Decidability Questions for Bisimilarity of Petri Nets and Some Related Problems

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Abstract.
The main result is undecidability of bisimilarity for labelled (place/transition) Petri nets. The same technique applies to the (prefix) language equivalence and reachability set equality, which yields stronger versions with simpler proofs of already known results. The paper also shows decidability of bisimilarity if one of the nets is deterministic up to bisimilarity. Another decidability result concerns semilinear bisimulations and extends the result of [CHM93] for Basic Parallel Processes (BPP).

1 Introduction

Bisimulations play an important role in the theory of parallelism and concurrency (cf. e.g.[M89]). An interesting question is decidability of bisimilarity for various classes of (models of) processes (see e.g. [CHS92],[CHM93] for recent results). In fact, *BPP* of [CHM93] are a special subclass of Petri nets. For the general (place/transition labelled) Petri nets, the problem was mentioned as open e.g. in [ABS91].

Using the halting problem for Minsky counter machines, this paper shows the undecidability of the problem even if restricted to Petri nets with a fixed static structure and two unbounded places.

A similar technique applies also to (prefix) language equivalence and equality of reachability sets. The undecidability for these problems is known from [B73],[H75], [H76]; the proofs use Hilbert's 10th problem and Petri nets weakly computing polynomials. The technique shown in this paper not only yields significantly simpler proofs but also puts restrictions sufficient for undecidability (fixed static structure, number of unbounded places) which the mentioned proofs fail to do.

This paper also shows some decidability results. The decidability of bisimilarity for one-to-one labelled (or "unlabelled") Petri nets was clear from the reducibility of (prefix) language equivalence of these nets to the reachability problem (cf. [H75], [M84]). We show here another reduction, which allows an easy generalization for the nets which are deterministic up to bisimilarity.

Another subclass of labelled Petri nets for which the decidability has been known is the above mentioned BPP of [CHM93] (isomorphic to Petri nets where each transition has one input place only). In fact, the decidability result there is more general: it applies to the subclass where the bisimulation equivalence is a congruence w.r.t. (nonnegative vector) addition. This paper further extends the mentioned result: it is shown that the existence of a semilinear bisimulation is sufficient for the decidability and that any congruence is semilinear.

Section 2 contains basic definitions, Section 3 the undecidability results, Section 4 the decidability results. Section 5 contains additional remarks (e.g. the relation to vector addition systems) and some hints for further work.

2 Definitions

$\mathcal{N}$ denotes the set of nonnegative integers, A^* the set of finite sequences of elements of A.

A (labelled) static net is a tuple (P, T, F), (P, T, F, L) respectively, where P and T are finite disjoint sets of *places* and *transitions* respectively,
$F : (P \times T) \cup (T \times P) \longrightarrow \mathcal{N}$ is a *flow function* (for $F(x, y) > 0$, there is an *arc* from x to y with *multiplicity* $F(x, y)$) and $L : T \longrightarrow A$ is a *labelling* (attaches an action name - from a set A - to each transition). By L we also denote the homomorphic extension $L : T^* \longrightarrow A^*$.

A (labelled) Petri net is a tuple $N = (S, M_0)$, where S is a (labelled) static net and M_0 is an *initial marking*, a *marking* M being a function $M : P \longrightarrow \mathcal{N}$. (A marking gives the number of *tokens* for each place). A transition t is *enabled* at a marking M, $M \xrightarrow{t}$, if $M(p) \geq F(p, t)$ for every $p \in P$. An enabled transition t may *fire* at a marking M yielding marking M', $M \xrightarrow{t} M'$, where $M'(p) = M(p) - F(p, t) + F(t, p)$ for all $p \in P$. In the natural way, the definitions can be extended for sequences of transitions $\sigma \in T^*$.

Reachability set of a Petri net N is defined as
$\mathcal{R}(N) = \{M \mid M_0 \xrightarrow{\sigma} M \ for\ some\ \sigma \in T^*\}$.
A *place* $p \in P$ is *unbounded* if for any $k \in \mathcal{N}$ there is $M \in \mathcal{R}(N)$ s.t. $M(p) > k$.

