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by

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Process-Algebraic Interpretations of Positive Linear and Relevant Logics

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Abstract

We investigate the use of positive linear and relevant logics to provide logical accounts of static process structure, and combinations of relevance and modality to account also for dynamic behaviour. A general notion of model is introduced, based on which three examples are given, using Milner's synchronous process calculus SCCS. The structure of models is enriched by prefixing operators to cover also dynamic behaviour. Logically dynamic behaviour is captured by adding past and future modal operators. The resulting logic is given sound and complete axiomatisations and shown to conservatively extend the positive fragment of linear logic. Finally the induced interpretations of formulas on process terms are characterised, and axiomatisations are given which are sound and complete with respect to validity in the process-based interpretations. The completeness proofs are based on rewriting and provide procedures for deciding validity and consistency of formulas with respect to the process-based interpretations.

1 Introduction

In this paper we study interpretations of positive, propositional fragments of linear and relevant logics in terms of process algebras. The basic idea is similar in spirit to Urquhart's semilattice interpretation of relevant logics [27]: Parallel composition furnishes a binary operation $\times$ and relative to an element x the implication is interpreted as the operation that transforms properties by left multiplication with x. That is,

$$x \models \phi \to \psi \text{ iff for all } y, \text{ if } y \models \phi \text{ then } x \times y \models \psi \qquad (1)$$

Thus $\to$ expresses a relativisation of properties to properties of parallel contexts. As parallel composition is usually assumed to be commutative the restriction to

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left-multiplication is harmless. With this definition the implication can be used as a general handle to address the difficult problem of deriving compositional theories for concurrency. One rule for parallel composition is sufficient, namely:

$$\frac{x \models \phi \to \psi \quad y \models \phi}{x \times y \models \psi}$$

The problem of compositionality has thus been transformed into the problem of verifying implicative properties. As a property-transformer this implication has numerous well-known relatives in Computer Science. The classical example is the weakest preconditions of Dijstra [9]. In concurrency close relatives are the weakest inner environments of Larsen [17] and the doubly relativised turnstiles of Stirling [26].

The relation to relevant and linear logics arise in the following way: Parallel composition is usually assumed to be associative as well as commutative, and to possess an identity 1. For instance in CCS (Milner [20]) the identity is NIL; in SCCS (Milner [19]) it is 1; and in theoretical CSP (Brookes et al [6]) it is RUN for $\times$ the operator $||$ and STOP for $\times$ the operator $|||$. With this structure the implication of (1) corresponds naturally to the consequence relation $\phi_1, \ldots, \phi_n \models \psi$ that holds whenever $x_1 \models \phi_1, \ldots, x_n \models \phi_n$ implies $x_1 \times \cdots \times x_n \models \psi$ where the empty product is 1. Observe that the following structural rules are validated:

$$\text{Reflexivity:} \quad \phi \models \phi$$

$$\text{Permutation:} \quad \frac{\phi_1, \ldots, \phi_n \models \gamma}{\psi_1, \ldots, \psi_n \models \gamma} \quad (\psi_1, \ldots, \psi_n \text{ a permutation of } \phi_1, \ldots, \phi_n)$$

$$\text{Cut:} \quad \frac{\phi_1, \ldots, \phi_n \models \phi \quad \psi_1, \ldots, \psi_m, \phi, \gamma_1, \ldots, \gamma_k \models \psi}{\psi_1, \ldots, \psi_m, \phi_1, \ldots, \phi_n, \gamma_1, \ldots, \gamma_k \models \psi}$$

Moreover with the intended interpretation of $\times$ as parallel composition the following structural rules will in general fail:

$$\text{Contraction:} \quad \frac{\phi_1, \ldots, \phi_n, \phi, \phi \models \psi}{\phi_1, \ldots, \phi_n, \phi \models \psi}$$

$$\text{Weakening:} \quad \frac{\phi_1, \ldots, \phi_n \models \psi}{\phi_1, \ldots, \phi_n, \phi \models \psi}$$

so the consequence relation is indeed linear in the sense of Girard [13].

Related observations have been made in the context of Petri nets by a number of authors (Brown [7], Gehlot and Gunter [12], Marti-Oliet and Meseguer [18], Winskel and Engberg [11]). Abadi and Plotkin [1] uses an implication related to ours to account for the assumption-guarantee principle for safety properties (c.f. Pnueli [24]). Other ways of relating linear logic to concurrency have also been tried: Abramsky and Vickers [3] uses quantales, a topological variant of linear

logic, to account for notions of process testing; and Abramsky and Jagadeesan [2] uses dataflow networks to interpret proofs in linear logic.

With the interpretation (1) the intensional (or multiplicative, in Girard's terminology [13]) fragment expresses purely structural properties of processes and is thus by itself of little interest. Extensional, or additive, connectives as well as modalities in the style of Hennessy-Milner logic [15] are needed in order to capture also the dynamic behaviour of processes. In section 2 we present a generalisation of Urquhart's semilattice model which is capable of capturing a wide range of positive linear and relevant logics in a uniform way. In section 3 we give two initial examples of interpretations based on Milner's synchronous calculus SCCS [19], one of positive (multiplicative and additive) linear logic and one of the positive fragment of the standard relevant system R. A third example also based on SCCS is given in section 4. This example serves as the basis for addressing the issue of completeness in the remaining parts of the paper. It is considered both in terms of its operational semantics: transition systems under a variant of testing equivalence (De Nicola and Hennessy [22]), and in terms of its equational theory. The latter, presented in section 5, extends models in a straightforward way, and in section 6 a modal logic is introduced which is sound and complete with respect to extended models and which conservatively extends positive linear logic. In section 7 the induced interpretations on process terms is characterised and shown themselves to induce the expected behavioral preorders. In section 8 the axiomatisation is specialised to the process-based interpretations, and in section 9 these specialised axiomatisations are shown complete and decidable. The completeness proof uses a rewriting strategy, and the key normalisation Theorem is proved in section 10.

2 Models for positive relevant logics

We first consider the problem of adding extensional conjunction $\wedge$ and disjunction $\vee$. The set of *positive formulas* is defined by the abstract syntax

$$\phi ::= X \mid t \mid \phi \rightarrow \phi \mid \phi \wedge \phi \mid \phi \vee \phi$$

where X ranges over atomic propositions. The intensional connectives are the (intensional) truthhood constant t and the implication $\rightarrow$, and the extensional connectives are $\wedge$ and $\vee$. We sometimes also have occasion to consider the fusion- or times-operator $\circ$. We generally assume $\rightarrow$ to have least binding power.

For the semantics it is well known (c.f. Urquhart [27], Dunn [10]) that the standard set-theoretic interpretation of $\wedge$ as intersection and $\vee$ as union is problematic in the context of relevant logics. The ternary relation model (Routley and Meyer [25]) remedies this under the assumption, however, that $\wedge$ and $\vee$ are distributive, and this assumption fails for linear logic. The solution we propose is to add a semilattice structure, relating the semilattice operation $\sqcap$ to process-algebraic choice operators, and requiring $\sqcap$ to be preserved by multiplication and atomic propositions.

Definition 2.1 (Frame, Model). A *frame* is a structure $F = (S, \sqcap, \times, 1)$ where

1. $1 \in S$,
2. $(S, \sqcap)$ is a semilattice,
3. $(S, \times, 1)$ is a monoid,
4. $\times$ distributes over $\sqcap$. That is, $x \times (y \sqcap z) = (x \times y) \sqcap (x \times z)$ for all $x, y, z \in S$.

A set $B \subseteq S$ is a *filter*, if for all $x, y \in S$, $x, y \in B$ iff $x \sqcap y \in B$. A *model* (based on F) is a pair $M = (F, V)$ where F is a frame and V is a valuation which for each propositional letter X gives a filter $V(X)$.

Our usage of the term filter is slightly nonstandard: usually filters are assumed to be neither empty nor improper. Note that B is a filter iff

1. $x \in B$ and $x \leq y$ implies $y \in B$, and
2. $x, y \in B$ implies $x \sqcap y \in B$.

The partial ordering $\leq$ on models is derived in the usual way: $x \leq y$ iff $x \sqcap y = x$. The relation of satisfaction, $x \models_M \phi$, is defined in the following way (indexing by M is elided when M is understood from the context):

$x \models X$ iff $x \in V(X)$,

$x \models t$ iff $1 \leq x$,

$x \models \phi \wedge \psi$ iff $x \models \phi$ and $x \models \psi$,

$x \models \phi \vee \psi$ iff $x \models \phi$ or $x \models \psi$ or there are $x_1, x_2 \in S$ such that $x_1 \sqcap x_2 \leq x$, $x_1 \models \phi$ and $x_2 \models \psi$,

$x \models \phi \rightarrow \psi$ iff for all $y \in S$, $y \models \phi$ only if $x \times y \models \psi$.

The fusion connective can be added by the condition

$x \models \phi \circ \psi$ iff there are $x_1, x_2 \in S$ such that $x_1 \times x_2 \leq x$, $x_1 \models \phi$ and $x_2 \models \psi$.

The filter property for atomic propositions extends to the full languages. This property is used extensively in the proof of soundness below.

Proposition 2.2 (The Filter Property) *For all* ϕ, $\{x \in S_M \mid x \models_M \phi\}$ *is a filter.*

PROOF: An easy structural induction. For $\vee$ suppose first that $x \models \phi \vee \psi$ and $x \leq y$. If $x \models \phi$ or $x \models \psi$ then $y \models \phi \vee \psi$ by the induction hypothesis, and if there are x_1, x_2 such that $x_1 \sqcap x_2 \leq x$, $x_1 \models \phi$ and $x_2 \models \psi$ then $x_1 \sqcap x_2 \leq y$ so $y \models \phi \vee \psi$. Conversely suppose that $x, y \models \phi \vee \psi$. If $x, y \models \phi$ or $x, y \models \psi$ then $x \sqcap y \models \phi \vee \psi$ by the induction hypothesis. If $x \models \phi$ and $y \models \psi$, say, then immediately $x \sqcap y \models \phi \vee \psi$. If $x \models \phi$ and $y_1, y_2 \leq y$, $y_1 \models \phi$ and $y_2 \models \psi$ then $x \sqcap y_1 \models \phi$ by the induction hypothesis, so indeed $x \sqcap y \models \phi \vee \psi$. The other cases are similar.

For $\rightarrow$, if $x \models \phi \rightarrow \psi$, $x \leq y$, and $z \models \phi$ then $x \times z \models \psi$ so by monotonicity of $\times$ and the induction hypothesis also $y \times z \models \psi$. So $y \models \phi \rightarrow \psi$. Conversely if $x, y \models \phi \rightarrow \psi$ and $z \models \phi$ then $x \times z, y \times z \models \psi$ so also $(x \times z) \sqcap (y \times z) = (x \sqcap y) \times z \models \psi$. Thus $x \sqcap y \models \phi \rightarrow \psi$ as desired.

The remaining cases are easy exercises. □

Let $\mathcal{M}$ be a class of models. A formula ϕ is *$\mathcal{M}$-valid*, if $1 \models_M \phi$ for all $M \in \mathcal{M}$. If $\mathcal{M}$ is the class of all models, ϕ is said to be *universally valid*. The appropriate logic for axiomatising universal validity is the positive fragment of linear logic axiomatised by the following Hilbert-type system (Avron [5]):

Axioms	**I**	$\phi \rightarrow \phi$ (reflexivity)	
	B	$(\psi \rightarrow \gamma) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \gamma))$	(transitivity)
	C	$(\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow (\psi \rightarrow (\phi \rightarrow \gamma))$	(permutation)
	$\wedge$-Intro	$(\phi \rightarrow \psi) \wedge (\phi \rightarrow \gamma) \rightarrow (\phi \rightarrow \psi \wedge \gamma)$	
	$\wedge$-Elim1	$\phi \wedge \psi \rightarrow \phi$	
	$\wedge$-Elim2	$\phi \wedge \psi \rightarrow \psi$	
	$\vee$-Intro1	$\phi \rightarrow \phi \vee \psi$	
	$\vee$-Intro2	$\psi \rightarrow \phi \vee \psi$	
	$\vee$-Elim	$(\phi \rightarrow \gamma) \wedge (\psi \rightarrow \gamma) \rightarrow (\phi \vee \psi \rightarrow \gamma)$	
	t1	t	
	t2	$t \rightarrow (\phi \rightarrow \phi)$	
Rules	**Detachment**	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}$	
	Adjunction	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}$	

Fusion can be conservatively added (c.f. Avron [5]) by the axioms

o1	$\phi \rightarrow (\psi \rightarrow (\phi \circ \psi))$
o2	$((\phi \circ \psi) \rightarrow \gamma) \rightarrow (\phi \rightarrow (\psi \rightarrow \gamma))$

Let $\vdash_{LL^+} \phi$ if ϕ is provable in this system.

Theorem 2.3 (Soundness and Completeness, LL^+) $\vdash_{LL^+} \phi$ *iff* ϕ *is universally valid.*

PROOF: Soundness is proved as usual by showing the axioms valid and the rules validity preserving. Completeness is proved by a modification of the Henkin-style construction standard in relevance logic (c.f. Dunn [10]). Let an LL^+-*theory* be any set T of formulas for which

1. $\phi \in T$ and $\vdash_{LL^+} \phi \to \psi$ implies $\psi \in T$, and
2. $\phi, \psi \in T$ implies $\phi \wedge \psi \in T$.

We then define a canonical model $M(LL^+)$ by letting S be the set of all LL^+-theories, $\sqcap$ intersection, 1 the set of all LL^+-theorems, and defining the multiplication $\times$ and the valuation V by

$$\begin{aligned} T_1 \times T_2 &= \{\psi \mid \exists \phi \in T_2 . \phi \to \psi \in T_1\} \\ V(X) &= \{T \mid T \text{ an } LL^+\text{-theory and } X \in T\} \end{aligned}$$

Clearly $\sqcap$, 1 and V are well-defined. For $\times$ suppose that $\psi \in T_1 \times T_2$ and that $\vdash_{LL^+} \psi \to \gamma$. Then there is some $\phi \in T_2$ such that $\phi \to \psi \in T_1$. By **B**, **C** and detachment, $\phi \to \gamma \in T_1$ too so $\gamma \in T_1 \times T_2$. Secondly if $\psi_1, \psi_2 \in T_1 \times T_2$ then there are $\phi_1, \phi_2 \in T_2$ such that $\phi_1 \to \psi_1, \phi_2 \to \psi_2 \in T_1$. Then $\phi_1 \wedge \phi_2 \in T_2$. By $\wedge$-Elim (1 and 2), **B** and detachment, we obtain $\phi_1 \wedge \phi_2 \to \psi_1, \phi_1 \wedge \phi_2 \to \psi_2 \in T_1$, so by $\wedge$-Intro, $\phi_1 \wedge \phi_2 \to \psi_1 \wedge \psi_2 \in T_1$ too, so $\psi_1 \wedge \psi_2 \in T_1 \times T_2$ as desired.

