Axiomatising Linear Time Mu-calculus

Roope Kaivola*

Laboratory for Foundations of Computer Science
University of Edinburgh
The King's Buildings, Edinburgh EH9 3JZ, United Kingdom

Abstract. We present a sound and complete axiomatisation for the linear time mu-calculus νTL, a language extending standard linear time temporal logic with fixpoint operators. The completeness proof is based on a new bi-aconjunctive non-alternating normal form for νTL-formulae.

1 Introduction

This paper solves the problem of providing a complete natural axiomatisation for the linear time mu-calculus νTL. The logic νTL is an extension of standard linear time temporal logic TL by fixpoint operators [3], allowing the expression of all ω-regular properties (for surveys of the area, see [5, 13]). It is expressively equivalent to Wolper's extended temporal logic ETL [21, 22]. However, requiring only the single *nexttime* temporal operator, the fixpoint-based νTL is syntactically more elegant than ETL, which requires an infinite family of temporal operators.

Although νTL is syntactically concise and straightforward, the axiomatisation problem for it has turned out to be rather intricate. The main culprit for this is the minimal fixpoint operator μ, or more exactly, the prevention of infinite regeneration of minimal fixpoints when trying to build a model for a consistent formula.

Previously the axiomatisation of νTL has been addressed by at least Lichtenstein [8] and Dam [4]. A closely related question, axiomatising the modal mu-calculus, a fixpoint formalism similar to νTL but interpreted over branching structures, has been examined by Kozen [7] and Walukiewicz [19, 20].

Generalising, there have been two approaches to showing the satisfiability of a consistent formula, the essential problem of the completeness proof of an axiomatisation. First, one may try to devise a method of constructing a model directly from a given consistent formula. In this line, Kozen [7] introduced the concept of aconjunctivity, restricting the structure of minimal fixpoint formulae to make it easier to build a model of a consistent formula, and showed the completeness of an axiomatisation of the modal mu-calculus restricted to the aconjunctive fragment of the language. The same approach was pursued by Lichtenstein in [8] to show the completeness of an axiomatisation of νTL restricted to a class of aconjunctive formulae.

* e-mail: Roope.Kaivola@dcs.ed.ac.uk

Another approach, and the one adopted here, is defining some normal form for formulae, and showing that any formula can be provably transformed to this normal form. If we know how to build a model for a consistent formula in this form, the satisfiability of any consistent formula has been shown. Pushing formulae to the normal form can be done inductively, by providing for every operator of the language a corresponding transformation. In the context of a related calculus $S1S$, the monadic second order theory of one successor, this approach was already used early by Siefkes in [12]. In the context of νTL, Dam [4] used Büchi automata -like normal forms to show the completeness of an axiomatisation of νTL containing an 'impure' axiom stating that a formula and its normal form are equivalent.

The axiomatisation of νTL used here is essentially the same as in [7, 8]. The completeness proof is based on a new normal form, the *bi-aconjunctive non-alternating form* for νTL formulae. The crucial property of such formulae is that not only is it easy to construct a model of a consistent formula, but the same holds also of its negation. In our opinion, the remarkable thing about the normal form and the completeness proof here is that after the expressive equivalence of the full νTL and the fragment in the normal form has been established by a purely semantic argument (section 3), the semantic equivalence can be lifted to the level of provability rather elegantly on the basis of what is already known about aconjunctivity (section 4).

Very recently, Igor Walukiewicz has presented a completeness proof for an axiomatisation of the modal mu-calculus [20], based on *disjunctive* normal forms, resembling nondeterministic tree automata. This result naturally carries over from the modal mu-calculus to the the linear mu-calculus νTL, as well. However, the proof involves an extremely complex argument using games between tableaux and a priority technique to create a winning strategy in a game. In this respect the easy negatability of formulae in the bi-aconjunctive non-alternating normal form, a property the disjunctive normal form lacks, makes the approach here considerably more straightforward. Bar one observation that was used in passing in Walukiewicz's work and has been adopted here to give a more elegant solution (see proof of Lemma 36), the work presented in the current paper was carried out independently. It should be pointed out that the argument presented here for νTL does not appear to carry over to the modal case, so Walukiewicz's proof is more general in this sense.

2 Preliminaries

Let us recall definitions of linear mu-calculus syntax and semantics, and introduce some related notation. The language is built from propositions, standard boolean connectives, the minimal fixpoint operator μ, and a temporal operator, the *nexttime* $\odot$. The maximal fixpoint operator ν is introduced as a derived operator. To keep the presentation simple, we assume that all the models on which νTL-formulae are interpreted are infinite. This means that we do not need separate weak and strong nexttime operators.

Definition 1. Σ^* (Σ^ω) is the set of finite (infinite) strings of elements of Σ, $|s|$ is the length of s, and ϵ the empty string. If $s = a_1 \dots a_n \dots$, s_i is the element a_i. Ord is the class of ordinals, and $\preceq$ their standard ordering.

Definition 2. Fix a countable set $\mathcal{Z}$ of propositions. The formulae of νTL are defined by the abstract syntax:

$$\phi ::= z \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \odot\phi \mid \mu z.\phi$$

where z varies over $\mathcal{Z}$. In $\mu z.\phi$, each occurrence of z in ϕ is required to be *positive*, i.e. in the scope of an even number of negations. The derived operators $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\top$, $\bot$ are as usual, and νz stands for $\nu z.\phi = \neg\mu z.\neg\phi[\neg z/z]$. The symbol σ refers to both the μ and ν-operators. The notation $\phi \trianglelefteq \phi'$ means that ϕ' is a subformula of ϕ.

An occurrence of a variable z in a formula ϕ is *bound* iff it is within a subformula $\sigma z.\phi' \trianglerighteq \phi$ and *free* otherwise. If $\phi, \phi_1, \dots, \phi_n$ are formulae and $z_1, \dots, z_n$ variables, $\phi[\phi_1/z_1, \dots, \phi_n/z_n]$ is the result of simultaneously substituting each ϕ_i for all free occurrences of z_i in ϕ. If some free variable z' of ϕ_i would be captured by a fixpoint $\sigma z'$ of ϕ in the substitution, the bound variable z' in ϕ is systematically renamed.

An occurrence of a variable z in a formula ϕ is *guarded* iff it is in a subformula of the type $\odot\phi'$. A formula ϕ is guarded iff for every fixpoint subformula $\sigma z.\phi'$ of ϕ, every occurrence of z in ϕ' is guarded.

Definition 3. A *model* is an infinite sequence of sets of propositions, $M \in (2^{\mathcal{Z}})^\omega$.