(Prefix) language of a labelled Petri net N is defined as
$\mathcal{L}(N) = \{w \in A^* \mid M_0 \xrightarrow{\sigma} \ for\ some\ \sigma\ with\ L(\sigma) = w\}$.

Given two labelled static nets $(P_1, T_1, F_1, L_1), (P_2, T_2, F_2, L_2)$, a binary relation $R \subseteq \mathcal{N}^{P_1} \times \mathcal{N}^{P_2}$ is a *bisimulation* if for all $M_1 R M_2$:
- for each $t \in T_1$, $M_1 \xrightarrow{t} M_1'$, there is $t' \in T_2$ s.t. $L_1(t) = L_2(t')$ and $M_2 \xrightarrow{t'} M_2'$, where $M_1' R M_2'$
and conversely
- for each $t' \in T_2$, $M_2 \xrightarrow{t'} M_2'$, there is $t \in T_1$ s.t. $L_1(t) = L_2(t')$ and $M_1 \xrightarrow{t} M_1'$, where $M_1' R M_2'$.

Two labelled Petri nets N_1, N_2 are *bisimilar* if there is a bisimulation relating their initial markings. Notice that if N_1, N_2 are bisimilar, then $\mathcal{L}(N_1) = \mathcal{L}(N_2)$.

3 Undecidability Results

A counter machine C with nonnegative counters $c_1, c_2, ..., c_m$ is a program
$1 : COMM_1;\ 2 : COMM_2;\ ......\ ;\ n : COMM_n$ where $COMM_n$ is $HALT$-command and $COMM_i$ $(i = 1, 2, ..., n-1)$ is a command of the type
$1/\ c_j := c_j + 1;\ goto\ k$ or
$2/\ if\ c_j = 0\ then\ goto\ k_1\ else\ (c_j := c_j - 1;\ goto\ k_2)$
$(1 \leq k, k_1, k_2 \leq n,\ 1 \leq j \leq m)$.
The set BS of *branching states* is defined as $BS = \{i \mid COMM_i$ is of the type $2\}$.

It is well-known (cf. [M67]) that there is a fixed ("universal") counter machine C with two counters c_1, c_2 such that it is undecidable for given input values x_1, x_2 of c_1, c_2 whether C halts.

Consider a counter machine C, with input values $x_1, x_2, ..., x_m$, in the above notation. We describe a construction of the *basic net* N_C which simulates C in a weak sense. By adding (x, y) we mean increasing $F(x, y)$ by 1 (mostly it means from 0 to 1 - adding one arc), unless otherwise stated. If $F(x, y)$ is not mentioned explicitly, it is equal to 0.

Let $c_1, c_2, ..., c_m$ (the counter part) and $s_1, s_2, ..., s_n$ (the state part) be places of N_C .

For $i = 1, 2, ..., n-1$ add the following transitions and arcs.

If $COMM_i$ is of the type

$1/\ c_j := c_j + 1;\ goto\ k$: add t_i with (s_i, t_i), (t_i, c_j), (t_i, s_k),

$2/\ if\ c_j = 0\ then\ goto\ k_1\ else\ (c_j := c_j - 1;\ goto\ k_2)$:
add t_i^Z (Z for zero) with (s_i, t_i^Z), (t_i^Z, s_{k_1}), and

t_i^{NZ} (NZ for non-zero) with (s_i, t_i^{NZ}), (c_j, t_i^{NZ}), (t_i^{NZ}, s_{k_2}).

The initial marking will consist of the input values $x_1, x_2, ..., x_m$ in $c_1, c_2, ..., c_m$, 1 token in s_1, 0 in the other places, which completes the construction of N_C.

N_C can simulate C in a natural way but (only) transitions t_i^Z can "*cheat*", i.e. fire although the relevant c_j was not 0.

Adding an *dc-transition* (*dc* for "definitely cheating") to N_C for some $i \in BS$ means adding a new transition t with (s_i, t), (c_j, t),(t, c_j),(t, s_{k_1}) (j, k_1 taken from $COMM_i$).
Notice that such t has the same effect as t_i^Z but firing it always means cheating.

Now we establish the main theorems.

