ a subpasting scheme of X, then replacing $domF$ by $codF$ in X gives another n-pasting scheme embedded in n-space, and also in X, replacing $domF$ by a single cell gives an n-pasting scheme

2. if F and F' are different (n+1)-cells such that $domF$ and $domF'$ are both subpasting schemes of X, then in X, replacing both $domF$ and $domF'$ by single cells gives an n-pasting scheme.

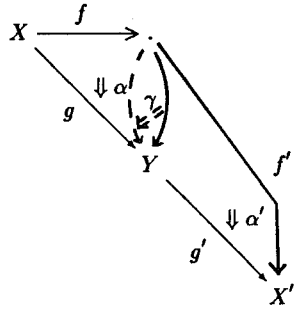
Note the difference in the two parts of condition 1 of domain replacement. By definition, X is a pair $(\overline{X}, j_X)$ consisting of an n-pasting scheme $\overline{X}$ and a monomorphism $j_X : C(X) \to G$. Since G is locally spherical, $domF$ is a pair $(\overline{domF}, j_{domF})$ consisting of a spherical n-pasting scheme $\overline{domF}$ and a monomorphism j_{domF}; similarly for $codF$. We assume that $\overline{domF}$ is a subpasting scheme of $\overline{X}$ and that j_X restricts to j_{domF}. If we replace $\overline{domF}$ by $\overline{codF}$ in $\overline{X}$, we would have an n-embedding $\overline{X'}$, and if we make the corresponding replacement of j_{domF}, we would have a new map $j_{X'}$ in *n-Computad*. The first part of condition 1 demands that $\overline{X'}$ is an n-pasting scheme and that $j_{X'}$ is a monomorphism, and hence, by composition with g, embeds $\overline{X'}$ into (n+1)-space. The second part of condition 1 only demands that the replacement of $\overline{domF}$ by a single cell is an n-pasting scheme.

Domain replacement is a strong condition. It is not true of all possible composites of (n+1)-cells, as illustrated by Example 3.11. So, one would dearly like to weaken it, but it is not possible to weaken it enough to capture all possible composites while still allowing the inductive argument outlined in Remark 3.8. Nevertheless, it is evidently true of all 2-pasting schemes, by Proposition 2.4, and we shall show by Theorem 4.14 and Proposition 7.1 that it is true of all 3-pasting schemes too. In Section 5, we shall re-express it in terms of horizontal pastings of (n+1)-cells.

The apparent asymmetry in the definition of domain replacement is, in the presence of the n-cell extension condition, illusory. One could say that that an (n+1)-embedding (G, g) satisfies the *codomain replacement* condition if (G, g) with the reverse orientation of (n+1)-space satisfies domain replacement. Corollary 4.8 shows that in the presence of the n-cell extension condition, the domain and codomain replacement conditions are equivalent. So we can now make a definition that is provably symmetric.

Definition 3.10 An *(n+1)-pasting scheme* is an (n+1)-embedding that satisfies the n-cell extension condition and the domain replacement condition.

Example 3.11 This example of a 3-pasting scheme will be used for several purposes. It will exhibit the need for and the independence of the various parts of the domain replacement condition. Also, it will be used to show that there exist composites of 4-cells that do not form 4-pasting schemes, and that there exist 4-pasting schemes that fail the fourth requirement of Johnson's "loop-freeness" condition. Consider the following diagram: it is to be understood to contain two 3-cells F and F'. The idea is to imagine F' as sitting in space below and to the right of F, and to imagine γ as directed vertically downwards. The domain of F is α, and the codomain is given by γ and $\overline{\alpha}$, which is not explicitly indicated. The domain of F' is given by α' and γ, and the codomain is given by $\overline{\alpha'}$, which is not depicted explicitly.

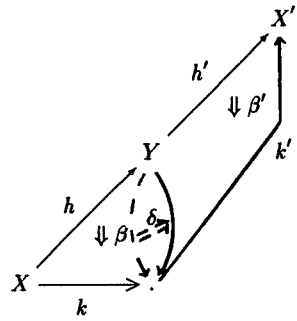


(3.2)

We shall denote this 3-pasting scheme by F_1. Moreover, we shall denote the somewhat trivial 3-pasting scheme given by removing γ from (3.2) by F_0.

Now consider the following diagram, which is similar to the first. It is to be understood to contain two 3-cells H and H'. The idea is that H sits to the left of and below H', and that δ is directed vertically upwards. The domain of H is given by β and δ, and

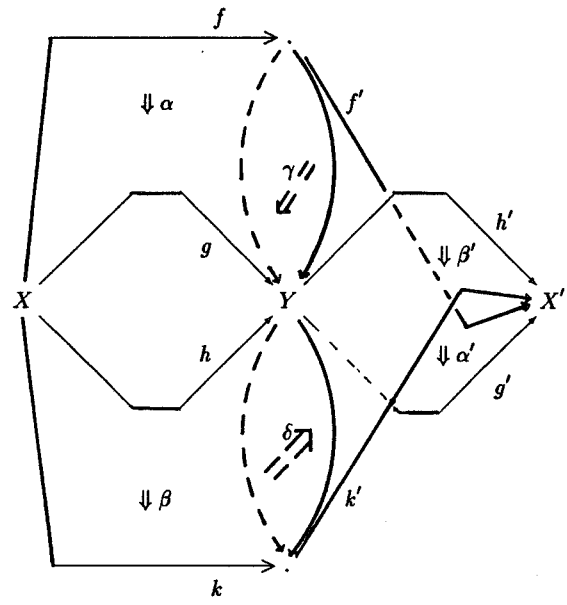
the codomain is given by $\overline{\beta}$, which is not depicted explicitly. The domain of H' is β', and the codomain is given by δ and $\overline{\beta'}$, which is not depicted explicitly.



(3.3)

We shall denote this 3-pasting scheme by H_1; and we shall denote the 3-pasting scheme given by removing δ from (3.3) by H_0.

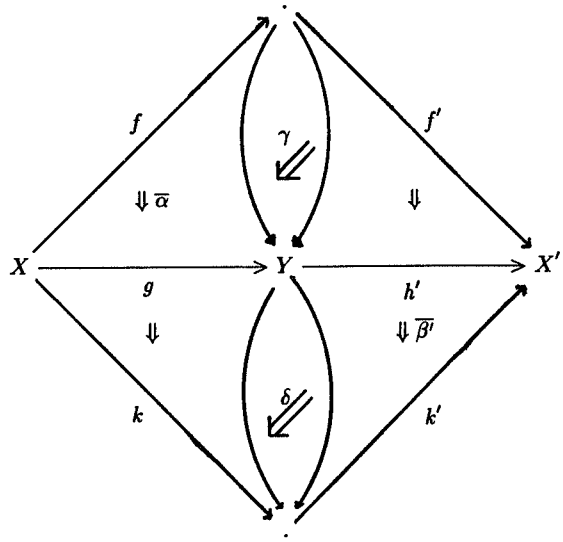
Now identify (3.2) and (3.3) at X, Y and X' by gradually curving (in a piecewise linear way) the two diagrams. This gives the following picture.



(3.4)

To the left side of the picture, we add two new 2-cells. The domain of one of them is given by g and $cod\,\delta$, and its codomain is k. The domain of the other is given by f and $cod\gamma$, and its codomain is h. Then, these two new 2-cells, together with $\overline{\alpha}$ and β enclose a region K with domain given by $\overline{\alpha}$ and one of the new 2-cells, and with codomain given by β and the other new 2-cell. Dually, on the right side of the picture, we can form another 3-cell K' whose domain includes $\overline{\beta'}$ and whose codomain includes α'.