The set of states of M satisfying a formula ϕ, denoted $\|\phi\|_M$, is defined by $\|z\|_M = \{i \in N \mid z \in M_i\}$, $\|\neg\phi\|_M = N \setminus \|\phi\|_M$, $\|\phi \wedge \phi'\|_M = \|\phi\|_M \cap \|\phi'\|_M$, $\|\odot\phi\|_M = \{i \in N \mid i+1 \in \|\phi\|_M\}$, $\|\mu z.\phi\|_M = \bigcap\{W \subseteq N \mid \|\phi\|_{M[W/z]} \subseteq W\}$, where $M[W/z]$ is defined by: $M[W/z]_i = M_i \cup \{z\}$ if $i \in W$, $M[W/z]_i = M_i \setminus \{z\}$ if $i \in N \setminus W$.

A formula ϕ is *true at state i of M*, denoted by $M, i \models \phi$, iff $i \in \|\phi\|_M$. A formula ϕ is *universally valid*, denoted by $\models \phi$, iff $M, m \models \phi$ for all models M and all states m of M. A formula ψ is *satisfiable* iff there exists a model M and a state m of M such that $M, m \models \phi$.

Definition 4. For all ordinals $\alpha \in$ Ord, the *fixpoint approximants* $\mu^\alpha z.\phi$ and $\nu^\alpha z.\phi$ are defined inductively by: $\mu^0 z.\phi = \bot$, $\nu^0 z.\phi = \top$, $\sigma^{\alpha+1} z.\phi = \phi[\sigma^\alpha z.\phi/z]$, $\mu^\lambda z.\phi = \bigvee_{\alpha \prec \lambda} \mu^\alpha z.\phi$ and $\nu^\lambda z.\phi = \bigwedge_{\alpha \prec \lambda} \nu^\alpha z.\phi$, where λ is a limit ordinal.

Proposition 5 (Knaster-Tarski). *$\mu z.\phi = \bigvee_\alpha \mu^\alpha z.\phi$ and $\nu z.\phi = \bigwedge_\alpha \nu^\alpha z.\phi$.*

Definition 6. The *closure* of a formula ϕ, denoted cl(ϕ), is the minimal set of formulae that contains ϕ and fulfils: if $\psi \wedge \psi' \in \text{cl}(\phi)$ then $\psi, \psi' \in \text{cl}(\phi)$, if $\neg\psi \in \text{cl}(\phi)$ or $\odot\psi \in \text{cl}(\phi)$ then $\psi \in \text{cl}(\phi)$, and if $\mu z.\psi \in \text{cl}(\phi)$ then $\psi[\mu z.\psi/z] \in \text{cl}(\phi)$.

Using the derived operators, every formula can be expressed in a form where negations are applied only to atomic propositions, i.e. in the positive normal form as defined e.g. in [13].

name	application	name	application
$\vee\boldsymbol{L}$	$\dfrac{i,\ \Gamma \cup \{\phi \vee \phi'\},\ d}{i,\ \Gamma \cup \{\phi\},\ \quad d}$	$\vee\boldsymbol{R}$	$\dfrac{i,\ \Gamma \cup \{\phi \vee \phi'\},\ d}{i,\ \Gamma \cup \{\phi'\},\ \quad d}$
$\wedge$	$\dfrac{i,\ \Gamma \cup \{\phi \wedge \phi'\},\ d}{i,\ \Gamma \cup \{\phi, \phi'\},\ d}$		
$\boldsymbol{\sigma}$	$\dfrac{i,\ \Gamma \cup \{\sigma z.\phi\},\ d}{i,\ \Gamma \cup \{u\},\ \quad d \cdot (u, \sigma z.\phi)}$ **1**	$\boldsymbol{U}$	$\dfrac{i,\ \Gamma \cup \{u\},\ \quad d}{i,\ \Gamma \cup \{\phi[u/z]\},\ d}$ **2**
$\odot$	$\dfrac{i,\ \quad \Gamma \cup \{\odot\phi_1, \ldots, \odot\phi_k\},\ d}{i+1,\ \{\phi_1, \ldots, \phi_k\},\ \quad d}$ **3**		

Note: **1:** u does not appear in d **2:** $d(u) = \sigma z.\phi$
3: $\Gamma \subseteq \mathcal{Z} \cup \{\neg z \mid z \in \mathcal{Z}\}$.
In each rule, Γ is disjoint from the other set

Fig. 1. Tableau rules

Definition 7. A formula ϕ is in *positive normal form* (abbr. *pnf*) iff it only contains atomic propositions, their negations, and the $\vee, \wedge, \odot, \mu$ and ν-operators.

If ϕ is a formula, pnf(ϕ) is the unique formula in positive normal form obtained from ϕ by pushing negations inwards using DeMorgan's laws and the rules $\neg\odot\phi = \odot\neg\phi$, $\neg\mu z.\phi = \nu z.\neg\phi[\neg z/z]$ and $\neg\nu z.\phi = \mu z.\neg\phi[\neg z/z]$.

Next, we give a tableau-like account of truth in a model, related to [2, 8, 15, 17]. For this, the notation is extended with definition constants and lists [14].

Definition 8. Fix a set $\mathcal{U}$ of *definition constants.* The notion of an *extended formula* is as that of a formula, but allowing definition constants in place of free atomic propositions. A *definition list* is a finite sequence $d = (u_1, \phi_1) \ldots (u_n, \phi_n)$ where every $u_i \in \mathcal{U}$ and ϕ_i is an extended formula, all u_i are distinct, and if u occurs in ϕ_i, then $u = u_j$ for some $j < i$. For every u_i, define $d(u_i) = \phi_i$. We say that u_i is *active* in ϕ iff either u_i occurs in ϕ, or there is some u_j, $i < j$, such that u_j occurs in ϕ and u_i is active in $d(u_j)$. If ϕ is an extended formula and d a definition list, $\phi[d]$ is defined by $\phi[\epsilon] = \phi$ and $\phi[d \cdot (u, \psi)] = (\phi[\psi/u])[d]$. If Γ is a set of extended formulae, $\Gamma[d] = \{\phi[d] \mid \phi \in \Gamma\}$.

Definition 9. Let ϕ be a formula in pnf. A *tableau* T for ϕ is an infinite sequence $T = (i_1, \Gamma_1, d_1)(i_2, \Gamma_2, d_2) \ldots$ where

- every $i_j \in N$, Γ_j is a finite set of extended formulae in pnf, and d_j is a definition list containing all definition constants in Γ_j,
- every $(i_{j+1}, \Gamma_{j+1}, d_{j+1})$ is derived from (i_j, Γ_j, d_j) by applying one of the rules in figure 1, and
- $(i_1, \Gamma_1, d_1) = (1, \{\phi\}, \epsilon)$.

We say that $j \in N$ is a $\odot$*-point* of T iff the $\odot$-rule is applied at point j of T. For every $j \in N$, the rule applied at point j induces a *dependency relation* $\rightarrow \subseteq \Gamma_j \times \Gamma_{j+1}$ by:

- if the rule is not $\odot$, the formula in Γ_j to which the rule is applied depends on the resulting formulae (e.g. $\phi \vee \phi' \rightarrow \phi$ for $\vee L$) and $\psi \rightarrow \psi$ for every other $\psi \in \Gamma_j$
- if the rule is $\odot$, $\odot\phi \rightarrow \phi$ for every formula of the form $\odot\phi \in \Gamma_j$.