To check the monoid properties we first prove commutativity. For this it suffices to show $\vdash_{LL^+} \phi \to ((\phi \to \psi) \to \psi)$, so that if $\phi \in T$ then $(\phi \to \psi) \to \psi \in T$ too. But $\vdash_{LL^+} (\phi \to \psi) \to (\phi \to \psi)$ so the result follows by **C** and detachment. For the identity of 1 assume first that $\psi \in 1 \times T$. Then there is a $\phi \in T$ such that $\phi \to \psi \in 1$. But then $\vdash_{LL^+} \phi \to \psi$ so $\psi \in T$ as desired. Conversely if $\phi \in T$ then by **I** also $\phi \in 1 \times T$. For associativity of $\times$ assume that $\gamma \in T_1 \times (T_2 \times T_3)$. Then there is a $\psi \in T_2 \times T_3$ such that $\psi \to \gamma \in T_1$, and thus a $\phi \in T_3$ such that $\phi \to \psi \in T_2$. By **B**, $(\phi \to \psi) \to (\phi \to \gamma) \in T_1$ so $\phi \to \gamma \in T_1 \times T_2$, and thus $\gamma \in (T_1 \times T_2) \times T_3$.

It remains to show $\times$ distributive. The containment $T_1 \times (T_2 \sqcap T_3) \subseteq (T_1 \times T_2) \sqcap (T_1 \times T_3)$ is clear. For the converse containment let $\psi \in T_1 \times T_2$ and $\psi \in T_1 \times T_3$. Then there are $\phi_2 \in T_2$ and $\phi_3 \in T_3$ such that $\phi_2 \to \psi, \phi_3 \to \psi \in T_1$. By $\vee$-Intro, $\phi_2 \vee \phi_3 \in T_2 \sqcap T_3$, and by $\vee$-Elim, $\phi_2 \vee \phi_3 \to \psi \in T_1$, giving the result.

We have thus shown the canonical model indeed to be a model. The proof is then completed by showing that $\phi \in T$ iff $T \models_{M(LL^+)} \phi$ using induction in the structure of ϕ. For atomic propositions and $\wedge$ the result is immediate. For t, if $t \in T$ and $\phi \in 1$, i.e. $\vdash_{LL^+} \phi$ then $\vdash_{LL^+} t \to \phi$ by t2, **C** and detachment, so $\phi \in T$. Thus $T \models t$. Conversely if $1 \subseteq T$ then $t \in T$ by t1.

For $\vee$ assume that $\phi \vee \psi \in T$. Let T_1 (T_2) be the least LL^+-theory containing ϕ (ψ). Note that $T_1 = \{\gamma \mid \vdash_{LL^+} \phi \to \gamma\}$ and similarly for T_2. It follows that $T_1 \sqcap T_2$

is the least LL^+-theory containing $\phi \vee \psi$, so $T_1 \sqcap T_2 \leq T$. But then $T \models \phi \vee \psi$ as by the induction hypothesis $T_1 \models \phi$ and $T_2 \models \psi$. Conversely assume that $T \models \phi \vee \psi$. If $T \models \phi$ or $T \models \psi$ we are done by the induction hypothesis, so let instead $T_1 \sqcap T_2 \leq T$, $T_1 \models \phi$ and $T_2 \models \psi$. By the induction hypothesis $\phi \in T_1$ and $\psi \in T_2$, so $\phi \vee \psi \in T_1 \sqcap T_2$ whence $\phi \vee \psi \in T$ too.

For $\rightarrow$ let $\phi \rightarrow \psi \in T$ and $T_1 \models \phi$ and we must show $T \times T_1 \models \psi$. By the induction hypothesis, $\phi \in T_1$, so $\psi \in T \times T_1$ and the result follows by the induction hypothesis. For the converse direction let $T \models \phi \rightarrow \psi$. Let T_1 be the least LL^+-theory containing ϕ. Then $T_1 \models \phi$ by the induction hypothesis, so $T \times T_1 \models \psi$. Thus $\psi \in T \times T_1$ by the induction hypothesis, and it follows that there is some $\gamma \in T_1$ such that $\gamma \rightarrow \psi \in T$. But then $\vdash_{LL^+} \phi \rightarrow \gamma$ so $\phi \rightarrow \psi \in T$ too as desired. □

Note that as LL^+-theories include the empty theory it follows from the proof of Theorem 2.3 that LL^+ is sound and complete with respect to models that contain an element 0 which is zero for both $\sqcap$ and $\times$. Other well-known relevance logics, both stronger and weaker than LL^+, are obtained by corresponding variations on frame-conditions and axiomatisations (c.f. Dam [8]). An important example is the positive fragment, R^+, of the standard relevant system R (c.f. Dunn [10]). This system is axiomatised by adding to the axioms for LL^+ the two axioms:

S $\quad (\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \gamma))$

Distribution $(\phi \vee \psi) \wedge \gamma \rightarrow (\phi \wedge \gamma) \vee (\psi \wedge \gamma)$

For the semantics an R^+-*frame* is a frame F with the following two properties:

1. For all $x \in S_F$, $x \times x \leq x$,
2. Whenever $x \sqcap y \leq z$, $x \not\leq z$ and $y \not\leq z$ then there are $x' \geq x$ and $y' \geq y$ such that $x' \sqcap y' = z$.

Condition 1 is referred to as *semi-idempotency*. Condition 2 is very close to the standard notion of distributivity in semilattices: Whenever $x \sqcap y \leq z$ then there are $x' \geq x$ and $y' \geq y$ such that $x' \sqcap y' = z$, regardless of whether $x \leq z$ or $y \leq z$ or not. For unital semilattices in particular, the definitions coincide. An R^+-*model* is a model which is based on an R^+-frame.

Theorem 2.4 (Soundness and Completeness, R^+) $\vdash_{R^+} \phi$ *iff* ϕ *is* $\mathcal{M}$*-valid where* $\mathcal{M}$ *is the class of all* R^+*-models.*

PROOF: The soundness of **S** and Distribution is proved as usual. For completeness all that is needed is to check that the canonical model $M(R^+)$ constructed as in the proof of Theorem 2.3 validates the two extra model conditions.

For semi-idempotency let T be an R^+-theory and $\psi \in T \times T$. Then there is some $\phi \in T$ such that $\phi \rightarrow \psi \in T$ too. The following derivation shows that $\vdash_{R^+} (\phi \rightarrow \psi) \wedge \phi \rightarrow \psi$ which is sufficient to establish the result:

1. $(\phi \to \psi) \wedge \phi \to (\phi \to \psi)$ $\wedge$-Elim1
2. $((\phi \to \psi) \wedge \phi \to \phi) \to ((\phi \to \psi) \wedge \phi \to \psi)$ 1, by **S**
3. $(\phi \to \psi) \wedge \phi \to \psi$ 2, by $\wedge$-Elim2

For distributivity let $T_1 \sqcap T_2 \leq T$, $T_1 \not\leq T$ and $T_2 \not\leq T$. Let T_1' (T_2') be the least R^+-theory containing $T_1 \cup T$ ($T_2 \cup T$). We must show that $T_1' \sqcap T_2' \subseteq T$. If $\phi \in T_1' \sqcap T_2'$ then there are $\phi_1 \in T_1'$ and $\phi_2 \in T_2'$ such that $\vdash_{R+} \phi_1 \vee \phi_2 \to \phi$. If $\phi_i \in T_i'$, $i \in \{1,2\}$ then there is some $\psi_i \in T_1$ and $\gamma_i \in T$ such that $\vdash_{R+} \psi_i \wedge \gamma_i \to \phi_i$. Let $\gamma = \gamma_1 \wedge \gamma_2$. It follows that $\vdash_{R+} (\psi_1 \wedge \gamma) \vee (\psi_2 \wedge \gamma) \to \phi$. But then by Distribution also $\vdash_{R+} (\psi_1 \vee \psi_2) \wedge \gamma \to \phi$. But both $\psi_1 \vee \psi_2$ and γ are in T so $\phi \in T$ too. □

3 Synchronous processes as models, I

In this section we give two examples of models based on a fragment of Milner's SCCS [19] under simulation and bisimulation equivalence (c.f. Hennessy and Milner [15]). The fragment involved contains synchronous parallel composition together with choice, prefixing and a unit process. Terms $p \in P^+$ in this fragment are given by the following abstract syntax:

$$p ::= 1 \mid a.p \mid p + p \mid p \times p$$

where a ranges over an abelian group L of *labels*. We assume generally that L is infinite and use a multiplicative notation for the group operations: $\cdot$ is label multiplication and e is the unit. The operational semantics of process terms is given by the transition relation $\xrightarrow{a}$ determined by the following axioms and rules:

$$1 \xrightarrow{e} 1 \qquad a.p \xrightarrow{a} p$$

$$\frac{p \xrightarrow{a} p'}{p + q \xrightarrow{a} p'} \qquad \frac{q \xrightarrow{a} p'}{p + q \xrightarrow{a} p'} \qquad \frac{p \xrightarrow{a} p' \quad q \xrightarrow{b} q'}{p \times q \xrightarrow{a \cdot b} p' \times q'}$$

Models are constructed from terms by quotienting under suitable behavioral congruence relations.

Definition 3.1 (Simulation, Bisimulation) A binary relation R on process terms is a *simulation* [1], if pRq implies

1. whenever $q \xrightarrow{a} q'$ then $p \xrightarrow{a} p'$ and $p' R q'$ for some term p'.

If whenever pRq then (1) holds and in addition its converse

2. whenever $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ and $p' R q'$ for some term q',

[1] In fact, according to [15] a *reverse* simulation.

then R is a *bisimulation.* If there is a simulation (bisimulation) R such that pRq then p *simulates* q, $p \sqsubseteq_s q$ (p and q are *bisimulation equivalent*, $p \simeq_b q$). If $p \sqsubseteq_s q$ and $q \sqsubseteq_s p$ then p and q are *simulation equivalent*, $p \simeq_s q$.

It is well known that bisimulation equivalence is strictly finer than simulation equivalence [15]. It is not difficult to verify that $\sqsubseteq_s$ is a precongruence and $\simeq_b$ a congruence with respect to the operations on terms. For $\simeq$ one of $\simeq_s$, $\simeq_b$ we can then form the quotient structure $P^+/\simeq$ in the obvious way by letting

$$\begin{aligned} [p]_\simeq \sqcap [q]_\simeq &= [p+q]_\simeq \\ [p]_\simeq \times [q]_\simeq &= [p \times q]_\simeq \\ 1 &= [1]_\simeq \end{aligned}$$

Theorem 3.2 *1. $P^+/\simeq_b$ is a frame.*

2. $P^+/\simeq_s$ is an R^+-frame when label multiplication is idempotent.

PROOF: Most of the checks involved are standard. The only exceptions are semi-idempotency and distributivity for $P^+/\simeq_s$. For semi-idempotency it suffices to show that $\{(p \times p, p) \mid p \in P^+\}$ is a simulation: If $p \xrightarrow{a} p'$ then $p \times p \xrightarrow{a} p' \times p'$ by idempotency of label multiplication.

For distributivity note first that $\sqsubseteq_s$ coincides with the induced semilattice ordering, for $p + q \sqsubseteq_s p$ holds always, and $p \sqsubseteq_s p + q$ iff $p \sqsubseteq_s q$. Let then $\mathrm{init}(p) = \{a \mid \exists p'.p \xrightarrow{a} p'\}$ and $p/a = \{p' \mid p \xrightarrow{a} p'\}$. Assume that $p+q \sqsubseteq_s r$, $p \not\sqsubseteq_s r$ and $q \not\sqsubseteq_s r$. Then

1. either $\mathrm{init}(r) \not\subseteq \mathrm{init}(p)$ or $\mathrm{init}(r) \subseteq \mathrm{init}(p)$ and there is some $a \in \mathrm{init}(r)$ and $r' \in r/a$ such that for all $p' \in p/a$, $p' \not\sqsubseteq_s r'$, and

2. the same for q.

Then $\mathrm{init}(r) \cap \mathrm{init}(p) \neq \emptyset$ and $\mathrm{init}(r) \cap \mathrm{init}(q) \neq \emptyset$, for $\mathrm{init}(r) \neq \emptyset$ and if for instance $\mathrm{init}(r) \cap \mathrm{init}(p) = \emptyset$ then $q \sqsubseteq_s r$. By the semilattice properties of $+$ we can use the $\sum$-notation for finite, nonempty sums. Let now for $a \in \mathrm{init}(p) \cap \mathrm{init}(r)$

$$p_a = \sum\{a.r' \mid r' \in r/a \text{ and for some } p'' \in p/a,\ p'' \sqsubseteq_s r'\}$$

and then $p' = \sum_{a \in \mathrm{init}(p) \cap \mathrm{init}(r)} p_a$. Define q' similarly. Clearly the sums involved are finite. Furthermore assume that for all $a \in \mathrm{init}(p) \cap \mathrm{init}(r)$ and for all $r' \in r/a$ there is no $p'' \in p/a$ such that $p'' \sqsubseteq_s r'$. Then, as $p+q \sqsubseteq_s r$, $\mathrm{init}(p) \cap \mathrm{init}(r) \subseteq \mathrm{init}(q)$ and for all $r' \in r/a$ there is some $q' \in q/a$ s.t. $q' \sqsubseteq_s r'$. Moreover, whenever $a \in \mathrm{init}(r) \setminus \mathrm{init}(p)$, $a \in \mathrm{init}(q)$ and the same holds, as $p + q \sqsubseteq_s r$. But then $q \sqsubseteq_s r$—a contradiction. Hence the sums are also nonempty, and p', q' are well-defined. Clearly $p \sqsubseteq_s p'$ and $q \sqsubseteq_s q'$. Also $p' + q' \simeq_s r$, for if $r \xrightarrow{a} r'$ then $p' + q' \xrightarrow{a} r'$ and if $p' + q' \xrightarrow{a} r'$ then $r \xrightarrow{a} r'$. □

In Theorem 3.2.2 it suffices to assume that the label structure forms a semilattice as there is no need for inverses. In the context of process algebra the

assumption of idempotency of label multiplication is then often realistic: It is appropriate for instance for multiway synchronisation.