Having made these constructions, we have a 3-pasting scheme, which we shall call G. The domain of G is given by α, β', and two of the new 2-cells. That it is a 3-embedding is easy; to check the 2-cell extension condition, observe that it is true for any 2-cell in the domain, then by removing a topmost 3-cell, say F, one can see that it is true for all 2-cells in $codF$, etcetera. For instance, the 2-pasting scheme given by starting with $domG$ and removing F and H' is depicted by



(3.5)

where the unlabelled 2-cells are two of the new 2-cells. The domain replacement condition can be most easily seen as a result of Theorem 4.14.

The 3-pasting scheme G has several 3-sequences, i.e. sequences of all the 3-cells in G such that the first has domain on $domG$, then removing it from G, one continues inductively. A formal definition appears in Corollary 4.6. Two important 3-sequences in G are H', K', F, F', K, H and F, K, H', H, K', F'. Observe that every 3-sequence must have both F and H' occurring before F' and H.

If one removes γ from G, then one still has a pasting scheme which we shall call G_1. A 3-sequence in G_1 is given by H', K', F_0, K, H, i.e. the replacement of the subsequence F, F' in our first 3-sequence above by the singleton F_0. Dually, if one removes δ from G, one also has a pasting scheme G_2, and G_2 has a 3-sequence given by F, K, H_0, K', F'. However, if one removes both γ and δ simultaneously from G, then one no longer has either a composite or a pasting scheme: any 3-sequence must have both F_0 before H_0 and H_0 before F_0, and the 2-cell extension condition fails. The latter is easiest seen as a

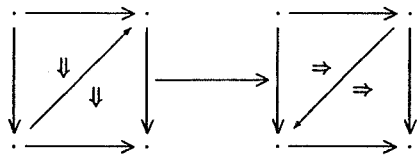
consequence of Proposition 4.3.

Given all of these definitions, one may have a 4-embedding with domain G_1, and with two 4-cells Y and Z. The domain of Y is F_0 and its codomain is F_1. The domain of Z is H_1 and its codomain is H_0. So the codomain of the 4-embedding is G_2. The 4-composite given by composing first the evident extension of Y followed by that of Z is certainly possible in any 4-category, but the 4-embedding does not satisfy condition 1 of domain replacement since Z is a topmost 4-cell. Note that the 4-embedding does satisfy the 3-cell extension condition. So this shows both that condition 1 of domain replacement is not redundant, and that there exist composites of 4-cells that do not form 4-pasting schemes.

Also, one may have a 4-embedding with domain G, and with 4-cells whose domains are F_1 and H_1, and with codomains given by new copies of the interiors of F_1 and H_1. This would satisfy the 3-cell extension condition and condition 1 of domain replacement, but it would not satisfy condition 2.

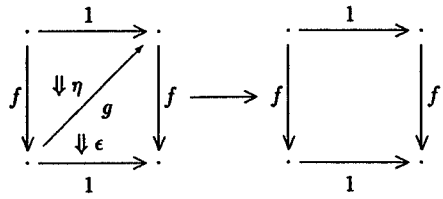
Finally, supposing one reverses the direction of γ in G, one would still have a 3-pasting scheme, and it may appear as part of the domain of a 4-cell in a 4-embedding; but then, the 4-embedding would fail to satisfy the fourth (and most substantial other than no direct loops) requirement of Johnson's "loop-freeness" condition, as one may take H_1 to be the subpasting scheme to act as a counterexample. Again, the 4-embedding could still satisfy the 3-cell extension condition and could in fact form a 4-pasting scheme with just one 4-cell and hence even satisfy the no direct loops condition.

Example 3.12 We now give an example of a 3-pasting scheme with just one 3-cell that does not satisfy Johnson's "no direct loops" condition since its underlying graph has a cycle.

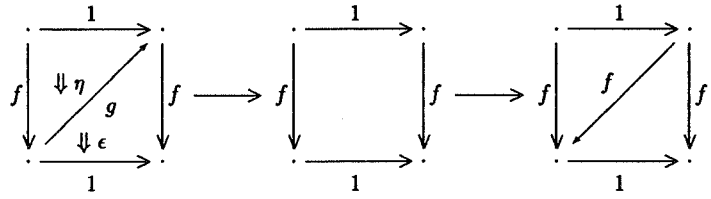


(3.6)

Such a scheme could arise in practice as follows. In [13], Street defines a "2-adjunction" in a 3-category. Some of the data is depicted by



where the unmarked interior region represents an identity 2-cell. Then, if one appends an identity 3-cell, one obtains



where the unmarked interior regions represent identity 2-cells. In fact, Vaughan Pratt observed some time ago that (3.7) itself fails the no direct loops condition, but his argument is somewhat more complex than ours.

Finally, to end this section, we need four more definitions, which have been used above for n, in order to allow us to continue the induction.

Definition 3.13 An *(n+1)-pasting diagram* $((G,g),h)$ in an (n+1)-computad H is an (n+1)-pasting scheme (G,g) together with an (n+1)-computad morphism $h : C(G) \to H$. The *domain* of an (n+1)-pasting diagram $((G,g),h)$ is given by $domG$ and the restriction of h to $domG$; dually for the *codomain*. We shall call an (n+1)-pasting diagram $((G,g),h)$ simply an *(n+1)-pasting scheme in H* if h is a monomorphism in the category $(n+1)$-*Computad*, i.e. if h is injective on k-cells for all k=0 to n+1.

Definition 3.14 An (n+1)-embedding (G, g) is *spherical* if $domG$ and $codG$ are spherical n-pasting schemes and $domG \cup codG$ is homeomorphic to the n-sphere.

Definition 3.15 An (n+1)-computad G is *locally spherical* if each (n+1)-cell together with its domain and codomain is a spherical (n+1)-pasting scheme in G containing one (n+1)-cell.

Definition 3.16 An *(n+1)-pasting inclusion* into an (n+1)-pasting scheme (G, g) is an n-computad inclusion j into G such that the composite gj is an (n+1)-pasting scheme for which every (n+1)-cell is an (n+1)-cell of (G, g). An *(n+1)-subpasting scheme* is an equivalence class of (n+1)-pasting inclusions under the evident equivalence.

4 Simple topological results

The main result of this section is Theorem 4.5, which states that given any (n+1)-pasting scheme (G, g) with at least one interior region, there exists an interior region F such that $domF$ is a subpasting scheme of $domG$, and such that if one removes $d(F)$ from G, then one has a new (n+1)-pasting scheme containing one fewer interior regions. We now give a series of Propositions leading to that result.

Definition 4.1 An (n+1)-embedding (G, g) is *acyclic* if there exists no alternating sequence $F_0, x_o, F_1, \ldots, F_m$ with m>0, with each F_i an (n+1)-cell and each x_i an n-cell of (G, g), with $F_0 = F_m$, and x_i an element of $outF_i \cap inF_{i+1}$.

Acyclicity is the highest dimensional part of Johnson's "no direct loops" condition, and it is the only part of the condition that we keep, as we shall show that it is a consequence of the n-cell extension condition. Example 3.12 shows that we do not require lower dimensional parts of the no direct loops condition.

Proposition 4.2 If an (n+1)-embedding (G, g) is acyclic and has at least one interior region, then there exists an interior region F for which $domF$ is a subpasting scheme of $domG$.

Proof: Let F_0 be any interior region. If every n-cell of inF lies on $domG$, then since $domF$ is spherical, it follows that $domF$ is a subpasting scheme of $domG$. If not, then let x_0 be an n-cell of inF that is not on $domG$. It must therefore be an element of $outF_1$ for some interior region F_1. Continue inductively. Since (G,g) is finite and acyclic, there must exist some m for which $domF_m$ is a subpasting scheme of $domG$. □

We shall call any interior region F of (G,g) such that $domF$ is a subpasting scheme of $domG$ a *topmost* region of (G,g).

Proposition 4.3 If an (n+1)-embedding (G,g) satisfies the n-cell extension condition, then it is acyclic.