For any $n \in N$, a sequence $\phi_0, \phi_1, \ldots$ is a *dependency sequence* from point n iff every $\phi_i \in \Gamma_{n+i}$ and $\phi_i \rightarrow \phi_{i+1}$ relative to the rule applied at point $n+i$. A tableau is *proper* iff there is no $n \in N$, $u \in \mathcal{U}$ and infinite dependency sequence $\phi_0, \phi_1, \ldots$ from point n such that $d_n(u) = \mu z.\phi$ for some ϕ, and $\phi_i = u$ for infinitely many $i \in N$. A tableau *agrees* with a model M iff for every $\odot$-point j of T and every $z \in \mathcal{Z}$, if $z \in \Gamma_j$ then $z \in M_{i_j}$, and if $\neg z \in \Gamma_j$ then $z \notin M_{i_j}$.

Notice that as a special case, the $\odot$-rule allows deriving $(i+1, \emptyset, d)$ from $(i, \emptyset, d)$ for any i and d.

Proposition 10. *Let ϕ be a guarded formula in pnf and M a model. Then $M, 1 \models \phi$ iff there is a proper tableau T for ϕ agreeing with M.*

Proof. Standard, see e.g. [15, 17, 8, 14].

3 Normal forms

In this section we introduce a normal form for νTL-formulae, the bi-aconjunctive non-alternating form, and show that the fragment of νTL consisting of formulae in this normal form has the same expressive power as the whole νTL.

The concept of aconjunctivity was introduced by Kozen [7] as a technical restriction, stating intuitively that in a formula $\mu z.(\ldots \phi \wedge \phi' \ldots)$ the minimal fixpoint μz cannot be regenerated in both ϕ and ϕ'.

Definition 11. Let $\sigma z.\phi$ be a formula, and ϕ' a subformula of it. We say that z is *active in* ϕ' iff either

- there is a free occurrence of z in ϕ', or
- there is a free occurrence of some z' in ϕ' such that ϕ' is a subformula of $\sigma z'.\phi''$, $\sigma z'.\phi''$ is a subformula of $\sigma z.\phi$, and z is active in $\sigma z'.\phi''$.

Definition 12. Let $\sigma z.\phi$ be a formula in pnf. We say that $\sigma z.\phi$ is *aconjunctive with respect to* z iff there is no subformula $\phi_1 \wedge \phi_2$ of $\sigma z.\phi$ such that z is active in both ϕ_1 and ϕ_2. A formula ϕ in pnf is *aconjunctive* iff every subformula of type $\mu z.\phi'$ of ϕ is aconjunctive with respect to z. An arbitrary formula ϕ is aconjunctive iff pnf(ϕ) is.

The new concept of bi-aconjunctivity requires not only that a formula itself is aconjunctive, but also that its dual is, as well.

Definition 13. A formula ϕ is *bi-aconjunctive* iff ϕ and $\neg\phi$ are aconjunctive.

In the same way that the alternation of existential and universal quantification leads to arithmetical and analytical hierarchies in recursion theory, the alternation of minimal and maximal fixpoints in mu-calculi leads to a hierarchical classification of formulae [6, 11]. In the normal form we are working with formulae that are extremely low in this hierarchy, and do not have any essential alternation of fixpoints.

Definition 14. Let ϕ be a formula in pnf. We say that ϕ is *non-alternating* iff

- there are no formulae $\nu z.\phi'$ and $\mu z'.\phi''$ such that $\phi \trianglelefteq \nu z.\phi' \triangleleft \mu z'.\phi''$ and z occurs free in $\mu z'.\phi''$, and
- there are no formulae $\mu z.\phi'$ and $\nu z'.\phi''$ such that $\phi \trianglelefteq \mu z.\phi' \triangleleft \nu z'.\phi''$ and z occurs free in $\nu z'.\phi''$,

An arbitrary formula ϕ is non-alternating iff pnf(ϕ) is.

Let us now define the normal form that forms the basis of the completeness proof in next section.

Definition 15. We say that a formula ϕ is in the *bi-aconjunctive non-alternating normal form* (abbreviated *banan-form*) iff

- ϕ is guarded,
- ϕ is bi-aconjunctive, and
- ϕ is not alternating.

To understand the motivation behind this normal form, it may help to relate it to automata on infinite objects. The automata having an exact correspondence with νTL-formulae in banan-form are a restricted form of weak alternating automata [10] on strings.[1]

Notice some easy properties of formulae in the normal form.

Lemma 16. *Let ϕ, ϕ' be formulae in banan-form. Then $\neg\phi$, $\odot\phi$, $\phi \wedge \phi'$ and $\phi[\phi'/z]$ are in banan-form.*

Proof. Straightforward.

The rest of this section is dedicated to showing the expressive equivalence of the class of formulae in banan-form and the whole νTL, i.e. showing that for an arbitrary formula ϕ, there is a formula ϕ' in the banan-form such that $\models \phi \Leftrightarrow \phi'$. Let us see first that alternation is not essential.

Lemma 17. *For any formula ϕ, there exists a non-alternating formula ϕ' such that $\models \phi \Leftrightarrow \phi'$.*

[1] The restriction is: in accepting (rejecting) components, $\vee$ ($\wedge$) branching is allowed only when at least one of the successors belongs to a strictly lower component.

name	application	name	application
$\vee$	$\dfrac{\Gamma \cup \{\psi \vee \psi'\}}{\Gamma \cup \{\psi\} \qquad \Gamma \cup \{\psi'\}}$	$\wedge$	$\dfrac{\Gamma \cup \{\psi \wedge \psi'\}}{\Gamma \cup \{\psi, \psi'\}}$
$\boldsymbol{\mu}$	$\dfrac{\Gamma \cup \{\mu x.\psi\}}{\Gamma \cup \{\psi[\mu x.\psi/x]\}}$	$\odot$	$\dfrac{\Gamma \cup \{\odot\psi_1, \ldots, \odot\psi_k\}}{\{\psi_1, \ldots, \psi_k\}}$ **1**
Note:	**1:** $\Gamma \subseteq \mathcal{Z} \cup \{\neg z \mid z \in \mathcal{Z}\}$. In each rule, Γ is disjoint from the other set		

Fig. 2. Tableau tree rules

Proof. Take any formula ϕ. By [8, Thm. 7.7] [21, Thm. 2.13] νTL and $S1S$, the monadic second-order theory of one successor, are expressively equivalent, i.e. there is an $S1S$-formula ϕ_S equivalent to ϕ.

As a consequence of the McNaughton's theorem [9], we know that $S1S$ and $WS1S$, the weak monadic second-order theory of one successor, are expressively equivalent (see e.g. [16, Thm. 4.6]), i.e. there is a $WS1S$-formula ϕ_{WS} equivalent to ϕ_S, hence equivalent to ϕ.

