Building domains from graph models

Wesley Phoa
Laboratory for the Foundations of Computer Science
James Clerk Maxwell Building, The King's Buildings
Edinburgh EH9 3JZ, Scotland

Abstract

In this paper we study partial equivalence relations (PERs) over graph models of the λ-calculus. We define categories of PERs that behave like predomains, and like domains. These categories are *small* and *complete*; so inside them, we can solve domain equations and construct polymorphic types. Upper, lower and convex powerdomain constructions are also available, as well as interpretations of subtyping and bounded quantification. Rather than performing explicit calculations with PERs, we work inside the appropriate *realizability topos*: this is a model of constructive set theory in which PERs can be regarded simply as special kinds of sets. In this framework, most of the definitions and proofs become quite simple and attractive. They illustrate some general techniques in 'synthetic domain theory' that rely heavily on category theory; using these methods, we can obtain quite powerful results about classes of PERs even when we know very little about their internal structure.

1 Introduction

Classical domain theory is an abstract theory, based on mathematical analogies. We model semantic domains by partial orders; we find order-theoretic analogues for the notions of information content and approximation; and the vague concept of 'computability' is replaced by the requirement of continuity. The resulting theory works well, but is unwieldy: features such as dependent types, nondeterminism and polymorphism can be difficult to model, and the grafted-on treatment of effectivity (via countable bases) is quite artificial.

Now, we already have nice untyped models of computation: for example, the recursive functions; the graph models of Plotkin and Scott; term models for the lambda calculus; algebras of combinator expressions. It could be argued that computation really goes on at this primitive untyped level—type disciplines are for the language designer and the programmer, 'semantic scaffolding' to make their jobs easier, but in the machine itself, type information is discarded.

So what we would like to do is begin with such a model of computation and construct a concrete theory of domains over it. Things like continuity would then be *consequences* of the computability built into the framework, rather than being part of the definition.

Taking this view, it's natural to begin with a (partial or total) combinatory algebra A and think of types as partial equivalence relations ('PERs') on A. That is, to specify a type or domain X we must specify

- which elements of A can meaningfully be regarded as (representing) elements of X; and
- which elements of A, thought of as elements of X, should be regarded as equal.

When A is (indices for) the partial recursive functions, this view of 'domains as PERs' has been followed up by several authors: see for example [Rosolini 1987], [Phoa 1990a], [Freyd *et al.* 1990]. It leads to rich and well-behaved categories of domains.

A very useful point of view is to think of such PERs as objects in the effective topos (see [Hyland 1982]). We can then use all the machinery of category theory and topos theory to manipulate our domains, but still have set-theoretic methods and intuitions available—a topos is just a constructive universe of sets, and so the PERs simply correspond to a particular class of 'sets' in this topos, called the *modest sets*. Polymorphism now has a natural 'set-theoretic' interpretation, see [Robinson 1989].

Once we have constructed a category of domains, we can use it to model some programming language. We should then try and relate this denotational semantics to an operational semantics for the language. To this end, it may be useful to look, not just at the partial recursive functions, but at other models which capture other operational intuitions. Amadio, and also Abadi and Plotkin (in [Abadi and Plotkin 1990]), have studied PERs over the D_∞ model and related models, and obtained good results; but the situation is still not well understood. Their results can't be understood inside a topos-theoretic framework yet.

Another class of interesting models are the term models. They seem important because they can actually be used to (partly) specify implementations of languages—computers can manipulate terms, whereas it's not practical for a computer to deal with Gödel numbers for partial recursive functions, or continuous functions. We think of elements of a term model as primitive machine instructions or data values, and we have a notion of equality, but not of reduction. Unfortunately, term models seem difficult to work with. Certain term models do in fact give rise to categories of domains; but the results are weaker, and harder to obtain, than those described below. See [Phoa 1991a].

In this paper we will be working with the r.e. graph model R (though the results are valid for any graph model). But this isn't a term model; why are we interested in it?

The first reason is that the graph model is a lot easier to work with. Term models are not so tractable—and there's always the risk that one's going to get caught up in the theory of reduction, which would be a disaster—whereas working with R feels much more like ordinary mathematics.

Secondly, though, the r.e. graph model does have intrinsic interest. For example, in some ways it mirrors our computational intuition a lot better than the D_∞ model; or at least, it captures different aspects of our computational intuition, ones that reach back to classical recursion theory. The hope is that properties of the model will be reflected in the behaviour of the domains we construct from it.

And the r.e. graph model *is* a sort of term model, or at least a hard model: not for the pure λ-calculus, but for Scott's language LAMBDA. (This is, roughly speaking, the λ-calculus plus strong definition by cases—see [Scott 1976]. If you prefer combinators, Beeson's language **PCA** is essentially the same thing: see [Beeson 1985].) And LAMBDA is a very natural language to consider. It's a kind of untyped version Plotkin's PCF; it contains some arithmetic and choice, and then the full machinery of the λ-calculus; thought of as an abstract machine language, it is almost practical.

Interestingly, the results in this paper can be used to prove adequacy theorems about translations of PCF into the bare untyped λ-calculus: see [Phoa 1992b]. So our remarks about capturing 'operational intuition' are not pure speculation.

There is an important question, however. What equality on terms is forced by R? We know that two λ-expressions denote the same element of a graph model precisely when they have the same Böhm tree (see [Barendregt 1984, chapter 19] for an exposition). But, as Beeson points out, the corresponding problem for LAMBDA doesn't seem to have been studied. One would expect it to be similarly related to some kind of deep operational equivalence; recent unpublished work of Kennaway, Klop, Sleep and de Vries on infinite term rewriting suggests a suitable notion of Böhm tree for LAMBDA-expressions, and this might be the right starting-point.

Plan of the paper

Section 2 contains some general categorical results that we'll need. First we recall the definition of *dominance*; a dominance is an object Σ of truth-values in the topos which can be used to define a well-behaved class of partial maps. Next we list some results about *replete* objects, due to Hyland and, independently, to Taylor: see [Hyland 1991] and [Taylor 1991]. The category of replete objects with respect to Σ is the smallest full reflective subcategory of the topos that contains Σ; in a realizability topos, if Σ is taken to be a modest set then every replete object is modest. Replete objects have 'all possible' good properties, and in a sense, the problem is to find out what these properties are.

Finally we study the *link relation*, defined for an arbitrary object X (a 'set' in the realizability topos) as follows:

$$x \lhd x' \quad \longleftrightarrow \quad \exists f \in X^{\Sigma}.\, f(\bot) = x \wedge f(\top) = x'$$

This is always reflexive, but it is not in general a partial order on X. However, we show that the full subcategory of objects which *are* partially ordered by the link relation is reflective; it follows that any replete object is partially ordered by the link relation.

In section 3 we prove results specific to the graph model R; we work entirely with the corresponding realizability topos $\mathcal{E}$. A suitable dominance Σ is defined and the 'lift' of a modest set (its 'Σ-partial map classifier') is explicitly described. It is a general fact that replete objects are closed under lifting; so, for example, they are closed under the formation of computable partial function spaces. For our choice of Σ, it turns out that the link relation is closely related to the inclusion relation on R; for example, the 'bottom' element added by the lift construction is indeed a least element under the link relation.

The remaining task is to formulate a good notion of chain completeness. As we would like *all* maps between 'chain-complete' objects to preserve suprema of chains, we must be careful to pick the right definition: a good notion seems to be that of *uniform completeness*. We first define a certain object $\bar{\omega}^+$, the 'generic chain with a sup and upper bound'; we then call an object X uniformly complete if, speaking loosely, every chain has a sup and chains in X with upper bounds correspond to maps $\bar{\omega}^+ \to X$. The full subcategory of uniformly complete objects is reflective, and hence every replete object is uniformly complete.

Though replete objects X with a least element under the link relation have the fixed point property (there is a least fixed point operator $X^X \to X$), it is not clear that if we regard X as a PER, fixed points can be calculated by the fixed point combinator in the underlying graph model. A solution to this problem is sketched; full details appear in [Phoa 1991a]. This result is not necessary to obtain a good category of domains, but it is useful in certain applications. For example, in [Phoa 1992b] a theorem relating PCF to the untyped λ-calculus is proved using the results in the present paper; the fact that fixed points in our domains may be calculated in 'the same way' as fixed points in the untyped λ-calculus is a crucial part of the argument.

