Lax naturality through enrichment*

Dedicated to Robert Gordon on the occasion of his 60th birthday

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Abstract

We develop the relationship between algebraic structure and monads enriched over the monoidal biclosed category LocOrd_l of small locally ordered categories, with closed structure given by $\mathrm{Lax}(A, B)$. We state the theorem, give a series of examples, and incorporate an account of sketches and contravariance into the theory. This was motivated by C.A.R. Hoare's use of category theoretic structures to model data refinement.

1 Introduction

In 1987, C.A.R. Hoare wrote a draft paper, "Data refinement in a categorical setting" [10] in which he used category theory to provide an abstract formalism for his development of data refinement over the previous twenty years [9]. The notion of data refinement is central to the programming method called stepwise refinement proposed by Wirth [19], and gave rise to work on abstract data types such as the IOTA programming system developed by Nakajima, Honda and Nakahara [16]. As Hoare said in [10], there was evidently a unified body of category theory underlying his constructions, but he was unaware of the details.

Prompted by Hoare's question, the second author of this paper wrote an article "An algebraic formulation for data refinement" [14] in which he gave

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a partial answer by use of monads on enriched categories, and the connection between monads and universal algebra. However, the answer was only partial because the theory of monads on enriched categories was insufficient at the time to encapsulate all of Hoare's constructions. Specifically, there was an account of monads enriched over a symmetric monoidal closed category (see [13], or [17] for an account directed towards computer scientists); but no account in which the symmetry axiom is dropped. Dropping assymmetry is essential because Hoare used lax transformations which are inherently assymetric (see Section 2). That in turn prompted the second author to write a series of articles, [5], [6] and [4], with Robert Gordon to extend the relationship between enriched monads and algebraic structure by dropping symmetry. That has now been completed, but the articles are accessible to only a tiny readership as they are written in the setting of categories enriched over biclosed bicategories, and assume some knowledge of that setting. Moreover, although establishing the central abstract result required, they are not immediately reducible to Hoare's question without further analysis as there are special features of the enriched categories implicit in his work.

So, in this paper, we give a more broadly accessible account of algebraic structure on categories enriched over the particular monoidal biclosed category LocOrd_l: that is the category of small locally ordered categories, together with the tensor product $\otimes$ (described in Section 2) for which $-\otimes A$ has right adjoint given by $\mathrm{Lax}(A,-)$, where $\mathrm{Lax}(A,B)$ is given by locally ordered functors and lax transformations. Of course, LocOrd_l was the monoidal biclosed category implicitly used by Hoare, as explained in [14]. Although motivated by Hoare's constructions, this is inherently a mathematical paper rather than a computer science one, as we give an abstract treatment of small locally ordered categories with algebraic structure. The special features of LocOrd_l that we use arise primarily in studying sketches and contravariance. Note that this paper does not definitively establish the connection to the computer science problem of data refinement, as Hoare's development remains to be fleshed out and properly analysed.

We were pleasantly surprised to discover (Proposition 8.2) that small locally ordered categories, locally ordered functors, and lax transformations that pointwise have left adjoints, form a 2-category (but see [2] for a more general result). In fact, it is technically delicate but not difficult to prove that they form a locally finitely presentable 2-category. A mild variant of that formed the basis for Hoare's account of structures of mixed variance, such as the closure in a cartesian closed locally ordered category. We were also pleased (Theorem 7.3) to see that a careful analysis of cotensors allows us to deduce that the universal property of the free algebra on a sketch extends to account for structure respecting lax transformations: that is not simply an assertion that the evident forgetful functor enriches.

The paper is organized as follows. In Section 2, we describe the enrichment we require and give the basic category theoretic definitions. In Sections 3 and 4, we describe the abstract mathematical result we require. In Section

5, we adapt it to our particular setting, and spell it out more explicitly. Section 6 is devoted to examples. Section 7 gives an account of sketches, and Section 8 accounts for contravariance, thus completing our analysis.

2 $\mathcal{V}$-monads for monoidal biclosed $\mathcal{V}$

A monoidal category $\mathcal{V}$ is called *biclosed* if for every object X of $\mathcal{V}$, both $-\otimes X\colon \mathcal{V}_\circ \to \mathcal{V}_\circ$ and $X \otimes -\colon \mathcal{V}_\circ \to \mathcal{V}_\circ$ have right adjoints, denoted $[\,X, -\,]_r$ and $[\,X, -\,]_l$ respectively. For monoidal biclosed locally small $\mathcal{V}$, it is evident how to define $\mathcal{V}$*-categories*, $\mathcal{V}$*-functors* and $\mathcal{V}$*-natural transformations*, yielding the 2-category $\mathcal{V}$-*Cat* of small $\mathcal{V}$-categories. The category $\mathcal{V}_\circ$ enriches to a $\mathcal{V}$-category with hom given by $[\,X, Y\,]_r$. Note that $[\,X, Y\,]_r$ cannot be replaced by $[\,X, Y\,]_l$ here, using the usual conventions of Kelly's book [11]. One can speak of *representable* $\mathcal{V}$-functors and there is an elegant theory of $\mathcal{V}$-adjunctions, see for instance [5]. If $\mathcal{V}_\circ$ is complete, then it is shown in [18] that $\mathcal{V}$-*Cat* is complete. So we can speak of the Eilenberg-Moore $\mathcal{V}$-category for a $\mathcal{V}$-monad. If $\mathcal{V}$ is also cocomplete, there is an elegant theory of limits and colimits in $\mathcal{V}$-categories generalising the situation for symmetric monoidal closed $\mathcal{V}$, see for instance [6].

Spelling out the situation for monads, a $\mathcal{V}$*-monad* on a $\mathcal{V}$-category $\mathcal{C}$ consists of a $\mathcal{V}$-functor $T\colon \mathcal{C} \to \mathcal{C}$ and $\mathcal{V}$-natural transformations $\eta\colon 1 \to T$ and $\mu\colon T^2 \to T$ satisfying the usual three axioms. The Eilenberg-Moore $\mathcal{V}$-category T-*Alg* has as objects the $T_\circ$-algebras, where $T_\circ$ is the ordinary monad underlying T, on the ordinary category $C_\circ$ underlying C. Given algebras $\langle A,\, a\rangle$ and $\langle B,\, b\rangle$, the hom object $T\text{-}\mathsf{Alg}(\langle A,\, a\rangle, \langle B,\, b\rangle)$ is the equaliser in $\mathcal{V}$ of the diagram

$$\begin{tikzcd}
\mathcal{C}(A,B) \arrow[rr, "\mathcal{C}(a,B)"] \arrow[dr, "T"'] & & \mathcal{C}(T(A),B) \\
& \mathcal{C}(T(A),T(B)) \arrow[ur, "\mathcal{C}(T(A),b)"'] &
\end{tikzcd} \tag{1}$$

Composition in T-Alg is determined by that in $\mathcal{C}$.

There is a forgetful $\mathcal{V}$-functor $U\colon T\text{-}\mathsf{Alg} \to \mathcal{C}$, and it has a left adjoint. If $\mathcal{C}$ is complete, then so is T-Alg, and U preserves limits.