Moreover, from the results of [1] and [10] relating $WSnS$ and non-alternating fixpoint calculi, we can see as a special case that $WS1S$ and alternation-free νTL are expressively equivalent, i.e. there is a non-alternating νTL-formula ϕ' equivalent to ϕ_{WS}, therefore equivalent to ϕ.

An alternative argument establishing the claim is by mapping a νTL-formula directly to a Büchi-automaton [17, 4], mapping this to an ETL-formula [18], and mapping the ETL-formula back to a non-alternating νTL-formula [8, section 7.5.7].

Next we show that for any non-alternating formula ϕ there is an equivalent formula ϕ' in the banan-form. For this, let us first show the claim for formulae with only minimal fixpoints.

Definition 18. Let ϕ be a guarded formula in pnf without any subformulae of the form $\nu z.\psi$. A *tableau tree* Tr for ϕ is a finite tree labelled with sets of formulae, such that

- The root of Tr is labelled with $\{\phi\}$.
- The sons of each internal node of Tr are derived by applying one of the rules in figure 2. Depending on the rule applied at a node t, t is called a $\vee, \wedge, \mu$ or $\odot$-node, respectively.
- Each leaf t of Tr is labelled with a set $\Gamma \cup \{\odot\psi_1, \ldots, \odot\psi_k\}$, $k \geq 0$, where $\Gamma \subseteq \mathcal{Z} \cup \{\neg z \mid z \in \mathcal{Z}\}$, and the same set labels a $\odot$-node earlier on the path from the root to the current node. If a leaf t is labelled with $\emptyset$, it is a *proper* leaf, otherwise it is a *loop* leaf. The earlier $\odot$-node with the same label is called the *loop node corresponding to* t, and denoted by $l(t)$.

It is easy to see that for any guarded ϕ with only minimal fixpoints, we can produce a tableau tree Tr for ϕ by just applying the derivation rules in any order. Each branch of the tree must eventually reach a leaf, since there are only finitely many different sets of formulae that can be produced by the derivations.

Definition 19. Let ϕ be a guarded formula in pnf without any subformulae of the form $\nu z.\psi$, and Tr a tableau tree for ϕ. For any node t of Tr, define the formula Δ_t by $\Delta_t = \bigwedge(\Gamma \cap (\mathcal{Z} \cup \{\neg z \mid z \in \mathcal{Z}\}))$, where Γ is the label of t.

Fix a distinct fresh variable z_t for every $\odot$-node t of Tr. Define a formula ϕ_t for every node t of Tr inductively by:

- if t is a $\vee$-node, $\phi_t = \phi_{t_1} \vee \phi_{t_2}$, where t_1 and t_2 are the sons of t
- if t is a $\wedge$ or μ-node, $\phi_t = \phi_{t'}$, where t' is the son of t
- if t is a $\odot$-node and is not the loop node for any leaf, then $\phi_t = \Delta_t \wedge \odot\phi_{t'}$, where t' is the son of t
- if t is a $\odot$-node and it is a loop node for some leaf, then $\phi_t = \mu z_t.\Delta_t \wedge \odot\phi_{t'}$, where t' is the son of t
- if t is a proper leaf, $\phi_t = \top$
- if t is a loop leaf, $\phi_t = z_{l(t)}$

Finally, define the formula ϕ_{Tr} as ϕ_t, where t is the root of Tr.

Lemma 20. *Let ϕ be a guarded formula in pnf without any subformulae of the form $\nu z.\psi$. Then there is a formula ϕ' in banan-form such that $\models \phi \Leftrightarrow \phi'$.*

Proof. Let Tr be tableau tree for ϕ. It is easy to see that ϕ_{Tr} is in banan-form. Let us show that $\models \phi \Leftrightarrow \phi_{Tr}$. By Prop. 10, for any model M,

- $M, 1 \models \phi$ iff there is a proper tableau $T = (i_1, \Gamma_1, d_1)\ldots$ for ϕ agreeing with M, and
- $M, 1 \models \phi_{Tr}$ iff there is a proper tableau $T' = (i'_1, \Gamma'_1, d'_1)\ldots$ for ϕ_{Tr} agreeing with M

Since ϕ has no subformulae of the form $\nu z.\psi$, any tableau T for ϕ is proper iff there is some $n \in N$ such that $\Gamma_j = \emptyset$ for all $j \geq n$. The same holds for ϕ_{Tr} and any tableau T' for ϕ_{Tr}. As it is easy to read a tableau T' for ϕ_{Tr} agreeing with M, from a tableau T for ϕ agreeing with M, and vice versa, this means that we can read a proper tableau T' for ϕ_{Tr} agreeing with M from any proper tableau T for ϕ agreeing with M, and vice versa.

A formula with only minimal fixpoints can be viewed as an alternating finite automaton on finite strings. The tableau tree construction corresponds then to mapping such automata to normal non-deterministic finite automata on finite strings.

Generalising this result to all non-alternating formulae is done inductively on the syntactic alternation depth of formulae. Notice that even non-alternating formulae, i.e. formulae without any proper alternation, can still be syntactically alternating.

Definition 21. Define the *syntactic alternation classes* Π_n and Σ_n for all $n \in N$ as the minimal sets fulfilling:

- $\Pi_0 = \Gamma_0$ is the set of all formulae in pnf without any fixpoint operators
- if ϕ is in pnf and has no subformulae of the form $\nu z.\psi$, and $\phi_1, \ldots, \phi_m \in \Sigma_n \cup \Pi_n$, then $\phi[\phi_1/z_1, \ldots, \phi_m/z_m] \in \Sigma_{n+1}$
- if ϕ is in pnf and has no subformulae of the form $\mu z.\psi$, and $\phi_1, \ldots, \phi_m \in \Sigma_n \cup \Pi_n$, then $\phi[\phi_1/z_1, \ldots, \phi_m/z_m] \in \Pi_{n+1}$

Lemma 22. *For any non-alternating formula ϕ, there exists a formula ϕ' in banan-form such that $\models \phi \Leftrightarrow \phi'$*

Proof. Any non-alternating formula ϕ can be trivially transformed into pnf, and further, into an equivalent guarded non-alternating formula by the transformations of [2, subsection 2.4]. Therefore it is enough to show the claim for all guarded non-alternating formulae in pnf. This is done by induction on the syntactic alternation depth hierarchy.
Induction basis: If $\phi \in \Pi_0 = \Gamma_0$, choosing $\phi' = \phi$ fulfils the claim.
Induction step: Let $\phi_1, \ldots, \phi_m \in \Sigma_n \cup \Pi_n$. By induction assumption, there are formulae $\phi'_1, \ldots, \phi'_m$ in banan-form such that $\models \phi_i \Leftrightarrow \phi'_i$ for all $1 \leq i \leq m$.