Theorem 3.1. *Bisimilarity as well as language equivalence are undecidable for labelled Petri nets, even if restricted to nets with a fixed static structure and 2 unbounded places.*

Proof. Let C be a (fixed) universal counter machine, with input values x_1, x_2, and N_C the basic net in the notation as above. To N_C, add places p, p' and a transition x with (s_n, x), (p, x). Take any one-to-one labelling L of transitions. For each $i \in BS$, add two $dc-$ transitions t_i', t_i'' with additional arcs (p, t_i'), (t_i', p'), (p', t_i''), (t_i'', p) and put $L(t_i') = L(t_i'') = L(t_i^Z)$.

Now take two copies N_1, N_2 of the arised net, N_1 with 1 token in p and 0 in p' (marking M_1), N_2 with 0 in p and 1 in p' (marking M_2). Notice that only c_1, c_2 are (possibly) unbounded.

If C halts (for input x_1, x_2): $L(\sigma)$ where σ is the correct (non-cheating) sequence ended by x belongs to $\mathcal{L}(N_1)$ and not to $\mathcal{L}(N_2)$.
Hence $\mathcal{L}(N_1) \neq \mathcal{L}(N_2)$, $\mathcal{L}(N_1) \not\subseteq \mathcal{L}(N_2)$, N_1, N_2 are not bisimilar.

If C does not halt: the union of the diagonal $\{(M, M)\}$ with the set of all (M', M''), where M', M'' differ on p, p' only and are reachable without cheating (from M_1, M_2 respectively), is a bisimulation containing (M_1, M_2).
Hence N_1, N_2 are bisimilar, $\mathcal{L}(N_1) = \mathcal{L}(N_2)$, $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$. □

Remark. Considering only language equivalence, we could use a simpler, "nonsymmetric", construction: N_1 without p' and $dc-$ transitions, N_2 with only one set of $dc-$ transitions moving the token from p' to p.

Theorem 3.2. *The containment and the equality problems for reachability sets of Petri nets are not decidable, even if restricted to nets with one of two fixed static structures and 5 unbounded places.*

Proof. Let C and N_C be as in Proof of Theorem 3.1. To N_C, add an $dc-$ transition t_i' for each $i \in BS$. Now add places COD,$HELP$,SC (step counter) and r_1,r_2; put 1 token in r_1, 0 in the others. Add (r_1, t),(t, r_2),(t, SC) for each (so far "made") transition t and (t_i^{NZ}, COD) for each t_i^{NZ}. Then add transitions

u_1,u_2,u_3, and
(COD, u_1),(r_2, u_1),(u_1, r_2),$(u_1, HELP)$ with $F(u_1, HELP) = 2$,
(r_2, u_2),(u_2, r_1),
$(HELP, u_3)$,(r_1, u_3),(u_3, r_1),(u_3, COD).
Hence each "$non - u_i$" transition "moves" the token from r_1 to r_2 and adds a token to SC; t_i^{NZ} transition, in addition, adds a token to COD. Before next firing of a $non - u_i$ transition, a sequence from $u_1^* u_2 u_3^*$ is performed (possibly) changing COD. Notice that the maximal change of COD (with $HELP$ empty) can be $COD := 2.COD$ or $COD := 2(COD + 1)$ (for t_i^{NZ}).
In the end, add places p with 1 token and p' with 0 tokens, a transition x with (s_n, x) and (p, t_i'), (t_i', p') for each $i \in BS$.

Now take two copies N_1, N_2 of the arised net, but N_1 will have an additional transition y and (s_n, y),(p, y),(y, p'). Trivially $\mathcal{R}(N_2) \subseteq \mathcal{R}(N_1)$. Also notice that only places c_1,c_2,COD,$HELP$,SC are (possibly) unbounded.

If C halts (for input x_1, x_2): N_1 can perform the correct (non-cheating) sequence finished by y with the maximal intermediate changes of COD. If N_2 "wants" to reach the same marking, it must fire the same number of transitions counted in SC; but, not having y, it must digress from the path of N_1 (N_2 cheats, i.e. uses some t_i^Z or t_i' instead of t_i^{NZ}) and can not reach at the same time the same value of COD (it is clear from the idea of COD as a binary number).
Hence $\mathcal{R}(N_1) \neq \mathcal{R}(N_2)$, $\mathcal{R}(N_1) \not\subseteq \mathcal{R}(N_2)$.