Proof: Suppose there is a cycle $F_0, x_0, \ldots, F_m$. Then, x_0 extends to an n-pasting scheme X from S to T in G, which in turn extends to $S(X)$ as in 3.5, enclosing the region $B(X)$. Recall that every n-cell of X is an element of $in(B(X))$, and $S(X)$ extends X through the exterior region of (G,g). So F_0 is outside $B(X)$ but F_1 is inside it. Since $F_0 = F_m$, it follows that there exists an i such that F_i is inside $B(X)$ but F_{i+1} is outside it, but this is a contradiction since every n-cell on X is oriented into $B(X)$, so cannot be an element of $outF_i$. □

In the presence of condition 1 of the domain replacement condition, Proposition 4.3 has a converse given by Proposition 4.9, but before proving that, we shall develop some of the consequences of Proposition 4.3.

Corollary 4.4 If an (n+1)-embedding (G,g) with at least one interior region satisfies the n-cell extension condition, then there exists an interior region F for which $domF$ is a subpasting scheme of $domG$.

Proof: Propositions 4.2 and 4.3. □

This leads us directly to the main theorem of this section. It allows us to use induction to show that any labelled (n+1)-pasting scheme has a composite.

Theorem 4.5 For any (n+1)-pasting scheme (G,g) with at least one interior region, there exists an interior region F such that $domF$ is a subpasting scheme of $domG$, and such that removing $d(F)$ from G gives a new (n+1)-pasting scheme containing one fewer interior regions.

Proof: This follows immediately from the definitions and Corollary 4.4 given the observation that any n-pasting scheme X from S to T in G that includes any n-cell in the domain of a topmost (n+1)-cell F necessarily includes every element of inF. □

Corollary 4.6 For any (n+1)-pasting scheme (G,g), the (n+1)-cells of (G,g) can be ordered $F_0 < F_1 \ldots < F_m$ such that there exists a sequence $G_0, G_1, \ldots, G_m$ of (n+1)-pasting schemes with $(G,g) = G_0$ and such that G_{i+1} is the result of removing $d(F_i)$ from G_i.

We shall call any such sequence an *(n+1)-sequence.*

Note that we have not yet used condition 2 of the domain replacement condition. It will be used later to show the unicity of the possible composites of a labelled pasting scheme.

Several other results follow from Proposition 4.3. One is that the apparent asymmetry in the definition of (n+1)-pasting scheme is illusory.

Proposition 4.7 Let (G,g) be an acyclic (n+1)-embedding. Then (G,g) satisfies domain replacement if and only if it satisfies codomain replacement.

Proof: Suppose (G,g) satisfies domain replacement. Let X be an n-pasting scheme from S to T in G and let F be an (n+1)-cell of G with $codF$ a subpasting scheme of X. We shall show that if one replaces $codF$ by $domF$ in X, one has an n-pasting scheme embedded in (n+1)-space. We proceed by induction on the number of (n+1)-cells of (G,g) that lie outside $B(X)$. If it is 1, then the (n+1)-cell must be F, so the replacement of $codF$ by $domF$ in X is $domG$, hence an n-pasting scheme embedded in (n+1)-space. If

it is m+1, then there exists an (n+1)-cell F_1 different to F and outside $B(X)$. If F_1 is not a topmost (n+1)-cell, then there exists F_2 outside $B(X)$ such that $outF_2 \cap inF_1$ is nonempty. Note that F_2 is not F since $codF$ is a subpasting scheme of X but F_2 lies outside $B(X)$. So, by induction, since (G,g) is acyclic, there exists a topmost (n+1)-cell F' of (G,g) that lies outside $B(X)$ and is not F. Then, if we remove $d(F)$ from G, we are done by induction. Thus we have proved part of codomain replacement. The rest follows directly from this part together with domain replacement. The converse is dual. □

Corollary 4.8 Let (G,g) be an (n+1)-embedding that satisfies the n-cell extension condition. Then (G,g) satisfies domain replacement if and only if it satisfies codomain replacement.

Proof: Propositions 4.3 and 4.7. □

Now we give the promised converse to Proposition 4.3.

Proposition 4.9 If an (n+1)-embedding (G,g) satisfies condition 1 of the domain replacement condition and is acyclic, then it satisfies the n-cell extension condition.

Proof: This follows by induction on the number of (n+1)-cells in (G,g). If (G,g) has no (n+1)-cells, the result is trivial since every n-cell lies on $domG$. If (G,g) has m+1 (n+1)-cells, let F be a topmost (n+1)-cell: one such exists by Proposition 4.2. Then, given an n-cell x, either x lies on $domF$, therefore on $domG$, or x is in G_1, the acyclic (n+1)-embedding given by removing $d(F)$ from G. Trivially, G_1 satisfies condition 1 of domain replacement, so by induction we are done. □

Corollary 4.10 Let (G,g) be an (n+1)-embedding that satisfies condition 1 of domain replacement. Then (G,g) satisfies the n-cell extension condition if and only if it is acyclic.

Proof: Propositions 4.3 and 4.9. □

Corollary 4.11 Let (G,g) be an (n+1)-pasting scheme and let X be an n-pasting scheme in G from S to T. Then the collection of (n+1)-cells of (G,g) that lie in $B(X)$ form an (n+1)-subpasting scheme of (G,g) with domain given by X and codomain given by $codG$.

Proof: Domain replacement and the fact of being an (n+1)-embedding are immediate. By Corollary 4.10, (G,g) is acyclic, and so our collection of (n+1)-cells is also acyclic, and hence by Corollary 4.10 again, satisfies the n-cell extension condition. □

Now, as indicated in Example 3.11, we shall exhibit, for the case of 3-embeddings, a simpler condition than domain replacement, but which in the presence of the 2-cell extension condition, is equivalent to it. Consider the following condition on an (n+1)-embedding (G,g)

(*) if x is a k-cell of G with k<n, then x appears in $c(F)$ for at most one interior region F of (G,g); and if x appears on $domG$, then it appears in $c(F)$ for no interior region of (G,g).

Proposition 4.12 Every (n+1)-pasting scheme (G,g) satisfies (*).

Proof: We proceed by induction on the number of (n+1)-cells of (G,g). If (G,g) has no (n+1)-cells, then the result is trivial. If (G,g) has m+1 (n+1)-cells, then let F be a topmost (n+1)-cell of (G,g) and let G_1 be the (n+1)-pasting scheme given by removing $d(F)$ from G. Then, by induction, G_1 satisfies condition (*). By definition of (n+1)-embedding, since F is a topmost (n+1)-cell, no k-cell in $d(F)$ appears in $c(F')$ for any (n+1)-cell F'. The result follows directly. □

In general, there is no converse to Proposition 4.12. The first 4-embedding in Example 3.11 satisfies both the 3-cell extension condition and condition (*), but it does not satisfy condition 1 of domain replacement. However, for n=3, we do have a converse.

Lemma 4.13 Let (G,g) be an acyclic 3-embedding that satisfies condition (*). Then if x is a k-cell on $codG$ for any k<2, then x either lies on $domG$ or x lies in $c(F)$ for some interior region F of (G,g).

Proof: The proof is by induction on the number of 3-cells in (G,g). If (G,g) has no 3-cells, the result is trivial. If (G,g) has m+1 3-cells, then by Proposition 4.2, (G,g) has a topmost region, say F. Note that if one replaces any spherical subpasting scheme of a 2-pasting scheme by another 2-pasting scheme with the same domain and codomain, then the result is a 2-pasting scheme. Moreover, by condition (*), no 0-cell or 1-cell in $c(F)$ lies on $domG$. Also, no 2-cell can lie on both $outF$ and $outE$. Hence, if one removes $d(F)$ from G, one obtains a new 3-embedding G_1. Trivially, G_1 is acyclic. Moreover, G_1 satisfies condition (*) and $codG_1 = codG$. So, by induction, we are done. □

Theorem 4.14 Let (G,g) be a 3-embedding that satisfies the 2-cell extension condition. Then (G,g) satisfies the domain replacement condition if and only if it satisfies condition (*).