Take any ϕ in pnf which has no subformulae of the form $\nu z.\psi$. By Lemma 20, there exists a formula ϕ' in banan-form such that $\models \phi \Leftrightarrow \phi'$. By Lemma 16, $\phi'[\phi'_1/z_1, \ldots, \phi'_m/z_m]$ is in banan-form. Since $\models \phi[\phi_1/z_1, \ldots, \phi_m/z_m] \Leftrightarrow \phi'[\phi'_1/z_1, \ldots, \phi'_m/z_m]$, the claim holds for $\phi[\phi_1/z_1, \ldots, \phi_m/z_m]$, and therefore for the class Σ_{n+1}.

Take then any $\phi \in \Pi_{n+1}$, and define $\phi' = \text{pnf}(\neg\phi)$. It is easy to see that $\phi' \in \Sigma_{n+1}$. By the above there is a ϕ'' in banan-form such that $\models \phi' \Leftrightarrow \phi''$, implying $\models \phi \Leftrightarrow \neg\phi' \Leftrightarrow \neg\phi''$. Furthermore, by Lemma 16, $\neg\phi''$ is in banan-form, so the claim holds for ϕ, and therefore for the class Π_{n+1}.

Now the expressive equivalence of the whole νTL and the fragment of formulae in banan-form is immediate.

Theorem 23. *For any formula ϕ, there exists a formula ϕ' in banan-form such that $\models \phi \Leftrightarrow \phi'$.*

Proof. Direct from Lemmas 17 and 22.

It needs to be pointed out that the previous theorem does not imply that for every ϕ there is a ϕ' in banan-form such that $\models \sigma z.\phi \Leftrightarrow \sigma z.\phi'$: although $\sigma z.\phi$ would be well-defined, i.e. although z would occur only positively in ϕ, this does not necessarily hold of ϕ'. In the mapping of Lemma 20, the positivity of free variables in the formula is preserved. However, in the mapping of Lemma 17 from an arbitrary formula to an equivalent non-alternating formula this does not appear to be possible.

4 Axiomatisation

This far we have operated purely on the semantic level. Let us define now the axiomatic system, essentially the same as in [7, 8], and show its completeness on the basis of the semantic expressive equivalence shown in the previous section.

Definition 24. We say that a formula ϕ is *provable* and write $\vdash \phi$, iff it is derivable in the following deductive system.

Axiom schemas:

ax1 All propositional tautologies

ax2 $\odot(\phi \Rightarrow \psi) \Rightarrow (\odot\phi \Rightarrow \odot\psi)$

ax3 $\odot\phi \Leftrightarrow \neg\odot\neg\phi$

ax4 $\phi[\mu z.\phi/z] \Rightarrow \mu z.\phi$

Rules of inference:

modus ponens: from ϕ and $\phi \Rightarrow \psi$ infer ψ

necessitation: from ϕ infer $\odot\phi$

fixpoint induction: from $\phi[\psi/z] \Rightarrow \psi$ infer $\mu z.\phi \Rightarrow \psi$

We say that a formula ϕ is *consistent* iff not $\vdash \neg\phi$.

Showing the soundness of this axiom system is easy.

Theorem 25 (Soundness). *If* $\vdash \phi$ *then* $\models \phi$.

Proof. All instances of the axiom schemas are clearly universally valid, and the modus ponens and necessitation rules validity-preserving. To see that also the fixpoint induction rule preserves universal validity, assume that $\not\models \mu z.\phi \Rightarrow \psi$. As by Prop. 5, $\mu z.\phi = \bigvee_\alpha \mu^\alpha z.\phi$, there is an α such that $\not\models \mu^\alpha z.\phi \Rightarrow \psi$ but $\models \mu^{\alpha'} z.\phi \Rightarrow \psi$ for all $\alpha' \prec \alpha$. This α cannot be 0 or a limit ordinal. Consequently, there are M, m such that $M, m \models \mu^\alpha z.\phi \wedge \neg\psi$, and by definition of μ^α, $M, m \models \phi[\mu^{\alpha-1} z.\phi/z] \wedge \neg\psi$. But as $\models \mu^{\alpha-1} z.\phi \Rightarrow \psi$ and z occurs only positively in ϕ, $\models \phi[\mu^{\alpha-1} z.\phi/z] \Rightarrow \phi[\psi/z]$, implying $M, m \models \phi[\psi/z] \wedge \neg\psi$, i.e. $\not\models \phi[\psi/z] \Rightarrow \psi$.

For the completeness proof, we need some technical lemmas.

Lemma 26 (Substitution). *If* $\vdash \phi$, *then* $\vdash \phi[\phi'/z]$.

Proof. Induction on the length of the proof of $\vdash \phi$.

Let us show first that the axiomatisation is complete in the class of all aconjunctive formulae. This follows easily from Kozen's results for the modal mu-calculus [7]. However, as the result is extended slightly in Lemma 35, a proof of it is sketched down here, as well. The formulation of the proof presented here is due to Stirling.

Definition 27. A tableau $T = (i_1, \Gamma_1, d_1)(i_2, \Gamma_2, d_2)\ldots$ is *consistent* iff $\bigwedge \Gamma_j[d_j]$ is consistent for every $j \geq 1$.

Lemma 28. *Let ϕ be a guarded formula in pnf. If there is a proper consistent tableau T for ϕ, then ϕ is satisfiable.*

Proof. Given a proper consistent tableau $T = (i_1, \Gamma_1, d_1)(i_2, \Gamma_2, d_2)\ldots$ for ϕ, define a model M by: for every $k \in N$, $M_k = \Gamma_j[d_j] \cap \mathcal{Z}$, where j is the unique $\odot$-point of T such that $i_j = k$. Then $M, 1 \models \phi$ by Prop. 10.

name	application	name	application
$\boldsymbol{\sigma'}$	$\dfrac{i,\ \Gamma \cup \{\sigma z.\phi\},\ d,\ d^s}{i,\ \Gamma \cup \{u\},\ d',\ d^{s\prime}}$ **1**	$\boldsymbol{U\nu}$	$\dfrac{i,\ \Gamma \cup \{u\},\ d,\ d^s}{i,\ \Gamma \cup \{\phi[u/z]\},\ d,\ d^s}$ **2**
$\boldsymbol{U\mu}$	$\dfrac{i,\ \Gamma \cup \{u\},\ d,\ d^s}{i,\ \Gamma \cup \{\phi[u/z]\},\ d,\ d^{s\prime}}$ **3**		
Note:	**1**: u does not appear in d, $d' = d \cdot (u, \sigma z.\phi)$, $d'_s = d_s \cdot (u, \sigma z.\phi)$. **2**: $d(u) = \nu z.\phi$ **3**: if $d = (u_1, \sigma z_1.\phi_1) \dots (u_n, \sigma z_n.\phi_n)$, $u = u_m$, $d(u_m) = \mu z.\phi$ and $d^s(u_m) = \mu z.(\phi \wedge \alpha)$, then $d^{s\prime}(u_i) = d^s(u_i)$ for $1 \leq i < m$, $d^{s\prime}(u_i) = d(u_i)$ for $m < i \leq n$, and $d^{s\prime}(u_m) = \mu z.(\phi \wedge \alpha \wedge \neg \bigwedge \Gamma[d^x])$ where $d^x(u_i) = d^s(u_i)$ for $1 \leq i \leq m$, $d^x(u_i) = d(u_i)$ for $m < i \leq n$		

Fig. 3. Strong tableau rules

It is easy to see that for any tableau element (i, Γ, d), if $\bigwedge \Gamma[d]$ is consistent, then some tableau rule can be applied to (i, Γ, d) to yield an element (i', Γ', d') so that $\bigwedge \Gamma'[d']$ is consistent. Therefore, it is easy to construct a consistent tableau T for any consistent formula ϕ. However, there is nothing in this construction to guarantee that the resulting T would be proper as well as consistent. To this purpose we use a technique similar to Kozen's [7] for strengthening minimal fixpoints, based on the following lemma.