The frame structures of $P^+/\simeq_b$ and $P^+/\simeq_s$ gives rise to natural interpretations of positive formulas as in section 2. Modalities to take account of dynamic behaviour can be added in the style of Hennessy-Milner logic [15] in the following way: For the case of simulation we can add modalities $[a]\phi$ with the interpretation:

$[p] \models [a]\phi$ iff for all p' such that $p \xrightarrow{a} p'$, $[p'] \models \phi$.

It is easy to verify that this definition is good, and that the filter property is maintained. For bisimulation we can add also the dual operator $<a>\phi$:

$[p] \models <a>\phi$ iff for some p', $p \xrightarrow{a} p'$ and $[p'] \models \phi$

provided that all occurrences of $<a>$ is in the scope of some $[a']\phi$.

4 Synchronous processes as models, II

The connection to linear and relevant logics established by results such as Theorem 3.2 is a very weak one: It only establishes soundness of the induced interpretations. In this section we introduce an example for which completeness can be established too. Thus this gives a technically precise sense in which linear logic is *exactly* the logic of static process structure.

The operational setting is a variation on that of the previous section. Instead of the sum-operator $+$ we allow the formation of a finite set P of process terms as a process term itself. The intended meaning of set formation is as an internal, or uncontrollable choice operator in contrast to the controllable choice involved in $+$, and we use $\oplus$ as its binary correlate (c.f. Hennessy [14]). Furthermore a constant 0 for deadlock is added. The set of process terms thus obtained is denoted by $P^{\oplus}$. A structured operational semantics in the style of CCS and SCCS can be given (c.f. Dam [8]). Here, however, we prefer a style akin to that of Hennessy and Plotkin [16]. The relations p may Λ and p must Λ where Λ is a finite set of labels, and the successor operations p after a, are defined inductively as follows:

may:

$$\frac{e \in \Lambda}{1 \text{ may } \Lambda} \qquad \frac{a \in \Lambda}{a.p \text{ may } \Lambda}$$

$$\frac{p \text{ may } \Lambda \quad p \in P}{P \text{ may } \Lambda} \qquad \frac{p \text{ may } \Lambda_1 \qquad q \text{ may } \Lambda_2}{p \times q \text{ may } \{a \cdot b \mid a \in \Lambda_1, b \in \Lambda_2\}}$$

must:

$$\frac{e \in \Lambda}{1 \text{ must } \Lambda} \qquad \frac{a \in \Lambda}{a.p \text{ must } \Lambda}$$

$$\frac{p_1 \text{ must } \Lambda_1 \quad \cdots \quad p_n \text{ must } \Lambda_n}{\{p_1, \ldots, p_n\} \text{ must } \Lambda_1 \cup \cdots \cup \Lambda_n} \qquad \frac{p \text{ must } \Lambda_1 \qquad q \text{ must } \Lambda_2}{p \times q \text{ must } \{a \cdot b \mid a \in \Lambda_1, b \in \Lambda_2\}}$$

after:

0 after $a = \emptyset$

$$1 \text{ after } a = \begin{cases} \{1\} & \text{if } a = e \\ \emptyset & \text{otherwise} \end{cases}$$

$$a.p \text{ after } b = \begin{cases} \{p\} & \text{if } a = b \\ \emptyset & \text{otherwise} \end{cases}$$

P after $a = \bigcup_{p \in P} p$ after a

$p \times q$ after $a = \bigcup_{(a_1,a_2):a_1 \cdot a_2 = a} \{p' \times q' \mid p' \in p \text{ after } a_1, q' \in q \text{ after } a_2\}$

For notational convenience we derive the relation "can" by p can a iff p may $\{a\}$ and the predicate "live" by p live iff p must Λ for some set Λ. Two behavioral preorders on processes are considered. The first, $\sqsubseteq_1$, emphasizes safety properties, and the second, $\sqsubseteq_2$, emphasizes liveness properties, viewing liveness as the absence of deadlock.

Definition 4.1 (Behavioral Preorders $\sqsubseteq_1$ and $\sqsubseteq_2$)

1. The preorder $\sqsubseteq_1$ on process terms is the largest (under containment) for which $p \sqsubseteq_1 q$ implies

   a. for all labels a, if p can a then q can a and p after $a \sqsubseteq_1 q$ after a.

2. The preorder $\sqsubseteq_2$ is the largest for which $p \sqsubseteq_2 q$ implies

   a. for all Λ, if p must Λ then q must Λ, and

   b. for all a, if p live and q can a then p can a and p after $a \sqsubseteq_2 q$ after a.

3. For $i \in \{1,2\}$, $p \simeq_i q$ iff $p \sqsubseteq_i q$ and $q \sqsubseteq_i p$.

Both preorders can be characterised as testing preorders along the lines of De Nicola and Hennessy [22] (Dam [8]). The preorder $\sqsubseteq_1$ is the familiar language containment preorder, and the preorder $\sqsubseteq_2$ is related to the bisimulation preorders considered e.g. by Milner [21]. Note that for the present language condition 4.1.2.a is a consequence of 4.1.2.b. To see this note that p must Λ iff p live and $\{a \mid p \text{ can } a\} \subseteq \Lambda$. It is not hard to verify that both $\sqsubseteq_1$ and $\sqsubseteq_2$ are precongruences with respect to the operations on terms, and the quotient structures $P^{\oplus}/\simeq_i$ are then formed as in section 3 by associating to $\sqcap$ the internal choice operator $\oplus$.

Theorem 4.2 *For $i \in \{1,2\}$, $P^{\oplus}/\simeq_i$ is a frame.*

PROOF: A consequence of the Algebraic Characterisation Theorem 5.6 below. □

Note that for the case $i = 1$, the preorder $\sqsubseteq_1$ corresponds to the converse of the induced semilattice ordering.

5 Synchronous algebras

In this section we extend the notion of frame to account more fully for the static and dynamic behaviour of processes, and arrive at the following equational presentation of processes:

Definition 5.1 (Synchronous Algebras) A *synchronous algebra* (over a given label group L) is a structure $A = (S, \sqcap, 0, \times, 1, L)$ where

1. $(S, \sqcap, \times, 1)$ is a frame, and
2. the following equations hold for all $x, y \in S$ and labels $a, b \in L$:
   a. $x \times 0 = 0$
   b. $a.(x \sqcap y) = (a.x) \sqcap (a.y)$
   c. $(a.x) \times (b.y) = (a \cdot b).(x \times y)$
   d. $e.1 = 1$

If in addition 0 is greatest with respect to the induced semilattice ordering $\leq$ then A is a *safety*, or *type 1 synchronous algebra*, and if 0 is least with respect to $\leq$ then A is a *liveness*, or *type 2 synchronous algebra.*

It is not hard to verify that $P^{\oplus}/\simeq_i$ forms a type i synchronous algebra for both $i = 1$ and $i = 2$. We go on to show that safety algebras characterise processes under $\sqsupseteq_1$, and that liveness algebras similarly characterise processes under $\sqsubseteq_2$. First representation theorems for the initial algebras are proved. In view of the uncontrollable nature of $\oplus$ it is natural to expect members of the initial algebras to be represented as appropriately closed sets of strings of labels.

Definition 5.2 (Paths, Normal Paths) Assume that L and $\{0, 1\}$ are disjoint.

1. A *path* σ is a member of $L^* \cdot \{0, 1\}$. A path σ is *normal* if $e1$ is not a suffix of σ.
2. If $\sigma = \alpha j$, $\alpha \in L^*$ and $j \in \{0, 1\}$, then $\mathrm{pre}(\sigma) = \alpha$ and $\mathrm{suf}(\sigma) = j$.
3. Normal paths are ordered by $\sigma_1 \leq \sigma_2$ iff either
   a. $\sigma_1 = \sigma_2$,
   b. $\mathrm{suf}(\sigma_1) = \mathrm{suf}(\sigma_2) = 0$ and $\mathrm{pre}(\sigma_1)$ is a prefix of $\mathrm{pre}(\sigma_2)$, or
   c. $\mathrm{suf}(\sigma_1) = 0$, $\mathrm{suf}(\sigma_2) = 1$, and $\mathrm{pre}(\sigma_1)$ is a prefix of $\mathrm{pre}(\sigma_2)(e^n)$ for some $n \geq 0$.

A set of paths Σ is *normal*, if all $\sigma \in \Sigma$ are normal. Below Σ is assumed to range over normal sets. For the initial safety algebra elements are represented by downwards closed normal sets, and for the initial liveness algebra by upwards closed normal sets. A set Σ is *downwards* or *1-closed* if whenever $\sigma \in \Sigma$ and $\sigma' \leq \sigma$ then $\sigma' \in \Sigma$. Dually Σ is *upwards* or *2-closed* if whenever $\sigma \in \Sigma$ and $\sigma \leq \sigma'$ then $\sigma' \in \Sigma$. For $i \in \{1,2\}$ the i-closure of a set Σ is denoted $\mathrm{cl}_i(\Sigma)$. If $\Sigma = \mathrm{cl}_i(\Sigma')$ for some finite set Σ' then Σ is *i-finitely generated* (i-f.g.). If Σ is i-f.g. then there is a least set Σ' generating Σ. The representations of the initial algebras are built using nonempty, closed, and f.g. sets Σ.

Next the operations on paths and normal sets are defined. Path prefixing is defined by $a.\sigma = a\sigma$ whenever either $a \neq e$ or $\sigma \neq 1$, and $e.1 = 1$. Multiplication $\times$ of paths is defined inductively by letting 0 be zero and 1 be unit for $\times$, and then $a_1\sigma_1 \times a_2\sigma_2 = (a_1 \cdot a_2).(\sigma_1 \times \sigma_2)$. The constants and operations on sets are given by

$$0_1 = \{0\},\ 0_2 = \{\sigma \mid \sigma \text{ a normal path}\}$$

$$1_1 = \{1\} \cup \{e^n 0 \mid n \in \omega\},\ 1_2 = \{1\}$$

$$a.(\Sigma)_1 = \{a.\sigma \mid \sigma \in \Sigma\} \cup \{0\},\ a.(\Sigma)_2 = \{a.\sigma \mid \sigma \in \Sigma\}$$

$$\Sigma_1 \oplus_i \Sigma_2 = \Sigma_1 \cup \Sigma_2$$

$$\Sigma_1 \times_i \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$$

Let then D_i, $i \in \{1,2\}$, be the algebra obtained by taking the set of all i-f.g., i-closed and nonempty normal sets Σ together with the constants and operations as just defined. Note that the induced semilattice ordering is $\supseteq$ for both $i = 1$ and $i = 2$. In the safety case this corresponds to the converse of the well-known Hoare-ordering: $\Sigma_1 \supseteq \Sigma_2$ iff for all $\sigma_2 \in \Sigma_2$ there is a $\sigma_1 \in \Sigma_1$ such that $\sigma_2 \leq \sigma_1$; and in the liveness case to the Smyth-ordering: $\Sigma_1 \supseteq \Sigma_2$ iff for all $\sigma_2 \in \Sigma_2$ there is a $\sigma_1 \in \Sigma_1$ such that $\sigma_1 \leq \sigma_2$.

Theorem 5.3 (Representation of Initial Algebras) *For $i \in \{1,2\}$, D_i is (up to isomorphism) the initial type i synchronous algebra.*

PROOF: It is very easy to verify for both $i = 1$ and $i = 2$ that indeed D_i is a type i synchronous algebra. To prove the result we then need for every type i synchronous algebra A to establish a unique homomorphism $f : D_i \to A$.

First, let gen_i denote the operator that given each i-closed set Σ gives its least generating set. The following equations hold:

1. $a.(\Sigma)_i = \mathrm{cl}_i\{a.\sigma \mid \sigma \in \mathrm{gen}_i(\Sigma)\}$
2. $\Sigma_1 \oplus_i \Sigma_2 = \mathrm{cl}_i(\mathrm{gen}_i(\Sigma_1) \cup \mathrm{gen}_i(\Sigma_2))$
3. $\Sigma_1 \times_i \Sigma_2 = \mathrm{cl}_i\{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \mathrm{gen}_i(\Sigma_1), \sigma_2 \in \mathrm{gen}_i(\Sigma_2)\}$

Note that any map $f : D_i \to A$ determines a map $f^\dagger$ from finite, nonempty sets of normal paths to A, defined by

$$\begin{aligned} f^\dagger(\{\sigma_1, \ldots, \sigma_n\}) &= f(\mathrm{cl}_i\{\sigma_1, \ldots, \sigma_n\}) \\ &= f(\mathrm{cl}_i\{\sigma_1\} \oplus_i \cdots \oplus_i \mathrm{cl}_i\{\sigma_n\}) \end{aligned}$$

for $n \geq 1$. Further, f is a homomorphism iff $f^\dagger$ satisfies

i. $f^\dagger\{\sigma_1, \ldots, \sigma_n\} = f^\dagger\{\sigma_1\} \oplus_A \cdots \oplus_A f^\dagger\{\sigma_n\}$, $n \geq 1$,

ii. $f^\dagger\{0\} = 0_A$,

iii. $f^\dagger\{1\} = 1_A$,

iv. $f^\dagger\{a\sigma\} = a.(f^\dagger\{\sigma\})_A$,

and any such $f^\dagger$ determines f. The only-if direction is straightforward, and clearly conditions i.–iv. defines $f^\dagger$, so if f is a homomorphism it is also unique. It remains to check existence. Note that $f^\dagger$ has the properties

a. $f^\dagger\{a.\sigma\} = a.(f^\dagger\{\sigma\})_A$,

b. $f^\dagger\{\Sigma\} = \sum\!\!\!\!\!\!\sum_A \{f^\dagger\{\sigma\} \mid \sigma \in \Sigma\}$, for Σ finite,

c. $f^\dagger\{\sigma_1 \cdot \sigma_2\} = f^\dagger\{\sigma_1\} \times_A f^\dagger\{\sigma_2\}$.