Proof: First assume that (G,g) satisfies condition (*). If one replaces any spherical subpasting scheme of a 2-pasting scheme by another 2-pasting scheme with the same domain and codomain, then the result is a 2-pasting scheme. So it suffices to show that if X is a 2-pasting scheme from S to T in (G,g), and F is a 3-cell in (G,g) with $domF$ lying on X, then no k-cell in $c(F)$ lies on X. The collection of 3-cells of (G,g) that lie outside $B(X)$ forms a 3-embedding with domain $domG$ and codomain X. We shall call this 3-embedding X'. Since (G,g) satisfies the 2-cell extension condition, it is acyclic, so X' is also acyclic. Moreover, X' also satisfies condition (*). So, by Lemma 4.13, for each k<2, every k-cell on $X = codX'$ occurs either on $domG$ or in $c(F')$ for some F' outside $B(X)$. So, since (G,g) satisfies (*), it follows that no k-cell on X lies in $c(F)$. For k=2, no k-cell can lie on both $outF$ and $outX$. The converse holds by Proposition 4.12. □

Theorem 4.14 allows us to reformulate the definition of a 3-pasting scheme as follows.

Definition 4.15 A 3-pasting scheme is a 3-embedding (G,g) that satisfies the 2-cell extension condition and condition (*).

5 Horizontal pasting

In Section 4, we showed that for any (n+1)-pasting scheme, there is at least one (n+1)-sequence of (n+1)-cells from the domain to the codomain of the pasting scheme. That will be used to show that every labelled pasting scheme has at least one composite. Note that we never used condition 2 of domain replacement in order to deduce the existence of such a sequence. Now, we do use condition 2 to provide the topological ideas required to prove unicity of a composite.

Definition 5.1 An *horizontal (n+1)-pasting* is an (n+1)-embedding (G, g) such that every interior region F has domain a subpasting scheme of $domG$ and codomain a subpasting scheme of $codG$; and such that if one simultaneously replaces every interior region F by a single n-cell from $dom(domF)$ to $cod(domF)$, then one has an n-pasting scheme G_h. We shall call G_h the *horizontal n-pasting scheme* of (G, g).

Remark 5.2 Every horizontal (n+1)-pasting evidently satisfies the n-cell extension condition. In fact, for n=1 to 4, every horizontal n-pasting is an n-pasting scheme: because (as will be shown in Proposition 7.1 for the case of n=4) if (G_1, g_1) and (G_2, g_2) are (n-1)-pasting schemes with $domG_2 = codG_1$ but with (G_1, g_1) and (G_2, g_2) otherwise disjoint, then the union of (G_1, g_1) and (G_2, g_2) is an (n-1)-pasting scheme. The result follows by induction on the number of (n-1)-cells. However, for n=5, there exist horizontal n-pastings that are not n-pasting schemes: condition 1 of domain replacement fails. A counterexample exists with just two 5-cells and four 4-cells, given by an analysis of Example 3.11.

Definition 5.3 An *(n+1)-whisker* is an (n+1)-pasting scheme that contains precisely one (n+1)-cell.

Observe that an (n+1)-whisker is necessarily an horizontal (n+1)-pasting. So, we can now reformulate domain replacement as follows.

Proposition 5.4 An (n+1)-embedding (G, g) satisfies the domain replacement condition if and only if for every n-pasting scheme X from S to T in G,

1. if F is an (n+1)-cell with $domF$ a subpasting scheme of X, then X together with F is an (n+1)-whisker.

2. if F and F' are different (n+1)-cells with $domF$ and $domF'$ both subpasting schemes of X, then X together with F and F' is an horizontal (n+1)-pasting.

6 Composition and the Interchange Law

In this section, we shall define pasting composition and horizontal composition, state and prove the interchange law for horizontal and vertical (top-level) composition, and state and prove the main result of the paper, that every labelled pasting scheme has a unique pasting composite.

Definition 6.1 A *labelling* of an (n+1)-embedding (G, g) in an (n+1)-category A is an (n+1)-computad morphism from $C(G)$ to A.

Note that Definition 6.1 incorporates the idea of labelling for both (n+1)-pasting schemes and horizontal (n+1)-pastings.

Notation 6.2 Given an (n+1)-category A, we shall denote by A_h the n-category given by all of the elements of A and all of the structure of A except for the n-source, n-target, and n-composition structure. We shall call A_h the *horizontal n-category* of A.

Observe that A_h is not the ordinary underlying n-category of A. For instance, if A is a 2-category, then the non-trivial arrows of A_h are the non-trivial 1-cells and 2-cells of A: in this case, our use of the term *horizontal category* agrees with that in the literature for 2-categories. If A is an (n+1)-category, the non-trivial n-cells of A_h are the non-trivial n-cells and (n+1)-cells of A. Recall that in Definition 5.1, we also defined G_h where (G, g) is an horizontal (n+1)-pasting. Given a labelling a of (G, g) in an (n+1)-category A, we shall denote the restriction of a to the n-pasting scheme G_h by a_h.

Definition 6.3 An *horizontal composite* of a labelled horizontal (n+1)-pasting $((G, g), a)$ in an (n+1)-category A is the n-pasting composite in A_h of the labelled n-pasting scheme (G_h, a_h).

Note that, by induction, an horizontal composite has n-domain and n-codomain in A given by the n-pasting composites of $domG$ and $codG$ with the restricted labellings. Moreover, since every whisker is an horizontal (n+1)-pasting, every labelled whisker has a unique horizontal composite. We shall refer to this simply as the composite of the labelled whisker.

Definition 6.4 A *vertical composite* in an (n+1)-category A is precisely an n-composite, i.e. a top level composite, in A.

Observe that the definitions of horizontal and vertical composition for (n+1)-categories agree with those for 2-categories if n=1.

We now move towards the interchange law, and using that, state and prove our n-categorical pasting theorem.

Lemma 6.5 Let $((G_1, g_1), a_1)$ and $((G_2, g_2), a_2)$ be labelled whiskers containing (n+1)-cells F_1 and F_2 respectively, such that $domG_2 = codG_1$, and $domF_2 = codF_1$, and (G_1, g_1) and (G_2, g_2) are otherwise disjoint, and the labellings agree on $codG_1$. Then the vertical composite of the composites of the labelled whiskers equals the composite of the labelled whisker given by the vertical composite of F_1 and F_2.

Proof: Take any composite of $((G_1, g_1), a_1)$. By induction, it is expressible in terms of k-composition for k=0 to n-1 in A_h. Take such an expression. If one replaces $a_1(F_1)$ by $a_2(F_2)$ in this expression, then by the n-categorical pasting theorem, one obtains an expression for the composite of $((G_2, g_2), a_2)$. Since the k-sources and k-targets of $a_1(F_1)$ and $a_2(F_2)$ agree for k=0 to n-1, it follows from the interchange laws between n-composition and k-composition for each k<n that the vertical composite of these expressions equals the composite determined by replacing $a_1(F_1)$ in the original expression by the vertical composite of $a_1(F_1)$ and $a_2(F_2)$, i.e. the composite of the labelled whisker as claimed. □

Theorem 6.6 (The interchange law)

Let $((G_1, g_1), a_1)$ and $((G_2, g_2), a_2)$ be labelled horizontal (n+1)-pastings such that $domG_2 = codG_1$, and the labellings agree on $codG_1$, and such that there is a bijection

$\natural : (n+1)\text{-}cells((G_1, g_1)) \to (n+1)\text{-}cells((G_2, g_2))$ such that for each (n+1)-cell F_1 in (G_1, g_1), we have $codF_1 = dom\natural F_1$. Then the vertical composite of the horizontal composites of $((G_1, g_1), a_1)$ and $((G_2, g_2), a_2)$ equals the horizontal composite of the labelled horizontal (n+1)-pasting given by the vertical composite of each pair $(F_1, \natural F_1)$.