Lemma 29. *If $\psi \wedge \mu z.\phi$ is consistent and z does not occur free in ψ, then $\psi \wedge \phi[\mu z.(\phi \wedge \neg\psi)/z]$ is consistent.*

Proof. If $\phi[\mu z.(\phi \wedge \neg\psi)/z] \wedge \psi$ is inconsistent, $\vdash \phi[\mu z.(\phi \wedge \neg\psi)/z] \Rightarrow \neg\psi$, hence $\vdash \phi[\mu z.(\phi \wedge \neg\psi)/z] \Rightarrow \mu z.(\phi \wedge \neg\psi)$. By fixpoint induction rule then $\vdash \mu z.\phi \Rightarrow \mu z.(\phi \wedge \neg\psi)$, implying $\vdash \mu z.\phi \Rightarrow \neg\psi$, i.e. $\mu z.\phi \wedge \psi$ is inconsistent.

Definition 30. Let ϕ be a formula in pnf. A *strong tableau* T for ϕ is an infinite sequence $T = (i_1, \Gamma_1, d_1, d_1^s)(i_2, \Gamma_2, d_2, d_2^s)\dots$ where

- every (i_j, Γ_j, d_j) is as in Def. 9, and d_j^s is a definition list such that if $d_j = (u_1, \sigma z_1.\phi_1)\dots(u_n, \sigma z_n.\phi_n)$, $d_j^s = (u_1, \sigma z_1.\phi_1 \wedge \alpha_1)\dots(u_n, \sigma z_n.\phi_n \wedge \alpha_n)$ for some formulae α_i (possibly $\top$),
- every $(i_{j+1}, \Gamma_{j+1}, d_{j+1}, d_{j+1}^s)$ is derived from $(i_j, \Gamma_j, d_j, d_j^s)$ by one of the rules $\vee L$, $\vee R$, $\wedge$ or $\odot$, which are as in figure 1 or by σ', $U\nu$ or $U\mu$ in figure 3, and
- $(i_1, \Gamma_1, d_1, d_1^s) = (1, \{\phi\}, \epsilon, \epsilon)$.

A strong tableau T being *proper* is defined as in Def. 9. T is *consistent* iff for every $j \in N$, $\bigwedge \Gamma_j[d_j^s]$ is consistent.

Lemma 31. *Let ϕ be an aconjunctive formula in pnf, and T a (strong) tableau for ϕ. For every $j \in N$ and $u \in \mathcal{U}$ such that $d_j(u) = \mu z.\phi'$ for some z and ϕ', the constant u is active in at most one formula $\psi \in \Gamma_j$.*

Proof. The claim holds trivially for the first element of T. All the (strong) tableau rules except $\wedge$ clearly preserve the validity of the claim, and $\wedge$ preserves it thanks to the aconjunctivity of ϕ.

Lemma 32. *Let ϕ be an aconjunctive formula in pnf. If ϕ is consistent, there is a consistent strong tableau T for ϕ.*

Proof. Let (i, Γ, d, d^s) be an element of a strong tableau such that $\bigwedge \Gamma[d_s]$ is consistent. It is easy to see that if any of rules $\wedge$, $\odot$, σ' or $U\nu$ can be applied to it, then for the resulting element $(i', \Gamma', d', d^{s\prime})$, $\bigwedge \Gamma'[d^{s\prime}]$ is consistent. By Lemmas 29 and 31 the same holds for the $U\mu$ rule. If $\vee L$ and $\vee R$ rules can be applied to (i, Γ, d, d^s), then at least one of them yields a $(i', \Gamma', d', d^{s\prime})$ such that $\bigwedge \Gamma'[d^{s\prime}]$ is consistent. As some rule is always applicable, this means that we can construct a consistent strong tableau T for ϕ, starting from the element $(1, \{\phi\}, \epsilon, \epsilon)$.

Lemma 33. *Let ϕ be a guarded formula in pnf, and T a strong tableau for ϕ. If T is consistent, then T is proper.*

Proof. Assume that $T = (i_1, \Gamma_1, d_1, d_1^s)\dots$ is not proper, and take the smallest $m \in N$ such that for some $k \in N$ and $n \geq m$, $d_k = (u_1, \sigma z_1.\phi_1)\dots(u_n, \sigma z_n.\phi_n)$, $d_k(u_m) = \mu z_m.\phi_m$, and $u_m \in \Gamma_j$ for infinitely many j. For every $j \in N$ define $\Gamma'_j = \{\psi \in \Gamma_j \mid u_m \text{ not active in } \psi\}$.

As ϕ is guarded, there is an infinite sequence of indices $j_1, j_2\dots$ such that the $U\mu$-rule is applied to u_m at point $j_h - 1$ of T for every $h \in N$. By Lemma 31 this means that for all $h \in N$, $\Gamma_{j_h-1} = \Gamma'_{j_h-1} \cup \{u_m\}$ and $\Gamma'_{j_h-1} = \Gamma'_{j_h}$, implying $\vdash \bigwedge \Gamma'_{j_h-1}[d^s_{j_h-1}] \Rightarrow \bigwedge \Gamma'_{j_h}[d^s_{j_h}]$ and $\vdash u_m[d^s_{j_h}] \Rightarrow \neg \bigwedge \Gamma'_{j_h}[d^s_{j_h}]$. By the choice of m we can assume without loss of generality that for every $m' < m$, if $d_k(u_{m'}) = \mu z_{m'}.\phi_{m'}$ then $u_{m'} \notin \Gamma_j$ for all $j \geq j_1$, meaning that the $U\mu$-rule is not applied to any of $u_1, \dots, u_{m-1}$ at any point $j \geq j_1$. Remembering the above, this implies that $\vdash u_m[d^s_j] \Rightarrow \neg \bigwedge \Gamma'_{j_h}[d^s_{j_h}]$ for all $j \geq j_h$ and all $h \in N$. Furthermore, $\Gamma'_{j_h}[d^s_{j_h}] \subseteq \mathrm{cl}(\Gamma_{j_1}[d^s_{j_1}])$ for all $h \in N$.