In b. $\sum\!\!\!\!\!\!\sum$ denotes the finite internal sum operator. There is now little difficulty in verifying the homomorphism properties of f. First $f(0_i) = f^\dagger\{0\} = 0_A$, and $f(1_i) = f^\dagger\{1\} = 1_A$. Next

$$\begin{aligned} f(a.(\Sigma)_i) &= f^\dagger(\mathrm{gen}_i(a.(\Sigma)_i)) \\ &= f^\dagger\{a.\sigma \mid \sigma \in \mathrm{gen}_i(\Sigma)\} \quad \text{(by 1.)} \\ &= \sum\!\!\!\!\!\!\sum_A \{f^\dagger\{a.\sigma\} \mid \sigma \in \mathrm{gen}_i(\Sigma)\} \quad \text{(by b.)} \\ &= \sum\!\!\!\!\!\!\sum_A \{a.(f^\dagger\{\sigma\})_A \mid \sigma \in \mathrm{gen}_i(\Sigma)\} \quad \text{(by a.)} \\ &= a.(\sum\!\!\!\!\!\!\sum_A \{f^\dagger\{\sigma\} \mid \sigma \in \mathrm{gen}_i(\Sigma)\})_A \quad \text{(by equational reasoning)} \\ &= a.(f^\dagger(\mathrm{gen}_i(\Sigma)))_A \quad \text{(by b.)} \\ &= a.(f(\Sigma))_A \end{aligned}$$

For the internal sum operator:

$$\begin{aligned} f(\Sigma_1 \oplus_i \Sigma_2) &= f^\dagger(\mathrm{gen}_i(\Sigma_1 \oplus_i \Sigma_2)) \\ &= f^\dagger(\mathrm{gen}_i(\Sigma_1) \cup \mathrm{gen}_i(\Sigma_2)) \quad \text{(by 2.)} \\ &= f^\dagger(\mathrm{gen}_i(\Sigma_1)) \oplus_A f^\dagger(\mathrm{gen}_i(\Sigma_2)) \quad \text{(by b.)} \\ &= f(\Sigma_1) \oplus_A f(\Sigma_2) \end{aligned}$$

Finally for parallel composition:

$$\begin{aligned}
f(\Sigma_1 \times_i \Sigma_2) &= f^\dagger(\mathrm{gen}_i(\Sigma_1 \times_i \Sigma_2)) \\
&= f^\dagger(\mathrm{gen}_i(\Sigma_1) \times \mathrm{gen}_i(\Sigma_2)) \quad \text{(by 3.)} \\
&= \Sigma_A\{f^\dagger\{\sigma_1 \cdot \sigma_2\} \mid \sigma_1 \in \mathrm{gen}_i(\Sigma_1), \sigma_2 \in \mathrm{gen}_i(\Sigma_2)\} \quad \text{(by b.)} \\
&= \Sigma_A\{f^\dagger\{\sigma_1\} \times_A f^\dagger\{\sigma_2\} \mid \sigma_1 \in \mathrm{gen}_i(\Sigma_1), \sigma_2 \in \mathrm{gen}_i(\Sigma_2)\} \quad \text{(by c.)} \\
&= (f^\dagger(\mathrm{gen}_i(\Sigma_1))) \times_A (f^\dagger(\mathrm{gen}_i(\Sigma_2))) \quad \text{(by equational reasoning)} \\
&= f(\Sigma_1) \times_A f(\Sigma_2).
\end{aligned}$$

The check that f is monotone is straightforward. We have thus established the homomorphism property of f, and the proof is complete. □

As $P^\oplus$ is a term algebra there are unique homomorphisms $[\![\cdot]\!]_i$ from $P^\oplus$ to D_i for $i \in \{1,2\}$. These homomorphisms are used to produce isomorphisms between $P^\oplus/\simeq_i$ and D_i thus establishing the Algebraic Characterisation Theorem below. For this purpose it suffices to show that $[\![\cdot]\!]_i$ is *fully abstract*, meaning for $i = 1$ that $p \sqsubseteq_1 q$ iff $[\![p]\!]_1 \subseteq [\![q]\!]_1$, and for $i = 2$ that $p \sqsubseteq_2 q$ iff $[\![p]\!]_2 \supseteq [\![q]\!]_2$. To prove full abstraction the operational structure of processes is mimicked using the representations D_i. For a set Σ define

Σ may Λ iff $\sigma = a.\sigma'$ for some $\sigma \in \Sigma$, $a \in \Lambda$, and path σ',

Σ must Λ iff for all $\sigma \in \Sigma$ there is some $a \in \Lambda$ and path σ' such that $\sigma = a.\sigma'$,

σ after $a = \{\sigma' \mid \exists \sigma \in \Sigma.\ \sigma = a.\sigma'\}$.

The following lemma relates the operational structure of terms and that of their representations.

Lemma 5.4 *1. p may Λ iff $[\![p]\!]_1$ may Λ*

2. If p can a then $[\![p$ after $a]\!]_1 = [\![p]\!]_1$ after a

3. p must Λ iff $[\![p]\!]_2$ must Λ

4. If p live and p can a then $[\![p$ after $a]\!]_2 = [\![p]\!]_2$ after a

PROOF: All four statements are proved by an essentially straightforward structural induction. For instance for 4 assume that p can a and that p live. Thus $p \neq 1$. For the remaining cases we calculate:

$$\begin{aligned}
[\![1 \text{ after } a]\!]_2 &= [\![1]\!]_2 && \text{as } p \text{ can } a \text{ iff } a = 1 \\
&= [\![1]\!]_2 \text{ after } a \\
[\![b.p \text{ after } a]\!]_2 &= [\![p]\!]_2 && \text{as } b.p \text{ can } a \text{ iff } a = b \\
&= [\![b.p]\!]_2 \text{ after } a
\end{aligned}$$

$$
\begin{aligned}
[\![p \oplus q \text{ after } a]\!]_2 &= [\![(p \text{ after } a) \cup (q \text{ after } a)]\!]_2 \\
&= [\![p \text{ after } a]\!]_2 \cup [\![q \text{ after } a]\!]_2
\end{aligned}
$$

$$
\begin{aligned}
&[\![p \text{ after } a]\!]_2 \\
&\quad = [\![\cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{p_1' \times p_2' \mid p_1' \in p_1 \text{ after } a_1, p_2' \in p_2 \text{ after } a_2\}]\!]_2 \\
&\quad = \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{[\![p_1' \times p_2']\!]_2 \mid p_1' \in p_1 \text{ after } a_1, p_2' \in p_2 \text{ after } a_2\}_2 \\
&\quad = \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{[\![p_1']\!]_2 \times [\![p_2']\!]_2 \mid p_1' \in p_1 \text{ after } a_1, p_2' \in p_2 \text{ after } a_2\}_2 \\
&\quad = \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{[\![p_1 \text{ after } a_1]\!]_2 \times [\![p_2 \text{ after } a_2]\!]_2\} \\
&\quad = \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{[\![p_1]\!]_2 \text{ after } a_1 \times [\![p_2]\!]_2 \text{ after } a_2\} \\
&\quad = [\![p]\!]_2 \text{ after } a
\end{aligned}
$$

□

It thus remains to prove that the behavioral preorders $\sqsubseteq_i$ on terms induce the appropriate orderings $\subseteq$, $\supseteq$ on the representations. This is done in two steps, using the "may", "must", and "after" relations on D_i to induce orderings $\sqsubseteq_i$ on D_i as in Definition 4.1.

Lemma 5.5 (Full Abstraction)

1. $p \sqsubseteq_1 q$ *iff* $[\![p]\!]_1 \subseteq [\![q]\!]_1$,
2. $p \sqsubseteq_2 q$ *iff* $[\![p]\!]_2 \supseteq [\![q]\!]_2$,

PROOF: Note first that $p \sqsubseteq_i q$ iff $[\![p]\!]_i \sqsubseteq_i [\![q]\!]_i$ by Lemma 5.4. For 1 it remains to show that $\Sigma_1 \sqsubseteq_1 \Sigma_2$ iff $\Sigma_1 \subseteq \Sigma_2$ where Σ_1, Σ_2 are 1-closed, and for 2 that $\Sigma_1 \sqsubseteq_2 \Sigma_2$ iff $\Sigma_1 \supseteq \Sigma_2$ where Σ_1, Σ_2 are 2-closed.
1. Assume first that $\Sigma_1 \sqsubseteq_1 \Sigma_2$ and that $\sigma_1 \in \Sigma_1$. If $\sigma_1 = 0$ then $\sigma_1 \in \Sigma_2$ as Σ_2 is 1-closed. If $\sigma_1 = 1$ then $\sigma_1 \in \Sigma_2$ as otherwise either Σ can e fails or else there will, as Σ_1 is 1-f.g., be a maximum n such that $(\Sigma_1 \text{ after } e^n)$ can e. Here the "after"-operation is extended to finite strings in the obvious way by $\Sigma \text{ after } (a_1 \cdots a_n) = ((\cdots(\Sigma \text{ after } a_1)\cdots) \text{ after } a_n)$. If $\sigma_1 = a\sigma_1'$ then Σ_1 can a so also Σ_2 can a whence by the induction hypothesis, $\sigma_1' \in \Sigma_2$ after a. But then $\sigma_1 \in \Sigma_2$ and we are done. For the converse implication is suffices to show that $\subseteq$ satisfies the conditions of Definition 4.1.1. That is, if $\Sigma_1 \subseteq \Sigma_2$ and Σ_1 can a then Σ_2 can a and Σ_1 after $a \subseteq \Sigma_2$ after a.
2. Suppose $\Sigma_1 \sqsubseteq_2 \Sigma_2$ and that $\sigma_2 \in \Sigma_2$. If $\sigma_2 = 0$ then Σ_2 is not live so neither is Σ_1 whence $\sigma_2 \in \Sigma_1$. Suppose $\sigma_2 = 1$. If $1 \notin \Sigma_1$ then either Σ_1 can e fails or else there is a maximal n such that $(\Sigma_1 \text{ after } e^n)$ can e. The first case leads to a contradiction whether Σ_1 live or not. For the second case, if for some $m \leq n$, Σ_1 after e^m is not live then $1 \in \Sigma_1$ as Σ_1 is 2-closed. Otherwise Σ_1 after $e^{n+1} \sqsubseteq_2 \Sigma_2$ after e^{n+1} but $(\Sigma_2 \text{ after } e^{n+1})$ can e which fails for Σ_1, a contradiction. Suppose finally that $\sigma_2 = a\sigma_2'$. If Σ_1 is not live then $\sigma_2 \in \Sigma_1$. If Σ_1 is live, as Σ_2 can a then Σ_1 can a and $\sigma_2' \in \Sigma_1$ after a by the induction hypothesis. But then $\sigma_2 \in \Sigma_1$ as desired. The converse implication is a straightforward check that the conditions of Def. 4.1.2 are satisfied. □

Corollary 5.6 (Algebraic Characterisation Theorem)

1. $P^{\oplus}/\simeq_1$ *is (up to isomorphism) the initial safety synchronous algebra with* $\sqsupseteq_1/\simeq_1$ *the induced ordering,*

2. $P^{\oplus}/\simeq_2$ *is (up to isomorphism) the initial liveness synchronous algebra with* $\sqsubseteq_2/\simeq_2$ *the induced ordering.*

PROOF: By the Full Abstraction Lemma. □

6 A modal linear logic of processes

In this section the language of positive formulas is extended by indexed future modalities $\rhd_a$ and past modalities $\lhd_a$. This choice of connectives allows a simple and elegant logical account of the structure of synchronous algebras, in particular the interplay between the static operations of multiplication and internal choice, and prefixing, expressing the dynamic capabilities of processes.

A synchronous algebra A is extended to a *model* $M = (A, V)$ by, as in Definition 2.1, adjoining a valuation V for which $V(X)$ is a filter in A for each propositional letter X. The relation of satisfaction is then defined by adding to the conditions of section 2 the following two conditions for the modal operators:

$$x \models_M \rhd_a \phi \text{ iff there is a } y \in S_M \text{ such that } a.y \leq x \text{ and } y \models_M \phi,$$

$$x \models_M \lhd_a \phi \text{ iff } a.x \models_M \phi.$$

In section 7 below the interpretations on process terms induced by these conditions are investigated. Note here that the filter property extends to the modal language. For the future modalities, $x, y \models \rhd_a \phi$ iff there are x', y' such that $a.x' \leq x$, $a.y' \leq y$ and $x', y' \models \phi$ iff there are x', y' such that $a.x' \sqcap a.y' = a.(x' \sqcap y') \leq x \sqcap y$ and $x' \sqcap y' \models \phi$ (by the induction hypothesis) iff $x \sqcap y \models \rhd_a \phi$. The past modalities are similar.

To axiomatise validity with respect to the class of all safety and liveness models respectively LL^+ is extended by the following axioms and rules concerning the modal operators.

Axioms	$\rhd_a$-$\vee$	$\rhd_a(\phi \vee \psi) \to \rhd_a \phi \vee \rhd_a \psi$
	$\lhd_a$-$\wedge$	$\lhd_a \phi \wedge \lhd_a \psi \to \lhd_a(\phi \wedge \psi)$
	$\to$-$\lhd_a$-$\rhd_a$	$\phi \to \lhd_a \rhd_a \phi$
	$\rhd_a$-$\lhd_a$-$\to$	$\rhd_a \lhd_a \phi \to \phi$
	a-b-**synchronisation**	$\lhd_a(\rhd_b \phi \to \psi) \leftrightarrow (\phi \to \lhd_{a.b} \psi)$
Rules	$\rhd_e$-**necessitation**	$\dfrac{\phi}{\rhd_e \phi}$

$\lhd_e$-**necessitation** $\quad \dfrac{\phi}{\lhd_e \phi}$

$\rhd_a$-**monotonicity** $\quad \dfrac{\phi \to \psi}{\rhd_a \phi \to \rhd_a \psi}$

$\lhd_a$-**monotonicity** $\quad \dfrac{\phi \to \psi}{\lhd_a \phi \to \lhd_a \psi}$

Write $\vdash_{PL} \phi$ if ϕ is provable in this extension of LL^+. Of the new axioms and rules most are entirely straightforward. The axiom $\rhd_a$-$\vee$ expresses the existential nature of the future modality and similarly the axiom $\lhd_a$-$\wedge$ expresses the universal nature of the past modality. The rules express the expected necessitation and monotonicity properties; thus the distributivity of $\rhd_a$ over $\vee$ and $\lhd_a$ over $\wedge$ is derivable. The axioms $\to$-$\lhd_a$-$\rhd_a$ and $\rhd_a$-$\lhd_a$-$\to$ are less obvious; they express a degree of duality between the future and past modalities. Finally the axiom a-b-synchronisation is the axiom that captures the dynamic properties of parallel composition. We note a few theorems of PL for future reference.