If C does not halt : the only difference is the transition y. If N_1 uses it in some firing sequence (it can be only once), it must have cheated; after performing the same sequence with one t_i' instead of t_i^Z, N_2 reaches the same marking by firing x (instead of y).
Hence $\mathcal{R}(N_1) = \mathcal{R}(N_2)$, $\mathcal{R}(N_1) \subseteq \mathcal{R}(N_2)$. □

Remark. The construction can be slightly modified so that also the reachability sets restricted to unbounded places have the same properties.

4 Decidability Results

4.1 Deterministic Nets

First we consider one-to-one labelled (or "unlabelled") Petri nets. It is clear that the bisimilarity problem is the same as the language equivalence problem in that case; the latter is known to be recursively equivalent to the reachability problem (cf. [H75]) which is known to be decidable from ([M84]). (The *reachability problem* is to decide for a given Petri net N and a marking M whether $M \in \mathcal{R}(N)$.)

Although the result is known, we show here another reduction of the language equivalence to the reachability problem. The proof here is simpler than that in [H75] and allows a straightforward generalization. First of all, we recall a standard decidable generalization of the reachability problem. We take the following auxiliary definition and lemma from [J90]:

Definition. Let $S = (P, T, F)$ be a (static) net. *Language* L_S is the set of formulas defined as follows:
1/ there is one variable $\mathcal{M}$ for elements of $\mathcal{N}^P$;
2/ a *term* is either atomic, $\mathcal{M}(p)$ or c, where $p \in P$, $c \in \mathcal{N}$, or of the form $t_1 + t_2$, where t_1, t_2 are terms;
3/ a *formula* is either atomic, $t_1 < t_2$ or $t_1 \leq t_2$, where t_1, t_2 are terms, or is of the form $f_1 \& f_2$, where f_1, f_2 are formulas. The semantics is natural.

For a concrete marking M, $f(M)$ denotes the instance of f in which M is substituted for $\mathcal{M}$.

Lemma. There is an algorithm with the following specification:
Input: A Petri net $N = (S, M_0)$ and a formula $f \in L_S$,
Output: YES if there is $M \in \mathcal{R}(N)$ s.t. $f(M)$ is true, NO otherwise.

It is easy to show that we can extend L_S allowing also formulas $\neg f$ and $f_1 \vee f_2$ without losing decidability. Notice that formulas like $M \xrightarrow{t}$, $\neg\ M \xrightarrow{t}$ are easily expressible in L_S.

A useful thing at considering bisimulations are "*bisimulation games*" (using C.Stirling's notion).

In case of two labelled Petri nets N_1, N_2, we can define the game as follows: Player 1 chooses one of the nets and fires an enabled transition (with some label). Player 2 replies by firing a transition with the same label in *the other* net. And again, Player 1 chooses one of the nets ... If Player 2 has no possible answer, he loses.

It is not difficult to see that Player 1 has a winning strategy if and only if N_1, N_2 are not bisimilar.

Take now two labelled Petri nets
$N_1 = (P_1, T_1, F_1, L_1, M_{01})$, $N_2 = (P_2, T_2, F_2, L_2, M_{02})$ with the same label set A and consider the following construction of a Petri net N:

Take union of N_1, N_2 (simply put N_1, N_2 beside each other).

Add a place p with 1 token and for any $a \in A$ add places p_a^1, p_a^2 with 0 tokens. To any transition t (from both T_1 and T_2) add its duplicate t' (the same inputs and outputs, the same label).
For any $t \in T_1$: add (p, t),(t, p_a^2),(p_a^1, t'),(t', p), where $L_1(t) = a$.
For any $t \in T_2$: add (p, t),(t, p_a^1),(p_a^2, t'),(t', p), where $L_2(t) = a$.

In fact, we have modelled the bisimulation game in one *(game) net* N. Player

1's winning corresponds to the situation where a token is in some p_a^i ($i = 1, 2$) and no transition $t' \in T_i$ with label a is enabled. It is clear that such situation can be desribed in the (enriched) language L_S for the net N and hence its reachability is decidable. In general, reachability of a winning situation does not mean that Player 1 has a winning strategy; Player 2 could possibly avoid the situtation by choosing more clever answers before. But if the labelling is one-to-one, he could not answer in a different way, hence the reachability of a winning situation really means the winning strategy.