Proof: This follows by induction on the number of (n+1)-cells in (G_1, g_1) and by Lemma 6.5. If (G_1, g_1) has no (n+1)-cells, the result is trivial. If (G_1, g_1) has one (n+1)-cell, the result is given by Lemma 6.5. If (G_1, g_1) has m+1 (n+1)-cells, take any horizontal composite of $((G_1, g_1), a_1)$. By induction, it is given by an (n-1)-composite, whose arguments consist of the composite of a labelled whisker and an horizontal composite of an horizontal (n+1)-pasting containing m (n+1)-cells. So, by the interchange law for n-composition and (n-1)-composition together with induction, it follows that the vertical composite of horizontal composites as in the statement of the theorem is given by an (n-1)-composite, whose arguments consist of the vertical composite of composites of two labelled whiskers and an horizontal composite of an (n+1)-pasting containing m (n+1)-cells. Moreover, the two labelled whiskers match as in the statement of Lemma 6.5. So, by Lemma 6.5 and by induction on n applied to Theorem 6.16, we are done. □

Corollary 6.7 Let $((G, g), a)$ be a labelled (n+1)-pasting scheme that is also an horizontal (n+1)-pasting, and assume that (G, g) contains at least one (n+1)-cell. Then its horizontal composite is the vertical composite of labelled whiskers. Moreover, the whiskers may be taken in any order.

Proof: We proceed by induction on the number of (n+1)-cells in (G, g). If (G, g) has one (n+1)-cell, the result is trivial. If (G, g) has m+1 (n+1)-cells, choose any one of them, say F. Then consider the vertical composite of the evident labellings of (G_1, g_1) and (G_2, g_2), where (G_2, g_2) is given by removing $domF$ from G and replacing it by another copy of $codF$, with the labelling of the new (n+1)-cell to be given by the identity, and where (G_1, g_1) is given similarly, by replacing $codF'$ for every other (n+1)-cell F' appropriately. By the interchange law and since (G_2, g_2) has m (n+1)-cells, we have the result. The final claim follows from the fact that we have allowed any (n+1)-cell F to

give a top whisker. □

Now we proceed to our n-categorical pasting theorem. For simplicity of exposition, we shall first prove a weak version of the result: this weak version allows us to use a simple but quite strong definition of composite of a labelled n-pasting scheme, which we shall call a strong composite. Afterwards, we shall discuss the more general natural definition of composite, which we shall call an n-pasting composite. At that point, we can state our n-categorical pasting theorem in its natural level of generality, and we shall use the weak result to prove it.

Definition 6.8 A *strong composite* of a labelled (n+1)-pasting scheme $((G,g),a)$ is the vertical composite of the composites of labelled whiskers determined by any (n+1)-sequence of (G,g) if (G,g) has at least one (n+1)-cell. If (G,g) has no (n+1)-cells, it is any n-pasting composite of $domG$.

Lemma 6.9 Let $((G,g),a)$ be a labelled (n+1)-pasting scheme containing exactly two (n+1)-cells F and F', each with domain on $domG$. Then both (n+1)-sequences of (G,g) determine the same (n+1)-cell in A, and this agrees with the horizontal composite of $((G,g),a)$.

Proof: By condition 2 of domain replacement, (G,g) is an horizontal (n+1)-pasting. So, by Corollary 6.7, we have the result. □

Theorem 6.10 (A weak n-categorical pasting theorem)

For any positive natural number n, every labelled n-pasting scheme in an n-category A has a unique strong composite.

Proof: If n=1, the result is trivial. So we may assume the result is true for n and prove it for n+1. Existence is immediate since every (n+1)-pasting scheme has an (n+1)-sequence by Corollary 4.6 and since every labelled whisker has a unique composite. For unicity, we proceed by induction. If (G,g) contains at most one (n+1)-cell, unicity holds by induction on n and by the definition of the composite of a labelled whisker. If (G,g) has

m+1 (n+1)-cells, suppose we have two (n+1)-sequences in (G, g). Either they commence with the same (n+1)-cell, in which case we are done by induction, or they commence with different (n+1)-cells. In the latter case, the result follows by induction and Lemma 6.9. □

We now make precise the general notion of n-pasting composite. The idea is that in composing a labelled n-pasting scheme, we could conceivably not merely find a sequence of whiskers, then evaluate the vertical composite of them, but we could take an horizontal composite of each equivalence class of a partition of them, then the vertical composite of these horizontal composites. Moreover, this process could be refined many times. Of course, the interchange law essentially shows us that we still determine the same composite as any strong composite. However, we now make that precise.

Definition 6.11 An (n+1)-pasting scheme (G, g) is called *indecomposable* if for some k≤n, (G, g) consists of one k-cell and its domain and codomain.

Notation 6.12 Given an (n+1)-pasting scheme (G, g), we will define $dom^k G$ inductively by $dom^n G = domG$ and $dom^k G = dom^k(domG)$ if k<n. We define $cod^k G$ dually.

Definition 6.13 A *simple decomposition* of an (n+1)-pasting scheme (G, g) consists of two (n+1)-subpasting schemes (G_1, g_1) and (G_2, g_2) such that

1. for some k≤n, $dom^k G_2 = cod^k G_1$, and (G_1, g_1) and (G_2, g_2) are otherwise disjoint
2. $(G_1, g_1) \cup (G_2, g_2) = (G, g)$ and
3. neither (G_1, g_1) nor (G_2, g_2) is a subpasting scheme of the other.

We define the notion of decomposition of an (n+1)-pasting scheme (G, g) by induction on the sum over all k≤n of the number of k-cells in (G, g).

Definition 6.14 A *decomposition* of an (n+1)-pasting scheme (G, g) consists of

1. (G, g), if (G, g) is indecomposable
2. a simple decomposition of (G, g) together with decompositions of (G_1, g_1) and (G_2, g_2) otherwise.

We now define the notion of an (n+1)-pasting composite of a labelled (n+1)-pasting scheme $((G,g),a)$ by induction on the sum over all k≤n of the number of k-cells in (G,g) and using the notion of a decomposition of (G,g).

Definition 6.15 An *(n+1)-pasting composite* of a labelled (n+1)-pasting scheme $((G,g),a)$ consists of the result of choosing a decomposition of (G,g) and evaluating

1. the labelling of the highest dimensional non-trivial cell in (G,g), if (G,g) is indecomposable
2. the k-composite in A of the (n+1)-pasting composites of (G_1,g_1) and (G_2,g_2) labelled by the restriction of a, otherwise.

Theorem 6.16 (An n-categorical pasting theorem)
For every positive natural number n, every labelled n-pasting scheme in an n-category A has a unique n-pasting composite.