Proposition 6.1 (Theorems of PL)

1. $\vdash_{PL} \rhd_a(\phi \vee \psi) \leftrightarrow \rhd_a \phi \vee \rhd_a \psi$
2. $\vdash_{PL} \lhd_a(\phi \wedge \psi) \leftrightarrow \lhd_a \phi \wedge \lhd_a \psi$
3. $\vdash_{PL} \rhd_{a^{-1} \cdot b}(\phi \to \psi) \to (\rhd_a \phi \to \rhd_b \psi)$

PROOF: For 1 and 2 use $\rhd_a$-$\vee$ and $\lhd_a$-$\wedge$ for one direction, and for the other the monotonicity rules together with the axioms for $\wedge$ and $\vee$. The following derivation establishes 3.

1.	$\psi \to \lhd_b \rhd_b \psi$	by $\to$-$\lhd_b$-$\rhd_b$
2.	$(\phi \to \psi) \to (\phi \to \lhd_b \rhd_b \psi)$	1, by transitivity, detachment
3.	$(\phi \to \lhd_b \rhd_b \psi) \to \lhd_{a^{-1} \cdot b}(\rhd_a \phi \to \rhd_b \psi)$	by $a^{-1} \cdot b$-a-synchronisation
4.	$(\phi \to \psi) \to \lhd_{a^{-1} \cdot b}(\rhd_a \phi \to \rhd_b \psi)$	2,3, by transitivity, detachment
5.	$\rhd_{a^{-1} \cdot b}(\phi \to \psi) \to \rhd_{a^{-1} \cdot b} \lhd_{a^{-1} \cdot b}(\rhd_a \phi \to \rhd_b \psi)$	4, by $\rhd_{a^{-1} \cdot b}$-monotonicity
6.	$\rhd_{a^{-1} \cdot b}(\phi \to \psi) \to (\rhd_a \phi \to \rhd_b \psi)$	5, by $\rhd_{a^{-1} \cdot b}$-$\lhd_{a^{-1} \cdot b}$-$\to$, transitivity, detachment

□

As the satisfaction conditions do not refer to the constant 0 and as in the absence of 0, safety and liveness algebras are each others duals, it is not surprising that soundness for safety algebras entails soundness for liveness algebras as well. For completeness this is slightly more subtle as in this case an interpretation for 0 must be provided.

Theorem 6.2 (Soundness and Completeness, PL) *The following statements are equivalent:*

1. $\vdash_{PL} \phi$,

2. ϕ *is* $\mathcal{M}$*-valid where* $\mathcal{M}$ *is the class of all models based on safety algebras,*

3. ϕ *is* $\mathcal{M}$*-valid where* $\mathcal{M}$ *is the class of all models based on liveness algebras.*

PROOF: The proof extends the corresponding proof for LL^+. Soundness is proved as usual. For instance for $\rhd_a$-$\vee$ assume that $x \models \rhd_a(\phi \vee \psi)$. Then there is an x' such that $x' \models \phi \vee \psi$ and $a.x' \leq x$. If $x' \models \phi$ or $x' \models \psi$ then we are done. Otherwise let $x'_1 \sqcap x'_2 \leq x'$, $x'_1 \models \phi$ and $x'_2 \models \psi$. Then $a.x'_1 \models \rhd_a\phi$ and $a.x'_2 \models \rhd_a\psi$ so $(a.x'_1) \sqcap (a.x'_2) \models \rhd_a\phi \vee \rhd_a\psi$ and then $x \models \rhd_a\phi \vee \rhd_a\psi$ by the filter property, as $(a.x'_1) \sqcap (a.x'_2) = a.(x'_1 \sqcap x'_2) \leq a.x' \leq x$. As another example consider a-b-synchronisation. Suppose $x \models \lhd_a(\rhd_b\phi \to \psi)$. Then $a.x \models \rhd_b\phi \to \psi$. Let $y \models \phi$ and we must show $x \times y \models \lhd_{a\cdot b}\psi$. Now $b.y \models \rhd_b\phi$ so $(a.x) \times (b.y) = (a \cdot b).(x \times y) \models \psi$. Thus $x \times y \models \lhd_{a\cdot b}\psi$ as desired. Soundness of the converse implication, and of the remaining axioms and rules is established in a similar manner.
Completeness, safety algebras. A canonical model construction is given, based on the completeness proof for positive linear logic, Theorem 2.3. *PL-theory* are defined analogously to LL^+-theories: They are closed under implications provable in PL, and adjunction. In the present case we restrict attention to nonempty PL-theories. The valuation V and operations $\sqcap$ and $\times$ are unchanged. The constant 0 is the set of all PL formulas, and 1 is the set of all PL theorems. Finally prefixing is defined by $a.T = \{\phi \mid \lhd_a\phi \in T\}$. It is not hard to check that the constants and operations are well-defined. Clearly 1 and 0 are nonempty PL-theories, and by the proof of Theorem 2.3 $\sqcap$ and $\times$ map PL-theories to PL-theories. To see they also preserve nonemptiness suppose $\phi \in T_1$ and $\psi \in T_2$. Then $\phi \vee \psi \in T_1 \sqcap T_2$. For $\times$ note that

$$\vdash_{LL^+} \phi \to (\psi \to ((\phi \to (\psi \to \gamma)) \to \gamma))$$

so that $\psi \to ((\phi \to (\psi \to \gamma)) \to \gamma) \in T_1$ whence $(\phi \to (\psi \to \gamma)) \to \gamma \in T_1 \times T_2$. To verify the well-definedness of prefixing suppose $\vdash_{PL} \phi \to \psi$ and $\phi \in a.T$. Then $\lhd_a\phi \in T$ so by $\lhd_a$-monotonicity, $\lhd_a\psi \in T$ too. Hence $\psi \in a.T$ as desired. Also if $\phi, \psi \in a.T$ then $\lhd_a\phi, \lhd_a\psi \in T$ so $\lhd_a\phi \wedge \lhd_a\psi \in T$, and then by $\lhd_a$-$\wedge$, $\lhd_a(\phi \wedge \psi) \in T$ as well. Hence $\phi \wedge \psi \in a.T$. For nonemptiness suppose that $\phi \in T$. Then by $\to$-$\lhd_a$-$\rhd_a$ also $\lhd_a\rhd_a\phi \in T$, so $\rhd_a\phi \in a.T$, and we have completed the well-definedness check.

To check that the canonical structure forms a safety algebra we know from the completeness proof for LL^+ that it forms a frame. In addition equations (i)-(iv), Definition 5.1 must be checked. Trivially $0 \times T \subseteq 0$. For the other direction let ϕ be an arbitrary formula. As T is nonempty (!) we can find some $\psi \in T$. Then $\psi \to \phi \in 0$ so that $\phi \in 0 \times T$. So it only remains to check the properties relating to prefixing. For equation (ii) we obtain $\phi \in a.(T_1 \sqcap T_2)$ iff

$\rhd_a\phi \in T_1$ and $\rhd_a\phi \in T_2$ (by the above observation) iff $\phi \in (a.T_1) \sqcap (a.T_2)$. For equation (iii) assume first that $\psi \in (a.T_1) \times (b.T_2)$. Then for some $\phi \in b.T_2$, $\phi \to \psi \in a.T_1$. Then $\rhd_a(\phi \to \psi) \in T_2$. By $\rhd_b$-$\lhd_b$-$\to$ transitivity and detachment we obtain $\vdash_{PL} (\phi \to \psi) \to (\rhd_b\lhd_b\phi \to \psi)$, so $\rhd_b\lhd_b\phi \to \psi \in a.T_1$, and then $\lhd_a(\rhd_b\lhd_b\phi \to \psi) \in T_1$. Then by a-b-synchronisation, $\lhd_a\phi \to \lhd_{a\cdot b}\psi \in T_1$ as well, thus $\lhd_{a\cdot b}\psi \in T_1 \times T_2$. But then $\psi \in (a \cdot b).(T_1 \times T_2)$ as desired. For the converse inclusion, assume that this holds, thus $\lhd_{a\cdot b}\psi \in T_1 \times T_2$. Then for some $\phi \in T_2$ does $\phi \to \lhd_{a\cdot b}\psi \in T_1$, and then by a-b-synchronisation, $\lhd_a(\rhd_b\phi \to \psi) \in T_1$ as well, so that $\rhd_b\phi \to \psi \in a.T_1$. Also $\rhd_b\phi \in b.T_2$ as we saw above and thus $\psi \in (a.T_1) \times (b.T_2)$ as needed. Equation (iv) is left as an easy exercise.

Finally we need to check that $\phi \in T$ iff $T \models \phi$. This part of the proof is common to both the safety and the liveness case. The proof is by induction in the structure of ϕ, and for all connectives except the modal ones the proof is identical to the corresponding part in the proof of completeness for LL^+, Theorem 2.3. For the modal connectives:

$\phi = \rhd_a\phi'$. If $T \models \phi$ then there is some nonempty PL-theory T' s.t. $T' \models \phi'$ and $a.T' \subseteq T$. By the induction hypothesis $\phi' \in T'$ thus $\rhd_a\phi' \in a.T'$ by the above observation and then $\rhd_a\phi' \in T$. Conversely, if $\rhd_a\phi' \in T$ then $a.\mathrm{th}\{\phi'\} \subseteq T$ where $\mathrm{th}\{\phi'\}$ is the least PL-theory containing the set $\{\phi'\}$. By the induction hypothesis, $\mathrm{th}\{\phi'\} \models \phi'$ so $T \models \rhd_a\phi'$.

$\phi = \lhd_a\phi'$. If $T \models \phi$ then $a.T \models \phi'$ and by the induction hypothesis, $\phi' \in a.T$ whence $\lhd_a\phi' \in T$. Conversely, if $\lhd_a\phi' \in T$ then $\phi' \in a.T$ and by the induction hypothesis $a.T \models \phi'$—i.e. $T \models \lhd_a\phi'$.

The proof for the safety case is then complete, for if $\not\vdash_{PL} \phi$ then $\phi \notin 1$, thus $1 \not\models \phi$.

Completeness, liveness algebras. This part of proof is a simple adaptation of the completeness proof for safety algebras. Here we can take $0 = \emptyset$ and proceed as above. It suffices to note that the required properties of 0 holds in this case. □

We can now show PL to be a conservative extension of LL^+ by embedding general models as in section 2 into models based on liveness algebras in a way that preserves satisfaction.

Theorem 6.3 *PL is a conservative extension of LL^+.*

PROOF: If $\not\vdash_{LL^+} \phi$ for some positive formula ϕ then we find a general model M such that $1_M \not\models \phi$, by 2.3. Moreover, as we noted, M may be assumed to contain an element 0 which is zero for both $\sqcap$ and $\times$. We can turn M into a model M' based on a liveness algebra by defining $a.x = x$ for all $a \in L$. Then it is a simple induction to verify that for all elements x, $x \models_M \phi$ iff $x \models_{M'} \phi$ for all positive formulas ϕ. But then $1_{M'} \not\models \phi$ so $\not\vdash_{PL} \phi$ by Theorem 6.2, and we are done. □

7 The process-based interpretations

The general semantics of positive modal formulas of section 6 induce through the semantical mappings $[\![\cdot]\!]_i$ interpretations on process terms themselves by $p \models_i \phi$ iff $[\![p]\!]_i \models \phi$. In this section these induced satisfaction relations are investigated further. It is shown, in particular, that the induced satisfaction relations respects the corresponding behavioral preorders on processes, and indeed characterises them, provided

1. only closed formulas are considered, and
2. a constant $\underline{0}$, interpreted over 0, is added to the language of positive modal formulas.

Formulas with these two properties are called *extended, closed formulas*; they form the main topic of the remaining part of the paper. The constant $\underline{0}$ is interpreted in the expected way: $x \models_M \underline{0}$ iff $0 \leq x$. Note that for safety algebras $\underline{0}$ denotes the singleton set $\{0\}$, and for liveness algebras $\underline{0}$ is the extensional truthhood constant. For a discussion of the problems involved in extending the soundness and completeness results of the preceding section to the extended language see Dam [8].

We first consider the interpretation of extended closed formulas in terms of the initial safety and liveness algebras. Note that the satisfaction conditions for conjunction, linear implication, and past modalities are given in purely structural terms, i.e. they do not refer to the ordering $\leq$ corresponding for the initial algebras to the behavioral preorders on terms. Hence no characterisation of the initial algebra interpretations are needed in these cases.

Proposition 7.1 (Initial Safety Algebra Interpretation) *Let* $\Sigma \in D_1$.

1. $\Sigma \models t$ *iff* Σ *may* Λ *implies* $e \in \Lambda$, *and* Σ *may* $\{e\}$ *implies* Σ *after* $e \models t$,
2. $\Sigma \models \phi \vee \psi$ *iff for all* $\sigma \in \Sigma$, $cl_1\{\sigma\} \models \phi$ *or* $cl_1\{\sigma\} \models \psi$,
3. $\Sigma \models \rhd_a \phi$ *iff*
   a. $\Sigma' \models \phi$ *for some* $\Sigma' \in D_1$,
   b. Σ *may* Λ *implies* $a \in \Lambda$, *and* Σ *may* $\{a\}$ *implies* Σ *after* $a \models \phi$,
4. $\Sigma \models \underline{0}$ *iff there is no* Λ *for which* Σ *may* Λ.