Our leading example of monoidal biclosed $\mathcal{V}$ is given as follows. Given locally ordered categories A and B, and locally ordered functors $H, K\colon A \to B$ a *lax transformation* $\alpha\colon H \Rightarrow K$ from H to K consists of, for each object a of A, an arrow $\alpha_a\colon H(a) \to K(a)$, such that for every arrow $f\colon a \to b$ in A, we have

$$\begin{tikzcd} H(a) \arrow[r, "\alpha_a"] \arrow[d, "H(f)"'] \arrow[dr, phantom, "\geq"] & K(a) \arrow[d, "K(f)"] \\ H(b) \arrow[r, "\alpha_b"'] & K(b). \end{tikzcd} \tag{2}$$

Dually, an *oplax transformation* from H to K has the same data but with the inequality of (2) reversed. Given lax transformations $\alpha, \beta\colon H \Rightarrow K$, we say $\alpha \leq \beta$ if $\alpha_a \leq \beta_a$ in B for every a in A. Thus we have a locally ordered category $\mathrm{Lax}(A, B)$ given by locally ordered functors and lax transformations. Our monoidal biclosed category $\mathcal{V}$ consists of the category of small locally ordered categories and locally ordered functors, together with that monoidal structure (described in Gray's book [8], Theorem I,4.9.) for which, for each object A of $\mathcal{V}$, $-\otimes A$ has right adjoint $\mathrm{Lax}(A, -)$. We call this tensor product the *Gray* tensor product. To give an explicit definition of it, let A and B be small locally ordered categories. Then we define $A \otimes B$ to be the locally ordered category

- whose objects are pairs $\langle a, b\rangle \in |\,A\,| \times |\,B\,|$;
- where a 1-cell from $\langle a, b\rangle$ to $\langle a', b'\rangle$ is a finite sequence of non-identity 1-cells alternating between 1-cells in A and 1-cells in B such that the subsequence of 1-cells in A is a directed path in A from a to a' and the subsequence of 1-cells in B is a directed path in B from b to b' (we often write the sequence of 1-cells as a sequence of pairs of a 1-cell and an object, as below);
- whose identity on $\langle a, b\rangle$ is the empty sequence;
- whose composition is given by first concatenating the sequences, then factoring according to the composition and identities in A and B; and
- whose local order is generated by the local order on A and that on B and

$$\begin{tikzcd} \langle a, b\rangle \arrow[r, "\langle f, b\rangle"] \arrow[d, "\langle a, g\rangle"'] \arrow[dr, phantom, "\geq"] & \langle a', b\rangle \arrow[d, "\langle a', g\rangle"] \\ \langle a, b'\rangle \arrow[r, "\langle f, b'\rangle"'] & \langle a', b'\rangle \end{tikzcd}$$

  for any $f\colon a \to a'$ and $g\colon b \to b'$.

For example,

$$
\begin{tikzcd}
\langle a,\, b\rangle \arrow[r, "f_1\colon a \to a_1"] & \langle a_1,\, b\rangle \arrow[d, "g_1\colon b \to b_1"] & \\
& \langle a_1,\, b_1\rangle \arrow[r, "f_2\colon a_1 \to a'"] & \langle a',\, b_1\rangle \arrow[d, "g_2\colon b_1 \to b"] \\
& & \langle a',\, b'\rangle
\end{tikzcd}
$$

with $f_i \neq id$, $g_i \neq id$ for $i = 1, 2$, is a 1-cell from $\langle a,\, b\rangle$ to $\langle a',\, b'\rangle$. The sequence alternates between maps in A and maps in B. The two subsequences are $a \overset{f_1}{\to} a_1 \overset{f_2}{\to} a'$ and $b \overset{g_1}{\to} b_1 \overset{g_2}{\to} b'$. To compose this 1-cell and $\langle a',\, b'\rangle \overset{h\colon b' \to b''}{\to} \langle a',\, b''\rangle$, we must compose g_2 with h. Then if that composite is id, delete it from the sequence.

A variant of the Gray tensor product is now appearing extensively in category theory, for instance in [7]. The Gray tensor product, as defined here, is not symmetric, but it is biclosed: $A \otimes -$ has right adjoint given by $\mathrm{Oplax}(A, -)$, where $\mathrm{Oplax}(A, B)$ is the locally ordered category of locally ordered functors from A to B and oplax transformations between them. We will denote this monoidal biclosed category by LocOrd_l.

3 Locally finitely presentable $\mathcal{V}$-categories

A monoidal biclosed category $\mathcal{V}$ is called *locally finitely presentable as a closed category* if $\mathcal{V}_\circ$ is locally finitely presentable, if the unit I of $\mathcal{V}$ is finitely presentable, and if $x \otimes y$ is finitely presentable whenever x and y are.

LocOrd_l is locally finitely presentable as a closed category. Henceforth, all monoidal biclosed categories to which we refer will be assumed to be locally finitely presentable as closed categories.

A $\mathcal{V}$-category $\mathcal{C}$ is said to have *finite tensors* if for every finitely presentable object x of $\mathcal{V}$, and for every object A of $\mathcal{C}$, $[\,x, \mathcal{C}(A, -)\,]_r\colon \mathcal{C} \to \mathcal{V}$ is representable, i.e., if there exists an object $x \otimes A$ of $\mathcal{C}$ together with a natural isomorphism $[\,x, \mathcal{C}(A, -)\,]_r \cong \mathcal{C}(x \otimes A, -)$. The $\mathcal{V}$-category $\mathcal{C}$ has *finite cotensors* if it satisfies the dual condition that for every finitely presentable x in $\mathcal{V}$ and every A in $\mathcal{C}$, the $\mathcal{V}_t$-functor $[\,x, \mathcal{C}(-, A)\,]_l\colon \mathcal{C}^{op} \to \mathcal{V}_t$ is representable, where $\mathcal{V}_t$ is the dual of $\mathcal{V}$; i.e., there exists an object $x \pitchfork A$ of $\mathcal{C}$ together with a natural isomorphism $[\,x, \mathcal{C}(-, A)\,]_l \cong \mathcal{C}(-, x \pitchfork A)$. A $\mathcal{V}$-category $\mathcal{C}$ is *cocomplete* whenever $\mathcal{C}_\circ$ is cocomplete, $\mathcal{C}$ has finite tensors, and $x \otimes -\colon \mathcal{C}_\circ \to \mathcal{C}_\circ$ preserves colimits for all finitely presentable x. A cocomplete $\mathcal{V}$-category $\mathcal{C}$ is *locally finitely presentable* (lfp) if $\mathcal{C}_\circ$ is locally finitely presentable, $\mathcal{C}$ has finite cotensors, and $x \pitchfork -\colon \mathcal{C}_\circ \to \mathcal{C}_\circ$ preserves filtered colimits for all finitely presentable x.

For any $\mathcal{V}$ that is locally finitely presentable as a closed category, $\mathcal{V}$ is necessarily a locally finitely presentable $\mathcal{V}$-category. So, in particular, LocOrd_l is a locally finitely presentable LocOrd_l-category.