Finally, we mention various constructions supported by our 'predomains' (the replete objects): polymorphic types, domain equations, powerdomains and subtyping/bounded quantification. Details appear in other papers, and in [Phoa 1990b]; there is nothing special about the graph model domains in this regard.

At the end of the paper, there are some remarks and suggestions about future lines of research.

All our results are applicable, not just to the r.e. graph model R, but to any graph model. It's not clear whether they can be extended to more general classes of models; I hope this is possible, but new techniques will have to be discovered. Contrast this with [Phoa 1991a]: the methods used there apply to any continuous model, and in particular any graph model; unfortunately the results obtained are weaker.

This paper presupposes a fair amount of category theory; in particular, the reader should be familiar with the view of toposes as 'models of constructive set theory'. A familiarity with PERs, and with realizability toposes and modest sets, would also be useful. Readers may find the full version of [Phoa 1991a] helpful, and more detailed expository accounts are in preparation by several authors. I hope that once the theory is in a more developed state, a simpler presentation will be possible.

Technical note 1

It should be emphasised again that almost all the definitions and proofs that follow are meant to be understood *in the internal language of the topos.* So, for example, when we talk about a 'PER' we do not usually mean a 'real' PER, but rather an internal partial equivalence relation inside the topos.

Remember that to obtain results valid in a topos, we have to reason *constructively:* for example, we're not allowed to use the law of the excluded middle or the axiom of choice. This requires care since there are many statements which are classically, but not constructively, equivalent. By the way, the internal logic is not the only way to show that formulas are valid in a realizability topos: the final version of [Phoa 1991a] contains an example, applicable to this paper, of how to construct a realizer for a formula.

Technical note 2

We will be constructing categories which are small and complete, applying the adjoint functor theorem and so on 'as if we were working in the ordinary category of sets' (even though there are *no* small complete categories in the ordinary category of sets). These things are more subtle than they appear; for example, there are several different notions of completeness.

For the sake of clarity and brevity, we sweep these technical difficulties under the carpet. However, all the arguments we use can be suitably formalized. The reader who is unhappy with this should have a look at [Robinson 1989], and the references cited there. Those who wish to see 'external' proofs should refer to [Phoa 1990b, section 5.4.2], which describes how completeness results closely related to ours can be obtained in a framework which involves working with fibrations rather than in the internal logic.

2 Category-theoretic results

2.1 Realizability toposes: notation and remarks

There is no room for a proper account here. For general topos theory the best reference is [Johnstone 1977]; for the effective topos, and realizability toposes in general, see [Hyland 1982]; and for modest sets see [Hyland *et al.* 1990]. An exposition aimed at non-category theorists is [Rosolini 199?]; and a briefer summary of the relevant facts appears in [Phoa 1991a].

Let A be a (total or partial) combinatory algebra; for example, the set $\mathbb{N}$ of indices for the partial recursive functions, or a model for the untyped λ-calculus. A can be used to give a realizability interpretation for intuitionistic logic. In fact, one can use it to construct a model of constructive set theory; more precisely, A gives rise to a realizability topos $\mathcal{E}$: cf. [Hyland 1982]. For example, when $\mathsf{A} = \mathbb{N}$ we get the effective topos $\mathcal{E}\!f\!f$.

There is a full embedding $\nabla\colon \mathcal{S}ets \hookrightarrow \mathcal{E}$ of the ordinary category of sets into this realizability topos. It factors through a full subcategory $\mathcal{S}ep$ of $\mathcal{E}$, where $\mathcal{S}ep$ is the category of *A-valued assemblies* (also called *ω-sets*, in the case $\mathsf{A} = \mathbb{N}$: see [Longo and Moggi 199?]). And $\mathcal{S}ets$ and $\mathcal{S}ep$ are both reflective subcategories of $\mathcal{E}$.

This last statement follows from the fact that they can be characterized using the double negation topology on $\mathcal{E}$. That is: $\mathcal{S}ep$ is (equivalent to) the category of $\neg\neg$-separated objects in $\mathcal{E}$, and $\mathcal{S}ets$ is (equivalent to) the category of $\neg\neg$-sheaves. Also, $\neg\neg$-closed subobjects correspond in a certain sense to subsets in the ordinary (classical) sense; they are classified by the object $\nabla 2 \rightarrowtail \Omega$ (where Ω is the full subobject classifier).

As one might expect, $\neg\neg$-closed (or '$\neg\neg$-stable') predicates have an important role to play in the sequel. Note that if ϕ and ψ are $\neg\neg$-closed, then $\phi \vee \psi$ need not be; but there is a $\neg\neg$-stable notion of disjunction (already used in [Phoa 1991a] without much explanation) which sometimes serves as a replacement.

Lemma 2.1 *If ϕ and ψ are $\neg\neg$-closed then $\neg\neg(\phi \vee \psi) \leftrightarrow (\neg\phi \to \psi)$.*

Proof. $\nabla 2 \rightarrowtail \Omega$ is the subobject of $\neg\neg$-closed truth-values; we've assumed that $\phi, \psi \in \nabla 2$. Now as $\nabla 2$ is the Heyting algebra obtained from Ω via the $\neg\neg$-topology, implication and negation in $\nabla 2$ are the same as in Ω, and the disjunction $\phi \vee_{\neg\neg} \psi$ may be defined by

$$\neg\neg(\phi \vee \psi)$$

(the double negation of the usual disjunction). This is a general fact about topologies on the object of truth-values Ω: see [Johnstone 1977].

But because $\nabla 2$ is actually a Boolean algebra, $\phi \vee_{\neg\neg} \psi$ is equivalent to $\neg\phi \to \psi$ (and also to $\neg\psi \to \phi$). ■

The following result tells us that this really is a useful notion of disjunction. We won't cite it explicitly but it explains why some of our definitions are useful ones; for example, see the discussion on partial maps and lifting in section 3.

Corollary 2.2 *Let ϕ, ψ and ξ be $\neg\neg$-closed. If ϕ entails ξ and ψ entails ξ, then $\neg\phi \to \psi$ entails ξ.*

Proof. We know that $\phi \vee \psi$ entails ξ. But ξ is $\neg\neg$-closed and so $\neg\neg(\phi \vee \psi)$ entails ξ. ∎

PERs and modest sets

The combinatory algebra A gives rise to an object $\mathbb{A} \in \mathcal{E}$ (in the case $\mathsf{A} = \mathbb{N}$ we get the natural numbers object of $\mathcal{E}\!f\!f$). Now subsets of A give rise to $\neg\neg$-closed subobjects of $\mathbb{A}$; in particular, PERs on A give rise to $\neg\neg$-closed partial equivalence relations on $\mathbb{A}$. The quotient (in $\mathcal{E}$) of such a partial equivalence relation is called a **modest set**. Modest sets in fact lie in $\mathcal{S}ep$, as one might expect. Moreover, it is internally true that "any map between modest sets is tracked by an element of $\mathbb{A}$": this is the 'generalized Church's thesis' proved in [Hyland 1982].

In fact, the definition of 'modest set' makes sense in any slice topos $\mathcal{E}/I$; and one obtains a fibration $\mathcal{M}od$ over $\mathcal{E}$ which, when restricted to a fibration over $\mathcal{S}ep$, is complete. The important thing is that this fibration is, in a suitable sense, the 'externalization' of a certain internal category in $\mathcal{E}$.

Namely, let $\mathbb{M}\mathbf{od}$ be the internal category of $\neg\neg$-closed PERs on $\mathbb{A}$ and maps tracked by elements of $\mathbb{A}$; it is an internal full subcategory of $\mathcal{E}$ (see [Hyland 1988]) whose externalization is the ordinary category of modest sets. As a consequence, it is closed under all small (i.e. internally definable) limits in $\mathcal{E}$—in a sense which we have not made precise here. So, for example, one can use it to model polymorphic λ-calculi (though this last result actually uses the fact that $\mathbb{M}\mathbf{od}$ is an internal category in $\mathcal{S}ep$, not just $\mathcal{E}$; see [Hyland *et al.* 1990]).

2.2 Partial maps and dominances

It is not quite accurate to think of a realizability topos $\mathcal{E}$ as a 'universe of computable sets and functions'. After all, it contains a copy of the category of sets—whose logic is far from effective.