PROOF: 1. $\Sigma \models t$ iff $\Sigma \subseteq \mathrm{cl}_1(1)$ iff for all $\sigma \in \Sigma$, $\sigma \leq 1$, iff Σ may Λ implies $e \in \Lambda$ and Σ may $\{e\}$ implies Σ after $e = \Sigma \models t$.
2. Assume $\Sigma \models \phi \vee \psi$ and let $\sigma \in \Sigma$. Then $\Sigma \supseteq \mathrm{cl}_1\{\sigma\}$, so by the filter property also $\mathrm{cl}_1\{\sigma\} \models \phi \vee \psi$. Either $\mathrm{cl}_1\{\sigma\} \models \phi$ or $\mathrm{cl}_1\{\sigma\} \models \psi$ in which case we are done, or there are $\Sigma_1, \Sigma_2 \in D_1$ such that $\Sigma_1 \models \phi$, $\Sigma_2 \models \psi$ and $\Sigma_1 \cup \Sigma_2 \supseteq \mathrm{cl}_1\{\sigma\}$. But $\mathrm{cl}_1\{\sigma\}$ is *coprime* with respect to $\subseteq$, that is, whenever $\Sigma_1 \sqcap \Sigma_2 \subseteq \mathrm{cl}_1\{\sigma\}$ then either

$\Sigma_1 \subseteq \mathrm{cl}_1\{\sigma\}$ or $\Sigma_2 \subseteq \mathrm{cl}_1\{\sigma\}$. Hence by the filter property also in this case either $\mathrm{cl}_1\{\sigma\} \models \phi$ or $\mathrm{cl}_1\{\sigma\} \models \psi$. For the converse direction assume that for all $\sigma \in \Sigma$, either $\mathrm{cl}_1\{\sigma\} \models \phi$ or $\mathrm{cl}_1\{\sigma\} \models \psi$. As Σ is generated by a finite set Σ', Σ can be written as a finite union $\cup\{\mathrm{cl}_1(\sigma) \mid \sigma \in \Sigma'\}$. As each $\mathrm{cl}_1\{\sigma\} \models \phi \vee \psi$ by the filter property also $\Sigma \models \phi \vee \psi$.

3. Similar to 1.

4. $\Sigma \models \underline{0}$ iff $\mathrm{cl}_1(0) \supseteq \Sigma$ iff $\Sigma = \{0\}$ iff for no Λ, Σ may Λ. □

Note that the elements of the form $\mathrm{cl}_i\{\sigma\}, i \in \{1,2\}$, are exactly those elements of D_i that are coprime with respect to $\subseteq$. Thus the statement of Proposition 7.1.2 can be read as

$$\Sigma \models \phi \vee \psi \text{ iff for all coprime } \Sigma' \subseteq \Sigma,\ \Sigma' \models \phi \text{ or } \Sigma' \models \psi.$$

Thus the interpretation of disjunction with respect to the initial safety algebra is seen to be related to the interpretation of disjunction in Beth models for propositional intuitionistic logic, and to that of Allwein and Dunn's recent Kripke models for linear logic [4]. Similar comments applies to the initial liveness algebra interpretation:

Proposition 7.2 (Initial Liveness Algebra Interpretation) *Let* $\Sigma \in D_2$.

1. $\Sigma \models t$ *iff* Σ *must* $\{e\}$ *and* Σ *after* $e \models t$,
2. $\Sigma \models \phi \vee \psi$ *iff for all* $\sigma \in \Sigma$, $cl_2\{\sigma\} \models \phi$ *or* $cl_2\{\sigma\} \models \psi$,
3. $\Sigma \models \rhd_a \phi$ *iff* Σ *must* $\{a\}$ *and* Σ *after* $a \models \phi$,
4. $\Sigma \models \underline{0}$ *(always).*

PROOF: Similar to the proof of Proposition 7.1. □

Using Lemma 5.4 it is straightforward to derive from these two Propositions equivalent, operationally determined satisfaction conditions directly on process terms. In addition to the relations may, must, and after, a syntactic characterisation of the coprime elements is needed. The appropriate notion is that of a *trace*: a process term built using only 0, 1, prefixing and $\times$. The set $traces(p)$ of traces of p is defined in the obvious way by

$$traces(0) = \{0\},$$

$$traces(1) = \{1\},$$

$$traces(a.p) = \{a.q \mid q \in traces(p)\},$$

$$traces(p_1 \oplus p_2) = traces(p_1) \cup traces(p_2),$$

$$traces(p_1 \times p_2) = \{q_1 \times q_2 \mid q_1 \in traces(p_1), q_2 \in traces(p_2)\}.$$

The satisfaction condition on terms derived from Propositions 7.1.2 and 7.2.2 is then the following:

$$p \models_i \phi \vee \psi \text{ iff for all traces } q \text{ of } p, q \models_i \phi \text{ or } q \models_i \psi.$$

We then show that the initial algebra interpretations induce the appropriate orderings. Each member $\Sigma \in D_i$, $i \in \{1,2\}$, is generated by a least, finite set Σ'. The *characteristic formula*, $\mathrm{cf}(\Sigma)$, of Σ is then determined as the disjunction of the representation $\mathrm{cf}(\sigma)$ of each member σ of Σ where 0 is represented as $\underline{0}$, 1 as t, and prefixing $a.\cdot$ as the future modality $\rhd_a\cdot$.

Theorem 7.3 (Logical Characterisation Theorem) *For $i \in \{1,2\}$ and $\Sigma_1, \Sigma_2 \in D_i$ the following statements are equivalent:*

1. $\Sigma_2 \models \mathit{cf}(\Sigma_1)$.
2. $\Sigma_1 \supseteq \Sigma_2$.
3. *For all extended, closed ϕ, if $\Sigma_1 \models \phi$ then $\Sigma_2 \models \phi$.*

PROOF: 1 iff 2. Assume that $\Sigma_2 \models \mathrm{cf}(\Sigma_1)$. By an argument similar to that of the proof of Proposition 7.1.2 this holds iff $\mathrm{cl}_i\{\sigma_2\} \models \mathrm{cf}(\Sigma_1)$ whenever σ_2 is a member of the least generating subset of Σ_2. This is the case iff $\mathrm{cl}_i\{\sigma_2\} \models \mathrm{cf}\{\sigma_1\}$ for some member σ_1 of the least generating subset of Σ_1. We then just need to show that $\mathrm{cl}_i\{\sigma_2\} \models \mathrm{cf}\{\sigma_1\}$ iff $\sigma_2 \leq \sigma_1$ for the case $i = 1$, and $\mathrm{cl}_i\{\sigma_2\} \models \mathrm{cf}\{\sigma_1\}$ iff $\sigma_1 \leq \sigma_2$ for $i = 2$. This is shown by easy induction in the length of σ_1.
2 implies 3. By the filter property.
3 implies 2. By the implication 2 to 1 it follows that $\Sigma_1 \models \mathrm{cf}(\Sigma_1)$. Assuming 3 we obtain $\Sigma_2 \models \mathrm{cf}(\Sigma_1)$, so $\Sigma_1 \supseteq \Sigma_2$ by the implication 1 to 2. □

8 Axiomatisation

In the remaining part of the paper we give procedures for deciding validity of formulas with respect to the process-based interpretations. That is, procedures that, given an extended closed formula ϕ, decides if ϕ is *i-valid*, meaning that the unit process satisfies ϕ under the D_i interpretation, for $i \in \{1,2\}$. The procedures use a rewriting-based approach. We add a number of new axiom schemas which are used to rewrite arbitrary extended closed formulas into a normal form. Thus sound and complete axiomatisations are obtained as byproducts using this approach. In the present section these axiomatisations are presented and their soundness proved.

Note first that with respect to the initial algebra interpretations the extensional falsehood constant $\bot$, and consequently also an "intuitionistic" negation $\neg$ and the extensional truthhood constant $\top$, can be derived: Let $\bot \triangleq \lhd_a \rhd_b \underline{0}$ for some

fixed $a, b \in L$ such that $a \neq b$, $\neg\phi \triangleq \phi \to \bot$, and $\top \triangleq \neg\bot$. To see that this is admissible note that in both D_1 and D_2, if $a.\Sigma_1 \leq b.\Sigma_2$ then $a = b$. Hence for neither of the two interpretations can there be a Σ for which $\Sigma \models \lhd_a \rhd_b \phi$ when $a \neq b$. The expressive power of the derived negation is rather curious. For instance $\neg\phi$ expresses *i-unsatisfiability* of ϕ in the sense that there is no $\Sigma \in D_i$ for which $\Sigma \models \phi$, and $\neg\neg\phi$ consequently expresses *i-satisfiability*, or consistency: $\Sigma \models \phi$ for some $\Sigma \in D_i$. Using negation to express satisfiability in this way is used extensively in the axiomatisations below. The extensional truth- and falsehood constants are governed by the expected axioms:

$\bot$-**Elim**	$\bot \to \phi$
$\top$-**Intro**	$\phi \to \top$

Let $PL_{\top,\bot}$ be PL augmented with these two axioms. The following Proposition summarises some theorems and derived rules of $PL_{\top,\bot}$:

Proposition 8.1 (Theorems of $PL_{\top,\bot}$)

1. $\vdash_{PL_{\top,\bot}} (\phi \to \neg\psi) \to (\psi \to \neg\phi)$
2. $\vdash_{PL_{\top,\bot}} \phi \to \neg\neg\phi$
3. $\vdash_{PL_{\top,\bot}} \neg\neg\neg\phi \to \neg\phi$
4. $\vdash_{PL_{\top,\bot}} \neg\phi \to (\psi \to \neg\phi)$
5. $$\frac{\vdash_{PL_{\top,\bot}} \top \to \phi \quad \vdash_{PL_{\top,\bot}} (\phi \wedge \psi) \to \gamma}{\vdash_{PL_{\top,\bot}} \psi \to \gamma}$$

PROOF: 1: An instance of C. 2: Use 1 and I. 3: Use 1, 2 and B. 4: Use $\bot$-Elim, B and C. 5: By the $\wedge$-axioms and transitivity we obtain $\vdash_{PL_{\top,\bot}} (\top \wedge \psi) \to \gamma$, and then by the $\wedge$-axioms, transitivity and $\top$-Intro, $\vdash_{PL_{\top,\bot}} \psi \to \gamma$. □

The extended logics $PL(D_1)$ and $PL(D_2)$ are determined by adding the following axioms to the axioms and rules of $PL_{\top,\bot}$. The axioms common to both $PL(D_1)$ and $PL(D_2)$ are the following:

Distribution	$\phi \wedge (\psi \vee \gamma) \to (\phi \wedge \psi) \vee (\phi \wedge \gamma)$
$\rhd_a$-$\wedge$	$\rhd_a\phi \wedge \rhd_a\psi \to \rhd_a(\phi \wedge \psi)$
$\lhd_a$-$\vee$	$\lhd_a(\phi \vee \psi) \to \lhd_a\phi \vee \lhd_a\psi$
$\lhd_a$-$\rhd_a$-$\to$	$\lhd_a\rhd_a\phi \to \phi$
$\neg\rhd_a$	$\neg\phi \leftrightarrow \neg\rhd_a\phi$
$\to$-$\vee$	$(\rhd_a\phi \to \bigvee_{b\in\Lambda} \rhd_b\psi_b) \to \bigvee_{b\in\Lambda}(\rhd_a\phi \to \rhd_b\psi_b)$

where in $\to$-$\vee$ Λ ranges over finite, nonempty subsets of L. The additional axioms for $PL(D_1)$ are the following six:

s1	$\neg\neg\underline{0}$
s2	$(\neg\neg\phi) \rightarrow (\underline{0} \rightarrow \phi)$
s3	$\rhd_a\phi \wedge \rhd_b\psi \rightarrow \underline{0}$ (provided $a \neq b$)
s4	$(\neg\neg\phi \wedge \neg\neg\psi \wedge (\rhd_a\phi \rightarrow \rhd_b\psi)) \rightarrow \rhd_{a^{-1}.b}(\phi \rightarrow \psi)$
s5	$(\top \rightarrow \bigvee_{a\in A} \rhd_a\phi_a) \leftrightarrow (\underline{0} \wedge (\bigvee_{a\in A} \neg\neg\phi_a))$
s6	$(\neg\neg\phi \wedge (\rhd_a\phi \rightarrow \underline{0})) \rightarrow \underline{0}$

For $PL(D_2)$ the following four axioms are added instead:

l1	$\top \rightarrow \underline{0}$
l2	$(\phi \wedge \rhd_a\top) \rightarrow \rhd_a\lhd_a\phi$
l3	$(\neg\neg\phi \wedge (\rhd_a\phi \rightarrow \rhd_b\psi)) \rightarrow \rhd_{a^{-1}.b}(\phi \rightarrow \psi)$
l4	$\neg(\top \rightarrow \bigvee_{a\in A} \rhd_a\phi_a)$

We note a number of theorems of $PL(D_1)$ and $PL(D_2)$ for later use.

Proposition 8.2 (Theorems of $PL(D_1)$ and $PL(D_2)$) *Let* $\vdash_{PL(D)} \phi$ *if* $\vdash_{PL(D_i)} \phi$ *for both* $i = 1$ *and* $i = 2$.