4 Algebraic Structure

We now define *algebraic structure* on an lfp $\mathcal{V}$-category $\mathcal{C}$. Let $|\,\mathcal{C}_f\,|$ denote the discrete $\mathcal{V}$-category on the set of (isomorphism classes of) finitely presentable objects in $\mathcal{C}_\circ$. Then a *signature* on $\mathcal{C}$ consist of a $\mathcal{V}$-functor $S\colon |\,\mathcal{C}_f\,| \to \mathcal{C}$. For each $c \in |\,\mathcal{C}_f\,|$, $S(c)$ is called the object of *basic operations of arity* c.

From S, we construct $F(S)\colon \mathcal{C}_f \to \mathcal{C}$ as follows: set

$$\begin{array}{rcl} S_0 & = & J, \quad \text{the inclusion of } \mathcal{C}_f \text{ in } \mathcal{C} \\ S_{n+1} & = & J + \sum\limits_{d \in |\,\mathcal{C}_f\,|} \mathcal{C}(d, S_n(-)) \otimes S(d), \end{array}$$

and define

$$\begin{array}{l} \sigma_0\colon S_0 \to S_1 \quad \text{to be} \quad inj\colon J \to J + \sum_{d \in |\,\mathcal{C}_f\,|} \mathcal{C}(d, S_0(-)) \otimes S(d) \\ \sigma_n = J + \sum_{d \in |\,\mathcal{C}_f\,|} \mathcal{C}(d, \sigma_{n-1}(-)) \otimes S(d)\colon S_n \to S_{n+1} \\ F(S) = \underset{n<\omega}{colim}\, S_n. \end{array}$$

In many cases of interest, each σ_n is a monomorphism, so $F(S)$ is the union of $\{\, S_n \,\}_{n<\omega}$. For each c, we call $F(S)(c)$ the object of *derived c-ary operations*.

A signature is typically accompanied by equations between derived operations. So we define the *equations* of an algebraic theory with signature S to consist of a $\mathcal{V}$-functor $E\colon |\,\mathcal{C}_f\,| \to \mathcal{C}$ together with $\mathcal{V}$-natural transformations $\tau_1, \tau_2\colon E \to F(S)(K(-))$, where $K\colon |\,\mathcal{C}_f\,| \to \mathcal{C}_f$ is the inclusion.

Algebraic structure on $\mathcal{C}$ consists of a signature S, together with equations $\langle E,\, \tau_1,\, \tau_2 \rangle$. We will generally denote algebraic structure by $\langle S,\, E \rangle$, suppressing τ_1 and τ_2.

We now define the algebras for a given algebraic structure. Given a signature S, an *S-algebra* consists of an object A of $\mathcal{C}$ together with a map $\nu_c\colon \mathcal{C}(c, A) \to \mathcal{C}(S(c), A)$ for each c. So, an S-algebra consists of a carrier A and an interpretation of the basic operations of the signature. This interpretation extends canonically to the derived operations, giving an $F(S)(K(-))$-algebra, as follows:

$$\nu_0\colon \mathcal{C}(c, A) \to \mathcal{C}(S_0(c), A) \quad \text{is the identity;}$$

to give $\nu_{n+1}\colon \mathcal{C}(c, A) \to \mathcal{C}(S_{n+1}(c), A)$, using the fact that $\mathcal{C}(-, A)$ preserves colimits, is to give a map $\mathcal{C}(c, A) \to \mathcal{C}(c, A)$, which we will make the identity, and for each d in $|\,\mathcal{C}_f\,|$, a map

$$\mathcal{C}(c, A) \to [\, \mathcal{C}(d, S_n(c)), \mathcal{C}(S(d), A) \,]_r,$$

or equivalently, $\mathcal{C}(c, A) \otimes \mathcal{C}(d, S_n(c)) \to \mathcal{C}(S(d), A)$, which is given inductively by

$$\mathcal{C}(c,A) \otimes \mathcal{C}(d,S_n(c)) \xrightarrow{\nu_n \otimes id} \mathcal{C}(S_n(c),A) \otimes \mathcal{C}(d,S_n(c)) \xrightarrow{comp} \mathcal{C}(d,A) \xrightarrow{\nu} \mathcal{C}(S(d),A)$$

Given signature S and equations E, an $\langle S,\ E\rangle$*-algebra* is an S-algebra that satisfies the equations, i.e., an S-algebra $\langle A,\ \nu\rangle$ such that both legs of

$$\begin{tikzcd} \mathcal{C}(c,A) \arrow[r, "\nu_\omega"] & \mathcal{C}(F(S)(Kc),A) \arrow[r, shift left, "\mathcal{C}(\tau_{1c},A)"] \arrow[r, shift right, "\mathcal{C}(\tau_{2c},A)"'] & \mathcal{C}(E(c),A) \end{tikzcd}$$

agree.

Given $\langle S,\ E\rangle$-algebras $\langle A,\ \nu\rangle$ and $\langle B,\ \delta\rangle$, we define the hom object $\langle S,\ E\rangle$-$\mathsf{Alg}(\langle A,\ \nu\rangle, \langle B,\ \delta\rangle)$ to be the equaliser in $\mathcal{V}$ of

$$\begin{tikzcd} \mathcal{C}(A,B) \arrow[rr, "\{\,\mathcal{C}(c,-)\,\}_{c\in|\mathcal{C}_f|}"] \arrow[d, "\{\,\mathcal{C}(S(c),-)\,\}_{c\in|\mathcal{C}_f|}"'] & & \prod_c [\,\mathcal{C}(c,A),\mathcal{C}(c,B)\,]_r \arrow[d, "\prod_c[\,\mathcal{C}(c,A),\delta_c\,]_r"] \\ \prod_c [\,\mathcal{C}(S(c),A),\mathcal{C}(S(c),B)\,]_r \arrow[rr, "\prod_c[\,\nu_c,\mathcal{C}(S(c),B)\,]_r"'] & & \prod_c [\,\mathcal{C}(c,A),\mathcal{C}(S(c),B)\,]_r . \end{tikzcd} \quad (3)$$

This agrees with our usual universal algebraic understanding of the notion of homomorphism of algebras, internalising it to $\mathcal{V}$. $\langle S,\ E\rangle$-Alg can then be made into a $\mathcal{V}$-category in which composition is induced by that in $\mathcal{C}$. An arrow in the underlying ordinary category $\langle S,\ E\rangle$-$\mathsf{Alg}_\circ$ is an arrow $f\colon A \to B$ in $\mathcal{C}_\circ$ such that for all finitely presentable c, $f\nu_c(-) = \delta_c(f-)\colon \mathcal{C}(c, A) \to \mathcal{C}(S(c), B)$, i.e., an arrow in $\mathcal{C}_\circ$ that commutes with all basic c-ary operations for all c.

The main result of [4], which is central to our work is

Theorem 4.1 *A $\mathcal{V}$-category A is equivalent to $\langle S,\ E\rangle$-Alg for algebraic structure $\langle S,\ E\rangle$ on $\mathcal{C}$ if and only if there is a finitary $\mathcal{V}$-monad T on $\mathcal{C}$ such that A is equivalent to T-Alg.*

5 Algebraic Structure on Locally Ordered Categories

The central part of our abstract theorem, Theorem 4.1, can be expressed with considerably less abstract category theoretic terminology in the case of greatest interest to us. As mentioned earlier, LocOrd_l is locally finitely presentable as a closed category, and also as a LocOrd_l-category. Moreover, the finitely presentable objects of LocOrd_l *include* all the finite locally ordered categories, which are all that we require. In fact, there are some more finitely presentable objects in LocOrd_l, but we will not need them. Further, we only need the forward direction of the theorem. Finally, we can express the definition of algebraic structure with less explicit reference to the tensor product, and we can remove the explicit use of $F(S)$ without important loss of generality.