If we restrict ourselves to the ordinary category of modest sets (PERs), then it *is* true that all maps are computable; that is, all *total* functions. But (as usual in computer science) we will also be interested in partial functions; and because the notion of subobject in a realizability topos is so flexible, there are still many partial maps which would not normally be regarded as computable. So the first problem will be to isolate a class of partial maps which *can* be regarded as computable.

Building on ideas due to Mulry and Scott, Rosolini developed a clean category-theoretic framework for discussing 'good' classes of partial maps in a topos. The idea is to single out an appropriate subset Σ of truth-values—which may be regarded as the 'computable' truth-values. A subobject whose characteristic function takes values in Σ will be called a **Σ-subset**; it will turn out that (if we choose the right Σ) the partial maps whose domains are Σ-subsets are exactly the ones which can be regarded as computable.

Let $\Sigma \rightarrowtail \Omega$ be a subset of the object of truth-values. Then Σ is a **dominance** if

$$\top \in \Sigma\,;$$
$$p \in \Sigma \,\wedge\, (p = \top \rightarrow q \in \Sigma) \;\rightarrow\; (p \wedge q) \in \Sigma.$$

These conditions ensure that the class of partial maps whose domain is a Σ-subset forms a category: the category of **Σ-partial maps**.

If $X' \rightarrowtail X$ is classified by a map $X \rightarrow \Sigma$ we write $X' \subset_\Sigma X$; then to say that Σ is a dominance is equivalent to saying that

- for any X, $X \subset_\Sigma X$; and
- if $X \subset_\Sigma Y$ and $Y \subset_\Sigma Z$ then $X \subset_\Sigma Z$.

The first condition ensures that all total maps are Σ-partial maps; the second, that Σ-partial maps can be composed in the usual way. See [Rosolini 1987] for further details.

For example, let $2 = 1 + 1$. The elements $\top, \bot \colon 1 \rightarrow \Omega$ define a mono $2 \rightarrowtail \Omega$, and we can regard 2 as the set of *decidable* truth-values. Then 2 is easily verified to be a dominance—a decidable subset of a decidable subset is again a decidable subset. Unfortunately, the partial maps with decidable domains form too small a class to be useful, so we must find larger dominances to work with.

Once we've fixed a dominance Σ, we have a notion of partial map classifier. Given an object $X \in \mathcal{E}$, we say that another object $X_\bot$ is a **Σ-partial map classifier** for X if, for any object Y, there is a natural bijection between partial maps $Y \rightharpoonup X$ with domain a Σ-subset and (total) maps $Y \rightarrow X_\bot$. (The object $X_\bot$ is necessarily unique; it is also called the **lift** of X.)

There are several ways to construct the Σ-partial map classifier $X_\bot$; for example, we could let

$$X_\bot \quad = \quad \{t \subset X \mid \text{“}t \text{ has at most one element”} \wedge |\exists x . x \in t| \in \Sigma\}.$$

(That is, it's a subobject of the usual topos-theoretic partial map classifier.) But for the particular dominance Σ used in this paper we have a different, more explicit description of the lift.

Actually, it's convenient to have a slightly more general notion of dominance and lifting. Let $\mathbb{C}$ be a small (i.e. internal) full subcategory of $\mathcal{E}$ containing a subobject Σ of Ω (we usually have $\mathbb{C} = \mathbb{M}\mathbf{od}$). Then we know what it means for Σ to be a dominance relative to $\mathbb{C}$: express the second characterization of dominance in the internal language of $\mathcal{E}$, letting X, Y, Z range over objects of $\mathbb{C}$.

More precisely, we say Σ is a **dominance relative to** $\mathbb{C}$ if

$$\forall X \in \mathbb{C}_0.\, X \subset_\Sigma X$$

and

$$\forall X, Y, Z \in \mathbb{C}_0.\, (X \subset_\Sigma Y) \wedge (Y \subset_\Sigma Z) \to (X \subset_\Sigma Z)$$

are valid (note that the statement '$X \subset_\Sigma Y$', for X, Y objects of $\mathbb{C}$, can be expressed as a formula in the internal language).

We can also define the notion of **Σ-partial map classifier relative to** $\mathbb{C}$; the precise definition is left to the reader.

Replete objects

We will need some results due to Hyland (see [Hyland 1991], and also Taylor's paper [Taylor 1991]). Suppose we are given a modest set Σ in a realizability topos.

Theorem 2.3 (Hyland, Taylor) *There is a* smallest *internal full subcategory containing Σ and closed under (internally defined) limits in $\mathcal{E}$. Equivalently, there is a smallest internal full reflective subcategory of $\mathcal{E}$ containing Σ.*

The objects of this category are called **replete** objects: there are a number of alternative ways to define them. Every replete object is modest. In [Taylor 1991], Taylor suggests that the replete objects can be regarded as 'sober' in some sense; this is a tempting intuition to adopt, but is not (yet) fully justified. For example, even in the effective topos we do not yet have a familiar-looking definition of 'sober' which is equivalent to 'replete' (a candidate appears in [Rosolini 1987], but the conjecture stubbornly resists proof).

Hyland's results show that (under certain assumptions satisfied by the object Σ defined later on in this paper) the objects 2 and $\mathbb{N}$, as well as Σ, are replete. Furthermore, since the replete objects are closed under all small limits in $\mathcal{E}$, they form an exponential ideal. In fact in the case that Σ is a dominance, they are closed under the appropriate partial function spaces as well:

Theorem 2.4 (Hyland) *If X is a replete object, then so is $X_\perp$.*

It is a general principle that the category of replete objects is the one we are really interested in; it is, almost by definition, the best possible category to study. But replete objects are, in themselves, rather hard to grasp.

This paper illustrates a general technique for proving theorems about replete objects. Namely: show that the collection of modest sets having a particular property forms a full reflective subcategory of $\mathbb{M}$**od**; then it follows that every replete object has that property, since the replete objects form a subcategory of that category. Of course, this only works for properties stable under isomorphism (so not, e.g., in the situation in [Phoa 1991a]).

Note that to prove that a subcategory of $\mathbb{M}$**od** is reflective, we need only prove that it's 'complete': this is because of the adjoint functor theorem (see [Phoa 1990c]).

2.3 The link relation

Let $\mathcal{E}$ be a realizability topos, and fix a subobject $\Sigma \rightarrowtail \Omega$ such that $2 \rightarrowtail \Sigma \rightarrowtail \Omega$. For example, Σ could be a dominance containing 2, or a dominance relative to some internal full subcategory $\mathbb{C}$.

If X is any object of $\mathcal{E}$, we have a relation $\triangleleft$ on X defined by

$$x \triangleleft x' \quad \longleftrightarrow \quad \exists f \in X^{\Sigma}.\, f(\bot) = x \wedge f(\top) = x'$$

(we say that f **links** x and x'). Call this the **link relation**; it is clearly reflexive, though it is not in general a partial order. Also, observe that if $x \triangleleft x'$ in X and $f\colon X \to Y$ is any map, then $f(x) \triangleleft f(x')$.

The idea of looking at the link relation is closely related to that of studying the linked Σ-spaces defined in [Phoa 1990a]. However, there the partial order on objects was defined in a different way, and the link relation was introduced as a technical tool; here the link relation is taken as basic.

Lemma 2.5 *If $X \in \mathcal{S}ep$ then any map $\Sigma \to X$ is determined by its values on $\bot$ and $\top$. In particular, if $x \triangleleft x'$ then there is a* unique *$f \in X^{\Sigma}$ linking x and x'.*

Proof. This follows immediately from the fact that $2 \rightarrowtail \Sigma$ is $\neg\neg$-dense. ∎

Lemma 2.6 *The link relation on the limit of a diagram of $\neg\neg$-separated objects is pointwise.*

Proof. Let $F\colon \mathbb{D} \to \mathcal{S}ep$ be a diagram of separated objects, with limit cone

$$\left(\pi_d\colon X = \varprojlim F \to F(d)\right)_{d \in \mathbb{D}}$$

and suppose $x, x' \in X$ with $x \triangleleft x'$; then, since each π_d is monotone,

$$\forall d.\ \pi_d x \triangleleft \pi_d x'.$$

Conversely, suppose we have $x, x' \in X$ with $\pi_d x \triangleleft \pi_d x'$ for all d: that is,

$$\forall d.\ \exists f_d \in F(d)^{\Sigma}.\ f_d(\bot) = \pi_d x \ \wedge\ f_d(\top) = \pi_d x'.$$

By the previous lemma, the maps $(f_d)_{d \in \mathbb{D}}$ form a cone over F; so there is a (unique) map $f \in X^{\Sigma}$ such that $f_d = \pi_d f$, i.e.

$$\forall d.\ \pi_d f(\bot) = \pi_d x \ \wedge\ \pi_d f(\top) = \pi_d x'.$$

Hence $f(\bot) = x$ and $f(\top) = x'$; and this shows that $x \triangleleft x'$. ∎

Since we can't prove that every object is partially ordered by the link relation, we'll simply look at those which are. First, call an object $X \in \mathcal{S}ep$ **link-preordered** if the link relation $\triangleleft$ on X is transitive (i.e. a preorder).