Proof: For n=1, the result is given by associativity of composition in the category A. So we may assume the result true for n and prove it for n+1. First observe that every strong composite of a labelled (n+1)-pasting scheme is an (n+1)-pasting composite: if (G,g) has at most one (n+1)-cell, this holds by the definition of strong composite; if (G,g) has more than one (n+1)-cell, then a strong composite is a vertical composite of whiskers, each containing fewer cells than (G,g), so by induction we are done. So, by the weak (n+1)-categorical pasting theorem, we have existence. For unicity, it suffices by the weak (n+1)-categorical pasting theorem to show that every (n+1)-pasting composite is a strong composite. So suppose we have an (n+1)-pasting composite of $((G,g),a)$. If (G,g) is indecomposable or contains no (n+1)-cells, the result is trivial. If our pasting composite is a vertical composite of the (n+1)-pasting composites of $((G_1,g_1),a_1)$ and $((G_2,g_2),a_2)$, then by induction, it is the vertical composite of strong composites, so taking the (n+1)-sequence of (G,g) given by an (n+1)-sequence of (G_1,g_1) followed by an (n+1)-sequence of (G_2,g_2) gives the result. So suppose our composite is a k-composite of (n+1)-pasting composites for some k<n. At least one of (G_1,g_1) and (G_2,g_2) contains an (n+1)-cell F. So by induction on the number of (n+1)-cells in (G,g) and by the interchange law for

k-composition and n-composition in A, our k-composite equals the vertical composite of the extensions of whiskers of (G_1, g_1) and (G_2, g_2) along $domG_2$ and $codG_1$ respectively. □

Observe what the n-categorical pasting theorem does not say: it does not say that given a labelled n-pasting scheme $((G, g), a)$, one can take an n-ball in it, evaluate the n-ball in A, then replace the n-ball by a single n-cell in (G, g) labelled by its composite in A, then continue inductively. Example 3.11 shows that this is impossible even for n=3, as the replacement of F_1 and H_1 in G by F_0 and H_0 respectively is not a 3-pasting scheme. In the next section, we explore conditions on an n-pasting scheme under which it is possible, as this is a useful technique in practice.

7 Unifying and evaluating n+1 composites

In this section, we shall show that 3-pasting schemes include all possible composites of a given set of 3-cells in a 3-category. We shall also discuss further the failure of this result for higher n. Finally, we shall indicate stronger conditions on n-pasting schemes that allow more flexibility in evaluating a composite, and the effect of such conditions upon our pasting result.

Proposition 7.1 Every composite of a set of 3-cells in a 3-category A is expressible as a 3-pasting composite of a labelled 3-pasting scheme.

Proof: We use induction on the number of non-trivial 3-cells in the composite. If the composite has no nontrivial 3-cells, then it is a composite of 2-cells. So it is a composite of a labelled 2-pasting scheme, hence of a labelled 3-pasting scheme. If the composite has a single non-trivial 3-cell, then it must be expressible without using vertical composition, so it is a composite of a labelled whisker and hence a composite of a labelled 3-pasting scheme. So it suffices to show that a k-composite of labelled 3-pasting schemes $((G_1, g_1), a_1)$ and $((G_2, g_2), a_2)$ is expressible as a labelled 3-pasting scheme for k=0 to 2. First suppose (G_1, g_1) and (G_2, g_2) are composed by identifying $cod^0 G$ with $dom^0 G_2$. It is immediate that the result is a 3-pasting scheme by Definition 4.15. Second, if $cod^1 G_1$

is identified with $dom^1 G_2$, again by Definition 4.15, the result is immediate. Finally, for 2-composition, it suffices by induction and Corollary 4.6 to assume that (G_1, g_1) contains just one interior 3-cell, and we assume that $codG_1$ and $domG_2$ are identified. Again by Definition 4.15, the result is immediate. □

Example 3.11 illustrates why Proposition 7.1 fails in general for n=4. It is easy to write conditions on a 4-whisker which would allow it, subject to those conditions, to be appended to the top of a 4-pasting scheme and to have a new 4-pasting scheme as a result, but we can find nothing non-trivial. The difficulty is with domain replacement. It is easy to verify that if (G_1, g_1) and (G_2, g_2) are (n+1)-embeddings with $domG_2 = codG_1$ but otherwise disjoint, then their union is an (n+1)-embedding. Moreover, if (G_1, g_1) and (G_2, g_2) both satisfy the n-cell extension condition, then so does their union. Moreover, if they satisfy (*), then so does their union. The property that given any two (n+1)-cells F_1 and F_2, there exists an n-pasting scheme X from S to T such that precisely one of F_1 and F_2 is enclosed $B(X)$, also is preserved by composition. But none of these suffices to give domain replacement.

In fact, Example 3.11 suggests that it may be difficult to find conditions on labelled n-embeddings to include all composites, short of an assumption to the effect that the n-embedding can be decomposed into whiskers. For instance, Example 3.11 can be modified to show that one may have a 4-embedding with three 4-cells, such that one of them has the property that each of the others extends to a whisker that passes under its codomain, but which does not represent a composite.

Now, as promised at the end of Section 6, we study conditions on labelled pasting schemes that allow greater freedom in the way in which we may evaluate composites. Specifically, we should like to be able to evaluate them locally, by taking any n-ball in the pasting scheme and evaluating it, then substituting a single cell for it with the labelling determined by its composite. In general, that is impossible as illustrated by Example 3.11. However, there are reasonable conditions on a pasting scheme under which it is possible. For the case of n=1, the situation is trivial, so we shall restrict our attention to the case of n+1.

Definition 7.2 Given a (n+1)-embedding (G, g), a *directed (n+1)-ball* in (G, g) consists of a subset B of the set of (n+1)-cells of (G, g) such that, B together with the domains and codomains of its elements forms a spherical (n+1)-pasting scheme.

Definition 7.3 An (n+1)-embedding satisfies the *strong n-cell extension* condition if for every partition of the embedding into directed (n+1)-balls and every n-cell x not in the interior of any of the (n+1)-balls, x extends to an n-pasting scheme X in G with domain S and codomain T that respects the partition.

Observe that the strong n-cell extension condition implies the n-cell extension condition, since one may take the discrete partition. The converse is false, as illustrated by Example 3.11.

Definition 7.4 An (n+1)-embedding (G, g) satisfies the *strong domain replacement* condition if for every n-pasting scheme X in G from S to T,

1. if F is an (n+1)-cell in (G, g) with $domF$ a subpasting scheme of X, then replacing $domF$ by $codF$ in X gives another n-pasting scheme embedded in (n+1)-space
2. if B is a directed (n+1)-ball with $domB$ a subpasting scheme of X, then in X, replacing $domB$ by a single cell gives an n-pasting scheme
3. if B and B' are directed (n+1)-balls with empty intersection such that $domB$ and $domB'$ are both subpasting schemes of X, then in X, replacing both $domB$ and $domB'$ by single cells gives an n-pasting scheme.

By the same proof as for domain replacement in the presence of the n-cell extension condition, it follows that the asymmetry in the definition of strong domain replacement is illusory in the presence of the strong n-cell extension condition. Also, an acyclic 3-embedding satisfies strong domain replacement if it satisfies condition (*) because the replacement of any spherical subpasting scheme of a 2-pasting scheme by a single cell yields a 2-pasting scheme.

Definition 7.5 A *strong (n+1)-pasting scheme* is an (n+1)-embedding that satisfies the strong n-cell extension condition and the strong domain replacement condition.

Every strong (n+1)-pasting scheme is evidently an (n+1)-pasting scheme, but as Example 3.11 shows, the converse is in general false. However, every 2-pasting scheme is a strong 2-pasting scheme. Moreover, a 3-pasting scheme is strong if and only if it satisfies the strong 2-cell extension condition.

We now develop the extra topological machinery we require to support our more general notion of composite.

Lemma 7.6 Let (G,g) be an (n+1)-pasting scheme and let B be a directed (n+1)-ball in (G,g). Suppose further that $domB$ extends to an n-pasting scheme X in G from S to T such that no n-cell of X lies in the interior of B. Then there exists an (n+1)-sequence of (G,g) for which, if F and F' are elements of B, every (n+1)-cell that lies between them in the (n+1)-sequence is also an element of B.