By mild abuse of notation, let LocOrd_{lf} denote the full sub-LocOrd_l-category of LocOrd_l determined by the (isomorphism classes of the) finite locally ordered categories, and let $|\,\mathsf{LocOrd}_{lf}\,|$ denote its set of objects regarded as a discrete LocOrd_l-category. Now say a *signature* on LocOrd_l is a LocOrd_l-functor $S\colon |\,\mathsf{LocOrd}_{lf}\,| \to \mathsf{LocOrd}_l$, and call $S(c)$ the locally ordered category of *basic operations of arity* c. We then define

$$\begin{aligned} S_0 &= J, \quad \text{the inclusion of } \mathsf{LocOrd}_{lf} \text{ in } \mathsf{LocOrd}_l \\ S_{n+1} &= J + \sum_{d \in |\,\mathsf{LocOrd}_{lf}\,|} \mathrm{Lax}(d, S_n(-)) \otimes S(d), \end{aligned}$$

and define

$$\begin{aligned} &\sigma_0\colon S_0 \to S_1 \quad \text{to be} \quad inj\colon J \to J + \textstyle\sum_{d \in |\,\mathsf{LocOrd}_{lf}\,|} \mathrm{Lax}(d, S_0(-)) \otimes S(d) \\ &\sigma_n = J + \textstyle\sum_{d \in |\,\mathsf{LocOrd}_{lf}\,|} \mathrm{Lax}(d, \sigma_{n-1}(-)) \otimes S(d)\colon S_n \to S_{n+1}. \end{aligned}$$

Then we say *algebraic structure* on LocOrd_l consists of a signature S together with a LocOrd_l-functor $E\colon |\,\mathsf{LocOrd}_{lf}\,| \to \mathsf{LocOrd}_l$ of *equations* together with LocOrd_l-natural transformations $E \overset{\tau_1}{\underset{\tau_2}{\rightrightarrows}} S_n(K(-))$ for some n, where $K\colon |\,\mathsf{LocOrd}_{lf}\,| \to \mathsf{LocOrd}_{lf}$ is the inclusion. We denote such algebraic structure by $\langle S, E\rangle$.

Given a signature S, an S-*algebra* consists of a small locally ordered category A together with a locally ordered functor $\nu_c\colon \mathrm{Lax}(c, A) \to \mathrm{Lax}(S(c), A)$ for each finite c. An S-algebra extends to an S_n-algebra for each n, by induction as follows:

- $\nu_0\colon \mathrm{Lax}(c, A) \to \mathrm{Lax}(S_0(c), A)$ is the identity

- $\nu_{n+1}\colon \mathrm{Lax}(c, A) \to \mathrm{Lax}(S_{n+1}(c), A)$ is given using the inductive definition of S_{n+1}, by the identity functor from $\mathrm{Lax}(c, A)$ to $\mathrm{Lax}(c, A)$, and for each finite d, the functor

  $$\mathrm{Lax}(c, A) \to \mathrm{Lax}(\mathrm{Lax}(d, S_n(c)), \mathrm{Lax}(S(d), A))$$

  given by

  $$\begin{aligned} &\mathrm{Lax}(c, A) \xrightarrow{\nu_n} \mathrm{Lax}(S_n(c), A) \xrightarrow{\text{“postcomposition”}} \mathrm{Lax}(\mathrm{Lax}(d, S_n(c)), \mathrm{Lax}(d, A)) \\ &\xrightarrow{\mathrm{Lax}(id, \nu_d)} \mathrm{Lax}(\mathrm{Lax}(d, S_n(c)), \mathrm{Lax}(S(d), A)). \end{aligned}$$

  So, this takes a functor $h\colon c \to A$ to the functor from $\mathrm{Lax}(d, S_n(c))$ to $\mathrm{Lax}(S(d), A)$ taking $\varphi\colon d \to S_n(c)$ to $\nu(\nu_n(h) \circ \varphi)\colon S(d) \to A$.

An $\langle S,\, E\rangle$-*algebra* is an S-algebra such that both legs of

$$\mathrm{Lax}(c,A) \xrightarrow{\nu_n} \mathrm{Lax}(S_n(K(c)),A) \underset{\mathrm{Lax}(\tau_{2c},A)}{\overset{\mathrm{Lax}(\tau_{1c},A)}{\rightrightarrows}} \mathrm{Lax}(E(c),A)$$

agree.

The LocOrd_l-category $\langle S,\, E\rangle$-Alg is defined in the evident way, by restricting (3) appropriately. Then we have

Corollary 5.1 *For any algebraic structure* $\langle S,\, E\rangle$ *on* LocOrd_l, $\langle S,\, E\rangle$-Alg *is finitarily* LocOrd_l*-monadic over* LocOrd_l.

6 Examples

In this section, we give a series of examples, all implicit in Hoare's work, of algebraic structures $\langle S,\, E\rangle$ on LocOrd_l. Theorem 4.1 tells us that for each such algebraic structure, there is a finitary monad T on LocOrd_l for which the LocOrd_l-category $\langle S,\, E\rangle$-Alg of algebras for the given algebraic structure is equivalent to T-Alg.

For a particularly simple example of algebraic structure on LocOrd_l, consider

Example 6.1 $S\colon |\,\mathsf{LocOrd}_{lf}\,| \to \mathsf{LocOrd}_l$

$$S(c) = \begin{cases} 1 & \text{if } c = 1 \\ 0 & \text{otherwise,} \end{cases}$$

with $E(c) = 0$ for all c. An $\langle S,\, E\rangle$-algebra is then a small locally ordered category A, together with an endofunctor on it: this follows since $\mathrm{Lax}(1,A)$ is equivalent to A. This agrees with Hoare's example of a "functorial generator", which is one of his two leading examples.

Example 6.2 $S\colon |\,\mathsf{LocOrd}_{lf}\,| \to \mathsf{LocOrd}_l$

$$S(c) = \begin{cases} 1 & \text{if } c = 2\text{, where 2 is the discrete category on two elements,} \\ 0 & \text{otherwise,} \end{cases}$$

with $E(c) = 0$ for all c. An $\langle S,\, E\rangle$-algebra is a small locally ordered category A together with a locally ordered functor $\circ\colon A \times A \to A$.

Example 6.3 $S\colon |\,\mathsf{LocOrd}_{lf}\,| \to \mathsf{LocOrd}_l$

$$S(c) = \begin{cases} \mathbf{2} & \text{if } c = 1\text{, where } \mathbf{2} \text{ is the category } \{\, 0 \to 1 \,\} \\ 0 & \text{otherwise,} \end{cases}$$

and $E(c) = 0$ for all c.