Proposition 2.7 *The category of link-preordered objects is closed under limits.*

Proof. Let $X = \varprojlim X_i$; suppose we have $x, x', x'' \in X$ with $x \lhd x', x' \lhd x''$; we want to show that $x \lhd x''$. Note that the elements $x \in X$ can be described as certain tuples (x_i), where $x_i \in X_i$.

But if $x = (x_i), x' = (x'_i), x'' = (x''_i)$ then by 2.6,

$$x_i \lhd x'_i,\ x'_i \lhd x''_i$$

and hence

$$x_i \lhd x''_i$$

for all i; so $x \lhd x''$, again by 2.6. Hence X is link-preordered. ∎

If X is link-preordered, we can define an equivalence relation $\sim$ on X by

$$x \sim x' \quad \longleftrightarrow \quad x \lhd x' \wedge x' \lhd x.$$

Call $X \in \mathcal{S}ep$ **link-ordered** if $\lhd$ is a partial order on X; that is, if $\sim$ is the equality relation.

Proposition 2.8 *The category of link-ordered objects is closed under limits.*

Proof. Let $X = \varprojlim X_i$ and suppose we have $x, x' \in X$ with $x \sim x'$. If $x = (x_i), x' = (x'_i)$ then $x_i \sim x'_i$ and hence $x_i = x'_i$ for all i; so $x = x'$. ∎

We have not stated these results in their proper 'indexed' or fibred form; this is left to the reader. For example, we should really say that the full subfibration of $\mathcal{S}ep^2 \to \mathcal{S}ep$ consisting of families of link-preordered objects is closed under limits.

Suppose now that Σ is a modest set. In particular we have $\Sigma \rightarrowtail \nabla 2$, where $\nabla 2$ is the $\neg\neg$-closed subobject classifier. Recall from [Hyland *et al.* 1990] that $\mathcal{M}od \simeq \mathcal{S}ep \cap Orth(\nabla 2)$, where $Orth(\nabla 2)$ is the category of objects X such that—in an appropriate internal sense—every map $\nabla 2 \to X$ is constant.

Proposition 2.9 *Any link-ordered object must be modest; so the category of link-ordered objects is small. More precisely, it's the externalization of an internal subcategory of the internal category* $\mathbb{M}\mathbf{od}$.

Proof. It suffices to show that any link-ordered object X is orthogonal to $\nabla 2$. But any map $f \colon \nabla 2 \to X$ gives rise to two 'opposite' maps $\Sigma \to X$ arising from the two 'opposite' embeddings $\Sigma \rightarrowtail \nabla 2$; so, as X is link-ordered, both of these maps must be constant, and so f must have been constant (since $\Sigma \rightarrowtail \nabla 2$ is $\neg\neg$-dense).

Actually, one also needs to check that the condition of being link-ordered is definable in the internal language (so that one can internally define a full subcategory of $\mathbb{M}\mathbf{od}$); this is straightforward. ∎

The adjoint functor theorem for small categories now gives us:

Corollary 2.10 *The category of link-ordered objects is reflective in $\mathbb{M}\mathbf{od}$.*

It follows that if Σ is link-ordered, then every replete object must be link-ordered.

Note. We can't say that the reflection into the category of link-ordered objects is given by $X \mapsto X/{\sim}$; there may be maps $\Sigma \to X/{\sim}$ that don't come from maps $\Sigma \to X$.

2.4 The situation in the effective topos

We won't than summarise all the work relating to the effective topos; instead, we just sketch the particular approach taken in [Phoa 1990a]. It fits in best with the spirit of this paper, and in fact the results obtained there motivated many of the definitions given here.

Computable partial maps

The effective topos is constructed using the partial recursive functions. So it's not surprising that the correct notion of computable partial map corresponds to a generalization of the notion of partial recursive function—that is, we look at maps with 'r.e.' domain. Following Mulry and Scott, we define

$$\Sigma \;=\; \{p \in \Omega \,|\, \exists f \in 2^{\mathbb{N}}.\, p \leftrightarrow (\exists n. f(n) = \top)\}.$$

That is, Σ is the closure of the decidable truth-values $2 = \{\top, \bot\}$ under countable joins (recall that since all maps $\mathbb{N} \to 2$ come from recursive functions, these should really be thought of as joins of recursively indexed countable families). Σ is a dominance; it is called the **r.e. subobject classifier**.

The Σ-order, and predomains

If $X \in \mathcal{E}\!\mathit{ff}$ is any object of the effective topos, there is an intrinsic preorder on X called the **Σ-preorder**; it's defined by

$$x \leq x' \quad\longleftrightarrow\quad \forall f \in \Sigma^X.\, f(x) \leq f(x').$$

Call X a **complete Σ-space** if the Σ-preorder on X is a partial order, and X has sups of chains (in the internal sense, remember) under this partial order. Any complete Σ-space must be a modest set (a PER).

The category of complete Σ-spaces is very well-behaved; it is closed under lifting, and it is complete. The proof of the latter fact relies on the following result: if X is a complete Σ-space and $x, x' \in X$, then

$$x \leq x' \quad\longleftrightarrow\quad \exists f \in X^{\Sigma}.\, f(\bot) = x \wedge f(\top) = x'.$$

(See [Phoa 1990a], and compare 3.3.) That is, we use the fact that for complete Σ-spaces, the Σ-order coincides with the link relation.

It follows that we have an extremely good category of (pre)domains, with solutions of domain equations, polymorphic types, powerdomains and many other things. Observe that every replete object is necessarily a complete Σ-space.

Our goal in the rest of the paper is to isolate a similar class of link-ordered objects in the realizability topos constructed from R, that can serve as a good category of predomains. In fact there is a slight difference in emphasis. In $\mathcal{E}\!f\!f$, we can show that the complete Σ-spaces are closed under lifting, and so there was no need to consider the replete objects (though $\mathcal{R}ep$ is for deeper reasons the 'right' category anyway).

Here it is not clear that the 'chain-complete' objects are closed under lifting, so we are forced to work with the replete objects. That is, the focus of this paper—unlike that of [Phoa 1990a]—is on proving things about replete objects. We only construct other categories with this end in mind.

3 Domains in the graph model

3.1 The r.e. graph model

We'll quickly review the Plotkin-Scott construction of graph models for the λ-calculus; see [Scott 1976] or [Beeson 1985] for more details.

Let X be a set equipped with a coding of pairs $\langle\cdot,\cdot\rangle\colon X \times X \to X$ and a coding of finite subsets $\mathrm{Fin}X \to X$. Let PX be the powerset of X. We can define a binary application operation $PX \times PX \to PX$ as follows: if $X_1, X_2 \subseteq X$, put

$$X_1 X_2 \;=\; \{x \in X \mid \exists y \in X.(y \subset X_2) \wedge (\langle y, x\rangle \in X_1)\}$$

where '$y \subset X_2$' means that $y \in X$ codes a finite subset of X_2.

This—together with a few other definitions which are omitted here—turns PX into a model for the untyped λ-calculus. We call it a *graph model*; submodels of such models are also called graph models.

From now on, if $y \in X$ codes a finite subset $\{x_1, \ldots, x_n\}$, we will abuse notation and simply write $\{x_1, \ldots, x_n\} \in X$ in place of y. (But note that different codings of pairs or finite subsets will, in general, give rise to different models; so this is actually a more serious abuse than it looks.)

The natural numbers come equipped with a standard (bijective) pairing $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as well as a coding of finite subsets $\mathrm{Fin}\mathbb{N} \to \mathbb{N}$:

$$\{n_1, \ldots, n_k\} \;\mapsto\; \sum_{i=1}^{k} 2^{n_i}.$$

So $P\mathbb{N}$ becomes a model for the untyped λ-calculus.