It is clear that the same argument applies if the nets are *deterministic*, i.e. no reachable marking enables two different transitions with the same label. We can even allow the nets to be *deterministic up to bisimilarity* - in such a net, different transitions with the same label can be enabled but their firings lead to bisimilar results. It even suffices when *one* of the nets is deterministic up to bisimilarity; the winning strategy is then to follow the relevant path in the *the other* net, giving, in fact, no choice to Player 2. Hence we have the following theorem.

Theorem 4.1. *Bisimilarity is decidable for two labelled Petri nets, supposing one of them is deterministic up to bisimilarity (hence if one of them is deterministic, hence if one of them is one-to-one labelled).*

The decidability of the above mentioned problem(s) depends on the decidability of the reachability problem. In fact, they are as hard as this problem:

Lemma. The reachability problem is PTIME-reducible to the bisimilarity problem for one-to-one labelled Petri nets.

Proof. It is known (cf. e.g. [H75]) that the reachability problem is recursively (in fact, P-TIME) equivalent to the problem $SPZRP$ of finding out if a given (single) place can be empty (zero tokens) in some reachable marking. Take now an instance of $SPZRP$: a Petri net N and a place p. To N, add a transion t, a place r with 1 token and (r, t),(t, r). Take any one-to-one labelling on transitions. Now take two copies N_1, N_2 of the arised net but N_2 will have arcs (p, t),(t, p) in addition. It is clear that: there is $M \in \mathcal{R}(N)$ with $M(p) = 0$ if and only if N_1, N_2 are not bisimilar. □

Remark. The problem, whether a given net is deterministic, can be reduced to the coverability problem (and is at least as hard) - we do not need "exact" reachability but only the reachability of a covering (componentwise greater or equal) marking.

The problem, whether a given net is deterministic up to bisimilarity, can be reduced to the reachability problem: take two copies of the net and construct the above described "game" net. Player 1 can win iff the original net was not deterministic up to bisimilarity. Using $SPZRP$ (cf. the proof of the previous lemma), it is easy to show that this problem is as hard as reachability problem.

Remark. Generally, non-bisimilarity at finite-branching transition systems, of which Petri nets are a special case, is semi-decidable. (cf. e.g. [M89] and [CHS92]). In such cases, the semi-decidability of bisimilarity is sufficient (for decidability).

Hence we can make another, somewhat artificial, generalization. If, in the case of bisimilar nets, there is a (defending) strategy for Player 2 controlled by a finite automaton (it always inputs the "name" (not label) of the transition fired by Player 1 and outputs the name of the transition for Player 2 to fire) then bisimilarity is semi-decidable: generate successsively all finite automata, include each of them in the "game" net and verify if Player 1 can win.

4.2 Semilinear Bisimulations

Without loss of generality, we will now only consider bisimilarity for the case of both nets having the same static structure (they differ in initial markings only). Then, a bisimulation is, in fact, a relation on $\mathcal{N}^n$. An equivalence relation R on $\mathcal{N}^n$ will be called a *congruence* if uRv implies $(u+w)R(v+w)$ for any $w \in \mathcal{N}^n$ (addition taken componentwise).

As we already mentioned, the recent result of [CHM93] can be interpreted in Petri nets. BPP are isomorphic to labelled Petri nets, where each transition has exactly one (single) input place (for any $t \in T$ there is one $p \in P$ s.t. $F(p,t) = 1$ and $F(p',t) = 0$ for any $p' \neq p$). But the result is more general: for the class of Petri nets where the bisimulation equivalence (the greatest bisimulation) is a congruence, bisimilarity is decidable.

We extend this result using the notion of semilinear sets (cf. e.g. [GS66]). It turns out that any (above mentioned) congruence is semilinear.

Definition. A *set* $B \subseteq \mathcal{N}^k$ of k-dimensional nonnegative vectors is *linear* if there are vectors b (*basis*), $c_1, c_2, ..., c_n$ (*periods*) from $\mathcal{N}^k$ such that
$B = \{b + x_1c_1 + x_2c_2 + \ldots + x_nc_n \mid x_i \in \mathcal{N}, 1 \leq i \leq n\}$.
We then write $B = [b \mid c_1, c_2, ..., c_n]$.
B is a *semilinear set* if it is a finite union of linear sets.