1. $\vdash_{PL(D)} \rhd_a(\phi \wedge \psi) \leftrightarrow (\rhd_a\phi) \wedge (\rhd_a\psi)$
2. $\vdash_{PL(D)} \lhd_a(\phi \vee \psi) \leftrightarrow (\lhd_a\phi) \vee (\lhd_a\psi)$
3. $\vdash_{PL(D)} \phi \leftrightarrow \lhd_a\rhd_a\phi$
4. $\vdash_{PL(D)} t \leftrightarrow \lhd_e t$
5. $\vdash_{PL(D)} \neg\phi \rightarrow \neg\lhd_a\phi$
6. $\vdash_{PL(D_1)} (\neg\neg\phi) \leftrightarrow (\underline{0} \rightarrow \phi)$
7. $\vdash_{PL(D_1)} \neg\lhd_a\underline{0}$
8. $\vdash_{PL(D)} \neg\lhd_a\rhd_b\phi$, *provided* $a \neq b$
9. $\vdash_{PL(D_2)} \neg(\rhd_a\phi \wedge \rhd_b\psi)$, *provided* $a \neq b$

PROOF: 1: $\leftarrow$ by $\rhd_a$-$\wedge$, $\rightarrow$ by the $\wedge$-axioms, transitivity and $\rhd_a$-monotonicity. 2: Similar. 3: By $\rightarrow$-$\lhd_a$-$\rhd_a$ and $\lhd_a$-$\rhd_a$-$\rightarrow$. 4: By $\lhd_e$-necessitation and t1, $\vdash_{PL(D)} \lhd_e t$, so $\vdash_{PL(D)} t \rightarrow \lhd_e t$ by t2. By similar reasoning $\vdash_{PL(D)} t \rightarrow \rhd_e t$. Then by $\lhd_e$-monotonicity, transitivity and $\lhd_e$-$\rhd_e$-$\rightarrow$, $\vdash_{PL(D)} \lhd_e t \rightarrow t$. 5: By $\rhd_a$-$\lhd_a$-$\rightarrow$, transitivity and permutation, $\vdash_{PL(D)} (\neg\phi) \rightarrow (\neg\rhd_a\lhd_a\phi)$, so by $\neg\rhd_a$ and transitivity the result obtains. 6: $\rightarrow$ by s2. $\leftarrow$ by transitivity, permutation and s1. 7: Let $b \neq a$. An outline of the proof follows:

1. $\neg\rhd_b\underline{0} \rightarrow \neg\lhd_b\rhd_b\underline{0}$	by 5
2. $\neg\neg\lhd_b\rhd_b\underline{0} \rightarrow \neg\neg\rhd_b\underline{0}$	by standard reasoning
3. $(\underline{0} \rightarrow \lhd_b\rhd_b\underline{0}) \rightarrow (\underline{0} \rightarrow \rhd_b\underline{0})$	by 7
4. $\underline{0} \rightarrow \rhd_b\underline{0}$	by $\rightarrow$-$\lhd_b$-$\rhd_b$
5. $\neg\lhd_a\underline{0}$	by $\lhd_a$-monotonicity and def. $\neg$

8: Let $a \neq b$. For $PL(D_1)$ first, by s3, $\lhd_a$-monotonicity and distribution of $\lhd_a$ over $\wedge$, $\vdash_{PL(D_1)} \lhd_a\rhd_a\top \wedge \lhd_a\rhd_b\phi \rightarrow \lhd_a\underline{0}$. Then by $\rightarrow$-$\lhd_a$-$\rhd_a$, $\lhd_a$-$\rhd_a$-$\rightarrow$ and standard reasoning, $\vdash_{PL(D_1)} \lhd_a\rhd_b\phi \rightarrow \lhd_a\underline{0}$. But by 7 it then follows that $\vdash_{PL(D_1)} \neg\lhd_a\rhd_b\phi$.
9: The proof is outlined as follows:

1. $\rhd_a\phi \wedge \rhd_b\psi \rightarrow \rhd_a\phi \wedge \rhd_b\psi \wedge \rhd_a\top$	by standard reasoning
2. $\rhd_a\phi \wedge \rhd_b\psi \rightarrow \rhd_a\lhd_a(\rhd_a\phi \wedge \rhd_b\psi)$	by l2
3. $\rhd_a\phi \wedge \rhd_b\psi \rightarrow \rhd_a(\lhd_a\rhd_a\phi \wedge \lhd_a\rhd_b\psi)$	by distribution of $\lhd_a$ over $\wedge$
4. $\rhd_a\phi \wedge \rhd_b\psi \rightarrow \rhd_a(\lhd_a\rhd_a\phi \wedge \bot)$	by 6, assuming $a \neq b$
5. $\rhd_a\phi \wedge \rhd_b\psi \rightarrow \rhd_a\bot$	by standard reasoning
6. $\neg(\rhd_a\phi \wedge \rhd_b\psi)$	by $\neg\rhd_a$

□

Theorem 8.3 (Soundness of $PL(D_1)$ and $PL(D_2)$) *For all extended, closed formulas ϕ, if $\vdash_{PL(D_i)} \phi$ then $1_i \models \phi$.*

PROOF: The proof is largely routine, and relies on Propositions 7.1 and 7.2. A representative collection of cases is proved below.

$\rhd_a$-$\wedge$. If $\Sigma \models \rhd_a\phi$ and $\Sigma \models \rhd_a\psi$ then either $i = 1$ and $\Sigma = 0_1$, in which case $\Sigma \models \rhd_a(\phi \wedge \psi)$, or else $\Sigma = a.\Sigma'$ and Σ after $a \models \phi \wedge \psi$ by 7.1.3 and 7.2.3.

$\rightarrow$-$\vee$. Suppose $\Sigma \models \rhd_a\phi \rightarrow \bigvee_{b\in\Lambda} \rhd_b\psi_b$. First if there is no Σ' such that $\Sigma \models \rhd_a\phi$ then we are done, so assume not. Let $\Lambda' = \{c \mid \exists\sigma.c.\sigma \in \Sigma\}$. If $\Lambda' = \emptyset$ then in case $i = 1$ $\Sigma = 0_1$, and $\Sigma \models \rhd_a\phi \rightarrow \rhd_b\psi_b$ for all $b \in \Lambda$. In case $i = 2$ we obtain a contradiction. So $\Lambda' \neq \emptyset$. For each $c \in \Lambda'$ let $\Sigma_c = \mathrm{cl}_i\{\sigma \in \Sigma \mid \exists\sigma'.\sigma = c.\sigma'\}$. Then $\Sigma = \bigcup_{c\in\Lambda'} \Sigma_c$. It suffices to show that for each $c \in \Lambda'$ there is a $b \in \Lambda$ such that $\Sigma_c \models \rhd_a\phi \rightarrow \rhd_b\psi_b$. Fix $c \in \Lambda'$. Then $\Sigma_c \models \rhd_a\phi \rightarrow \bigvee_{b\in\Lambda} \rhd_b\psi_b$ by the filter property. Let $\Sigma_1 \models \rhd_a\phi$. We can assume that Σ_1 has the form $a.\Sigma_1'$. Hence $\Sigma_c \times \Sigma_1 \models \bigvee_{b\in\Lambda} \rhd_b\psi_b$, so $c \cdot a \in \Lambda$ and $\Sigma_c \times \Sigma_1 \models \rhd_{c\cdot a}\psi_{c\cdot a}$. But the chosen disjunct was independent of Σ_1, so we have verified that $\Sigma_c \models \rhd_a\phi \rightarrow \rhd_{c\cdot a}\psi_{c\cdot a}$.

s4. Suppose ϕ and ψ are both satisfiable in D_1 and that $\Sigma \models \rhd_a\phi \rightarrow \rhd_b\psi$. If $\Sigma = 0_1$ the proof is easily completed, so suppose not. If $\Sigma \neq (a^{-1} \cdot b).\Sigma'$ for some Σ' then there is some σ in the least generating set of Σ such that $\sigma = c.\sigma'$ for some σ' and $c \neq a^{-1} \cdot b$. Then $\mathrm{cl}_1\{\sigma\} \not\models \rhd_b\psi$, a contradiction. So indeed $\Sigma = (a^{-1} \cdot b).\Sigma'$ and whenever $\Sigma_1 \models \phi$ then $\Sigma' \times \Sigma_1 \models \psi$, that is, $\Sigma \models \rhd_{a^{-1}\cdot b}(\phi \rightarrow \psi)$.

s5. Suppose $\Sigma \models \top \rightarrow \bigvee_{a\in\Lambda} \rhd_a\phi_a$ and suppose for a contradiction that $\Sigma \neq 0_1$. Then for some b and σ, $b.\sigma \in \Sigma$. Pick any $c \notin \Lambda$. As L is infinite such

a c exists. Now $(b^{-1}\cdot c).0_1 \models \top$ so $\Sigma \times (b^{-1}\cdot c).0_1 \models \bigvee_{a\in\Lambda} \rhd_a\phi_a$. But then $\mathrm{cl}_1\{b.\sigma\} \times \mathrm{cl}_1\{(b^{-1}\cdot c).0\} = \mathrm{cl}_1\{c.0\} \models \rhd_a\phi_a$ by 7.1.2, but this is a contradiction by 7.1.3. For the second conjunct observe that $0_1 \models \bigvee_{a\in\Lambda} \rhd_a\phi_a$ so $0_1 \models \rhd_a\phi_a$ for some $a \in \Lambda$ by 7.1.2 and then ϕ_a is 1-satisfiable by 7.1.3. For the converse implication, if ϕ_a is 1-satisfiable then for any Σ, $0_1 \times \Sigma = 0_1 \models \bigvee_{a\in\Lambda} \rhd_a\phi_a$ by 7.1.2 and 3, so $0_1 \models \top \to \bigvee_{a\in\Lambda} \rhd_a\phi_a$.

l2. If $\Sigma \models \phi$ and $\Sigma \models \rhd_a\top$ then $\Sigma = a.\Sigma'$ for some Σ'. But then $\Sigma \models \rhd_a\lhd_a\phi$.

l4. If $\Sigma \models \top \to \bigvee_{a\in\Lambda} \rhd_a\phi_a$ then $\Sigma \times 0_2 \models \bigvee_{a\in\Lambda} \rhd_a\phi_a$, a contradiction. □

Note that the soundness of $PL(D_1)$, axiom s5 in particular, depends on the assumption that the underlying label group is infinite.

9 Completeness and decidability

In this section the completeness of $PL(D_1)$ and $PL(D_2)$ is proved, and it is shown how the proof determines decision procedures for the properties of i-validity and i-satisfiability. The proof has three ingredients. First a suitable notion of normal form is introduced. Secondly the i-valid and i-satisfiable normal forms are characterised in syntactic terms. Thirdly, and finally, we show that each formula is provably equivalent to a formula in normal form.

Definition 9.1 (Normal Form, Satisfiable Normal Form) The set satNF of *satisfiable normal forms* is defined inductively by

1. t, $\top$, $\underline{0} \in$ satNF,
2. if for each $a \in \Lambda$, $\phi_a \in$ satNF then $\bigvee_{a\in\Lambda} \rhd_a\phi_a \in$ satNF.

The set NF of *normal forms* is NF = satNF $\cup \{\bot\}$.

Proposition 9.2 *Let $i \in \{1,2\}$. The following statements are equivalent.*

1. $\phi \in$ satNF
2. $\vdash_{PL(D_i)} \neg\neg\phi$
3. $\Sigma \models \phi$ *for some* $\Sigma \in D_i$

PROOF: 1 iff 2. By soundness $\not\vdash_{PL(D_i)} \neg\neg\bot$ so we need just check $\vdash_{PL(D_i)} \neg\neg\phi$ whenever $\phi \in$ satNF. An easy structural induction suffices to establish this. For t and $\top$ use I or $\top$-Intro, t1 and 8.1.2. For $\underline{0}$ and $i = 1$ use I and 8.2.6, and for $i = 2$ use l1. For $\bigvee_{a\in\Lambda} \rhd_a\phi_a \in$ satNF use the induction hypothesis and $\neg\rhd_a$.
3 implies 1. If $\Sigma \models \phi$ then $\phi \in$ satNF.
2 implies 3. If $\vdash_{PL(D_i)} \neg\neg\phi$ then $1_i \models \neg\neg\phi$ by soundness. □

Next the valid normal forms are characterised. In order to be able to define valNF uniformly we assume here that $\underline{0}$ in the case $i = 2$ is defined by $\underline{0} \triangleq \top$.

Definition 9.3 (Valid Normal Forms) The set valNF of *valid normal forms* is defined inductively by

1. $t, \top \in$ valNF,
2. if $\bigvee_{a\in\Lambda} \rhd_a \phi_a \in$ satNF, $e \in \Lambda$, and $\phi_e \in$ valNF then $\bigvee_{a\in\Lambda} \rhd_a \phi_a \in$ valNF.

Proposition 9.4 *Let $i \in \{1,2\}$ and $\phi \in NF$. The following statements are equivalent.*

1. $\phi \in$ valNF
2. $\vdash_{PL(D_i)} \phi$
3. $1_i \models \phi$

PROOF: 1 iff 2. An easy structural induction in ϕ. Note $\not\vdash_{PL(D_i)} \bot$, $\not\vdash_{PL(D_i)} \underline{0}$ for $i = 1$, $\vdash_{PL(D_i)} t$, and $\vdash_{PL(D_i)} \top$. Let $\phi = \bigvee_{a\in\Lambda} \rhd_a \phi_a \in$ satNF and $\vdash_{PL(D_i)} \phi$. Then $1_i \models \phi$ by soundness, so $e \in \Lambda$ and $\vdash_{PL(D_i)} \phi_e$. By the induction hypothesis $\phi_e \in$ valNF so $\phi \in$ valNF as well. If conversely $\phi \in$ valNF and $\vdash_{PL(D_i)} \phi_e$ then $\vdash_{PL(D_i)} \phi$ by the induction hypothesis, $\rhd_e$-necessitation and $\vee$-Intro.
2 implies 3. By soundness.
3 implies 1. An easy structural induction in ϕ. □

The largest single step of the completeness proof is the normalisation theorem below. At first glance it might seem surprising that as sparse a vocabulary as the constants plus $\vee$ and $\rhd_a$ suffices to express the whole language. On the other hand we have already seen that in both D_1 and D_2 all occurrences of $\times$ are eliminable in favour of operators 0, 1, $\oplus$ and prefixing only.

Theorem 9.5 (Normalisation) *Let $i \in \{1,2\}$. There is an effective procedure which given any extended closed ϕ produces a $\phi' \in$NF such that $\vdash_{PL(D_i)} \phi \leftrightarrow \phi'$.*

PROOF: See section 10. □

Corollary 9.6 (Completeness of $PL(D_i)$) *Let $i \in \{1,2\}$ and $1_i \models \phi$ for ϕ an extended closed formula. Then $\vdash_{PL(D_i)} \phi$.*

PROOF: By soundness, the Normalisation Theorem and Proposition 9.4. □

In a similar fashion the decidability of the properties of i-validity and i-satisfiability is easily seen.