An $\langle S,\, E\rangle$-algebra is a small locally ordered category A together with two locally ordered endofunctors H and K on it, and an oplax transformation from H to K: this is because a locally ordered functor from $A \cong \mathrm{Lax}(1,A)$ to $\mathrm{Lax}(\mathbf{2},A)$ amounts to exactly the above. This is the second of Hoare's two leading examples, that of a "simulation generator".

Example 6.4 To obtain a small strict monoidal locally ordered category as an $\langle S,\, E\rangle$-algebra, first put

$$S(c) = \begin{cases} 1 & \text{if } c = 2 \\ 1 & \text{if } c = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then an S- algebra is a small locally ordered category A together with locally ordered functors $\circ\colon A \times A \to A$ and $I\colon 1 \to A$.

The equations apply to derived operations; so we need to describe S_2.

$$\begin{array}{l} S_0\colon \mathsf{LocOrd}_{lf} \to \mathsf{LocOrd}_l \quad \text{is the inclusion } J. \\ S_1(c) = c + \sum_{d \in |\mathsf{LocOrd}_{lf}|} \mathrm{Lax}(d,c) \otimes S(d), \end{array}$$

but for all d other than 0 and 2, $S(d) = 0$, and so $\mathrm{Lax}(d,c) \otimes S(d) = 0$; and $\mathrm{Lax}(0,c) = 1$ and $\mathrm{Lax}(2,c) = c \times c$. So

$$S_1(c) = c + S(0) + (c \times c) \otimes S(2) = c + 1 + c \times c.$$

The S-algebra $\langle A,\, I\colon 1 \to A,\, \circ\colon A \times A \to A\rangle$ yields an S_1-algebra $\langle A,\, \nu_1\rangle$ given by A, and the functor from $\mathrm{Lax}(c, A)$ to $\mathrm{Lax}(S_1(c), A)$ with

- first component $\mathrm{Lax}(c, A) \to \mathrm{Lax}(c, A)$ the identity
- second component $\mathrm{Lax}(c, A) \to \mathrm{Lax}(1, A)$ the constant at I
- third component $\mathrm{Lax}(c, A) \to \mathrm{Lax}(c \times c, A)$ sending h to the functor sending $\langle X,\, Y\rangle$ to $h(X) \circ h(Y)$.

Now,

$$S_2(c) = c + \sum_{d \in |\mathsf{LocOrd}_{lf}|} \mathrm{Lax}(d, S_1(c)) \otimes S(d).$$

Again, for all d other than 0 and 2, $S(d) = 0$, and so $\mathrm{Lax}(d, S_1(c)) \otimes S(d) = 0$; $\mathrm{Lax}(0, S_1(c)) = 1$, and $\mathrm{Lax}(2, S_1(c)) = S_1(c) \times S_1(c)$. So,

$$\begin{array}{rcl} S_2(c) & = & c + S(0) + (S_1(c) \times S_1(c)) \otimes S(2) \\ & = & c + 1 + S_1(c) \times S_1(c). \end{array}$$

Our S-algebra yields the S_2-algebra given by A and the locally ordered functor from $\mathrm{Lax}(c, A)$ to $\mathrm{Lax}(S_2(c), A)$ with

- first component $\mathrm{Lax}(c, A) \to \mathrm{Lax}(c, A)$ the identity
- second component $\mathrm{Lax}(c, A) \to \mathrm{Lax}(1, A)$ the constant at I
- third component $\mathrm{Lax}(c, A) \to \mathrm{Lax}(S_1(c) \times S_1(c), A)$ sending h to the functor sending a pair $\langle X,\, Y\rangle$ to $(\nu_1 h)(X) \circ (\nu_1 h)(Y)$.

Spelling out the third component in more detail, $S_1(c) = c + 1 + c \times c$. So there are nine classes of possibilities for the pair $\langle X, Y\rangle$, and we have $\mathrm{Lax}(c, A) \to \mathrm{Lax}((c + 1 + c \times c) \times (c + 1 + c \times c), A)$. On the first and second components of each copy of $S_1(c)$ respectively, our functor sends h to $X \mapsto h(X) \circ I$. On the second and first component respectively, h is sent to $X \mapsto I \circ h(X)$. On the first and third components, we have h sent to $\langle X, \langle Y, Z\rangle\rangle \mapsto h(X) \circ (h(Y) \circ h(Z))$. On the third and first components, h goes to $\langle\langle X, Y\rangle, Z\rangle \mapsto (h(X) \circ h(Y)) \circ h(Z)$. So we put

$$E(c) = \begin{cases} 2 & \text{if } c = 1, \\ 1 & \text{if } c = 3, \\ 0 & \text{otherwise,} \end{cases}$$

and define $\tau_1, \tau_2\colon E \to S_2$ with components

$$\tau_1, \tau_2\colon E(1) \longrightarrow S_2(1) = 1 + S(0) + S_1(1) \times S_1(1),$$

so that τ_1 is the constant on the first component of $S_2(1)$, and τ_2 sends one element to the first and second component of the copies of $S_1(1)$ and sends the other element to the second and first components of the copies of $S_1(1)$; and we define $\tau_1, \tau_2\colon E(3) \to S_2(3) = 3 + S(0) + S_1(3) \times S_1(3)$ by having τ_1 choose the first and third components of $S_1(3) = 3 + 1 + 3 \times 3$, thus an element of $3 \times (3 \times 3)$, and we choose that element by choosing the first, second and third components of 3 respectively; and similarly for τ_2 but with the third and first components of $S_1(3)$.

Then, an $\langle S, E\rangle$-algebra is precisely a small strict monoidal locally ordered category. With some effort, one can modify $\langle S, E\rangle$ so that an $\langle S, E\rangle$-algebra is a small monoidal locally ordered category; one can add symmetry too.

Example 6.5 To obtain a small locally ordered category with a terminal object as an $\langle S, E\rangle$-algebra, we first consider what an equational formulation of such a locally ordered category would entail. We require a small locally ordered category A, together with an object T of A, equivalently a functor from $1 = \mathrm{Lax}(0, A)$ to $A = \mathrm{Lax}(1, A)$. We also require for each object X of A, a map from X to T. This is given by the existence of a functor t from $A = \mathrm{Lax}(1, A)$ to $\mathrm{Lax}(\mathbf{2}, A)$, subject to equations to assert that the first component of t_X must be X, and the second component must be T. That is forced by commutativity of the diagrams

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[dr, "id"'] & \mathrm{Lax}(\mathbf{2}, A) \arrow[d, "\mathrm{Lax}(\partial_0, A)"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "t"] \arrow[d] & \mathrm{Lax}(\mathbf{2}, A) \arrow[d, "\mathrm{Lax}(\partial_1, A)"] \\ 1 \arrow[r, "T"'] & A. \end{tikzcd}$$

Observe that the presence of t also asserts that for every arrow $f\colon X \to Y$ in A, we have

$$\begin{tikzcd} X \arrow[r, "t_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & T \arrow[d, "id_T"] \\ Y \arrow[r, "t_Y"'] & T. \end{tikzcd} \tag{4}$$

But that is a harmless addition, as that is true for any map $f\colon X \to Y$ in a locally ordered category with terminal object T.