Proof: By the dual of Corollary 4.11, the (n+1)-cells of (G,g) not in $B(X)$ form a subpasting scheme of (G,g) with domain $domG$ and codomain X. So they yield an (n+1)-sequence S_X. Since B is a spherical (n+1)-pasting scheme, it has an (n+1)-sequence S_B. Moreover, since X does not enter the interior of B, every (n+1)-cell of B lies inside $B(X)$. So we may extend S_X by S_B. Finally, by domain replacement, induction, and Corollary 4.11, the remaining (n+1)-cells of (G,g) may be appended to give an (n+1)-sequence of the whole of (G,g). □

Remark 7.7 By an argument similar to that of Proposition 4.3, it is routine to verify that given a directed (n+1)-ball B in an (n+1)-embedding (G,g), if $domB$ extends to an n-pasting scheme X in G from S to T such that no n-cell of X lies in the interior of B, then (G,g) is "loop-free relative to" B, i.e. for any alternating sequence $F_0, x_0, \ldots, F_m$ of (n+1)-cells F_i and n-cells x_i of (G,g), with x_i an element of $outF_i \cap inF_{i+1}$, and with F_0 and F_m both in B, it follows that F_i is in B for every i. Just as in Proposition 4.9, the converse holds if (G,g) satisfies condition 1 of (strong) domain replacement. As suggested by the terminology, this new condition is closely related to Johnson's "loop-freeness" condition. More precisely, if (G,g) is the domain of a single (n+2)-cell, then our condition is a special case of Johnson's condition.

Proposition 7.8 If (G,g) is a strong (n+1)-pasting scheme and B is a directed (n+1)-ball in (G,g), then replacing B by a single (n+1)-cell in (G,g) gives a strong (n+1)-pasting scheme (G',g').

Proof: It is immediate that (G',g') is an (n+1)-embedding, that it satisfies the strong n-cell extension condition and conditions 2 and 3 of strong domain replacement. For condition 1 of strong domain replacement, let X be an n-pasting scheme in G' from S to T, and let F be an (n+1)-cell in (G',g') with $domF$ a subpasting scheme of X. Then, X is an n-pasting scheme in G. So, if F is an (n+1)-cell in (G,g), we are done. If not, then F must be our new (n+1)-cell. Then the result follows from Lemma 7.6 and domain replacement in (G,g) applied several times. □

Corollary 7.9 Given any partition of the (n+1)-cells of a strong (n+1)-pasting scheme (G,g) into directed (n+1)-balls, replacement of each directed (n+1)-ball by a single (n+1)-cell yields a strong (n+1)-pasting scheme.

Proof: Use Proposition 7.8 to replace one directed (n+1)-ball at a time. □

Corollary 7.9 gives us conditions under which we can evaluate a labelled (n+1)-pasting scheme by evaluating directed (n+1)-balls in any combination. However, for a particular purpose, we may only be interested in replacing a particular directed (n+1)-ball by a single (n+1)-cell. We can relax the above conditions a little in order to allow precisely that rather than the more general phenomenon.

Proposition 7.10 Let (G,g) be an (n+1)-pasting scheme and let B be a directed (n+1)-ball in (G,g). Suppose

1. $domB$ extends to an n-pasting scheme X from S to T in G such that no n-cell in X lies in the interior of B

2. for any n-pasting scheme X from S to T in G such that $domB$ is a subpasting scheme of X, the replacement of $domB$ in X by a single n-cell gives an n-pasting scheme

3. for any n-pasting scheme X from S to T in G such that $domB$ is a subpasting scheme of X and for any (n+1)-cell F of (G,g) not in B such that $domF$ is also a subpasting scheme of X, replacing both $domB$ and $domF$ by single cells gives an n-pasting scheme.

Then, in (G,g), replacing B by a single (n+1)-cell gives an (n+1)-pasting scheme.

Proof: The n-cell extension condition and condition 1 of domain replacement follow from Lemma 7.6. Everything else is immediate from the conditions. □

Corollary 7.11 Let (G,g) be a 3-pasting scheme and let B be a directed 3-ball in (G,g). Suppose moreover that $domB$ extends to a 2-pasting scheme X from S to T in G such that no 2-cell in X lies in the interior of B. Then, in (G,g), replacing B by a single 3-cell gives a 3-pasting scheme.

Remark 7.12 It is evident from Proposition 7.10 that if all (n+1)-pasting schemes that appear in an (n+2)-pasting scheme satisfy the conditions of the proposition, then part of domain replacement necessarily holds for that (n+2)-pasting scheme. Of course, one can strengthen the conditions of Proposition 7.10 mildly for (n+1)-pasting schemes to similarly dispense with condition 2 of domain replacement for an (n+2)-pasting scheme. However, even in the presence of condition (*), one still cannot simply discard domain replacement completely. Example 3.11 can be used to construct a counter-example.

Now, given the above topological machinery, we can define the notion of a weak composite of a labelled strong (n+1)-pasting scheme and prove that every such pasting scheme has a unique weak composite. We define the term weak composite inductively.

Definition 7.13 Given a labelled strong (n+1)-pasting scheme $((G,g),a)$ in an (n+1)-category A, a *weak composite* of $((G,g),a)$ is the value of

1. any (n+1)-pasting composite of $((G,g),a)$
2. any weak composite of the (n+1)-pasting composites determined by a partition of the (n+1)-cells of (G,g) into directed (n+1)-balls.

Observe that Corollary 7.9 has been used implicitly in the definition of weak composite: it is precisely Corollary 7.9 that allows us to speak of the weak composite in the second part of the definition, as we know that the partition of (G, g) yields a strong (n+1)-pasting scheme. The theorem follows directly from that.

Theorem 7.14 For every positive natural number n, every labelled strong (n+1)-pasting scheme $((G, g), a)$ in an (n+1)-category A has a unique weak composite.

Proof: By the (n+1)-categorical pasting theorem (Theorem 6.16), every labelled strong (n+1)-pasting scheme has an (n+1)-pasting composite. For the unicity, it suffices to show that a weak composite determined by taking a non-trivial partition of (G, g) into directed (n+1)-balls has the same value as some (n+1)-pasting composite. By induction, it suffices to assume that a non-trivial partition has only one non-trivial directed (n+1)-ball B. So take decompositions of B and of (G', g'), the (n+1)-pasting scheme given by replacing B in (G, g) by a single (n+1)-cell. Then, a decomposition of (G, g) is given by taking that of (G', g') and refining it by that of B. So, by the (n+1)-categorical pasting theorem, we are done. □

Corollary 7.15 Let $((G, g), a)$ be a labelled 3-pasting scheme that satisfies the strong 2-cell extension condition. Then $((G, g), a)$ has a unique weak composite.

Evidently, we can modify Theorem 7.14 mildly in order to capture the situation in which we wish only to evaluate a particular labelled directed (n+1)-ball, then to use that to take a composite.

Theorem 7.16 Let $((G, g), a)$ be a labelled (n+1)-pasting scheme in an (n+1)-category A and suppose (G, g) and a directed (n+1)-ball B satisfy the conditions of Proposition 7.10. Then the (n+1)-cell in A determined by evaluating B then evaluating the labelled (n+1)-pasting scheme given by replacing B in (G, g) by a single (n+1)-cell labelled by the (n+1)-pasting composite of B, is the (n+1)-pasting composite of $((G, g), a)$.

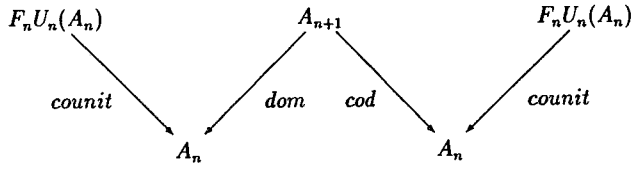
Proof: As for Theorem 7.14. □

Corollary 7.17 Let $((G,g),a)$ be a labelled 3-pasting scheme and let B be a directed 3-ball such that $domB$ extends to a 2-pasting scheme X from S to T with no 2-cell of X lying in the interior of B. Then if one evaluates B, then uses the result to evaluate $((G,g),a)$, the result is the composite of $((G,g),a)$.

8 Free n-categories

In [13], Street gives a definition of what it means for a given n-category to be free. His definition is not given in terms of the existence of a left adjoint. However, the following inductive procedure is implicit in his paper.