It has a submodel consisting of just the *recursively enumerable* subsets; we will call this the **r.e. graph model** (or, in this paper, just 'the graph model') and denote it by R. This is the model that we'll be looking at. We remark again, though, that the results hold for graph models in general.

3.2 Σ, and lifting

As in $\mathcal{E}\!f\!f$, we begin by describing a suitable notion of computable partial map; equivalently, by defining a dominance. The idea will be that the element $\varnothing \in \mathsf{R}$ represents 'divergence' or 'no result'. Further details may be found in [Phoa 1991a], where an analogous definition is used.

Let Σ be the R-valued assembly with underlying set $\{\top, \bot\}$ and

$$|\bot| = \varnothing, \quad |\top| = \{0\}.$$

This gives us an object Σ in the realizability topos $\mathcal{E}$—in fact, a modest set.

There are several other ways to define Σ; it will be useful to have one that makes sense in the internal language. Is there an internally definable $\neg\neg$-closed PER on $\mathbb{R}$ whose quotient is Σ? It's tempting to try defining

$$pRq \quad \longleftrightarrow \quad (p = q = \varnothing) \vee (p = q = \{0\})$$

but this PER is not $\neg\neg$-closed. So in accordance with lemma 2.1, we let R be the following $\neg\neg$-closed PER on $\mathbb{R}$:

$$pRq \quad \longleftrightarrow \quad \neg(p = q = \varnothing) \to (p = q = \{0\}).$$

Then $\Sigma \cong \mathbb{R}/R$.

In fact, $\Sigma \rightarrowtail \nabla 2$; as a consequence, if $X' \subset_\Sigma X$ and $x \in X'$ then 'codes for x in X' are the same thing as codes for x in X'; that is, a representative can be chosen for X' such that the inclusion $X' \rightarrowtail X$ is tracked by the identity (the denotation of $\lambda x.x$ in R). This will simplify our arguments.

We would like to prove that Σ is a dominance, so we can use it to characterize computable partial maps. It is; but the proof is very messy, and we don't need the result in full generality, so we only prove a special case here. Curious readers should consult the final version of [Phoa 1991a]; the method described there applies also in our situation.

Proposition 3.1 *Σ is a dominance relative to $\mathbb{M}$**od**.*

Proof. All we really have to show is that if $X \subset_\Sigma Y$ and $Y \subset_\Sigma Z$ then $X \subset_\Sigma Z$. Suppose $p \in \mathbb{R}$ tracks the classifier $Y \to \Sigma$ of X, and $q \in \mathbb{R}$ tracks the classifier $Z \to \Sigma$ of Y; then there is a map $Z \to \Sigma$ tracked by

$$\{\langle\{m_1, \ldots, m_k\} \cup \{n_1, \ldots, n_l\}, 0\rangle \,|\, \langle\{m_1, \ldots, m_k\}, 0\rangle \in q,\ \langle\{n_1, \ldots, n_l\}, 0\rangle \in p\}$$

that classifies $X \rightarrowtail Z$. ■

This proof, like most of the proofs in this paper, looks rather external. The rigorous internal versions would be basically the same, but with lots of extra λ-abstractions (since we would be establishing the validity of universally quantified formulas instead of particular instances). That is, in the external version we typically start with some r.e. sets (elements of R) and construct other r.e. sets from them; in the internal version (which talks about elements of the corresponding object $\mathbb{R} \in \mathcal{E}$ and internally defined operations on them), we'd have to explicitly effectivize the construction by exhibiting an appropriate enumeration operator. Well, this would be tedious, so we won't do it.

Now we'd like an explicit description of the lift of a modest set. That is, if we're given a modest set $Y \cong \mathbb{R}/R$ then we want to describe its lift as another modest set $Y_\perp \cong \mathbb{R}/R_\perp$ for some PER $R_\perp$ defined in terms of R.

First note that there's a predicate $\mathbb{R} \to \Sigma$ tracked by

$$\{\langle\{0\}, 0\rangle\}$$

that classifies the $\neg\neg$-closed subobject of $\mathbb{R}$ corresponding to

$$\{p \in \mathsf{R} | \, 0 \in p\} \ \subset \ \mathsf{R}.$$

Also, we'll need the fact that the operation

$$p \ \mapsto \ p - 1 \ = \ \{n - 1 | \, n \in p\}$$

is tracked by an element of R, namely

$$\{\langle\{n+1\}, n\rangle | \, n \in \mathbb{N}\};$$

so it gives rise to a map $\mathbb{R} \to \mathbb{R}$. Finally, recall that the operation of binary union is definable by an element of R.

The idea now is that $\varnothing R_\perp \varnothing$, and if pRp then $(\{0\} \cup (p+1)) R_\perp (\{0\} \cup (p+1))$. Intuitively, for a map taking values in $\mathbb{R}/R_\perp$, the result $\varnothing$ indicates divergence and 0 flags the convergence of a computation.

But again, we have to be careful to get something $\neg\neg$-closed. So if R is a PER, define a new PER $R_\perp$ by

$$pR_\perp q \quad \longleftrightarrow \quad \neg(p = q = \varnothing) \to ((p-1)R(q-1) \wedge 0 \in p \wedge 0 \in q).$$

(Observe that if R is a $\neg\neg$-closed partial equivalence relation then so is $R_\perp$.)

Note that there is a natural inclusion $\mathbb{R}/R \rightarrowtail \mathbb{R}/R_\perp$ tracked by the element

$$\{\langle\varnothing, 0\rangle\} \ \cup \ \{\langle\{n\}, n+1\rangle | n \in \mathbb{N}\} \in \mathsf{R}$$

of R.

Proposition 3.2 *If $Y \cong \mathbb{R}/R$, then $Y_\perp \cong \mathbb{R}/R_\perp$.*

Proof. First note that $Y \subset_\Sigma \mathbb{R}/R_\perp$; its classifying map is tracked by $\{\langle\{0\},0\rangle\}$.

Let $X' \subset_\Sigma X$ and suppose p tracks a map $X' \to Y$ and q tracks the classifier $X \to \Sigma$ of X'. Then the corresponding map $X \to \mathbb{R}/R_\perp$ is tracked by the following element $r \in \mathbb{R}$:

$$q \cup \{\langle\{m_1,\ldots,m_k\} \cup \{n_1,\ldots,n_l\}, r+1\rangle \mid \langle\{m_1,\ldots,m_k\},r\rangle \in p \wedge \langle\{n_1,\ldots,n_l\},0\rangle \in q\}.$$

Why? Well, suppose aRa, corresponding to an element $x \in X$; and suppose we apply r to a to get an element $b = ra \in \mathbb{R}$. The 'q' in the first half of the definition of r ensures that if $x \in X'$ then $0 \in b$; the second, more complicated, half of the definition ensures that if $x \in X'$ then $pa + 1 \subset b$. And if $\neg(x \in X')$, $b = \varnothing$.

Conversely, given a map $X \to \mathbb{R}/R_\perp$, we can obtain a partial map $X \rightharpoonup Y$ with domain a Σ-subset by pulling back along $Y \subset_\Sigma \mathbb{R}/R_\perp$. And it's straightforward to check that these operations are mutually inverse. ∎

Observe that $R_\perp$ has the property that $\varnothing R_\perp \varnothing$. We will call such PERs **canonically focal**. (This notation will be explained in the next section.) If a modest set is the quotient of such a PER, we will call it a **canonically focal** modest set. Note that the canonically focal modest sets need not be closed under isomorphism.

For some applications it is useful to define $R_\perp$ in a different, more complicated, way. We won't discuss the details here; we simply note that, as for Σ, there are many possible descriptions (see [Phoa 1991a] again).

We don't need to use the following result in this paper, but it is useful in more detailed investigations.

Proposition 3.3 *The fibration of canonically focal modest sets is closed under arbitrary products.*

Proof. Immediate from the fact that canonically focal sets form an exponential ideal and that canonically focal PERs are closed under arbitrary intersection: cf. [Hyland 1988, 2.7] or [Phoa 1991a, proposition 2.5.1]. (To see that the canonically focal sets form an exponential ideal, observe that if X is canonically focal then for any object Y, $\varnothing$ tracks a map $Y \to X$.) ∎

It follows from this that the canonically focal modest sets provide a model for the second order λ-calculus, and even the theory of constructions (as described in [Hyland and Pitts 1991], but here we can only model weak, not strong, sums of types indexed over kinds). They do not form a small complete category, of course, as they are not closed under arbitrary limits.