Finally, we must assert the unicity of t_X. To do that, it suffices to add the equation $t_T = id_T$, which amounts to the commutativity of

$$\begin{tikzcd} 1 \arrow[r, "T"] \arrow[d, "T"'] & A \arrow[d, "t"] \\ A \arrow[r, "\Delta"'] & \mathrm{Lax}(\mathbf{2}, A) \end{tikzcd}$$

and to assert that for each $f\colon X \to Y$, diagram (4) commutes: for that is always true in any category with terminal object T, and it follows from the commutativity that we have the desired unicity by putting $Y = T$ and since $t_T = id_T$. To express that commutativity, we need a new operation

$$(-)_t\colon \mathrm{Lax}(\mathbf{2}, A) \to \mathrm{Lax}(\mathbf{3}, A)$$

taking a map $f\colon X \to Y$ to a commutative triangle, together with equations to force that commutative triangle to be of the form

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "t_X"'] & & Y \arrow[dl, "t_Y"] \\ & T. & \end{tikzcd}$$

This amounts to the assertion that three diagrams commute. Let $\mathbf{3}$ have objects a, b and c, with maps from a to b and from b to c. Let $inc_1\colon \mathbf{2} \to \mathbf{3}$ choose a and b; let inc_2 choose a and c and let inc_3 choose b and c. Then we want

$$\begin{tikzcd}
\mathrm{Lax}(\mathbf{2},A) \arrow[r, "(-)_t"] \arrow[dr, "id"'] & \mathrm{Lax}(\mathbf{3},A) \arrow[d, "\mathrm{Lax}(inc_1,A)"] \\
& \mathrm{Lax}(\mathbf{2},A),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Lax}(\mathbf{2},A) \arrow[r, "(-)_t"] \arrow[d, "\mathrm{Lax}(\partial_0,A)"'] & \mathrm{Lax}(\mathbf{3},A) \arrow[d, "\mathrm{Lax}(inc_2,A)"] \\
A \arrow[r, "t"'] & \mathrm{Lax}(\mathbf{2},A)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Lax}(\mathbf{2},A) \arrow[r, "(-)_t"] \arrow[d, "\mathrm{Lax}(\partial_1,A)"'] & \mathrm{Lax}(\mathbf{3},A) \arrow[d, "\mathrm{Lax}(inc_3,A)"] \\
A \arrow[r, "t"'] & \mathrm{Lax}(\mathbf{2},A).
\end{tikzcd}$$

We can then encode these operations and equations in terms of algebraic structure, just as we did for small strict monoidal locally ordered categories, c.f.[3].

Example 6.6 The argument of Example 6.5 is representative: it extends without much effort to binary products (Hoare's "cartesian product"), an initial object and binary coproducts (Hoare's "discriminated union"). It does not seem to extend to small locally ordered categories with equalisers, the reason being that given a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "h"'] & B \arrow[d, "k"] \\
A' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & B'
\end{tikzcd}$$

with $f'h \leq kf$ and $g'h \leq kg$, there is no reason to expect it to extend to the equalisers of $\langle f, g\rangle$ and $\langle f', g'\rangle$.

7 Sketches

It follows from the previous section that many examples of small locally ordered categories with structure form LocOrd_l-categories of the form T-Alg for a finitary LocOrd_l-monad T on LocOrd_l. It follows that for any small locally ordered category L_0, one can consider the free T-algebra L on L_0; and lax transformations between locally ordered functors from L_0 to a T-algebra $\mathcal{M}$ lift uniquely to structure respecting lax transformations on L.

But to give a full account of Hoare's data refinement, we need a little more, as his starting point L_0 need not be just a small locally ordered category, but may also possess some more axioms, such as stating that a certain object be a product of two others.

In order to incorporate that into our account, we need a new definition, so that we can extend the notion of sketch to an "$\langle S,\, E\rangle$-sketch", for arbitrary algebraic structure $\langle S,\, E\rangle$. This is essentially a special case of Power and Wells' "E-sketches" [15] but in a somewhat more sophisticated setting.

Definition 7.1 *Given algebraic structure $\langle S,\, E\rangle$ on a locally finitely presentable $\mathcal{V}$-category $\mathcal{C}$, an $\langle S,\, E\rangle$-*sketch *consists of an object A of $\mathcal{C}$ together with a finite family of triples $\langle c_i,\, j_i\colon d_i \to F(S)c_i,\, \varphi_i\colon d_i \to A\rangle$ consisting of finitely presentable objects c_i and d_i, and maps j_i and φ_i in $\mathcal{C}$.*

We generally supress j_i, leaving it implicit from d_i, thus using the notation $\langle c_i,\, d_i,\, \varphi_i\rangle$.

Given a finite family $\mathcal{D}$ of pairs $\langle c_i,\, d_i\rangle$ (with j_i implicit), we will call an $\langle S,\, E\rangle$-sketch whose family of triples has first two components determined by $\mathcal{D}$ an *$\langle S,\, E,\, \mathcal{D}\rangle$-sketch.* A map of $\langle S,\, E,\, \mathcal{D}\rangle$-sketches from $\langle A,\, \{\, \varphi_i \,\}\rangle$ to $\langle B,\, \{\, \psi_i \,\}\rangle$ is a map $f\colon A \to B$ in $\mathcal{C}$ such that $f\varphi_i = \psi_i$ for each i. This gives a category $\langle S,\, E,\, \mathcal{D}\rangle$-$\mathsf{Sketch}$, which is finitarily monadic over $\mathcal{C}_\circ$: in fact, we can enrich it to a $\mathcal{V}$-category finitarily monadic over $\mathcal{C}$ by describing $\langle S_\mathcal{D},\, E_\mathcal{D}\rangle$ by

$$S_\mathcal{D}(c) = \begin{cases} \sum_i d_i & \text{if } c = 0, \\ 0 & \text{otherwise} \end{cases}$$

and $E_\mathcal{D}(c) = 0$ for all c.

Also, given $\mathcal{D}$, we define a category $\langle S,\, E,\, \mathcal{D}\rangle$-$\mathsf{Alg}$ with objects $\langle S,\, E\rangle$-algebras $\langle A,\, \nu\rangle$ together with, for each $\langle c_i,\, d_i\rangle$, a map $h_i\colon c_i \to A$. Note that the domain of h_i is c_i, not d_i, but each $h_i\colon c_i \to A$ determines the map $\nu(h_i) \circ j_i\colon d_i \to A$. The maps of $\langle S,\, E,\, \mathcal{D}\rangle$-$\mathsf{Alg}$ are evident. It is finitarily monadic over $\mathcal{C}_\circ$, and the evident functor $\mathcal{U}\colon \langle S,\, E,\, \mathcal{D}\rangle\text{-}\mathsf{Alg} \to \langle S,\, E,\, \mathcal{D}\rangle\text{-}\mathsf{Sketch}$ commutes with the forgetful functors to $\mathcal{C}_\circ$ and hence has a left adjoint $\mathcal{F}$ (See [1] Section 3.7 and [15]).