Since $\mathrm{cl}(\Gamma_{j_1}[d^s_{j_1}])$ is finite, there are some $h < l$ such that $\Gamma'_{j_h}[d^s_{j_h}] = \Gamma'_{j_l}[d^s_{j_l}]$. But then $\vdash \bigwedge \Gamma_{j_l-1}[d^s_{j_l-1}] \Rightarrow (u_m[d^s_{j_l-1}] \wedge \bigwedge \Gamma'_{j_l-1}[d^s_{j_l-1}]) \Rightarrow (\neg \bigwedge \Gamma'_{j_h}[d^s_{j_h}] \wedge \bigwedge \Gamma'_{j_l}[d^s_{j_l}]) \Rightarrow \bot$, implying that T is not consistent.

Proposition 34. *Let ϕ be a guarded aconjunctive formula. If ϕ is consistent, then ϕ is satisfiable.*

Proof. As $\vdash \phi \Leftrightarrow \mathrm{pnf}(\phi)$, we can assume that ϕ is in pnf. If ϕ is consistent, by Lemmas 32 and 33, there is a proper consistent strong tableau and therefore a proper consistent tableau T for ϕ. By Lemma 28 this means that ϕ is satisfiable.

Lemma 35. *Let ψ and χ be formulae such that*
- *$\psi \wedge \nu x.\chi$ is well-formed and consistent,*
- *ψ is guarded, aconjunctive and in pnf, and*
- *there exists a guarded aconjunctive formula $\bar{\chi}$ in pnf such that $\vdash \chi \Leftrightarrow \bar{\chi}$.*

name	application
$\boldsymbol{U\nu\chi}$	$\dfrac{i,\ \Gamma \cup \{u\},\qquad d,\ d^s}{i,\ \Gamma \cup \{u, \bar{\chi}[u/x]\},\ d,\ d^s}$ **1**
$\odot$	$\dfrac{i,\quad \Gamma \cup \{\odot\phi_1, \ldots, \odot\phi_k\},\ d,\ d^s}{i+1,\ \{\phi_1, \ldots, \phi_k\},\qquad d,\ d^s}$ **2**
Note:	**1**: $d(u) = \nu x.\chi$ and $U\nu\chi$ has not been applied after previous application of $\odot$ **2**: $\Gamma \subseteq \mathcal{Z} \cup \{\neg z \mid z \in \mathcal{Z}\} \cup U_\chi \cup \{\neg u \mid u \in U_\chi\}$ where $U_\chi = \{u \in U \mid d(u) = \nu x.\chi\}$ and for every $u \in U_\chi \cap \Gamma$ the $U\nu\chi$-rule has been applied to u after previous application of $\odot$.

Fig. 4. Modified strong tableau rules

Then $\psi \wedge \nu x.\chi$ is satisfiable.

Proof. Let us modify slightly the rules for a strong tableau by adding a new $U\nu\chi$-rule and modifying the $\odot$-rule as in figure 4, and by requiring that the $U\nu$-rule is not applied to a constant u such that $d(u) = \nu x.\chi$. Notice that as x does not necessarily occur only positively in $\bar{\chi}$, we can have negated occurrences of a constant u corresponding to $\nu x.\chi$ in a tableau.

Since $\vdash \chi \Leftrightarrow \bar{\chi}$ implies $\vdash \nu x.\chi \Leftrightarrow \chi[\nu x.\chi/x] \Leftrightarrow \bar{\chi}[\nu x.\chi/x]$ by Lemma 26, the $U\nu\chi$-rule preserves consistency. As in Lemma 32, the consistency of $\psi \wedge \nu x.\chi$ implies then the existence of a consistent strong tableau (with the modified rules) $T = (i_1, \Gamma_1, d_1, d_1^s)\ldots$ for $\psi \wedge \nu x.\chi$. As in Lemma 33 the consistency of T means that it is proper, as well.

Define a model M on the basis of T as in the proof of Lemma 28, and define a set $W \subseteq N$ by $W = \{k \in N \mid \exists j \in N, u \in \mathcal{U} : i_j = k, u \in \Gamma_j \text{ and } d_j(u) = \nu x.\chi\}$. For every $w \in W$, we can read from T a proper tableau witnessing $M[W/x], w \models \bar{\chi}$ by Lemma 10. Since $\vdash \chi \Leftrightarrow \bar{\chi}$, this implies by Theorem 25 that $M[W/x], w \models \chi$ for all $w \in W$. As $1 \in W$, this implies $M, 1 \models \nu x.\chi$. From T we can also read a proper tableau witnessing $M, 1 \models \psi$. Consequently, $M, 1 \models \psi \wedge \nu x.\chi$, as required.

The following lemma is the heart of the completeness proof. Essentially it shows that we can lift the expressive equivalence of the whole νTL and the fragment of formulae in banan-form from the level of semantics to the level of provability.

Lemma 36. *For any formula ϕ, there exists a formula ϕ' in banan-form such that $\vdash \phi \Leftrightarrow \phi'$.*

Proof. We show the claim by induction on the structure of the formula ϕ. As before, we assume that ϕ is written using just the $\wedge, \neg, \odot$ and μ-operators.
Induction basis: For an atomic ϕ, choosing $\phi' = \phi$ clearly fulfils the claim.

Induction step for $\wedge$, $\neg$, $\odot$: Suppose that for ϕ_1, ϕ_2 we have ϕ_1', ϕ_2' in banan-form such that $\vdash \phi_1 \Leftrightarrow \phi_1'$ and $\vdash \phi_2 \Leftrightarrow \phi_2'$. By Lemma 16 $\phi_1' \wedge \phi_2'$, $\neg\phi_1'$ and $\odot\phi_1'$ are in banan-form and clearly $\vdash \phi_1 \wedge \phi_2 \Leftrightarrow \phi_1' \wedge \phi_2'$, $\vdash \neg\phi_1 \Leftrightarrow \neg\phi_1'$ and $\vdash \odot\phi_1 \Leftrightarrow \odot\phi_1'$.
Induction step for μ: Suppose that for ϕ we have a ϕ' in banan-form such that $\vdash \phi \Leftrightarrow \phi'$. By Theorem 23, there exists a formula ψ in banan-form such that $\models \mu z.\phi \Leftrightarrow \psi$. If we have $\vdash \mu z.\phi \Leftrightarrow \psi$, the induction step is satisfied, as ψ is in banan-form. Suppose then that $\not\vdash \mu z.\phi \Leftrightarrow \psi$. This means that either

1 $\not\vdash \mu z.\phi \Rightarrow \psi$, or
2 $\not\vdash \psi \Rightarrow \mu z.\phi$

In case 1 we must have $\not\vdash \phi[\psi/z] \Rightarrow \psi$, as otherwise $\vdash \mu z.\phi \Rightarrow \psi$ could be derived by the fixpoint induction rule.[2] This means that $\phi[\psi/z] \wedge \neg\psi$ is consistent. As $\vdash \phi \Leftrightarrow \phi'$, by Lemma 26 $\vdash \phi[\psi/z] \Leftrightarrow \phi'[\psi/z]$, implying that $\phi'[\psi/z] \wedge \neg\psi$ is consistent. Since ϕ' and ψ are in banan-form, by Lemma 16 $\phi'[\psi/z] \wedge \neg\psi$ is in banan-form, hence guarded and aconjunctive. As it is consistent, by Prop. 34 it is satisfiable, i.e. there are M, m such that $M, m \models \phi'[\psi/z] \wedge \neg\psi$.