The following lemma is useful; its proof is left to the reader.

Lemma 3.4 *Let X and Y be modest. Any map $X \to Y_\perp$ has a canonical extension to a map $X_\perp \to Y_\perp$.*

This extension is 'strict'. To make this precise, we need to go back to look at the link relation.

3.3 The link relation vs. the inclusion relation

The inclusion relation on R gives us a $\neg\neg$-closed relation

$$\blacktriangleleft \rightarrowtail \mathbb{R} \times \mathbb{R}$$

Lemma 3.5 *This relation coincides with the link relation on* $\mathbb{R}$.

Proof. First suppose that we have $p, q \in \mathbb{R}$ with $p \triangleleft q$; that is, there exists $f\colon \Sigma \to \mathbb{R}$ with $f(\bot) = p$, $f(\top) = q$. Now f is tracked by some $a \in \mathbb{R}$; so it suffices to show that

$$\forall a \in \mathbb{R}\,((a\varnothing = p \wedge a\{0\} = q) \;\to\; p \blacktriangleleft q).$$

Note that this formula is $\neg\neg$-stable, so 'it has the same meaning as it does in $\mathcal{S}ets$' (see [Hyland 1982]). And it's certainly valid there.

Conversely, we must show that if $p \blacktriangleleft q$ then $p \triangleleft q$; that is, we must construct a map

$$\blacktriangleleft \to \mathbb{R}^{\Sigma}$$

which takes a pair $p, q \in \mathbb{R}$ with $p \blacktriangleleft q$ to a map $\Sigma \to \mathbb{R}$ linking them. It suffices to construct a suitable map

$$\blacktriangleleft \times \Sigma \to \mathbb{R}$$

Now in $\mathbb{R} \times \mathbb{R}$ we have

$$|\langle p, q\rangle| \;=\; \{2n \,|\, n \in p\} \,\cup\, \{2n+1 \,|\, n \in q\};$$

so the codes for elements of $\blacktriangleleft \rightarrowtail \mathbb{R} \times \mathbb{R}$ are r.e. sets r such that

$$2n+1 \in r \;\longrightarrow\; 2n \in r.$$

So codes for elements of $\blacktriangleleft \times \Sigma$ are r.e. sets of the form

$$\{2m \,|\, m \in r\}$$

or

$$\{2m \,|\, m \in r\} \,\cup\, \{1\};$$

(where r is an r.e. set satisfying the above condition) and the required map $\blacktriangleleft \times \Sigma \to \mathbb{R}$ is tracked by

$$\{\langle \{4n\}, n\rangle \,|\, n \in \mathbb{N}\} \,\cup\, \{\langle \{4n+2, 1\}, n\rangle \,|\, n \in \mathbb{N}\}.$$

$\blacksquare$

If $X \cong \mathbb{R}/R$ is a modest set, consider the relation $\blacktriangleleft$ on X defined by

$$x \blacktriangleleft x' \;\longleftrightarrow\; \exists p, p' \in \mathbb{R}\,([p]_R = x \wedge [p']_R = x' \wedge p \blacktriangleleft p').$$

(Note that this definition only makes sense once we have fixed a particular R such that $X \cong \mathbb{R}/R$.) By the lemma, this coincides with the link relation on X. So the link relation on a modest set has a very natural interpretation.

Observe that not all modest sets are link-ordered. For example, consider the modest set with underlying set $\{a, b, c\}$ (three arbitrary symbols) and

$$|a| = \{\{0\}\}, \quad |b| = \{\{0,1\}, \{2\}\}, \quad |c| = \{\{2,3\}\};$$

then $a \lhd b$ and $b \lhd c$ but not $a \lhd c$.

We showed in subsection 2.3 that the category of objects X for which the link relation was a partial order was a full reflective subcategory of the category $\mathcal{M}od$ of modest sets. So in fact the external category (fibration) of link-ordered objects is actually the externalization of the internal full subcategory of $\mathbb{M}\mathbf{od}$ consisting of the link-ordered modest sets. It will often be convenient to confuse the internal and external categories.

Call an object **focal** if it has a bottom element under the link relation. Again using the lemma, it's easy to see that any canonically focal object is focal. But note that, while canonically focal objects are injective with respect to Σ-subsets, there's no obvious reason why focal objects should be.

Proposition 3.6 *For any modest set X, $X_\perp$ is focal.*

Proof. Immediate from the explicit description of $X_\perp$ given above. ■

However, if X is link-ordered, there is no reason to suppose that $X_\perp$ should be: as an external fact about PERs on R, this seems obvious, but the constructive proof breaks down (try it!).

3.4 Chains and completeness

Consider the category of link-ordered objects for which the link order has sups of chains (in an internal sense—to labour the point again). Since the link order on a limit of such objects is pointwise, it's easy to see that this category is again small and complete. But this is not enough; we want to show that if X and Y are objects of this category then any map $f: X \to Y$ preserves sups of chains. And there seems to be no reason why this should be true.

To prove such a continuity principle, we need to move to a smaller category. First, let's take a closer look at what chains should be. Let ω be the R-valued assembly with underlying set $\{0, 1, 2, 3, \ldots\}$ and

$$|0| = \varnothing,\ |1| = \{1\},\ |2| = \{1,2\},\ |3| = \{1,2,3\},\ \ldots$$

This gives rise to an object ω of $\mathcal{E}$. We would like to think of this object as the 'generic chain'.

Indeed, if X is a link-ordered object, then maps $\omega \to X$ give rise to chains in X. But the converse is not true; there are 'bad' objects X in which we can find chains which do not come from maps $\omega \to X$.

For example, let X have underlying set $\{0, 1, 2, 3, \ldots\}$ and put

$$\begin{array}{rcllll} |0| & = & \{\{0\}, & \{1\}, & \{2\}, & \{3\}, \ldots\} \\ |1| & = & \{\{0,1\}, & \{1,2\}, & \{2,3\}, & \{3,4\}, \ldots\} \\ |2| & = & \{ & \{1,2,3\}, & \{2,3,4\}, & \{3,4,5\}, \ldots\} \\ |3| & = & \{ & & \{2,3,4,5\}, & \{3,4,5,6\}, \ldots\} \\ |4| & = & \{ & & & \{3,4,5,6,7\}, \ldots\} \end{array}$$

The elements of X clearly form an infinitely ascending chain; but the chains of their *codes* all terminate after a finite number, so there can be no map $\omega \to X$ corresponding to this chain.

To see how to exclude this and other pathologies, we need a few more definitions. Let $\bar{\omega}$ be the assembly with underlying set $\{0, 1, 2, 3, \ldots, \infty\}$ with $|n| = \{1, 2, \ldots, n\}$ as above and

$$|\infty| = \{1, 2, 3, \ldots\}.$$

It will be the 'generic chain with a supremum'. Also, let $\bar{\omega}^+$ be the assembly with underlying set $\{0, 1, 2, 3, \ldots, \infty, \top\}$ with $|n| = \{1, 2, \ldots, n\}$ and $|\infty| = \{1, 2, \ldots\}$ as before, and

$$|\top| = \{0, 1, 2, 3, \ldots\}.$$

It will be the 'generic chain with (a supremum and) an upper bound'.

Lemma 3.7 *Any map out of $\bar{\omega}^+$ is uniquely determined by its action on $1, 2, 3, \ldots$ and on $\top$.*

Proof. By $\neg\neg$-density, a map out of $\bar{\omega}^+$ is uniquely determined by its action on $1, 2, 3, \ldots$, ∞ and $\top$; but by continuity of application in the graph model, its action on $1, 2, 3, \ldots$ determines its action on ∞. ■

Let X be link-ordered. Call X a **uniformly complete** modest set if for every chain (x_n) in X

- there exists a map $f \in X^{\bar{\omega}^+}$ such that $f(n) = x_n$ and $f(\infty) = f(\top)$; and
- for every upper bound $x \in X$ for (x_n), there exists a map $f \in X^{\bar{\omega}^+}$ such that $f(n) = x_n$ and $f(\top) = x$.

By the lemma, these maps must necessarily be unique.