10 Proof of the Normalisation Theorem

The proof proceeds by cases and induction in the modal depth of formulas. Abbreviate $\vdash_{PL(D_i)} \phi \leftrightarrow \phi'$ by $\phi \equiv_i \psi$, or just $\equiv$ when i is understood from the context. We prove the slightly more general statement that for each extended closed ϕ there is an effectively computable $\phi' \in$ NF of a modal depth not exceeding that of ϕ such that $\phi \equiv_i \phi'$.

10.1 The safety case

Let first $\phi = \phi_1 \to \phi_2$, and assume that $\phi_1, \phi_2 \in \mathrm{NF}$. We proceed cases on ϕ_1 and when necessary also ϕ_2.

a. $\phi_1 = \bot$. Then $\phi \equiv \top$.

b. $\phi_1 = \top$. We proceed by cases on ϕ_2.

   i. $\phi_2 = \bot$. Then $\phi \equiv \bot$.

   ii. $\phi_2 = \top$. Then $\phi \equiv \top$.

   iii. $\phi_2 = \underline{0}$. Here $\phi \equiv \underline{0}$.

   iv. $\phi_2 = t$. Then $\phi \equiv \underline{0}$ by s5.

   v. $\phi_2 = \bigvee_{a\in A} \rhd_a \phi_a$. Here $\phi \equiv \underline{0}$, again by s5.

c. $\phi_1 = \underline{0}$. If $\phi_2 = \bot$ then $\phi \equiv \bot$. Otherwise $\phi_2 \in \mathrm{satNF}$ so $\vdash_{PL(D_1)} \phi$ by Proposition 9.2, so $\phi \equiv \top$ by $\top$-Intro and Propositions 8.2.6 and 8.1.4.

d. $\phi_1 = t$. Here $\phi \equiv t$.

e. $\phi_1 = \bigvee_{a\in\Lambda_1} \rhd_a \psi_a$. We proceed again by cases on ϕ_2.

   i. $\phi_2 = \bot$. Then $\phi \equiv \bot$.

   ii. $\phi_2 = \top$. Here $\phi \equiv \top$.

   iii. $\phi_2 = \underline{0}$. Note first that $\vdash_{PL(D_1)} \underline{0} \to \phi$. For $\vdash_{PL(D_1)} \phi_1 \to \neg\neg\underline{0}$, giving the result by s2 and permutation. For the converse implication $\vdash_{PL(D_1)} \phi \to \bigwedge_{a\in\Lambda_1}(\rhd_a\psi_a \to \underline{0})$ by standard reasoning. Secondly $\vdash_{PL(D_1)} \neg\neg\psi_a$ for all $a \in \Lambda_1$ by Proposition 9.2 so by s6 and Proposition 8.1.5 $\vdash_{PL(D_1)} \bigwedge_{a\in\Lambda_1}(\rhd_a\psi_a \to \underline{0}) \to \underline{0}$ and we're done.

   iv. $\phi_2 = t$. By standard reasoning we first obtain $\phi \equiv \bigwedge_{a\in\Lambda_1}(\rhd_a\psi_a \to \rhd_e t)$. As $\vdash_{PL(D_1)} \neg\neg\rhd_a\psi_a$ for each $a \in \Lambda_1$ and $\vdash_{PL(D_1)} \neg\neg\rhd_e t$ then by s4 $\phi \equiv \bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}}(\psi_a \to t)$. By the induction hypothesis we find for each $a \in \Lambda_1$ a $\gamma_a \in \mathrm{NF}$ such that $\psi_a \to t \equiv \gamma_a$ and thus $\phi \equiv \bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}}\gamma_a$. If some γ_a is $\bot$ then $\phi \equiv \bot$ by $\neg\rhd_{a^{-1}}$. Otherwise $\vdash_{PL(D_1)} \underline{0} \to \bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}}\gamma_a$ by Proposition 9.2, 8.2.6 and $\wedge$-Intro, so if Λ_1 has size greater than 1 then by s3, $\phi \equiv \underline{0}$. Otherwise let $\Lambda_1 = \{a\}$ and we obtain $\phi \equiv \rhd_{a^{-1}}\gamma_a$.

   v. $\phi_2 = \bigvee_{b\in\Lambda_2} \rhd_b\gamma_b$. By standard reasoning, $\phi \equiv \bigwedge_{a\in\Lambda_1}\bigvee_{b\in\Lambda_2}(\rhd_a\psi_a \to \rhd_b\gamma_b)$, and as in case (iv) we obtain $\phi \equiv \bigwedge_{a\in\Lambda_1}\bigvee_{b\in\Lambda_2} \rhd_{a^{-1}\cdot b}(\psi_a \to \gamma_b)$. So by the induction hypothesis we find for each pair a, b a $\delta_{a,b} \in \mathrm{NF}$ such that $\delta_{a,b} \equiv \psi_a \to \gamma_b$, and then $\phi \equiv \bigwedge_{a\in\Lambda_1}\bigvee_{b\in\Lambda_2} \rhd_{a^{-1}\cdot b}\delta_{a,b}$. Using Distribution

$$\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2}\bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}\cdot f(a)}\delta_{a,f(a)},$$

   and by $\neg\rhd_a$ we can assume that $\delta_{a,b} \neq \bot$ for all a, b. Fix $f : \Lambda_1 \to \Lambda_2$. Suppose there is some $a_1, a_2 \in \Lambda_1$ such that $a_1^{-1} \cdot f(a_1) \neq a_2^{-1} \cdot f(a_2)$.

Then $\bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}\cdot f(a)}\delta_{a,f(a)} \equiv \underline{0}$ by s3 and 9.2. If on the other hand $a_1 \cdot f(a_1) = a_2^{-1} \cdot f(a_2) = a_f$, say, for all $a_1, a_2 \in \Lambda$ then

$$\bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}\cdot f(a)}\delta_{a,f(a)} \equiv \rhd_{a_f} \bigwedge_{a\in\Lambda_1} \delta_{a.f(a)}.$$

We now apply the induction hypothesis and find some δ_{a_f} such that $\bigwedge_{a\in\Lambda_1} \delta_{a.f(a)} \equiv \delta_{a_f}$. It is now a simple matter to complete the rewriting using the tools already introduced.

This completes the case for $\phi = \phi_1 \to \phi_2$. Assume next that $\phi = \phi_1 \wedge \phi_2$, $\phi_1, \phi_2 \in$ NF. The only interesting case here is when $\phi_1 = \bigvee_{a\in\Lambda_1} \rhd_a \phi_a$ and $\phi_2 = \bigvee_{b\in\Lambda_2} \rhd_b \psi_b$. The other case are either already covered or in the case of t easily reducible to the present case. To rewrite into normal form first use distribution to obtain the form $\phi \equiv \bigvee_{a,b} \rhd_a \phi_a \wedge \rhd_b \psi_b$, and then 9.2 and s3 to obtain either $\phi \equiv \underline{0}$ if $\Lambda_1 \cap \Lambda_2$ is empty, or if not, $\phi \equiv \bigvee_{a\in\Lambda_1\cap\Lambda_2} \rhd_a(\phi_a \wedge \psi_a)$. The induction hypothesis is then used in a routine way to rewrite $\bigvee_{a\in\Lambda_1\cap\Lambda_2} \rhd_a(\phi_a \wedge \psi_a)$ into normal form.
The remaining cases are straightforward and left to the reader, as is the check that the size of ϕ does not increase under normalisation.

10.2 The liveness case

Let again $\phi = \phi_1 \to \phi_2$, $\phi_1, \phi_2 \in$ NF. The case for one of ϕ_1 or ϕ_2 equal to $\underline{0}$ need not be considered. The case for one of ϕ_1 or ϕ_2 equal to $\bot$ is trivial.

a. $\phi_1 = \top$. Procees by case on ϕ_2.

   i. $\phi_2 = \top$. Then $\phi \equiv \top$.

   ii. $\phi_2 = t$. Then $\phi \equiv \bot$ by l4, as $t \equiv \rhd_e t$.

   iii. $\phi_2 = \bigvee_{b\in\Lambda_2} \rhd_b \psi_b$. Here $\phi \equiv \bot$ by l4.

b. $\phi_1 = t$. Here $\phi \equiv \phi_2$.

c. $\phi_1 = \bigvee_{a\in\Lambda_1} \rhd_a \phi_a$. Proceed by cases on ϕ_2.

   i. $\phi_2 = \top$. Then $\phi \equiv \top$ by 8.1.4.

   ii. $\phi_2 = t$. We first obtain $\phi \equiv \bigwedge_{a\in\Lambda_1}(\rhd_a \phi_a \to \rhd_e t)$. By 9.2, $\rhd_a \phi_a$ is 2-satisfiable, so by l3 and 6.1.3 $\phi \equiv \bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}}(\phi_a \to t)$. By 8.2.9 we obtain $\phi \equiv \bot$ whenever Λ_1 contains more than one element. Otherwise let $\Lambda_1 = \{a\}$, and it is now a simple matter to apply the induction hypothesis to $\phi_a \to t$ and obtain the desired normal form.

   iii. $\phi_2 = \bigvee_{b\in\Lambda_2} \rhd_b \psi_b$. As in the proof for the safety case we obtain $\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2} \bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}\cdot f(a)}\delta_{a,f(a)}$ where each $\delta_{a,f(a)}$ is 2-satisfiable and in normal form. Fix $f : \Lambda_1 \to \Lambda_2$. If there is some $a_1, a_2 \in \Lambda_1$ such that $a_1^{-1} \cdot f(a_1) \neq a_2^{-1} \cdot f(a_2)$ then $\bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}\cdot f(a)}\delta_{a,f(a)} \equiv \bot$. Otherwise let $\Lambda_1 = \{a_f\}$ and then $\bigwedge_{a\in\Lambda_1} \rhd_{a^{-1}\cdot f(a)}\delta_{a,f(a)} \equiv \rhd_{a_f} \bigwedge_{a\in\Lambda_1} \delta_{a,f(a)}$, and the

rewriting into normal form can then be completed by the induction hypothesis and standard reasoning.

Assume next that $\phi = \phi_1 \wedge \phi_2$, $\phi_1, \phi_2 \in$ NF. The case for one ϕ_1 and ϕ_2 equal to $\top$ is trivial. For the rest:

a. $\phi_1 = t$. If $\phi_2 = t$ then $\phi \equiv t$ so assume instead that $\phi_2 = \bigvee_{b\in\Lambda_2} \rhd_b \psi_b$. Then $\phi \equiv \bigvee_{b\in\Lambda_2}(\rhd_e t \wedge \rhd_b \psi_b$, so if $e \notin \Lambda_2$ then $\phi \equiv \bot$ by 8.2.9. Otherwise $\phi \equiv \rhd_e(t \wedge \psi_b)$ which is easily rewritten into normal form using the induction hypothesis.

b. $\phi_1 = \bigvee_{a\in\Lambda_1} \rhd_a \phi_a$. The proof proceeds as in case (a).

The remaining cases and the check that normalisation does not increase size is left to the reader.

11 Conclusion

Our aim has been to investigate the use of linear and relevant logics as logical handles on the static structure of processes, and in this framework explore the use of modal operators to account for dynamic behaviour. We gave three examples, one of which was studied in detail, and a number of completeness and characterisation results was obtained concerning axiomatisations and the relationship to linear logic proper as well as to the computational interpretations of (term) models. Computationally the example is rather weak in that it lacks a suitable notion of controllable choice. It is an important issue for future work to extend our approach in this direction. In Dam [8] one such extension is pursued, sacrificing, however, the algebraic interpretation of formulas and the relationship to linear logic.

Basic to our approach is a concept of processes as a semilattice-ordered structure with the semilattice operation a choice operator required to be preserved by parallel composition. This notion of model is related to the ternary relation model for relevant logics of Routley and Meyer [25] (see Dam [8]) and to the models for BCK-logics of Ono and Komori [23]. There are also relations to algebraic models (such as quantales) of relevant and linear logics (c.f. Dunn [10], Abramsky and Vickers [3]). While preservation of choice by parallel composition is natural when parallel composition is synchronous, for asynchronous models this is likely to be too strong, and only monotonicity with respect to the induced semilattice ordering should be expected.

Interest has throughout focused on the implication. An alternative is use a fusion, or intensional conjunction operator as in e.g. Winskel [28]. There an operator $\otimes$ is introduced with the interpretation $p \models \phi_1 \otimes \phi_2$ iff $\exists p_1, p_2$ such that $p = p_1 \times p_2$, $p_1 \models \phi_1$ and $p_2 \models \phi_2$. As formulas involving $\otimes$ can only be used as assumptions no undesirable increase in the discriminating power of formulas results. Otherwise quotienting with respect to some suitable behavioral

equivalence $\simeq$ is needed, so that $p \models \phi_1 \otimes \phi_2$ iff $\exists p_1, p_2$ such that $p \simeq p_1 \times p_2$, $p_1 \models \phi_1$ and $p_2 \models \phi_2$. By using the implication instead this parametrisation on the equivalence $\simeq$ is avoided.

It may be of interest to consider process-based interpretations of connectives other than the ones we have considered above, notably the De Morgan negation $\sim$, the intensional sum-operator $+$, and the linear modalities ! and ?. Given just the monoid structure of models, by distinguishing a constant formula $\perp$ the double negation construction of Girard [13] (the phase semantics) applies, and full propositional linear logic can be interpreted. Relating to the intended interpretation of the monoid operation as parallel composition this interpretation is however of dubious practical value. Our general models can be extended to cover De Morgan negation by an approach similar to that of Ono and Komori: A subset of prime elements and an involution $(\cdot)^*$ is presupposed and then $\sim$ is interpreted by $x \models \sim\phi$ iff for all prime $y \geq x$, $y^* \not\models \phi$. For the linear ? a possibility is to add a binary relation R which is reflexive and for which

1. if $1Rx$ then $1 \leq x$,
2. if $(x \sqcap y)Rz$ then there are x_1, y_1 such that xRx_1, yRy_1 and $x_1 \sqcap y_1 \leq z$,
3. if $x \times yRz$ then there are x_1, y_1 such that xRx_1, yRy_1 and $x_1 \times y_1 \leq z$,

and then satisfaction is extended by $x \models ?\phi$ iff there is some y such that $1 \leq y \leq x$, y idempotent (i.e. $y \times y = y$) and for all z, if yRz then $z \models \phi$.

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