Since $\vdash \phi[\psi/z] \Leftrightarrow \phi'[\psi/z]$, by Theorem 25 $\models \phi[\psi/z] \Leftrightarrow \phi'[\psi/z]$, which implies $M, m \models \phi[\psi/z] \wedge \neg\psi$. By the choice of ψ, we know that $\models \psi \Leftrightarrow \mu z.\phi$, which implies $\models \phi[\psi/z] \Leftrightarrow \phi[\mu z.\phi/z]$, i.e. $\models \phi[\psi/z] \Leftrightarrow \mu z.\phi$. Consequently, $M, m \models \mu z.\phi \wedge \neg\psi$. But this contradicts $\models \mu z.\phi \Leftrightarrow \psi$, meaning that case 1 cannot hold.

In case 2, $\psi \wedge \neg\mu z.\phi = \psi \wedge \nu z.\neg\phi[\neg z/z]$ is consistent. As $\vdash \phi \Leftrightarrow \phi'$, by Lemma 26 $\vdash \neg\phi[\neg z/z] \Leftrightarrow \neg\phi'[\neg z/z]$. Since ϕ' is in banan-form, by Lemma 16 $\neg\phi'[\neg z/z]$ is in banan-form, hence guarded and aconjunctive. But then by Lemma 35 $\psi \wedge \nu z.\neg\phi[\neg z/z] = \psi \wedge \neg\mu z.\phi$ is satisfiable, contradicting $\models \mu z.\phi \Leftrightarrow \psi$, i.e. case 2 cannot hold either.

Consequently, $\vdash \mu z.\phi' \Leftrightarrow \psi$, which concludes the induction step.

Based on this lemma, the completeness of the axiomatisation follows easily.

Theorem 37. *If a formula ϕ is consistent, then ϕ is satisfiable.*

Proof. Direct from Lemma 36, Proposition 34, and Theorem 25.

Corollary 38 (Completeness). *If $\models \phi$ then $\vdash \phi$.*

Proof. If $\models \phi$, then $\neg\phi$ is not satisfiable, therefore not consistent, implying $\vdash \neg\neg\phi$, i.e. $\vdash \phi$.

Acknowledgements

I would like to thank Colin Stirling for many useful discussions. This work has been funded by the Academy of Finland, the British Council, the CVCP Overseas Research Students Awards Scheme and the Finnish Cultural Foundation.

[2] For the record, this in retrospect very natural observation had escaped us, leading to a more restricted and cumbersome solution, until we saw it used in passing in Igor Walukiewicz's work [20].

References

1. Arnold, A. & Niwiński, D.: Fixed Point Characterisation of Weak Monadic Logic Definable Sets of Trees, in Nivat, M. & Podelski, A. (eds.): *Tree Automata and Languages*, Elsevier, 1992, pp. 159-188
2. Banieqbal, B. & Barringer, H.: Temporal Logic with Fixed Points, in *Temporal Logic in Specification*, LNCS vol. 398, Springer-Verlag, 1989, pp. 62-74
3. Barringer, H. & Kuiper, R. & Pnueli, A.: A Really Abstract Concurrent Model and its Temporal Logic, in *Proc. of the 13th ACM POPL*, 1986
4. Dam, M.: Fixpoints of Büchi automata, in *Proceedings of the 12th FST & TCS*, LNCS vol. 652, Springer-Verlag, 1992, pp. 39-50
5. Emerson, E. A.: Temporal and Modal Logic, in van Leeuwen, J. (ed.): *Handbook of Theoretical Computer Science*, Elsevier/North-Holland, 1990, pp. 997-1072
6. Emerson, E. A. & Lei, C. L.: Efficient Model-Checking in Fragments of the Propositional Mu-calculus, in *Proc. of the First IEEE LICS*, 1986, pp. 267-278
7. Kozen, D.: Results on the Propositional μ-calculus, in *Theoretical Computer Science*, vol. 27, 1983, pp. 333-354
8. Lichtenstein, O.: *Decidability, Completeness, and Extensions of Linear Time Temporal Logic*, PhD thesis, The Weizmann Institute of Science, Rehovot, Israel, 1991
9. McNaughton, R.: Testing and Generating Infinite Sequences by a Finite Automaton, in *Information and Control*, vol. 9, 1966, pp. 521-530
10. Muller, D.E. & Saoudi, A. & Schupp, P.: Alternating Automata, the Weak Monadic Theory of the Tree, and its Complexity, in *Proc. of the 13th ICALP*, LNCS vol. 226, Springer-Verlag, 1986, pp. 275-283
11. Niwiński, D.: On Fixed Point Clones, in *Proc. of the 13th ICALP*, LNCS vol. 226, Springer-Verlag, 1986, pp. 402-409
12. Siefkes, D.: *Büchi's Monadic Second Order Successor Arithmetic, Decidable Theories I*, LNM vol. 120, Springer-Verlag, 1970
13. Stirling, C.: Modal and Temporal Logics, in Abramsky, S. & al. (eds.): *Handbook of Logic in Computer Science*, Oxford University Press, 1992, pp. 477-563
14. Stirling, C. & Walker, D.: Local Model Checking in the Modal Mu-calculus, in *Theoretical Computer Science*, vol. 89, 1991, pp. 161-177
15. Streett, R. S. & Emerson, E. A.: An Automata Theoretic Decision Procedure for the Propositional Mu-Calculus, in *Information and Computation*, vol. 81, 1989, pp. 249-264
16. Thomas, W.: Automata on Infinite Objects, in van Leeuwen, J. (ed.): *Handbook of Theoretical Computer Science*, vol. 2, North-Holland, 1989, pp. 133-191
17. Vardi, M. Y.: A Temporal Fixpoint Calculus, in *Proceedings of the 15th ACM Symposium on Priciples of Programming Languages*, 1988, pp. 250-259
18. Vardi, M. Y. & Wolper, P.: Reasoning about Infinite Computations, in *Information and Computation*, vol. 115, 1994, pp. 1-37
19. Walukiewicz, I.: On Completeness of the μ-calculus, in *Proc. of the 8th IEEE LICS*, 1993, pp. 136-146
20. Walukiewicz, I.: Completeness of Kozen's Axiomatisation of the Propositional μ-calculus, to appear in *Proc. of the 10th IEEE LICS*, 1995
21. Wolper, P.: *Synthesis of Communicating Processes from Temporal Logic Specifications*, PhD thesis, Stanford University, 1982
22. Wolper, P.: Temporal Logic can be More Expressive, in *Information and Control*, vol 56, 1983, pp. 72-99