Proposition 3.8 *In the situation just described, $f(\infty) = \sup_n x_n$. Furthermore, if $g: X \to Y$ is any map from X to another uniformly complete modest set, then*

$$g(\sup_n x_n) = \sup_n g(x_n).$$

Proof. The first statement follows easily from the characterization of the link relation in terms of the inclusion relation, and the monotonicity of application in the graph model.

As for the second statement: Let the chain (x_n) give rise to a map $f\colon \bar{\omega}^+ \to X$ with $f(\infty) = f(\top)$; then

$$g(x_n) = g(f(n)) \lhd g(f(\infty)) = g(\sup_n x_n)$$

since g is monotone. So since Y is uniformly complete, there exists a unique map $h\colon \bar{\omega}^+ \to Y$ such that $h(n) = g(x_n), h(\top) = g(f(\infty))$. And

$$h(\infty) = \sup_n g(x_n).$$

But gf is such a map; so by uniqueness, $h = gf$. So

$$g(\sup_n x_n) = g(f(\infty)) = h(\infty) = \sup_n g(x_n).$$

■

In particular, any map $f\colon \omega \to X$ extends uniquely to a map $\bar{f}\colon \bar{\omega} \to X$ and $\bar{f}(\infty) = \sup_n x_n$.

Proposition 3.9 *The uniformly complete modest sets form a small complete category.*

Proof. This is just like the proof of 2.7: Let $X = \varprojlim X_i$ where the X_i are uniformly complete; we want to show that X is uniformly complete.

Suppose (x_n) is a chain in X; then it gives rise to chains (x_{in}) in each X_i. To verify the first condition: since the X_i are uniformly complete, we get corresponding maps $\bar{\omega}^+ \to X_i$; these agree on each n; they also agree on $\top$ (since the value at $\top$ is the value at ∞, which is determined by the value at each n). So by uniqueness these maps form a cone; so we get a map $\bar{\omega}^+ \to X$ as desired. The second condition is verified in a similar way. ■

An example

It might be instructive to think about what's going on in $\mathbb{R}$, the modest set corresponding to R itself. If $a \in \mathsf{R}$, we know that

$$a\{1, 2, 3, \ldots\} = \bigcup_n a\{1, 2, \ldots, n\};$$

$\mathbb{R}$ is link-ordered and chain complete, and furthermore every chain in $\mathbb{R}$ corresponds to a map $\omega \to \mathbb{R}$ (this follows from the fact that all the equivalence classes are singletons). So the 'supremum map'

$$\mathbb{R}^\omega \to \mathbb{R}$$

is tracked by the element of R denoted by

$$\lambda x.\, x\{1, 2, 3, \ldots\}.$$

Proposition 3.10 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be any map (which must be tracked by an element $p \in \mathbb{R}$). Then*

$$\begin{array}{ccc} \mathbb{R}^{\omega} & \longrightarrow & \mathbb{R} \\ f^{\omega}\downarrow & & \downarrow f \\ \mathbb{R}^{\omega} & \longrightarrow & \mathbb{R} \end{array}$$

Proof. The statement is again $\neg\neg$-stable, so it follows immediately from continuity of application in the underlying model. ∎

3.5 A category of predomains

We know that the category of uniformly complete modest sets is quite well-behaved. All we need to verify now is that it actually contains the objects we would like it to. It certainly contains $\mathbb{N}$, since the order on that is discrete; and because it's complete, it's closed under the formation of total function spaces (as well as many other things; see 3.7). To show that it's closed under forming partial function spaces—spaces of computable partial maps—we need:

Conjecture 3.11 *If X is a uniformly complete modest set, so is $X_\perp$. (And so, in fact, $X_\perp$ is a uniformly complete focal modest set.)*

As an (external) statement about ordinary PERs on R, this seems as if it ought to be immediate; but unfortunately I haven't been able to find a constructively valid proof of the corresponding statement about PERs on the object $\mathbb{R} \in \mathcal{E}$, i.e. a proof valid in $\mathcal{E}$. In fact it is not even obvious that the conjecture should be true. So we need to take a different approach.

In fact, all we want is a small complete category, all of whose objects are uniformly complete modest sets, containing $\mathbb{N}$ and closed under the lift construction. The obvious candidate is the category of replete objects; since Σ is closed under countable joins (easy to check by direct calculation), $\mathbb{N}$ is replete. So we only have to check that replete objects are uniformly complete.

But the category of uniformly complete modest sets is a small complete category. So to show that the category of replete objects will serve as a category of predomains—that is, to show that every replete modest set is complete—we only need to establish the following:

Proposition 3.12 *Σ is a uniformly complete modest set.*

The proof is by direct calculation from the definition of Σ.

Remark

Perhaps this seems like cheating; we couldn't prove something about the category we're worried about so we moved to a smaller one (just as in the previous section). But it's better to think of things the other way round: The replete objects, by definition, 'morally' have all the good properties they could possibly have; so the problem is to establish what these good properties are. What we've shown in this section is that among them are the properties of being link-ordered and uniformly complete, which is enough to justify treating the replete objects as predomains.

3.6 Least fixed points

By the usual argument, if X is link-ordered and chain-complete, and has a least element, then any map $f: X \to X$ has a least fixed point; in fact, there is a least fixed point operator $X^X \to X$.

If X is link-ordered and canonically focal, then the element $\perp \in X$ for which $\varnothing$ is a code is clearly the bottom element of X. And then we see that if X is uniformly complete any map $X \to X$ has a least fixed point.

But we might want to say a little more. Suppose $p \in \mathsf{R}$ tracks $f: X \to X$; then we get a map $\omega \to X$ tracked by

$$\{\langle\{0,\ldots,n-1\},k\rangle \mid k \in p^n\varnothing\}.$$

It would be nice if the least fixed point of f were coded by

$$\mathsf{y}p = \bigcup_n p^n\varnothing$$

(recall that $\mathsf{y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$, or indeed any fixed point combinator, denotes the *least* fixed point combinator in any graph model: see [Scott 1976] or [Barendregt 1984, theorem 19.3.4]). That is, we'd like to be able to use a single fixed point combinator to compute fixed points in any domain. This would support the idea that the PERs only provide a type discipline, and that actual computations do not depend on the PER structure.

The result would follow if the extension $\bar{\omega} \to X$ were also tracked by f; but *a priori* there's no reason why we shouldn't have to change codes in passing to the extension. We will sketch a solution to this problem; details can be found in [Phoa 1991a] (which deals with a different model, but the techniques carry over). For brevity, we'll confuse closed λ-expressions with their denotations in R.

Define

$$\ulcorner n \urcorner = \lambda f.f^n\varnothing;$$

then

$$\mathsf{y} = \bigcup_n \ulcorner n \urcorner.$$

Call a PER R s-**complete** if it satisfies the following condition: for all $p, q \in \mathbb{R}$,

$$(\forall n.\,(p\ulcorner n \urcorner)R(q\ulcorner n \urcorner)) \longrightarrow (p\mathsf{y})R(q\mathsf{y}).$$

And call a modest set s-**complete** if it's isomorphic to the quotient of $\mathbb{R}$ by a complete PER.

The full subcategory of s-complete modest sets is closed under limits; so every replete object is s-complete. It's easy to show that if $\varnothing R \varnothing$ and R is s-complete then y can compute a fixed point of any endomorphism $\mathbb{R}/R \to \mathbb{R}/R$: cf. [Phoa 1991a, proposition 3.2.3]. And one can also check that this will be the *least* fixed point, with respect to the link order.

3.7 Applications of completeness

Polymorphic types

Since the categories we have defined are small and complete, they support the Hyland-Moggi interpretation of polymorphism: we can interpret polymorphic types as products indexed by the collection of all such PERs (or whatever), which is itself an object of $\mathcal{E}$. Observe that the objects-of-objects of the categories are subobjects of the object of $\neg\neg$-closed PERs, and hence are themselves objects of $\mathcal{S}ep$. They do not lie in $\mathcal{S}ets$, however; that is, they are not $\neg\neg$-sheaves, because the predicates that define these classes of PERs are not $\neg\neg$-closed predicates on the object of $\neg\neg$-closed PERs.

Because all the results have been expressed in the internal language, they are true in an appropriately 'indexed' sense; so there is also no problem modelling dependent types and a suitable version of the theory of constructions. See [Hyland 1988] or [Hyland *et al.* 1990] for more details.