Now, to incorporate all the data Hoare has in his L_0 into our account, we view L_0 as an $\langle S,\, E,\, \mathcal{D}\rangle$-sketch for appropriate $\langle S,\, E,\, \mathcal{D}\rangle$ and $L = \mathcal{F}(L_0)$. We need to do a little more work to obtain the result we seek, but first for an example

Example 7.2 Let $\langle S,\, E\rangle$ be the algebraic structure for small locally ordered categories with binary products. Let $\mathcal{D}$ consist of one pair $\langle 2,\, d\rangle$, where d is the free category on the graph

$$\cdot \leftarrow \cdot \rightarrow \cdot \quad ,$$

with the inclusion into $F(S)(2)$ yielding that part of $F(S)(2)$ that gives the product cone over two objects. Then, an object of $\langle S,\, E,\, \mathcal{D}\rangle$-$\mathsf{Sketch}$ is a small locally ordered category together with a diagram of the form

$$A \xleftarrow{f} C \xrightarrow{g} B.$$

An object of $\langle S,\, E,\, \mathcal{D}\rangle$-$\mathsf{Alg}$ is a small locally ordered category B with binary products, together with a pair $\langle X,\, Y\rangle$ of objects of B. A map in $\langle S,\, E,\, \mathcal{D}\rangle$-$\mathsf{Sketch}$ from A to $\mathcal{U}(B)$ is a locally ordered functor $H\colon A \to B$ that sends the diagram

$$A \xleftarrow{f} C \xrightarrow{g} B.$$

to the cone

$$X \xleftarrow{\pi_0} X \times Y \xrightarrow{\pi_1} Y,$$

so $H(A) = X$, $H(B) = Y$, and $H(C) = X \times Y$, with f and g sent to the projections. This determines a map in $\langle S,\, E,\, \mathcal{D}\rangle$-$\mathsf{Alg}$ from $\mathcal{F}(A)$ to B: it is a locally ordered functor that strictly preserves binary products and sends $\langle A,\, B\rangle$ to $\langle X,\, Y\rangle$. Since it preserves binary products strictly, it takes C to $X \times Y$ and respects the projections.

Now, to consider the behaviour of the construction $\mathcal{F}$ with respect to lax transformations, given any locally ordered category B, the locally ordered category $\overline{B} = \mathrm{Oplax}(\mathbf{2}, B)$ has the property that to give a locally ordered functor from any A to $\overline{B}$ is to give a pair of locally ordered functors $\langle G,\, F\rangle$ from A to B, together with a lax transformation $\alpha\colon G \Rightarrow F$, naturally in A. Moreover, it follows from general theory about monads that if B has a T-algebra structure, then $\overline{B}$ acquires a T-algebra structure, and the above universal property lifts to T-Alg.

So, in particular, to give a locally ordered functor from c to $\overline{B}$ is to give a pair $\langle h,\, k\rangle$ of locally ordered functors from c to $\overline{B}$ and a lax transformation $\rho\colon h \Rightarrow k$.

So, given an $\langle S,\, E\rangle$-algebra $\langle B,\, \nu\rangle$ together with, for each $\langle c_i,\, d_i\rangle$ in $\mathcal{D}$, a pair $\langle h_i,\, k_i\rangle$ of locally ordered functors from c_i to B and a lax transformation $\rho\colon h_i \Rightarrow k_i$, with corresponding locally ordered functor $\chi_i\colon c_i \to \overline{B}$, and given an $\langle S,\, E,\, \mathcal{D}\rangle$-sketch $\langle A,\, \{\, \varphi_i\colon d_i \to A\,\}\rangle$, the following correspond bijectively:

- a pair of locally ordered functors $G, F\colon A \to B$, and a lax transformation $\alpha\colon G \Rightarrow F$ such that $G(\varphi_i) = \nu(h_i) \circ j_i$, $F(\varphi_i) = \nu(k_i) \circ j_i$, and $\alpha(\varphi_i) = \nu(\rho_i) \circ j_i$
- an $\langle S,\, E,\, \mathcal{D}\rangle$-sketch morphism from $\langle A,\, \{\, \varphi_i\,\}\rangle$ to $\mathcal{U}(\langle \overline{B},\, \overline{\nu},\, \{\, \chi_i\,\}\rangle)$
- an $\langle S,\, E,\, \mathcal{D}\rangle$-algebra map from $\mathcal{F}(\langle A,\, \{\, \varphi_i\,\}\rangle)$ to $\langle \overline{B},\, \overline{\nu},\, \{\, \chi_i\,\}\rangle$
- a pair $\langle \overline{G},\, \overline{F}\rangle$ of $\langle S,\, E\rangle$-algebra maps from $\mathcal{F}(\langle A,\, \{\, \varphi_i\,\}\rangle)$ to $\langle B,\, \nu\rangle$

and a 2-cell of $\langle S,\, E\rangle$-algebras $\overline{\alpha}\colon \overline{G} \Rightarrow \overline{F}$ respecting the $\mathcal{D}$-structures of $\mathcal{F}(\langle A,\, \{\, \varphi_i\,\}\rangle)$ and the family $\langle h_i,\, k_i,\, \rho_i\rangle$.

Putting this all together, we have

Theorem 7.3 *Every sketch respecting lax transformation*

$$\alpha\colon G \Rightarrow F\colon \langle A, \{\varphi_i\}\rangle \to \mathcal{U}(B, \nu, h, k, \rho)$$

extends uniquely to structure respecting

$$\overline{\alpha}\colon \overline{G} \Rightarrow \overline{F}\colon \mathcal{F}(\langle A, \{\varphi_i\}\rangle) \to \langle B, \nu, h, k, \rho\rangle.$$

8 Contravariance

Finally, we require an account of contravariance. It is well known that the category of small cartesian closed categories does not enrich over Cat. So, it is usual to restrict to enrichment over the cartesian closed category of groupoids. However, when one has locally ordered structure, one can—and Hoare does—do something more delicate, which is to restrict not to lax transformations that are isomorphisms, but to total transformations as follows.

Definition 8.1 *Given locally ordered functors $G, F\colon A \to B$, a* total *transformation from G to F is a lax transformation $\alpha\colon G \Rightarrow F$ such that for every object X of A, $\alpha_X\colon G(X) \to F(X)$ has a left adjoint retraction, i.e., there exists a 1-cell $\ell_X\colon F(X) \to G(X)$ such that $\ell_X \circ \alpha_X = id_{G(X)}$ and $id_{F(X)} \leq \alpha_X \circ \ell_X$.*

It is well known that given the data for α such that each α_X has a left adjoint ℓ_X, then α is lax if and only if ℓ is oplax (see, for instance, [12]). For the analysis of this section, the assumption that the left adjoint be a retraction is unnecessary. However, it is part of Hoare's definition, so we include it. This also corresponds to the common domain theoretic notion of a projection-embedding pair.

Proposition 8.2 *The following data*

- *small locally ordered categories*
- *locally ordered functors*
- *total transformations*

with the evident composition, form a 2-category.