Theorem 4.2. *For the class of (couples of) labelled Petri nets where bisimilarity implies the existence of a semilinear bisimulation relating the initial markings, bisimilarity is decidable.*

Proof. Due to decidability of Presburger arithmetic (theory of addition) (see e.g. [O78]) it can be verified whether a given semilinear set is a bisimulation (w.r.t. two given nets); it is not difficult to verify that the conditions from the definition of a bisimulation can be then expressed by a Presburger formula.
Semi-decidability (and hence decidability) of bisimilarity is then clear: generate successively all semilinear sets and verify for each of them if it relates the initial markings and is a bisimulation (deciding the relevant Presburger formula). □

The rest of this section is devoted to the proof of the following theorem.

Theorem 4.3. *Any congruence ρ on $\mathcal{N}^n$ is semilinear (as a subset of $\mathcal{N}^{2n}$).*

The proof will rely on a double induction - on the dimension n and on a norm $\|\rho\| \in \mathcal{N}$. In fact, the proof is driven in a similar way to the proof in [CP67] (attributed to [R63]) that any congruence is finitely generated. But we will use a simpler notion of the norm (which would simplify their proof as well).

$\mathcal{Z}$ will denote the set of all integers. We will compare elements of $\mathcal{Z}^n$ according to the (partial) order $\leq$ defined componentwise in the natural way ($u \leq v$ iff it holds for all components).

First we deal with the case of one dimension.

Lemma. Any congruence $\rho \subseteq \mathcal{N} \times \mathcal{N}$ is semilinear.

Proof. The case of ρ being the identity relation is clear. Otherwise ρ is generated by one element $(r, r+m) \in \rho$, where $m > 0$ and $r+m$ is as small as possible (it is straightforward to verify it; it is an exercise in [CP67]). Then, in fact, $\rho = [(0,0) \mid (1,1)] \cup [(r,r) \mid (1,1),(0,m),(m,0)]$. □

In proving the general case, we will refer to the following well-known fact.

Lemma (Dickson). Any subset of $\mathcal{N}^r$, $r \in \mathcal{N}$ whose elements are pairwise incomparable is finite.

Proof. By induction on r, it is easy to show that any infinite sequence of elements of $\mathcal{N}^r$ has an infinite ascending subsequence. □

Notation. α, β, γ denote elements of $\mathcal{N}^n$, μ, ν elements of $\mathcal{Z}^n$ (possibly with indices). For any $\mu \in \mathcal{Z}^n$, $(\mu)_i$ denotes the i-th component of μ. $\mathbf{0}$ denotes the zero vector in $\mathcal{N}^n$, $\lambda_i \in \mathcal{N}^n$ the vector with the i-th component 1 and with the other components 0. For $\mu \in \mathcal{Z}^n$, $\mu^+ = max(\mu,\mathbf{0})$, $\mu^- = max(-\mu,\mathbf{0})$, where max is taken componentwise.

For any congruence $\rho \subseteq \mathcal{N}^n \times \mathcal{N}^n$, we denote by G_ρ the following subset of $\mathcal{Z}^n$: $G_\rho = \{\mu \mid \mu = \alpha - \beta \text{ for some } (\alpha,\beta) \in \rho)\}$.

Notice that $\mu^+, \mu^- \in \mathcal{N}^n$ and $\mu = \mu^+ - \mu^-$. In fact, (μ^+, μ^-) is the least element (according to $\leq$ taken componentwise) of the set $\{(\alpha,\beta) \in \mathcal{N}^n \times \mathcal{N}^n \mid \alpha - \beta = \mu\}$. Also notice that G_ρ is a subgroup of $\mathcal{Z}^n$ since ρ is a congruence.