Incidentally, no attempt has been made to prove any 'polymorphic parametricity' results here. Such theorems, for general full reflective subcategories of $\mathbb{M}\mathbf{od}$, are the subject of much current research.

Domain equations

There's not much to say here. Firstly, a small complete category which is complete must also be cocomplete (see [Hyland 1988]); in particular, we may solve domain equations in our category of domains by the usual inverse limit method, as described in [Plotkin and Smyth 1982]. The proof of the limit-colimit coincidence is virtually the same as in the classical case.

Secondly, in a small complete category there must be initial algebras for all endofunctors; and these provide solutions for the corresponding domain equations: [Hyland 1988]. That is, for the case of *covariant* domain equations, solutions exist for completely abstract reasons and there is no need to resort to the inverse limit method.

Convex powerdomains

The convex powerdomain CX of a (pre)domain X should be the free semilattice on X. Now $\mathcal{E}$ has free semilattices, and the category of replete modest sets is reflective in $\mathcal{E}$; so to construct CX, we simply form the free semilattice on X in $\mathcal{E}$ and reflect back into the category of replete modest sets. More details on the construction, and its logical aspects, appear in [Phoa 1990a] and [Phoa and Taylor 1989]. However, the technique used in [Phoa and Taylor 1989] to characterize the Egli-Milner order is not applicable now; so only the easy direction is known to be valid here.

It can be shown (as in [Phoa 1990a], [Phoa and Taylor 1989]) that if X is focal then CX is focal. However, if X is canonically focal there seems to be no reason why we should be able to take CX to be canonically focal.

Upper and lower powerdomains

Whereas the convex powerdomain construction relied on the fact that the category of replete modest sets was reflective in $\mathcal{E}$, the upper and lower powerdomains depend more on the order-theoretic nature of these objects—the fact that any replete modest set can be regarded as a poset in $\mathcal{E}$.

Note that certain foundational problems are raised here. What exactly is the category of posets in $\mathcal{E}$? It could be regarded as a fibration over $\mathcal{E}$; but it may be better to think of it as an $\mathcal{E}$-enriched category. Instead of entering into a long discussion about the theory of locally internal categories, we just gloss over these technical matters; the intuition, after all, is clear.

Proposition 3.13 *The category of replete modest sets is reflective in the category of posets in $\mathcal{E}$.*

Proof. The limit of any diagram of replete modest sets (calculated in $\mathcal{E}$) has the pointwise order; this shows that the inclusion into the category of posets preserves limits. The result now follows from the adjoint functor theorem. ■

The lower and upper powerdomains, if they exist, are free meet- and join-semilattices in the category of predomains. But these free constructions certainly exist in the category of posets, by the usual construction; so again, we simply have to reflect back into the category of replete modest sets. See [Phoa 1990c] for a few more details.

Subtypes

The fact that we are working in a full reflective subcategory of $\mathbb{M}\mathbf{od}$ means that we can also model subtyping and bounded quantification—provided we choose a particular reflection functor. This is quite a general fact; see [Phoa 1992a] for details. However, it is not clear how to integrate subtyping with recursive types.

4 Final remarks

Work so far on 'synthetic domain theory' has dealt with three kinds of questions[1], namely

1. The search for particular categories of domains.
2. Categorical abstraction and axiomatisation.
3. Applications to 'real' problems in programming language theory.

This paper has concentrated mainly on the first question, by constructing domains in a particular realizability topos. I hope that the methods used shed some light on the second question, by showing how the abstract categorical view interacts with one particular model. And, as I mentioned in the introduction, the results presented here also have some bearing on purely operational questions: see [Phoa 1992b].

It's obvious that the uniformly complete modest sets—and indeed, the link-ordered objects—are not fully understood; that's why we had to resort to talking about the replete modest sets. There is a great deal of work left to do. So it wouldn't be honest to say that the theory described here is simpler than classical domain theory; rather, we have freed ourselves of the complications of traditional domain theory, but the new approach has subtleties of its own.

Hopefully, once things are properly understood, everything will work out simply. But rather than speculate along these lines, I should briefly mention some other problems that have not been dealt with here. They are currently being studied.

- Is there a 'logic of synthetic domains' that makes sense here?
- What role could 'finite' elements play in the theory, if any?
- How are our domains related to ones constructed from 'real' term models?
- Can we characterize the 'operational' properties of our domains?

Acknowledgements

The work described here was part part of a Ph.D. dissertation written under the supervision of Martin Hyland; I must thank him for being so patient and helpful. I also benefited from discussions and correspondence with many other people—too many to list here. However, the support of Mike Fourman has been especially valuable.

This paper was first presented at the Computer Science Logic conference in Heidelberg (October 1990); I am grateful to the organizers for this opportunity

[1]I am indebted to one of the referees for drawing my attention to this classification.

and to the referees for their many useful comments. In revising the paper I was partly supported by the Australian Research Council.

While preparing the final version, I was funded by the Royal Commission for the Exhibition of 1851. I must thank two anonymous referees, whose trenchant remarks I found extremely useful.

References

M. Abadi, G. Plotkin (1990) A PER model of polymorphism and recursive types, in: Proc. of 5th Annual Symposium on Logic in Computer Science.

H. P. Barendregt (1984) *The Lambda-Calculus: Its Syntax and Semantics (revised edition)*, North-Holland.

M. Beeson (1985) *Foundations of Constructive Mathematics*, Springer.

P. Freyd, P. Mulry, G. Rosolini, D. S. Scott (1990) Extensional PERs, in: Proc. of 5th Annual Symposium on Logic in Computer Science.

J. M. E. Hyland (1982) The effective topos, in: *The L. E. J. Brouwer Centenary Symposium* (eds. A. S. Troelstra, D. van Dalen), North-Holland.

— (1988) A small complete category, *Annals of Pure and Applied Logic* 40, 135-165.

— (1991) First steps towards synthetic domain theory, in: *Category Theory, Proceedings, Como 1990* (eds. A. Carboni, M. C. Pedicchio, G. Rosolini), Springer LNCS 1488.

J. M. E. Hyland, A. M. Pitts (1989) The theory of constructions, categorical semantics and topos-theoretic models, in *Categories in Computer Science and Logic (Proc. Boulder 1987)*, *Contemp. Math.* **92**, 137–199.

J. M. E. Hyland, E. P. Robinson, G. Rosolini (1990) The discrete objects in the effective topos, *Proc. Lond. Math. Soc.* (3) 60, 1–36.

P. T. Johnstone (1977) *Topos Theory*, Academic Press.

G. Longo, E. Moggi (199?) Constructive natural deduction and its "ω-set" interpretation, LIENS–90–21, October 1990; to appear in *Math. Structures in Comp. Sci.*

W. K.-S. Phoa (1990a) Effective domains and intrinsic structure, in: Proc. of 5th Annual Symposium on Logic in Computer Science.

— (1990b) *Domain Theory in Realizability Toposes*, thesis, University of Cambridge; available as CST-82-91/ECS-LFCS-91-171, Department of Computer Science, University of Edinburgh.

— (1990c) Reflectivity for categories of synthetic domains, draft.

— (1991a) From term models to domains, in the Proceedings of Theoretical Aspects of Computer Software (Sendai, 1991), Springer Lecture Notes in Computer Science 526; final version, submitted.

— (1991b) Two results on set-theoretic polymorphism, in the Proceedings of Category Theory and Computer Science (Paris, 1991), Springer Lecture Notes in Computer Science.

— (1992a) Using fibrations to understand subtypes, to appear in *Proc. of the Durham Symposium on Applications of Category Theory to Computer Science*, London Mathematical Society.

— (1992b) PCF and the untyped λ-calculus, submitted.

W. K.-S. Phoa, P. Taylor (1989) The synthetic Plotkin powerdomain, draft.

G. D. Plotkin, M. B. Smyth (1982) The category-theoretic solution of recursive domain equations, *Siam Journal of Computing* vol. 11 no. 4.

E. Robinson (1989) How complete is PER? in: Proc. of 4th Annual Symposium on Logic in Computer Science.

G. Rosolini (1987) Categories and effective computations, in: Lecture Notes in Computer Science 283, Springer.

— (199?) About modest sets, to appear.

D. S. Scott (1976) Data types as lattices, *SIAM Journal of Computing* 5 (3), 522-587.

P. Taylor (1991) The fixed point property in synthetic domain theory, in: Proc. of 6th Annual Symposium on Logic in Computer Science.