Proof The only non-trivial part is to prove the interchange law; that amounts to showing that, given total transformations $\alpha\colon G \Rightarrow F\colon B \to \mathcal{C}$ and $\beta\colon K \Rightarrow H\colon A \to B$, the diagram

$$\begin{tikzcd} G\circ K \arrow[r, "\alpha\circ K"] \arrow[d, "G\circ\beta"'] & F\circ K \arrow[d, "F\circ\beta"] \\ G\circ H \arrow[r, "\alpha\circ H"'] & F\circ H \end{tikzcd}$$

commutes. Given X an object of A, lax naturality of α shows that we have

$$\begin{tikzcd}
G(K(X)) \arrow[r, "\alpha_{K(X)}"] \arrow[d, "G(\beta_X)"'] \arrow[dr, phantom, "\geq"] & F(K(X)) \arrow[d, "F(\beta_X)"] \\
G(H(X)) \arrow[r, "\alpha_{H(X)}"'] & F(H(X)).
\end{tikzcd} \tag{5}$$

But each β_X has a left adjoint ℓ_X, also a 1-cell in B. So lax naturality of α also asserts that we have

$$\begin{tikzcd}
G(H(X)) \arrow[r, "\alpha_{H(X)}"] \arrow[d, "G(\ell_X)"'] \arrow[dr, phantom, "\geq"] & F(H(X)) \arrow[d, "F(\ell_X)"] \\
G(K(X)) \arrow[r, "\alpha_{K(X)}"'] & F(K(X)),
\end{tikzcd} \tag{6}$$

but since $G(\ell_X) \dashv G(\beta_X)$ and $F(\ell_X) \dashv F(\beta_X)$, it follows that by taking the mates under these adjunctions of (6) we have

$$\begin{tikzcd}
G(H(X)) \arrow[r, "\alpha_{H(X)}"] \arrow[dr, phantom, "\leq"] & F(H(X)) \\
G(K(X)) \arrow[r, "\alpha_{K(X)}"'] \arrow[u, "G(\beta_X)"] & F(K(X)). \arrow[u, "F(\beta_X)"']
\end{tikzcd} \tag{7}$$

Putting (5) and (7) together gives the result.

We call this 2-category Tot.

Proposition 8.3 Tot *is locally finitely presentable as a 2-category.*

Proof Cotensors with $\mathbf{2}$ are as follows: given small locally ordered B, an object of $\mathbf{2} \pitchfork B$ is a reflection in B, i.e., a 1-cell that has a left adjoint retraction. A 1-cell in $\mathbf{2} \pitchfork B$ from h to h' amounts to a square of the form

$$\begin{tikzcd}
b \arrow[r, hook, shift left, "h"] \arrow[d, "f"'] \arrow[r, phantom, "\top"] & c \arrow[l, shift left] \arrow[d, "g"] \\
b' \arrow[r, hook, shift left, "h'"] \arrow[r, phantom, "\top"] & c' \arrow[l, shift left]
\end{tikzcd} \qquad \text{with } g \circ h \leq h' \circ f.$$

Then, $[\,\mathbf{2}, \mathrm{Tot}(A,B)\,] \cong \mathrm{Tot}(A, \mathbf{2} \pitchfork B)$.

The functor $\mathbf{2} \pitchfork -\colon \mathsf{LocOrd}_\circ \to \mathsf{LocOrd}_\circ$ preserves filtered colimits and limits, so has a left adjoint $\mathbf{2} \otimes -$. To describe $\mathbf{2} \otimes B$, first take the tensor in LocOrd_l, then freely make the maps of the form $\langle \mathbf{2},\, b\rangle$ into reflections, so for instance $\mathbf{2} \otimes \mathbf{2}$ looks like

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\cdot \arrow[r] \arrow[d, shift left=2, hook] \arrow[d, phantom, "\dashv"] \arrow[dr, phantom, "\leq"] & \cdot \arrow[d, shift left=2, hook] \arrow[d, phantom, "\dashv"] \\
\cdot \arrow[u, shift left=2] \arrow[r] & \cdot \arrow[u, shift left=2]
\end{tikzcd}.$$

We already know that $\mathsf{LocOrd}_\circ$ is locally finitely presentable. $\Box$

By Proposition 8.3, we have the full strength of our main abstract theorem, Theorem 4.1, immediately at our disposal: the definition of algebraic structure is as before, applied to the $\mathcal{V}$-category Tot, where $\mathcal{V}$ is Cat. Our analysis of sketches extends naturally too: the only property we used of $\overline{B}$ was that it was $\mathbf{2} \pitchfork B$. So we have a full account of Hoare's total simulations in this setting.

Example 8.4 [Closure of a cartesian category] First observe that for any LocOrd_l-enriched algebraic structure $\langle S,\, E\rangle$ on LocOrd_l, $\langle S,\, E\rangle$-$\mathsf{Alg}_\circ$ is finitarily monadic over $\mathsf{LocOrd}_\circ$. It is *not* automatically true, but nonetheless true in all our examples, that for any such $\langle S,\, E\rangle$, it restricts to Cat-enriched algebraic structure on Tot. So we shall only study closure.

To give that, the main data we need is a function that to each pair $\langle X,\, Y\rangle$ provides a diagram of the form

$$\begin{tikzcd}
& {[\,X, Y\,] \times X} \arrow[r, "ev"] \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & Y \\
{[\,X, Y\,]} & & X
\end{tikzcd}.$$

So we add

$$S(2) = \begin{tikzcd}
& \cdot \arrow[r] \arrow[dl] \arrow[dr] & \cdot \\
\cdot & & \cdot
\end{tikzcd}.$$

Then $\mathrm{Tot}(2, A) \to \mathrm{Tot}(S(2), A)$ is to make the above construction. Given reflections $X \mathrel{\substack{\xrightarrow{f} \\ \top \\ \xleftarrow[g]{}}} X'$ and $Y \mathrel{\substack{\xrightarrow{h} \\ \top \\ \xleftarrow[k]{}}} Y'$, we must verify that in a cartesian closed locally ordered category, we obtain a pointwise reflection between the diagrams

$$\begin{tikzcd}[column sep=small] & {[\,X,Y\,] \times X} \arrow[r, "ev"] \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & Y \\ {[\,X,Y\,]} & & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & {[\,X',Y'\,] \times X'} \arrow[r, "ev'"] \arrow[dl, "\pi_0'"'] \arrow[dr, "\pi_1'"] & Y' \\ {[\,X',Y'\,]} & & X' \end{tikzcd}$$

with the right adjoint part lax. The only piece of data that is not completely obvious is the map $[\,X, Y\,] \to [\,X', Y'\,]$ which is to have a left adjoint retraction; we define it to be

$$[\,X, Y\,] \xrightarrow{[\,g, h\,]} [\,X', Y'\,].$$

It has adjoint $[\,f, k\,]$, which is a retract. The laxness of the right adjoint part is trivial (in fact strict for π_0 and π_1); for ev, it amounts to

$$\begin{tikzcd} {[\,X,Y\,] \times X} \arrow[r, "{[\,g,h\,] \times f}"] \arrow[d, "ev"'] \arrow[dr, phantom, "\geq"] & {[\,X',Y'\,] \times X'} \arrow[d, "ev'"] \\ Y \arrow[r, "h"'] & Y', \end{tikzcd}$$

which in fact holds strictly in any cartesian closed category. For the unit, one wants and has exactly

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\geq"] & X' \arrow[d, "\eta"] \\ {[\,Y, Y \times X\,]} \arrow[r, "{[\,k, h \times f\,]}"'] & {[\,Y', Y' \times X'].} \end{tikzcd}$$

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