Definition. $\mu, \nu \in \mathcal{Z}^n$ are *compatible* if $(\mu)_i \geq 0$ iff $(\nu)_i \geq 0$ for all $i = 1, 2, \ldots, n$. A *subset* of $\mathcal{Z}^n$ is *compatible* if any pair of its elements is compatible. A set $C \subseteq \mathcal{Z}^n$ is a *basic set w.r.t.* a set $G \subseteq \mathcal{Z}^n$ if C is a finite subset of G such that every $\mu \in G$ can be written $\mu = c_1\mu_1 + c_2\mu_2 + \ldots + c_r\mu_r$, where $c_i \in \mathcal{N}$ (i=1,2,...,r) and $\{\mu_1, \mu_2, \ldots, \mu_r\}$ is a compatible subset of C.

Lemma. For any subgroup $G \subseteq \mathcal{Z}^n$ there is a basic set w.r.t. G.

Proof. In [CP67]. The main points are the following facts:
there are finitely many (at most 2^n) maximal compatible subsets of G,
Dickson's Lemma,
G is closed under addition and subtraction. $\square$

Now we define a norm $\|\rho\| \in \mathcal{N}$ in such a way that $\|\rho\| = 0$ will mean that ρ is the greatest (in set-inclusion sense) of all congruences with the same G_ρ.

Definition. The *norm (height)* $\|\alpha\|$ of a vector $\alpha \in \mathcal{N}^n$ is the sum of its components.
The *norm (height)* $\|\rho\|$ of a congruence $\rho \subseteq \mathcal{N}^n \times \mathcal{N}^n$ is defined
$\|\rho\| = min\{c \in \mathcal{N} \mid c = \|(\alpha_1 - \mu_1^+)\| + \|(\alpha_2 - \mu_2^+)\| + \ldots + \|(\alpha_k - \mu_k^+)\|$, where $\{\mu_1, \mu_2, \ldots, \mu_k\}$ is a basis set w.r.t. G_ρ and for every $i = 1, 2, \ldots, k$ there is β_i such that $(\alpha_i, \beta_i) \in \rho$ and $\alpha_i - \beta_i = \mu_i\}$.

Notice that $\|\rho\| = 0$ iff $(\mu_i^+, \mu_i^-) \in \rho$ for all $i = 1, 2, \ldots, k$.

Lemma. Any congruence $\rho \subseteq \mathcal{N}^n \times \mathcal{N}^n$ with $\|\rho\| = 0$ is linear.

Proof. $\|\rho\| = 0$ means that $(\mu_i^+, \mu_i^-) \in \rho$, $i = 1, 2, \ldots, k$, for a basis set $\{\mu_1, \mu_2, \ldots, \mu_k\}$ w.r.t. G_ρ. Then it is easy to verify that
$\rho = [(\mathbf{0},\mathbf{0}) \mid (\mu_1^+, \mu_1^-), \ldots, (\mu_k^+, \mu_k^-), (\lambda_1, \lambda_1), \ldots, (\lambda_n, \lambda_n)]$. $\square$

To finish the proof of the theorem, we show that the semilinearity of a congruence $\rho \subseteq \mathcal{N}^n \times \mathcal{N}^n$ with $\|\rho\| > 0$ follows from the semilinearity of all congruences on $\mathcal{N}^{n-1}$ and all congruences on $\mathcal{N}^n$ with lesser norm.

Definition. For a congruence $\rho \subseteq \mathcal{N}^n \times \mathcal{N}^n$ define the relations ρ_i ($i = 1, 2, \ldots, n$) as follows: $\rho_i = \{(\alpha, \beta) \in \mathcal{N}^n \times \mathcal{N}^n \mid (\alpha + \lambda_i, \beta + \lambda_i) \in \rho\}$.

Notice that $\rho \subseteq \rho_i$ and
$\{(\alpha, \beta) + (\lambda_i, \lambda_i) \mid (\alpha, \beta) \in \rho_i\} = \{(\alpha, \beta) \in \rho \mid (\alpha)_i > 0 \text{ and } (\beta)_i > 0\}$.

Lemma. ρ_i is a congruence for any $i, 1 \leq i \leq n$ and if $\|\rho\| > 0$ then $\|\rho_i\| < \|\rho\|$ for some i.