For each natural number n, it is possible to define an adjunction in the usual sense $F_n \dashv U_n : n\text{-}Cat \to n\text{-}Cmptd$, where $n\text{-}Cmptd$ is defined inductively as follows. Put $0\text{-}Cat = 0\text{-}Cmptd = Set$ with $F_0 = U_0 =$ the identity functor on the category of sets. A *weak (n+1)-computad* G consists of an object G_n of $n\text{-}Cmptd$, a set G_{n+1}, and two functions $dom, cod : G_{n+1} \to F_n(G_n)$ such that $dom\, dom = dom\, cod$ and $cod\, dom = cod\, cod$. A *morphism* of weak (n+1)-computads from (G_{n+1}, G_n, dom, cod) to (H_{n+1}, H_n, dom, cod) consists of a morphism $f_n : G_n \to H_n$ of weak n-computads and a function $f_{n+1} : G_{n+1} \to H_{n+1}$ such that $dom\, f_{n+1} = f_n\, dom$ and $cod\, f_{n+1} = f_n\, cod$. The category $(n+1)\text{-}Cmptd$ is the category of weak (n+1)-computads and morphisms of weak (n+1)-computads, with the evident composition. Given an (n+1)-category A, define the weak (n+1)-computad $U_{n+1}A = G$ by $G_n = U_n(A_n)$ where A_n is the underlying n-category of A, and G_{n+1} is the limit in the category of sets of the diagram



(8.1)

where A_n is here identified with its set of n-cells.

Each (n+1)-functor $f : A \to B$ induces a morphism $U_{n+1}f : U_{n+1}(A) \to U_{n+1}(B)$ in

the evident way. The induction is complete once one establishes

Proposition 8.1 The functor U_{n+1} has a left adjoint F_{n+1}.

Proof: The proof is implicit in [13] and was discussed by Street at the International Category Theory meeting in Louvain-la-Neuve in 1987. The construction for the case n=1 appears in Street's paper [12]. □

Proposition 8.2 An n-category is free in Street's sense if and only if it is in the image of F_n.

Proof: This is implicit in the definition of free n-category in [13]. □

In fact, with mild conditions on the (n+1)-computad G, the equivalence relations needed in the construcion of $F_{n+1}(G)$ have normal forms determined by rewrite systems. This analysis is part of current work by Eilenberg and Street.

A little more can be said to strengthen Proposition 8.1. It is routine to verify that all of the data for the adjunction between $(n+1)$-Cat and $(n+1)$-$Cmptd$ commutes with the evident forgetful functors to n-Cat and n-$Cmptd$ as in the diagram

$$\begin{array}{ccccccc}
(n+1)-Cat & \longrightarrow & n-Cat & \ldots & 1-Cat & \longrightarrow & 0-Cat \\
F_{n+1}\uparrow\dashv\downarrow U_{n+1} & & Fn\uparrow\dashv\downarrow U_n & & F_1\uparrow\dashv\downarrow U_1 & & F_0\uparrow\dashv\downarrow U_0 \\
(n+1)-Cmptd & \longrightarrow & n-Cmptd & \ldots & 1-Cmptd & \longrightarrow & 0-Cmptd
\end{array} \qquad (8.2)$$

So, assuming the existence of at least three strongly inaccessible cardinals, we may now define Cat_∞ to be the object of the large 2-category $[\omega^{op}, CAT]$ given by the top line of diagram (8.2). Similarly, we may define $Cmptd_\infty$ to that object of $[\omega^{op}, CAT]$ determined by the bottom line of diagram (8.2). So we may strengthen Proposition 8.1 to

Corollary 8.3 The collection $(F_n \dashv U_n | n < \omega)$ forms an adjuntion $F \dashv U$ in $[\omega^{op}, CAT]$.

Now we may define ∞-*Cat* to be the limit in CAT of the diagram Cat_∞. Similarly, define ∞-*Cmptd* to be the limit in CAT of the diagram $Cmptd_\infty$. Then, taking the limits induces an adjunction $F_\infty \dashv U_\infty : \infty\text{-}Cat \to \infty\text{-}Cmptd$.

Our category ∞-*Cat* does not quite agree with Street's ω-*Cat*. Ours is the full subcategory of his determined by those of his ω-categories for which every element is an n-cell for some n. This agrees with the notation of Al-Agl and Steiner. Nevertheless, we can still deduce

Corollary 8.4 An ∞-category is free in Street's sense if and only if it is in the image of F_∞.

We now seek to compare our definition of n-computad with the above. First note that at several points we have implicitly used the notion of the underlying (n+1)-computad of an (n+1)-category, for instance in defining a labelling of an (n+1)-pasting scheme in Definition 6.1. The underlying (n+1)-computad is subtly different to the underlying weak (n+1)-computad: where the latter uses $F_nU_n(A_n)$ as in diagram (8.1), the former uses *diags* of the underlying computad of A_n. So although for each n, there is an evident fully faithful functor J_n : *n-Computad* $\to$ *n-Cmptd* expressing *n-Computad* as a full subcategory of *n-Cmptd* (and this lifts to a map in $[\omega^{op}, CAT]$), U_n does not factor through J_n in general. Our n-computads amount to those weak n-computads whose domains and codomains are given by pasting diagrams, and so the underlying n-computad and weak n-computad of an n-category differ accordingly. Since we do not capture all possible composites in general, J_n generally expresses *n-Computad* as a proper subcategory of *n-Cmptd*. However, since we do capture all composites for n=1 to 3, it follows that we have

Proposition 8.5 For n=1 to 4, the functor J_n is an equivalence of categories.

There is a strong case that what we have called a weak computad would be better called simply a computad, and what we have called a computad would be better called a strong computad. Our choice in this paper has been just for brevity. However, it should be noted that our computads do incorporate a large class of the more general weak computads. For instance, it follows from Johnson and Street's work that all of the

simplexes are computads. Moreover, Examples 3.11 and 3.12 illustrate several pasting schemes that may form domains and codomains for higher dimensional computads such as those describing at least some of the higher level adjuncions. Finally, Johnson's "coherent situations" [4] may be discussed more directly in the geometrical terms of our computads.

9 Bibliography

[1] F.A.Al-Agl and R.Steiner, Nerves of multiple categories, preprint (1990)

[2] J.W.Gray, Formal Category Theory: Adjointness for 2-Categories, in Lecture Notes in Math 391 (Springer, Berlin, 1971)

[3] M.Johnson, Pasting diagrams in n-categories with applications to coherence theorems and categories of paths, Doctoral thesis, University of Sydney (1987)

[4] Michael Johnson, The combinatorics of n-categorical pasting, J. Pure Appl Algebra 62 (1989) 211-225

[5] Michael Johnson, Finding a representable family and the free construction on 1, International Category Theory Meeting 1990 (lecture)

[6] M.M.Kapranov and V.A.Voevodsky, Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders, preprint, Steklov Mathematical Institute (1990)

[7] G.M.Kelly and R.H.Street, Review of the elements of 2-categories, in Lecture Notes in Math 420 (Springer, Berlin, 1974) 75-103

[8] A.J.Power, A 2-categorical pasting theorem, J. Algebra 129 (1990) 439-445

[9] A.J.Power, An algebraic formulation for data refinement, in Lecture Notes in Comp. Science 442 (Springer, Berlin, 1990) 390-401

[10] A.J.Power, Coherence for bicategories with finite bilimits II (in preparation)

[11] M.B.Smyth and G.D.Plotkin, The category-theoretic solution to recursive domain equations, SIAM J. Computing 11 (1982) 761-783

[12] Ross Street, Limits indexed by category-valued 2-functors, J. Pure Appl Algebra 8 (1976) 149-181

[13] Ross Street, The algebra of oriented simplexes, J. Pure Appl. Algebra 49 (1987) 283-336