Proof. Realizing that $G_\rho = G_{\rho_i}$, the proof is straightforward. $\square$

Proof of Theorem 4.3. Due to the previous lemmas it suffices to consider the case $n > 1$ and $\|\rho\| > 0$. Without loss of generality suppose $\|\rho_1\| < \|\rho\|$. Due to the induction hypothesis, $\|\rho_1\|$ is semilinear. Adding (λ_1, λ_1) to bases of all linear sets (in a semilinear definition of $\|\rho_1\|$), we get, in fact, the semilinear set

$$S_1 = \{(\alpha, \beta) \in \rho \mid (\alpha)_1 > 0 \text{ } and \text{ } (\beta)_1 > 0\}$$

The set

$$S_2 = \{(\alpha, \beta) \in \rho \mid (\alpha)_1 = 0 \text{ } and \text{ } (\beta)_1 = 0\}$$

is isomorphic to a congruence on $\mathcal{N}^{n-1}$ and hence is semilinear according to the induction hypothesis. Due to symmetry, it now suffices to show semilinearity of

the set
$$S_3 = \{(\alpha, \beta) \in \rho \mid (\alpha)_1 = 0 \textit{ and } (\beta)_1 > 0\}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be all minimal elements of the set $\{\alpha \mid (\alpha, \beta) \in S_3$ for some $\beta\}$. (There are finitely many of them due to Dickson's lemma.) Let $\beta_1, \beta_2, \ldots, \beta_k$ be any vectors such that $(\alpha_i, \beta_i) \in S_3$ for $i = 1, 2, \ldots, k$. Let R be the linear set $R = [(\mathbf{0},\mathbf{0}) \mid (\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k), (\lambda_2, \lambda_2), \ldots, (\lambda_n, \lambda_n)]$.

Define $W = \{(\alpha, \beta) \mid$there is γ such that $(\alpha, \gamma) \in R$ and $(\gamma, \beta) \in S_1\}$. The semilinearity of W is clear because the domains of Presburger arithmetic predicates are semilinear (cf. [GS66]); but we have a simple subcase - it suffices to realize that the projection of a semilinear set to a "subspace" is semilinear and to show that for a semilinear S the set $\{\alpha \in S \mid (\alpha)_1 = (\alpha)_2\}$ is semilinear (which is relatively straightforward).

We now show that $W = S_3$ by which the proof will be finished. Because $R \subseteq \rho$, the inclusion $W \subseteq S_3$ is clear.
Take now any $(\alpha, \beta) \in S_3$. It means $\alpha \geq \alpha_j$ for some minimal α_j mentioned above. Hence $(\alpha, \beta_j + \alpha - \alpha_j) \in R \subseteq \rho$. By symmetry and transitivity $(\beta_j + \alpha - \alpha_j, \beta) \in \rho$ and, since $(\beta_j + \alpha - \alpha_j)_1 > 0$ and $(\beta)_1 > 0$, $(\beta_j + \alpha - \alpha_j, \beta) \in S_1$. Hence $(\alpha, \beta) \in W$.
□

5 Additional remarks

Recall that (*n-dimensional*) *vector addition systems* (*VASs*) are isomorphic to (reachability sets of) Petri nets (with n places) without self-loops (without both (p,t),(t,p) as arcs). Hopcroft and Pansiot in [HP79] introduce also $VASSs$ ($VASs$ with an additional finite state control); $n - dimVASS$ are, in fact, Petri nets with (at most) n unbounded places. They show that any $2 - dimVASS$ (unlike $3 - dim$) and any $5 - dimVAS$ (unlike $6 - dim$) is an effectively computable semilinear set; hence the equality problem is decidable for them. They also show how any $n - dimVASS$ can be simulated by an $(n+3) - dimVAS$; it can be done by a Petri net with $n + 2$ places (using self-loops).

It can be checked that our proof, in fact, shows the undecidability for (very restricted subclasses of) $5 - dimVASS$ and $8 - dimVAS$ (leaving the dimensions 3,4, resp. 6,7, open).

For bisimilarity and (prefix) language equivalence we have undecidability for (a very restricted subclass of) Petri nets with 2 unbounded places. My strong conjecture is that it is decidable for the case of 1 unbounded place, the proof of which I hope to give later. (E.g. I think that in that case bisimilarity implies the existence of a semilinear bisimulation).

It will be also interesting to find out the relation betweeen the deterministic nets and "semilinear bisimulations" and, in the whole, better explore the "decidability border" for bisimilarity.

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